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PAPER P-1197

ON MAXMIN AND MINMAX STRATEGIES IN MULTI-STAGE GAMES AND ATACM

Lowell Bruce Anderson
Jerome Bracken
James E. Falk
Jeffrey H. Grotte
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August 1976

IDA INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

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400 Army-Navy Drive, Arlington, Virginia 22202

IDA Independent Research

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER P-1197	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On MaxMin and MinMax Strategies in Multi-Stage Games and ATACM		5. TYPE OF REPORT & PERIOD COVERED Final 6. PERFORMING ORG. REPORT NUMBER P-1197
7. AUTHOR(s) Lowell Bruce Anderson, Jerome Bracken, James E. Falk, Jeffrey H. Grotte, Eleanor L. Schwartz		8. CONTRACT OR GRANT NUMBER(s) IDA Independent Research
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Defense Analyses Program Analysis Division 400 Army Navy Dr., Arlington, VA 22202		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE August 1976 13. NUMBER OF PAGES 60
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

This document is unclassified and suitable for public release.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Aircraft Allocation to Missions, Multi-Stage Games, Game Theory, General Purpose Forces, Tactical Air Forces, MaxMin, MinMax, Optimal Strategies, MaxMin Strategies, MinMax Strategies

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The purpose of this paper is to discuss some aspects of the use of MaxMin and MinMax strategies in the analysis of multi-stage games. This discussion is motivated by the development of the ACDA Tactical Air Campaign Model (ATACM). ATACM proposes the use of approximate MaxMin and MinMax strategies instead of optimal mixed

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) (equilibrium) strategies. Following the introduction, Chapter II gives a critique of the MaxMin, MinMax approach as implemented in ATACM. Chapter III treats several computational aspects of true MaxMin and MinMax strategies; and Chapter IV gives a result concerning MaxMin and MinMax values in adaptive and nonadaptive games.

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APPENDICES

I. INTRODUCTION

Lowell Bruce Anderson and Jerome Bracken

The purpose of this paper is to discuss some aspects of the use of MaxMin and MinMax strategies in the analysis of multi-stage games. This discussion is motivated by the development of the ACDA Tactical Air Campaign Model (ATACM), which is documented in References [9] and [10]. ATACM proposes the use of approximate MaxMin and MinMax strategies instead of optimal mixed (equilibrium) strategies. Chapter II, below, gives a critique of the MaxMin, MinMax approach as implemented in ATACM. Chapter III treats several computational aspects of true MaxMin and MinMax strategies; and Chapter IV gives a result concerning MaxMin and MinMax values in adaptive and nonadaptive games. A more detailed summary of these chapters follows.

Chapter II is a rather comprehensive critique of ATACM. The principal criticisms go to the heart of the MaxMin, MinMax approach as it is implemented in ATACM. First, if one player plays a conservative strategy, the other player upon observing this can drive the outcome down towards the payoff corresponding to that conservative strategy. Optimal mixed strategies, on the other hand, yield expected results that cannot be driven down towards a conservative payoff. Second, conservative strategies may give results that are insensitive to important force structure changes, while optimal mixed strategies could properly reflect the importance of these changes. And third, if the game being modeled is inherently stochastic (which air combat is), then MaxMin and MinMax strategies also yield only expected results, not guaranteed bounds as claimed in ATACM. Two other aspects of ATACM are also criticized: (1) the ATACM approximation procedure, and (2) the ATACM assessment procedure. Finally, some suggestions for improvement are made.[1]

If the optimal mixed strategies and game value were known, then knowing the true MaxMin and MinMax strategies could be useful additional information. The OPTSA models (References [3], [4], and [5]) calculate the optimal strategies and game values for the games they address; but, as currently programmed, they cannot calculate the MaxMin or MinMax strategies. Chapter III treats several aspects of the computation of MaxMin and MinMax strategies in multi-stage games. A new method for finding MaxMin and MinMax strategies for one-stage games is proposed. Computation of exact MaxMin and MinMax strategies for multi-stage games is discussed, and computation times are estimated.

MaxMin and MinMax strategies can be considered for several types of games, two of which are: (a) nonadaptive games, and (b) behavioral games, which in Reference [13] are shown to be equivalent to adaptive games. The relationship between the MaxMin and MinMax strategies of an adaptive game and the MaxMin and MinMax strategies of the corresponding nonadaptive game is discussed in Chapter IV.

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[1]It is appropriate to remark that ATACM shares with DYGAM (References [6] and [7]) the characteristic that the stated problem can be solved rigorously except for approximation error. Counterexamples for other models not having this guaranteed optimization philosophy are provided in References [8] and [11].

II. A CRITIQUE OF ATACM

Lowell Bruce Anderson

The ACDA Tactical Air Campaign Model (ATACM) is described in References [9] and [10]. Reference [9] claims that the specific features of ATACM are that ATACM permits

(1) as many as four user-defined aircraft types per side and as many as eight different missions per aircraft type;

(2) automatic generation of approximate, optimal enforceable aircraft allocation strategies as a function of stage for any subset of the missions for which user-specified fractions are not supplied;

(3) calculation of firm upper and lower bounds on the objective function value associated with the enforceable strategies employed;

(4) the option to use a weighted sum of three different objective functions as the criterion for generating the optimal strategies;

(5) the option to individually weight the Blue and Red contributions to these objective functions as a function of stage; and

(6) the option to specify fractional or numerical reinforcements for any aircraft type as a function of stage.

If the procedures used in ATACM provide useful information, then feature (1) above would be very significant and important, and would make ATACM the premier model in its field. On the other hand, feature (1) is largely irrelevant if ATACM does not provide useful information. Whether ATACM provides useful information or not depends on what one means by useful information, which in turn largely depends on the definitions and interpretations associated with features (2) and (3), and on the acceptability of the assessment methodology. In this paper, we concentrate on the definitions and assumptions associated with features (2) and (3) because the assessment methodology of ATACM could be changed if warranted.[1]

The key terms in feature (2) are "approximate, optimal, enforceable ... strategies." These terms are not directly defined in Reference [9]. However, it is clear from the details of References [9] and [10] that by "optimal enforceable strategies," the developers of ATACM mean MaxMin and MinMax strategies. This distinction is important because the standard definition of "optimal strategies" for a two-person zero-sum game (like the game in ATACM) is that optimal strategies are the (possibly mixed) equilibrium strategies.[2] Thus, the claim of feature (2) is, at best, misleading. Properly phrased, feature (2) should be stated as:

(2') ATACM generates strategies that are, in some sense, approximations to the MaxMin and MinMax strategies of the game played in ATACM.

This revised statement of feature (2) raises two questions: How worthwhile is it to generate MaxMin and MinMax strategies in lieu of optimal (equilibrium) strategies, which ATACM cannot generate? How good are ATACM's approximations to the MaxMin and MinMax strategies? The second question is related to feature (3), which claims that ATACM calculates firm upper and lower bounds on (properly phrased) the payoffs produced by the MaxMin and MinMax strategies. By "firm bounds" the developers of ATACM apparently mean true bounds, not tight

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[1] A few specific changes to the assessment methodology will be suggested below. Of course, a sufficient amount of changes in the assessment methodology could increase the computer running time to the point where it is no longer practical to use ATACM. Thus, the blanket statement above that "the assessment methodology could be changed" is, in general, an oversimplification.

[2] Accordingly, throughout this chapter we will use the term "optimal strategies" to mean the (possibly mixed) equilibrium strategies.

bounds. But it is trivial to calculate true bounds if one does not care how tight these bounds are, and true but very loose bounds can be quite useless.

In Section A we will discuss the limitations of considering only MaxMin and MinMax strategies, as is done in ATACM. In Section B we will discuss the approximation procedures used in ATACM. In Section C we will discuss some limitations of the assessment procedure used in ATACM. Finally, in Section D we will make a suggestion that might make ATACM a more useful model, provided that this suggestion can be implemented without greatly increasing the computer running time.

A. SOME ADVANTAGES AND LIMITATIONS OF MAXMIN AND MINMAX

MaxMin, MinMax, and optimal (equilibrium) strategies are equivalent for any two-person, zero-sum game with a saddlepoint. So for this section (only) suppose that the game under consideration does not have a saddlepoint.

1. A Discussion of the Claimed Advantages of MaxMin and MinMax

Two advantages of considering MaxMin and MinMax strategies, as opposed to optimal mixed strategies, are claimed in Reference [10]. These two advantages are summarized as follows: (1) MaxMin and MinMax strategies are pure strategies, and many military commanders might abhor the concept of randomization to decide each day's aircraft assignment. (2) The "game" of a war in Europe will be "played" once at most. Optimal mixed strategies guarantee to each side only that the side's expected payoff will not be less than a specific amount (the value of the game). The actual payoff to either side is a random variable which may be above or below the expected payoff. On the other hand, conservative play (MaxMin or MinMax, as appropriate) will guarantee to each side an actual payoff that is greater than or equal to the worst the side could receive with optimal mixed strategies although not as good as the expected payoff from optimal mixed strategies.

We believe that these claimed advantages are not as great as they first might appear. First, while commanders might not flip a coin to decide how to allocate their aircraft, they would attempt to avoid making decisions in a completely predictable manner. Indeed, they would attempt to exploit an enemy's predictability and they might even attempt to set up and fake out an enemy.[1] Playing optimal mixed strategies is not a perfect way to model each side's attempt to exploit his enemy's predictability and surprise him when appropriate. However, playing mixed strategies seems to us to be a better way to reflect these characteristics of war rather than playing that each side uses his conservative pure strategy throughout the war. Thus, if there is a significant difference between the MaxMin value and the MinMax value (so that there is much to be gained by surprise), playing optimal mixed strategies may well be more realistic, not less realistic, than playing conservative MaxMin and MinMax strategies.

The second argument above, that MaxMin and MinMax strategies are more appropriate than optimal mixed strategies for a game (or war) that will be played (or fought) only once, is a long-standing point of discussion in game theory. If the payoffs to each player satisfy typical axioms for utilities (such as in von Neumann and Morgenstern, Reference [18], or as in Luce and Raiffa, Reference [15]), then the situation is clear: optimal mixed strategies are more appropriate than MaxMin and MinMax strategies. For example, if the payoffs in the example of Reference [10] are in terms of utilities to Blue, then Blue is indifferent between an expected payoff of 3.4 and a certain payoff of 3.4, and he prefers either to a certain payoff of 2.0.

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[1]"Surprise" is a *principle of war*, "conservative play" is not.

The problem is that it is relatively much easier to model a physical occurrence such as Blue minus Red firepower delivered, than it is to determine Blue's and Red's utilities for delivering firepower.

What one should do if it is not known whether the payoffs satisfy the axiomatic conditions of utilities is not clear, and we will certainly not resolve that issue here. But there is an intuitive belief that if a game is played many times and no one play of the game strongly affects the end result, then the conditions of utility are "approximately satisfied" and the expected payoff is a reasonable measure. (The developers of ATACM agree with this intuition in Reference [10].) On the other hand, if the game is played only once and there are significant differences in possible outcomes, then the conditions of utility might not be satisfied.[1]

While the war will be fought (at most) once, aircraft allocation decisions will not be made only once. A commander could decide to re-allocate his aircraft for each raid on each day of the war, and he could make one allocation in one part of the theater and another in another part. For example, a 30-day war with three raids per day into two areas of the theater could result in 180 allocations. Thus, as in many plays of one game, the commander has many distinct allocation decisions. So in this sense, the sequential game of aircraft allocation is intuitively similar to many plays of one game and expected payoff would be the preferred measure. On the other hand, it is possible that an "unlucky" decision on the first raid of the first day could have a dominant impact on the rest of the war,

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[1]It should be noted that this structure is intuitive, not formal, because a game played many times can be thought of as one large game with many sub-games inside of it, and the large game is then played only once. But given this intuitive structure (as opposed to a formal structure), one can ask how the sequential game of aircraft allocations fits into the intuition.

no matter what decisions are made for the rest of the war. To the extent that this dominance can occur, a sequential game is intuitively similar to one play of one game. Considering only this argument, expected payoff may not be clearly preferred over MaxMin and MinMax as a measure of effectiveness, but it is not clearly inferior either. Accordingly, the validity of the second claimed advantage of MaxMin and MinMax over optimal mixed strategies is also in doubt.

Another facet of the second claimed advantage of the MaxMin, MinMax approach is that, while the assessment portion of ATACM is deterministic, actual combat is not deterministic. Thus, even if the entries in the payoff matrix are truly the expected outcomes of an air war, the MaxMin of these entries is the MaxMin of expected results, and the MinMax is the MinMax of expected results. Accordingly, whether a commander plays a MaxMin strategy or a mixed strategy, all he can count on is an expected result of non-deterministic combat, not a certainty. Either way, to quote from Reference [10], "...the outcome is enforceable by the two sides only in an expected value sense," and so this claimed advantage of MaxMin and MinMax over optimal mixed strategies is really no advantage at all.

2. Some Limitations of MaxMin and MinMax

The discussion above attempts to counter the two claimed advantages for considering MaxMin and MinMax strategies in lieu of optimal mixed strategies. Combining these counter arguments gives a major limitation of using MaxMin and MinMax strategies: *A side will try to exploit its enemy's predictability if it has the opportunity and there is a payoff from doing so.* If one side sticks to its conservative play MaxMin strategies, then the other side could observe this over the course of the war and allocate its aircraft specifically against that MaxMin strategy (instead of using its own MinMax strategy).

This would push the payoff to the MaxMin side down toward the MaxMin value.

The hypothetical example of Reference [10] is given below:

		Red		
		CAS	ABA	INT
Blue	CAS	4	1	6
	ABA	3	5	2
	INT	2	0	2

	.8	.2	0	
.4	4	1	6	1
.6	3	5	2	2
0	2	0	2	0
	4	5	6	

MaxMin = 2, MinMax = 4, expected value = 3.4

If Blue plays his MaxMin strategy (ABA) on each day of the war, then Red can observe this and fly INT instead of CAS. This strategy may push the payoff to Blue towards 2.0. If so, Blue will have to either accept a payoff nearer 2.0 or fly CAS instead. How close the payoff is to 2.0 depends on when Red starts flying INT and on the details of the assessment procedure. But it is not at all clear that Red would see Blue flying ABA each day and yet never fly INT. And it is not at all clear that if Red started flying INT, Blue would still continue to fly ABA (in the example game).

In summary, given that each side must make a decision day after day, it is not reasonable that one side would see the other play its conservative strategy and not adjust its own strategy accordingly to drive the payoff toward the MaxMin (or MinMax) value. Playing mixed strategies may be a reasonable way to model the case where both sides continually adjust their strategies to anticipate the moves of the other.

A second limitation of considering MaxMin and MinMax strategies is that, since these are pure strategies, they may be insensitive to important force structure changes. For example, suppose that two force structures are under consideration and for both force structures the Blue MaxMin strategy is to fly all ABA while the optimal mixed strategy is to fly both CAS and ABA. Suppose further that the second force structure results in a slightly lower payoff from flying ABA but a much greater payoff from flying CAS. Then the MaxMin structure would prefer the first force structure over the second. A numerical example, following Reference [10], is as follows:

BLUE FORCE STRUCTURE NO. 1

		CAS	ABA	INT	
	CAS	4	1	6	1
Blue	ABA	3	5	2	2
	INT	2	0	2	0
		4	5	6	

MaxMin = 2, MinMax = 4, MaxMin vs. MinMax Payoff = 3
Optimal mixed payoff = 3.4

BLUE FORCE STRUCTURE NO. 2

		CAS	ABA	INT	
	CAS	400	0	600	0
Blue	ABA	2	500	1	1
	INT	1	0	1	0
		400	500	600	

MaxMin = 1, MinMax = 400, MaxMin vs. MinMax Payoff = 2
Optimal mixed payoff > 200

In the above example, force structure 1 has both a better MaxMin and a better MaxMin versus MinMax payoff to Blue than force structure 2 does, yet force structure 2 would appear to be much better for Blue.

The numerics of the above example are not important. What is important is that a MaxMin, MinMax approach can overlook the capability of general purpose aircraft to fly any one of several missions without the enemy knowing in advance which mission will be flown. Accordingly, the MaxMin, MinMax approach can give an unrealistic advantage to a special purpose aircraft that might be only slightly better on one mission and much worse on all other missions than an alternative general purpose aircraft. It may even be possible that if a special purpose aircraft is bought by the MaxMin side in place of a general purpose aircraft, then that side's enemy might more easily force the outcome of the war down towards the MaxMin value.

Finally, there is the problem of how one uses the MaxMin and MinMax strategies and values. If an analyst is comparing two force structures, he might prefer a force with much higher MaxMin value and a slightly lower game value when compared with an alternative force. However, ATACM does not permit such a comparison because it cannot compute the optimal (mixed) strategies or the game value. Instead, the developers of ATACM seem to suggest considering the value procedure by playing the MaxMin strategy versus the MinMax strategy (conservative play on both sides). But this "conservative play payoff" does not depend on any of the possible payoffs of the game (except for itself), other than that it must be above the MaxMin payoff and below the MinMax payoff. That is, changing one entry in the (complete) game payoff matrix can make this payoff as high as the MinMax or as low as the MaxMin. It seems to us that this "conservative play payoff" is an arbitrary number and the only justification for considering this payoff as a measure is the one implied by the developers of ATACM: that the two commanders would actually use MaxMin and MinMax strategies. We believe this to be a weak argument for the reasons given above.

B. LIMITATIONS OF THE ATACM APPROXIMATION PROCESS

ATACM makes two approximations in calculating the MaxMin and MinMax values, and it gives the strategies that produce these approximations. Three points concerning these approximations are important.

First, these two approximations are made whether or not the game has a saddlepoint (and so the discussion below applies whether or not the game has a saddlepoint).

Second, the approximate values are not necessarily close to the true MaxMin or MinMax values. Let m and M denote the true MaxMin and MinMax values, respectively, of the game; and let $\overline{m}$ and $\overline{M}$ denote the ATACM approximations to these values. It is true that

$$\overline{m} \leq m \leq M \leq \overline{M} .$$

Thus, if $\overline{m}$ is close (in some sense) to $\overline{M}$, then not only are the approximations of $\overline{m}$ to m and $\overline{M}$ to M good, but m and M are close to each other and so are necessarily close to the game value (produced by the optimal, possibly mixed, strategies) which lies between them. Thus if, for a particular case, $\overline{m}$ is close to $\overline{M}$, then all the arguments (pro and con) dealing with MaxMin and MinMax in Section A are irrelevant, and ATACM produces useful information for that case. However, if $\overline{m}$ is not close to $\overline{M}$, then ATACM cannot indicate the relative closeness of $\overline{m}$, m, M, and $\overline{M}$. It may be that $\overline{m}$ is close to m and $\overline{M}$ is close to M, but m is not close to M; or it may be that m = M and the approximations are terrible; or it may be that none of the quantities are close to any other.

Third, the strategies produced by ATACM are not the MaxMin and MinMax strategies. The strategies produced are those used to obtain $\overline{m}$ and $\overline{M}$. In general, the relationship between those strategies and the true MaxMin and MinMax strategies is not clear. Thus, when $\bar{m}$ is not close to $\bar{M}$, $\bar{m}$ may not be close to m, and the ATACM "approximate MaxMin" strategy may not be similar to the true MaxMin strategy (the same holds for the MinMax strategies).

The two approximations made in ATACM are discussed on pages 30 through 34 of Reference [9]. We have no rigorous basis for commenting on the first approximation, using linear interpolation to generate a first pass approximated strategy, but intuitively this seems like a reasonable approximation. However, as Reference [9] states, a second approximation is required to obtain the strategies that yield $\bar{m}$ and $\bar{M}$. This second approximation is where significant inaccuracies can occur.

If Blue is the MaxMin player, the second approximation requires, for the computation of the "approximate MaxMin" strategy, that at several stages Blue resources be rounded down to the nearest grid points and that Red's resources be rounded up to the nearest grid points. (The grid points form a grid over Blue and Red inventories, and the location of the grid points are input to ATACM.) In the example given in Appendix A of Reference [9], Blue has two types of aircraft, Red has one type; the grid points are at 0, 333, 667, and 1000 for Blue plane type 1 and at 0, 200, and 400 for Blue plane type 2--Red's grid points are at 0, 400, 800, and 1200 for its plane type 1. Thus, in the extreme case, if ATACM were computing the MinMax value and Blue had 332 aircraft of type 1 and 199 aircraft of type 2 while Red had 401 aircraft of type 1, ATACM would round this to no Blue aircraft of either type and to 800 Red aircraft. We suspect that this type of approximation would generally lead to very poor bounds. In Section D we suggest an improvement to this type of approximation.

Results of three ATACM runs are given on pages A-33 through A-41 of Appendix A, Reference [9]. Our interpretation of these runs may be in error, but it appears as if the three runs were made with the same inputs except for the initial number of Blue and Red aircraft. The measure of effectiveness used is cumulative Blue air plus ground firepower minus cumulative Red air plus ground firepower for a two-day (stage) war.[1] The results are:

Trial	Red MinMax minus Blue MaxMin ($\bar{M} - \bar{m}$)
1	20,250
2	11,042
3	16,072

The units of $\bar{M}$ and $\bar{m}$ are important, and they appear to be closer to tons rather than ounces of firepower. For example, in the ATACM data the aircraft of type 1 deliver 2 units of CAS firepower per sortie and they have a sortie rate of 2.0 on CAS mission. Thus it would take over 2,531 Blue aircraft of type 1 to deliver 20,250 units of firepower in two days, assuming no attrition whatsoever. (But for Trial 1, there are only 400 Blue aircraft of type 1 in the theater on day 1, with 100 replacements on day 2, and Blue aircraft of type 2 cannot deliver CAS firepower.) Similarly, it would take over 1,687 Red aircraft to deliver 20,250 units of firepower in two days with the data in ATACM, assuming no attrition. (But for Trial 1, there are only 500 Red aircraft in the theater on day 1 with 200 replacements on day 2.)[2]

Based on the arguments above, we believe that $\bar{M} - \bar{m}$ is a reasonable measure of whether or not ATACM produces useful

[1] Blue's air plus ground firepower is weighted by 1/2 on day 1 and the value of residual Blue and Red aircraft of type 1 is 8 units of firepower (Blue aircraft of type 2 have no residual value).

[2] With bounds this wide, it seems to us that no meaning whatsoever can be attached to the value produced by the "approximate MaxMin" strategy versus the "approximate MinMax" strategy.

information. Clearly the value of $\bar{M} - \bar{m}$ depends on the data used as well as on the model, but for the sample data given in Appendix A of Reference [9] it appears that ATACM does not produce useful information.

C. LIMITATIONS OF THE ATACM ASSESSMENT PROCESS

It should be noted that developers of ATACM did not intend to make major contributions to the (assessment) modeling of air combat. For example, page 12 of Reference [9] states: "In consideration of these limitations, ATACM is purposely structured so that other, alternative assessment methodologies can be implemented with minimal programming effort." ATACM uses features from other models in order to build an assessment methodology on which the optimization methodology could operate. Accordingly, there are no significant new features in the assessment methodology of ATACM. As mentioned in our introduction, we have not thoroughly reviewed the assessment methodology of ATACM, and so the limitations and suggestions for improvement given below are not necessarily complete; however, they should be relatively easy to implement in the ATACM computer program.

1. Engagements

It may be that no one ordered set of engagements for air combat is absolutely correct, and it may be that several different orderings of different engagements are reasonably acceptable. However, the particular set of engagements played in ATACM has three logical deficiencies which might render it unreasonable. These deficiencies are that--

(1) Air Base Defense (ABD) aircraft cannot engage enemy Rear (i.e., air base) SAM Suppressor (RSS) aircraft, and Battlefield Defense (BD) aircraft cannot engage either enemy RSS aircraft or enemy Forward (i.e., battlefield) SAM Suppressor (FSS) aircraft. ABD aircraft should be able to engage enemy RSS aircraft and BD aircraft should be able to engage both enemy RSS and enemy FSS aircraft.

(2) BD aircraft cannot engage enemy ABA aircraft. Perhaps there is some way for ABA aircraft to reach enemy air bases by not going over the FEBA; but this should be an option instead of a fixed rule in the model, and if the ABA aircraft are not vulnerable to enemy BD aircraft, then it is hard to see why ATACM assumes they are vulnerable to enemy forward (battlefield) SAMs. BD aircraft should be able to engage enemy ABA aircraft, and likewise ABAE (ABA Escort) aircraft should be able to engage enemy BD aircraft.

(3) In ATACM, Blue forward SAMs shoot at Red CAS aircraft, then Blue BD aircraft shoot at Red CAS aircraft, then Red CAS aircraft deliver ordnance on ground units. But if the Blue forward SAMs are near (or behind) the ground units they are protecting, it is hard to see how air-to-air engagements between CAS and BD aircraft could occur between the time that the SAMs shoot at the CAS aircraft and the time that the CAS aircraft shoot back at the units the SAMs are defending. This order and the corresponding order for ABA and ABD aircraft and rear SAMs should be changed.

2. Attrition Equations

Our main comment on the equations in ATACM is that these equations have been pieced together out of various other air combat models and no justification is given in Reference [9] as to why these equations, and not others, were selected for use in ATACM. Page 16 of Reference [10] gives one sentence of justification, namely: "Initially we propose using the more detailed VECTOR equations for most interactions to be modeled." But it is not clear in Reference [10] what is meant by "detail," why detail is desired, which equations the VECTOR equations are more detailed than, which interactions the VECTOR equations should not be used for, and what equations should be used for these other interactions. We believe that the key word in the above sentence from Reference [10] is "Initially." We recommend to a potential user of ATACM that he not accept the equations initially used in ATACM (we believe they have several important limitations), and that he select, from the various types of attrition equations that are available, those which he feels are most appropriate for the interactions modeled in ATACM. Various suitable attrition equations can be found in References [1], [2], [3], [12], and [14]. Section (2) of Reference [1] gives a reasonable menu of homogeneous attrition equations suitable for modeling air combat. The other references provide details (such as how to handle heterogeneous forces) on these equations.

Finally, we note that (a) the method for considering sortie rates described on page 17 of Reference [4] will not work if the sortie rates are less than 1.0 (and it is only an approximation if the sortie rates are greater than 1.0) (see Reference [3] for details); (b) aircraft in ATACM are sheltered proportionally, not by priority; (c) ABA aircraft in ATACM know which shelters are empty and they attack only full shelters (however, aircraft in the open are apparently not attacked preferentially over aircraft in shelters); and (d) ATACM does not play Quick Reaction Alert aircraft.

D. A SUGGESTION FOR IMPROVING ATACM

If the assessment methodology were sufficiently improved, we would have only one criticism of ATACM; namely, the MaxMin, MinMax methodology of ATACM is meaningless if $\overline{M} - \overline{m}$ is large. And $\overline{M} - \overline{m}$ must necessarily be large for those sets of input data for which the difference between the true MinMax and true MaxMin values (M-m) is large (i.e., the games being played do not have saddlepoints, and are not close to having saddlepoints). For these sets of data, we believe that the MaxMin, MinMax approach of ATACM should not be used. Either a more aggregated model such as OPTSA (Reference [3]) is required, or the grid points should be used to develop approximate optimal strategies in a manner similar to that described in Reference [7].

Since it is generally not known in advance whether or not $M - m$ is small, we recommend either using OPTSA or implementing concepts similar to those described in Reference [7] over improving the MaxMin, MinMax approach in ATACM.[1] However, if for a particular set of input data M - m were small, then ATACM would be a worthwhile model to determine optimal strategies (for these data) provided that--

(1) $\overline{M} - \overline{m}$ were shown to be small, and

(2) the computer running time of ATACM were not excessive.

But, we believe that even if M = m, $\overline{M} - \overline{m}$ will not usually be small if the rounding procedure described here (and on page 32 of Reference [9]) is used.

Using the notation defined in Reference [9],[2] as well as new notation defined below, we suggest the following alternative to ATACM's rounding procedure. The rounding procedure is used in ATACM to compute a bound on $TP_{t+1}(Z_{ij}(X_t))$. $TP_{t+1}(Z_{ij}(X_t))$ could not be computed directly because $Z_{ij}(X_t)$ is not in general a grid point. So $Z_{ij}(X_t)$ is rounded to nearest "lower right" grid point, denoted by $\hat{Z}_{ij}(X_t)$, and $TP_{t+1}(\hat{Z}_{ij}(X_t))$ is used in place of $TP_{t+1}(Z_{ij}(X_t))$. We recommend "rounding the strategies" but not "rounding the resources." That is, let

$S^r_t(X_t)$ = Red's one-stage MinMax strategy corresponding not to X_t, but to the grid point *closest* to X_t (not necessarily the "upper left" grid point).

Let ATACM make a "first pass" as described on pages 30 and 32 of Reference [9]. On the second pass compute $TP_{t+1}(Z_{ij}(X_t))$

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[1]The description of the DYGAM model in References [6] and [7] emphasizes using the grid point approximations to allow many stages to be played. We believe that using approximations to allow playing multiple aircraft types and multiple missions per aircraft (as claimed in the features of ATACM) and playing range-payload tradeoffs are relatively much more important for air combat models.

[2]Except that we replace the erroneous notation $TP(X_t)$ and $\hat{S}(X_t)$ with $TP_t(X_t)$ and $\hat{S}_t(X_t)$, respectively.

by having Blue play his first pass strategy ($\hat{S}$) and by Red playing $S^r_{t+1}(Z_{ij}(X_t))$ on period t+1, and playing $S^r_{t+2}(X^r_{t+2})$ on period t+2, where

$$X^r_{t+2} = Z_{\hat{S}_{t+1} S^r_{t+1}(Z_{ij}(X_t))}(Z_{ij}(X_t)) ,$$

and playing $S^r_{t+3}(X^r_{t+3})$ on period t+3, where

$$X^r_{t+3} = Z_{\hat{S}_{t+2} S^r_{t+2}(X^r_{t+2})}(X^r_{t+2}) ,$$

and so on to the end of the war. That is, for each grid point in each period, the war would have to be re-fought forward to its end in order to calculate the payoff (in the same manner as the war is fought forward to determine the MaxMin versus MinMax payoffs as described on page 34 of Reference [9]).

This suggestion requires three comments. First, while it will not necessarily produce true bounds on the MaxMin and MinMax values of the game described in Reference [9], it will produce true bounds on the game where the commanders are required to make the same allocations when between grid points as they do on the nearest grid point. For example, if the grid points are 0, 400, 800, and 1200, and a commander has 548 aircraft, he thinks: "548 is about 400, so I'll do what is optimal for 400 aircraft" (but the assessments are calculated using 548 aircraft). Thus, true bounds would be obtained for a slightly revised game.

Second, this suggestion will not necessarily make $\bar{M} - \bar{m}$ small, even if M = m, because of the first pass approximation.

Third, this suggestion may violate condition (2) above, i.e., it may make the computer running time of ATACM excessive. But it is the only way that we currently see to enable ATACM to make $\bar{M} - \bar{m}$ small, given that M - m is small.

III. COMPUTATION OF MAXMIN AND MINMAX STRATEGIES IN MULTI-STAGE GAMES

Eleanor L. Schwartz

This chapter considers multi-stage games, MaxMin and MinMax strategies, and computation time. The OPTSA multi-stage game models are documented in detail in References [3] and [4]; the reader is assumed to have some familiarity with them. The framework is a game with a specified number of decision periods; at the beginning of each decision period Blue and Red each (simultaneously) make a decision from an input list of strategies. A stream of such decisions made by both players over the course of the game leads to a final payoff. Methods for finding decisions that in some sense optimize the payoff are sought; traditionally, an equilibrium point in randomized strategies has been sought. The idea of finding MaxMin and MinMax values and strategies as enforceable bounds on the final payoff has been proposed and implemented in ATACM (Reference [9]).

At all times the amount of computation needed to process a game must be considered. Models such as DYGAM (References [6] and [7]) and ATACM find approximations (that might or might not be close) to the randomized strategy equilibrium point and MaxMin and MinMax values, respectively, in order to process games of more stages than OPTSA can process. This approximation is made possible by computing in the state space of aircraft resources, rather than the space of possible decisions.

Section A derives an efficient way to find MaxMin and MinMax strategies of matrix games, bringing together the game-solving method of the Revised OPTSA Model (Reference [3]) with

MaxMin and MinMax ideas. Section B discusses MaxMin and MinMax strategies in multi-stage games. Section C derives computation time formulas for finding MaxMin and MinMax strategies for the game encountered in the OPTSA models.

A. COMPUTATION OF MAXMIN AND MINMAX STRATEGIES AND VALUES OF ONE-STAGE GAMES

The Revised OPTSA Model uses a game-solving algorithm that is able to find optimal randomized strategies for one-stage matrix games without computing all the payoff entries. As a result the total running time is greatly reduced from previous versions. This method can be adapted to find MaxMin and MinMax strategies of a matrix game without computing all the payoff entries.

The problem of finding the optimal randomized Blue strategy can be formulated as the linear program

$$\begin{aligned} &\text{maximize } \sigma \\ &\text{s.t.} \quad \sigma \leq \sum_{i=1}^{m} a_{ij} x_i , \quad j = 1 \text{ to } n \\ &\qquad \sum_{i=1}^{m} x_i = 1 \\ &\qquad x_i \geq 0 , \quad i = 1 \text{ to } m , \end{aligned} \tag{1}$$

where a_{ij} is the payoff to Blue when Blue chooses pure strategy i (1 to m) and Red chooses pure strategy j (1 to n), and x_i is the probability Blue plays pure strategy i. In the Revised OPTSA Model this LP is solved by considering a subset of the first n constraints, solving the relaxed LP, and checking the solution to see if any constraints not considered are violated. If none are, the current solution is optimal for the whole game. Otherwise the most violated constraint is added to the constraint set and the procedure repeated.

The MaxMin problem can be formulated as

$$
\begin{aligned}
& \text{maximize } \sigma \\
& \text{s.t.} \quad \sigma \leq \sum_{i=1}^{m} a_{ij} x_i , \qquad j = 1 \text{ to } n \\
& \qquad \sum_{i=1}^{m} x_i = 1 \qquad\qquad\qquad\qquad (2) \\
& \qquad\qquad x_i = 0 \text{ or } 1 , \quad i = 1 \text{ to } m.
\end{aligned}
$$

The additional constraints that the x_i be integer have been added to the original LP; therefore the optimal objective function value will be lower. Exactly one x_i will equal 1; the corresponding i is Blue's MaxMin strategy.

This integer program can be solved by an algorithm similar to the relaxation technique above. The difference is that in problem (1) each relaxed LP was solved by the dual simplex method; here a simple comparison test can be used. The algorithm is as follows. Let $J_0 \subset \{1,...,n\}$ be the set of Red pure strategies being considered at the current iteration, i_0 the current Blue optimal strategy, and σ_0 the current objective function value to the relaxed problem (2). This solution satisfies the constraints in J_0, i.e., for $j \varepsilon J_0$,

$$\sigma_0 \leq \sum_{i=1}^{m} a_{ij} x_i = a_{i_0 j} x_{i_0} = a_{i_0 j} .$$

Since

$$\sum_{i=1}^{m} a_{ij} x_i = a_{i_0 j} ,$$

for the feasibility test we only need to compute the elements

$$\{a_{i_0 j}, \ j \varepsilon J_0'\}$$

where

$$J_0' = \{1,\ldots,n\} - J_0 ,$$

and compare each element with σ_0. If

$$\sigma_0 \leq \sum_{i=1}^{m} a_{ij}x_i = a_{i_0 j}x_{i_0} = a_{i_0 j}$$

for each $j \in J_0'$, i.e., the current solution is also feasible for the constraints not explicitly considered, problem (2) has been solved, as addition of new constraints cannot increase the objective function value.

Suppose there is some $j_1 \in J_0'$ such that $\sigma_0 > a_{i_0 j_1}$. Let us assume also that

$$(\sigma_0 - a_{i_0 j_1}) \geq (\sigma_0 - a_{i_0 j}) , \quad j = 1 \text{ to } n ,$$

i.e., j_1 is the constraint most violated by the current solution. We can compute the column j_1 and augment the constraint set by letting $J_1 = J_0 \cup \{j_1\}$. We now solve the relaxed problem (2) with constraint set J_1. Let us assume that a minimum value, denoted by m_i^0, of each row over the columns of J_0 has been kept. Each element of column j_1 is tested in turn. If $a_{i_1 j} < m_i^0$, then $m_i^1 = a_{i_1 j}$; if not, $m_i^1 = m_i^0$. That is, given the minimum of each row over the set of columns J_0, we find the minimum of each row over the set of columns J_1. The maximum element of this set of row minima becomes the new optimal objective function value σ_1, and the row i_1 in which it appears becomes the new MaxMin strategy. The feasibility test is then performed again; stopping if σ_1 and i_1 are optimal for the whole problem; otherwise the procedure is repeated.

Problem (2) is easier to solve than problem (1) in that no simplex pivots are done, merely a comparison test. However, recall that solving problem (1) also found the Red optimal randomized strategy from the dual variables at optimality. The Red MinMax strategy and value cannot be directly found by solving problem (2). A similar but separate problem must be solved. By suitable storage of payoff entries, however, duplications in payoff computation can be avoided.

An example of the new method of solving problem (2) will clarify things. This game, where $m = 4$ and $n = 6$, was used as an example in the Revised OPTSA Model documentation. There is no saddlepoint. Looking at the whole matrix, the Blue MaxMin strategy is B4, with value 4.

	R1	R2	R3	R4	R5	R6
B1	5	8	1	2	4	5
B2	5	11	3	1	1	7
B3	3	3	4	6	1	7
B4	6	10	5	4	8	6

Step 1a. Arbitrarily let R1 be the first Red pure strategy to be tried, so $J_0 = \{R1\}$. Compute column R1 (elements a_{11} to a_{41}), resulting in the matrix below, where a circle around a payoff entry indicates that it has been computed.

	R1	R2	R3	R4	R5	R6
B1	5	8	1	2	4	5
B2	5	11	3	1	1	7
B3	3	3	4	6	1	7
B4	6	10	5	4	8	6

The minimum row values m_i^0 are just the a_{i1}, that is, 5, 5, 3, and 6. The maximum of these is 6, occurring in row B4, so $\sigma_0 = a_{41} = 6$ and $i_0 = 4$.

Step 1b. Perform the feasibility test. We know $\sigma_0 \leq a_{41}$; is $\sigma_0 \leq a_{4j}$, $j = 2$ to 6? Compute row B4, yielding the matrix below.

	R1	R2	R3	R4	R5	R6
B1	5	8	1	2	4	5
B2	5	11	3	1	1	7
B3	3	3	4	6	1	7
B4	6	10	5	4	8	6

Note that entry $a_{44} = 4 < \sigma_0 = 6$. Therefore Red pure strategy R4 should also be considered.

Step 2a. Let the constraint set $J_1 = J_0 \cup \{R4\} = \{R1, R4\}$. Compute column $R4 = \{a_{i4}, i = 1 \text{ to } 4\}$, yielding:

	R1	R2	R3	R4	R5	R6
B1	5	8	1	2	4	5
B2	5	11	3	1	1	7
B3	3	3	4	6	1	7
B4	6	10	5	4	8	6

We now compute new row minima m_i^1, by taking the minimum of m_i^0 and a_{i4}, $i = 1$ to 4, as in the table below.

i	Minimum of row i over columns $J_0 = m_i^0$	a_{i4}	Minimum of row i over columns $J_1 = m_i^1$
1	5	2	2
2	5	1	1
3	3	6	3
4	6	4	4

The maximum of the m_i^1 is 4 in row 4. Thus $\sigma_1 = 4$, which is less than $\sigma_0 = 6$, and i_1, the MaxMin strategy, is 4, which happens to be the same as i_0.

Step 2b. Perform the feasibility test. Is $\sigma_1 \leq a_{4j}$? (We know $\sigma_1 \leq a_{41}$, and $\sigma_1 \leq a_{44}$.) Since row 4 has already been computed we only need to look across and see that, indeed, 4 is the smallest element. Therefore the procedure stops; the MaxMin strategy is Blue pure strategy 4 and the MaxMin value is 4. Note that only 12 of the 24 payoff entries need to be computed.

This method is applicable to any matrix game. It could be a huge nonadaptive game where each row and column represents a stream of decisions over several stages, or it could be the small matrix occurring at one stage of a behavioral game. The OPTSA I games could be solved by the new method, with much less computation.

An important factor to consider is the relationship between the number of rows and columns that need to be computed to solve either problem (1) or problem (2) and the size of the payoff matrix. In one recent analysis that used OPTSA and solved games for many variations of a realistic set of data, only one or two rows and one or two columns were required to solve the majority of the games. This is probably because of dominations among the pure strategies which is due to the data and assessments used in OPTSA. Payoff matrices generated from realistic data may have enough domination among rows and columns so that the number of rows and columns that need to be computed grows very slowly with the size of the matrix. Even payoff matrices with independent, randomly generated elements frequently might not require computation of all rows and columns for solution. Exploration of this relationship could be considered in further research on multi-stage games.

B. COMPUTATION OF NONADAPTIVE AND ADAPTIVE MAXMIN AND MINMAX STRATEGIES AND VALUES OF MULTI-STAGE GAMES

Suppose there is a two-person zero-sum game of several stages, containing rules about when Blue and Red can make decisions and what information each side has at any decision point. Given these rules, suppose Blue wishes to formulate (at the beginning of the game) a decision policy to guarantee that the payoff is at least some value σ--no matter what Red does. Furthermore, suppose Blue wishes to find the maximum σ for which such a decision policy exists. Similarly, suppose Red wishes to find a decision policy to guarantee that the payoff is no more than τ, no matter what Blue does, and to find the minimum τ. The maximal σ is less than or equal to the minimal τ, and they may be equal. For a one-stage matrix game (or any game in normal form) the standard formulas

$$\sigma = \max_{i=1 \text{ to } m} \; \min_{j=1 \text{ to } n} a_{ij}$$

and

$$\tau = \min_{j=1 \text{ to } n} \; \max_{i=1 \text{ to } m} a_{ij}$$

yield the maximal σ and minimal τ, and their arguments $\hat{i}$ and $\hat{j}$, respectively, provide the Blue and Red decision policies. That is, $\hat{i}$ is such that

$$\min_{j=1 \text{ to } n} a_{\hat{i}j} = \sigma$$

and $\hat{j}$ is such that

$$\max_{i=1 \text{ to } m} a_{i\hat{j}} = \tau \ .$$

A matrix game such as this can be solved relatively quickly by the method in Section A above.

The nonadaptive multi-stage game of OPTSA I is easily transformed into a large one-stage matrix game where each strategy consists of a sequence of allocation choices, one for each period, that is to be played regardless of what the other side does. If there are b_k allocation choices available to Blue in decision period k, k = 1 to s, then the total number of Blue pure strategies for the large matrix game is $B = \prod_{k=1}^{s} b_k$. The matrix game has the following quantities associated with it.

- A Blue MaxMin strategy i_N and associated MaxMin value σ_N,
- A Red MinMax strategy j_N and associated MinMax value τ_N,
- A payoff $a_{i_N j_N}$ when each side plays its nonadaptive strategy.

  (Note that $\sigma_N \leq a_{i_N j_N} \leq \tau_N$.)

- A game value $g = \hat{x}^T A\hat{y}$, where $\hat{x}$ and $\hat{y}$ are the optimal Blue and Red randomized strategies for the nonadaptive game.

  (Note that $\sigma_N \leq g \leq \tau_N$.)

The value g is found in OPTSA I (with $\hat{x}$ and $\hat{y}$); one cannot say where g lies with respect to $a_{i_N j_N}$ (unless $\sigma_N = \tau_N$).

The multi-stage game solved in OPTSA II allows both sides at each stage to choose their strategy dependent on what both have done during preceding stages. Blue (and Red too) is allowed to formulate a *policy* based on all the freedom of decision he has. How do we find MaxMin and MinMax strategies and values for this game? First, note that *what will actually occur in* the game is a sequence of allocations by both sides that will lead to a final payoff. Therefore any outcome to be taken into consideration in solving an adaptive or behavioral game will also appear in the corresponding nonadaptive game.

Therefore, the maximal σ, call it σ_A, is at least σ_N, as there is a strategy for Blue (i.e., i_N) by which he is assured σ_N with certainty. Examples have been constructed where σ_A is strictly greater than σ_N. Similarly, for Red there is a $\tau_A \leq \tau_N$. Corresponding to σ_A and τ_A are behavioral strategies that take into account the decision rules of the game. (These strategies do not involve randomization.) The values provide bounds on the OPTSA II game value that might be tighter than the nonadaptive MaxMin and MinMax values. However, it is not possible in general to tell where the OPTSA II game value, the OPTSA I game value, the outcome from each side playing its conservative nonadaptive strategy (MaxMin or MinMax, as appropriate), and the outcome from each side playing its conservative adaptive strategy lie with respect to each other.

Putting the game of OPTSA II into normal form would involve listing all the policies for Blue and Red. This would result in a huge unwieldy matrix. However, the method used to find the optimal (randomized) behavioral strategies can be combined with the method in Section A above to find (in two passes) the values σ_A and τ_A and the associated MaxMin and MinMax behavioral strategies. Let us consider a game of three stages for an example, and let there be b_k "pure" choices for Blue and r_k for Red at stage k, k = 1 to 3. At the end of stage 2 there are $b_1b_2r_1r_2$ places where Blue could be, one for each possible history of choices. At each of these places Blue is faced with a one-stage matrix game for which the MaxMin strategy and value can be found by the method in Section A. This results in $b_1b_2r_1r_2$ values each guaranteed, conditioned on the place at the end of stage 2. Organize these into b_1r_1 matrix games, each $b_2 \times r_2$. For each of these matrix games, the MaxMin strategy will guarantee Blue the maximum value possible regardless of where Red moves, conditioned on the outcome at the end of stage one. This yields b_1r_1 MaxMin values, which are organized into one final game, the MaxMin strategy and value of which yield Blue's first period choice and the value σ_A. To find the Red MinMax strategies, the same $b_1r_1b_2r_2$ matrices at the end of stage 2 can be used, but the b_1r_1 matrices of MinMax values will not be the same as the matrices of MaxMin values; therefore some kind of two-pass procedure is needed.

The embedding procedure of the Revised OPTSA Model (Reference [2], Chapter II, Section E) can be used to find the adaptive MaxMin and MinMax strategies as efficiently as possible. With judicious programming the two passes of the procedure can be meshed to avoid unnecessary duplication in computation.

The motivation of both the DYGAM and ATACM models is to find quantities connected with the game of OPTSA II where the numbers b_k and r_k are so large that OPTSA II, or the adaptive MaxMin and MinMax finding procedure, would take too much computer time. They both make the approximation of condensing the "$b_1b_2r_1r_2$ places at the end of stage 2" into somewhat fewer places. DYGAM tries to find the optimal randomized behavioral strategies; ATACM tries to find the adaptive MaxMin and MinMax strategies. Because of this condensation of the state space, games with many more stages than 3 can be processed in a reasonable amount of computer time.

C. COMPUTATION TIME REQUIREMENTS

This section derives estimates of computation time formulas for several different games solved by OPTSA methods. The approach is similar to that used in Reference [3]. The formulas are in terms of number of days of combat simulated, since the running time of all the OPTSA models is essentially proportional to that number. Given the time it takes to simulate one day of combat in an assessment routine, an estimate of the time OPTSA would take to solve the game can be obtained. This can then reasonably be compared with the time of other models.

Let there be s stages and let b_k and r_k be the number of pure strategies available to Blue and Red, respectively, in stage k. Let stage k comprise d_k days of combat; thus a war has $D = \sum_{k=1}^{s} d_k$ days. The first days of the stages are decision days, which must be identical for Blue and Red. Let w and c be average estimates of the number of rows and columns that need to be computed to solve a matrix game. The actual number needed might vary somewhat with the input data and advance start. Assume w and c are independent of the size of the matrix (as has been discussed); also assume they are independent of stage. The two-pass MaxMin and MinMax procedure might yield values of the w and c different from those found in the procedure for finding the optimal randomized strategy.

1. Computing Randomized Strategies for Nonadaptive Games

There is one large matrix; the order is $\prod_{k=1}^{s} b_k$ rows by $\prod_{k=1}^{s} r_k$ columns. Call these numbers B and R, respectively. The number of payoff entries that need to be generated is $wR + cB - wc$. The last term will be insignificant if B and R are large. To find the number of *daily campaigns* needed, first note that the procedure described in Reference [3] can be applied. This involves not resimulating the whole war but only the latter periods where the pure strategy changes. That is, to compute a column of R elements one would think that RD daily campaigns would be required. However, this can be reduced considerably to $\sum_{k=1}^{s} d_k \prod_{\ell=1}^{k} r_\ell$. For example, if $s = 3$ and $r_\ell = n \; \forall \; \ell$, the formula reduces from n^3D to $nd_1 + n^2d_2 + n^3d_3$; if $d_i = \frac{D}{s}$ for $i = 1$ to s, the reduction is on the order of $\frac{s-1}{s}$. The corresponding formula for the number of daily campaigns needed to compute a row of B elements reduces from BD to $\sum_{k=1}^{s} d_k\left(\prod_{\ell=1}^{k} b_\ell\right)$. With w rows and c columns needed to be computed, the total formula becomes:

Number of daily campaigns that must be computed ≈

$$\underbrace{w\left[\sum_{k=1}^{s} d_k\left(\prod_{\ell=1}^{k} b_\ell\right)\right]}_{\text{campaigns per row}} + \underbrace{c\left[\sum_{k=1}^{s} d_k\left(\prod_{\ell=1}^{k} r_\ell\right)\right]}_{\text{campaigns per column}} \tag{3}$$

where

w = number of rows,

c = number of columns.

For some (though not most) assessment routines, the method of recomputing only the days where the strategy changes may not be valid. Thus wBD + cRD remains a general estimate, though an extremely conservative one.

2. Computing Randomized Strategies for Adaptive Games

This formula is derived in Reference [3]; we restate it here. For stage ℓ, let $a_\ell = wr_\ell + cb_\ell - wc$. This is an estimate of the number of payoff entries needed to solve a game at stage ℓ. The number of daily campaigns needed, using the "no unnecessary days recomputed" feature, becomes $a_1 d_1 + a_1 a_2 d_2 + \ldots + a_1 a_2 \ldots a_s d_s$ =

$$\sum_{k=1}^{s} d_k\left(\prod_{\ell=1}^{k} a_\ell\right). \tag{4}$$

The estimates for the MaxMin and MinMax games are exactly the same for randomized strategy ones, except w and c might be bigger if two passes are made. This is because *exactly* the same kind of relaxation technique is used; the difference is between simplex pivots and comparison tests, which is very small compared to payoff computation time. Thus formula (3) serves also for nonadaptive MaxMin and MinMax games, formula (4) for adaptive ones.

These formulas can be simplified. Let $b_\ell = r_\ell = n$ for all ℓ (ℓ=1 to s), and call $a_\ell = wr_\ell + cb_\ell - wc = a$ for all ℓ. Then $B = R = n^s$. Formula (3), the nonadaptive game, then reduces to $(w+c) \sum_{k=1}^{s} d_k n^k$. If d_s is large compared to the other d_k, as is often the case, the sum is dominated by its last term, $d_s n^s$. Formula (4) becomes $\sum_{k=1}^{s} a^k d_k$, which again is dominated by its last term $a^s d_s$. The ratio

$$\frac{\text{adaptive \# campaigns}}{\text{nonadaptive \# campaigns}} \approx \frac{a^s d_s}{(w+c)n^s d_s} = \frac{1}{(w+c)}\left(\frac{a}{n}\right)^s .$$

Given w and c, $\frac{a}{n}$ is almost independent of n. If $w = c = 2$, then $a = wn + cn - wc = 4n - 4$ so $a/n \approx 4$. As s gets larger the nonadaptive game quickly becomes much more favorable in terms of computation.

IV. PURE STRATEGIES IN ADAPTIVE AND NONADAPTIVE GAMES

James E. Falk and Jeffrey H. Grotte

In general, research in multi-stage games has dealt with mixed-strategy phenomena. The foregoing chapters on the other hand, raise questions concerning the features of pure strategies in multi-stage games. The beginnings of two parallel approaches to this problem are given in Appendices A and B. While the content of these appendices overlap to some extent, their philosophy and notations differ and consequently it is of interest to present them both.

The key theorem, common to both appendices, proves that if a multi-stage game has a nonadaptive pure strategy saddlepoint, then no adaptive strategy saddlepoint will yield a better MaxMin value to the maximizing player or a better MinMax value to the minimizing player. Appendix A also gives a condition for an adaptive strategy saddlepoint to imply the existence of a nonadaptive strategy saddlepoint and shows how to relate the adaptive and nonadaptive strategies when adaptive and nonadaptive saddlepoints exist. Some examples are also presented. Appendix B, in addition to the key theorem, discusses the effectiveness of nonadaptive strategies against pre-announced adaptive strategies.

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APPENDIX A

ON ADAPTIVE AND NONADAPTIVE PURE STRATEGIES IN ZERO-SUM SEQUENTIAL GAMES

by

Jeffrey H. Grotte

APPENDIX A

ON ADAPTIVE AND NONADAPTIVE PURE STRATEGIES IN ZERO-SUM SEQUENTIAL GAMES

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Let G be a two-person zero-sum sequential game; i.e., G comprises M stages. At each stage each player executes an action that he chooses from some set of actions available to him at that stage. At the end of the game, each player will receive some payoff based on both players' choices of actions. We will assume that each player's set of actions at any stage is finite and known by both players before the game begins. Further, a player's set of actions at any stage is independent of either player's previous actions. For the purposes of this paper, each player may play in one of two modes:

Nonadaptive pure strategy. The player must pick his actions for each stage before the game begins, and then play his sequence of actions regardless of how the game progresses.

Adaptive pure strategy. For each stage, the player waits to see what actions both sides have taken up to that stage before choosing his action for that stage. For this case, we will require that the player have perfect and complete knowledge of the past.[1]

We will look at two extremes and compare the results. The cases will be:

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[1]Others have examined adaptive pure strategies in other contexts. See for example, the majorant and minorant games of von Neumann [4] and the metagames of Howard [1]. The reader is also referred to Karr [2] and Kuhn [3]. (These references are listed at the end of this appendix.)

(1) Both players play nonadaptive strategies.

(2) Both players play adaptive strategies.

A remarkable fact is that, if we disallow randomization, both these cases can be modeled by normal form games.

Let $\{S_j^i\}$ i=1,2; j=1,...,M be the players' sets of actions for all stages, so that S_j^i is player i's action set at stage j. Denote by s_j^i some specific element of S_j^i. Thus a *play* of the game is

$$\left(s_1^1 s_2^1 \dots s_M^1;\ s_1^2 s_2^2 \dots s_M^2\right),\ s_j^i \in S_j^i \quad i=1,2;\ j=1,\dots,M\ .$$

The payoff of the game (from 2 to 1) we denote by

$$O\left(s_1^1 s_2^1 \dots s_M^1;\ s_1^2 s_2^2 \dots s_M^2\right)\ .$$

Nonadaptive Model

Let $R^i = \{\left(s_1^i s_2^i \dots s_M^i\right) \mid s_j^i \in S_j^i\}$ i=1,2 .

Then define the normal form game GN with the first player's pure strategies being the elements of R^1, and the second player's pure strategies being the elements of R^2. We define the outcome of the players' choices $r^i \in R^i$ i=1,2 in the following manner

$$\tilde{O}\left(r^1, r^2\right) \equiv O\left(s_1^1 s_2^1 \dots s_M^1;\ s_1^2 s_2^2 \dots s_M^2\right)$$

where

$$s_1^1 s_2^1 \dots s_M^1 = r^1$$

$$s_1^2 s_2^2 \dots s_M^2 \quad r^2\ .$$

Example: A two stage game

$$S_1^1 = \{a,b\} \quad S_2^1 = \{a,b\}$$

$$S_1^2 = \{p,q\} \quad S_2^2 = \{p,q\} .$$

Let

$$O_1 = O(aa,pp) \qquad O_9 = O(ba,pp)$$
$$O_2 = O(aa,pq) \qquad O_{10} = O(ba,pq)$$
$$O_3 = O(aa,qp) \qquad O_{11} = O(ba,qp)$$
$$O_4 = O(aa,qq) \qquad O_{12} = O(ba,qq)$$
$$O_5 = O(ab,pp) \qquad O_{13} = O(bb,pp)$$
$$O_6 = O(ab,pq) \qquad O_{14} = O(bb,pq)$$
$$O_7 = O(ab,qp) \qquad O_{15} = O(bb,qp)$$
$$O_8 = O(ab,qq) \qquad O_{16} = O(bb,qq)$$

then the normal form of this nonadaptive case is

	pp	pq	qp	qq
aa	O_1	O_2	O_3	O_4
ab	O_5	O_6	O_7	O_8
ba	O_9	O_{10}	O_{11}	O_{12}
bb	O_{13}	O_{14}	O_{15}	O_{16}

Figure 1

Adaptive Model

Although straightforward, the adaptive model requires somewhat more effort to formulate.

Define $F_1^1 \equiv S_1^1$ and $F_1^2 \equiv S_1^2$. For $j=2,\ldots,M$, let F_j^1 be the set of all functions f from

$$S_1^2 \times S_2^2 \times \ldots \times S_{j-1}^2 \text{ into } S_j^1 .$$

Similarly define F_j^2, $j=2,\ldots,M$.

Note that F_1^i is a set of *actions* while F_j^i, $j \neq 1$ is a set of functions. The finiteness of the S_j^i guarantees the finiteness of the F_j^i.

Now define

$$P^i = \left\{ \left(f_1^i f_2^i \ldots f_M^i\right) \mid f_j^i \in F_j^i \right\} \quad i=1,2 .$$

Now we can model the adaptive case by the normal form game GA where player 1's pure strategies are the elements of P^1 and where P^2 is the set of player 2's pure strategies. We evaluate the outcome of a choice $\rho^1 \in P^1$, $\rho^2 \in P^2$ as follows:

ρ^1 equals some sequence $f_1^1 f_2^1 \ldots f_M^1$ and

ρ^2 equals some sequence $f_1^2 f_2^2 \ldots f_M^2$.

Now, since $F_1^i = S_1^i$ for $i = 1,2$, then $f_1^i =$ some $s_1^i \in S_1^i$ $i=1,2$.

We can now find inductively

$$\begin{aligned} s_2^1 &= f_2^1(s_1^2) & s_2^2 &= f_1^2(s_1^1) \\ s_3^1 &= f_3^1(s_1^2 s_2^2) & s_3^2 &= f_3^2(s_1^1 s_2^1) \\ s_4^1 &= f_4^1(s_1^2 s_2^2 s_3^2) & s_4^2 &= f_4^2(s_1^1 s_2^1 s_3^1) \end{aligned}$$

and so on.

Define $\hat{O}(\rho^1,\rho^2) \equiv O(s_1^1 s_2^1 \ldots s_M^1;\ s_1^2 s_2^2 \ldots s_M^2)$.

Example

$$S_1^1 = \{a,b\} \quad S_2^1 = \{a,b\}$$

$$S_1^2 = \{p,q\} \quad S_2^2 = \{p,q\} \ .$$

Then

$$F_1^1 = \{a,b\} \qquad F_2^1 = \left\{{}^a/_a,\ {}^a/_b,\ {}^b/_a,\ {}^b/_b\right\}$$

where ${}^g/_h$ defines the function

$$f_2^1(p) = g \qquad f_2^1(q) = h \ .$$

Similarly $F_1^2 = \{p,q\}$, $F_2^2 = \left\{{}^p/_p,\ {}^p/_q,\ {}^q/_p,\ {}^q/_q\right\}$ where ${}^\ell/_m$ defines

$$f_2^2(a) = \ell \qquad f_2^2(b) = m \ .$$

Using our procedure for computing outcomes, and the notation from the nonadaptive example, it is easy to see that the normal form game we have thus determined is given in Figure 2.

Note the obvious: the set of outcomes of the game GN is precisely the set of outcomes of the game GA, not counting multiplicities.

<u>LEMMA 1</u>: a) $\max\limits_{\rho^1 \varepsilon P^1} \min\limits_{\rho^2 \varepsilon P^2} \hat{O}(\rho^1,\rho^2) \geq \max\limits_{r^1 \varepsilon R^1} \min\limits_{r^2 \varepsilon R^2} \tilde{O}(r^1,r^2)$

b) $\min\limits_{\rho^2 \varepsilon P^2} \max\limits_{\rho^1 \varepsilon P^1} \hat{O}(\rho^1,\rho^2) \leq \min\limits_{r^2 \varepsilon R^2} \max\limits_{r^1 \varepsilon R^1} \tilde{O}(r^1,r^2)$

<u>PROOF</u>: a) Let $\underline{r}^1$ be an element of R^1 at which the $\max\limits_{r^1 \varepsilon R^2} \min\limits_{r^2 \varepsilon R^2} \tilde{O}(r^1,r^2)$ is achieved. We know $\underline{r}^1$ represents the sequence

$$\underline{s}_1^1\ \underline{s}_2^1 \cdots \underline{s}_M^1 \ .$$

	$p\ {}^{p}/_{p}$	$p\ {}^{p}/_{q}$	$p\ {}^{q}/_{p}$	$p\ {}^{q}/_{q}$	$q\ {}^{p}/_{p}$	$q\ {}^{p}/_{q}$	$q\ {}^{q}/_{p}$	$q\ {}^{q}/_{q}$
$a\ {}^{a}/_{a}$	O_1	O_1	O_2	O_2	O_3	O_3	O_4	O_4
$a\ {}^{a}/_{b}$	O_1	O_1	O_2	O_2	O_7	O_7	O_8	O_8
$a\ {}^{b}/_{a}$	O_5	O_5	O_6	O_6	O_3	O_3	O_4	O_4
$a\ {}^{b}/_{b}$	O_5	O_5	O_6	O_6	O_7	O_7	O_8	O_8
$b\ {}^{a}/_{a}$	O_9	O_{10}	O_9	O_{10}	O_{11}	O_{12}	O_{11}	O_{12}
$b\ {}^{a}/_{b}$	O_9	O_{10}	O_9	O_{10}	O_{15}	O_{16}	O_{15}	O_{16}
$b\ {}^{b}/_{a}$	O_{13}	O_{14}	O_{13}	O_{14}	O_{11}	O_{12}	O_{11}	O_{12}
$b\ {}^{b}/_{b}$	O_{13}	O_{14}	O_{13}	O_{14}	O_{15}	O_{16}	O_{15}	O_{16}

Figure 2.

Select that

$$\underline{\rho}^1 = \underline{f}_1^1\ \underline{f}_2^1 \ldots \underline{f}_M^1$$

where

$$\underline{f}_1^1 = \underline{s}_1^1 \ .$$

$$\underline{f}_j^1(\cdot) = \underline{s}_j^1 \text{ for all arguments, } j=2,\ldots,M \ .$$

Observe that the $\underline{\rho}^1$ row of the game GA must have precisely the same outcomes as the $\underline{r}^1$ row of the game GN; therefore

$$\min_{\rho^2 \varepsilon P^2} \hat{O}(\underline{\rho}^1,\rho^2) = \min_{r^2 \varepsilon R^2} \tilde{O}(\underline{r}^1,r^2)$$

$$= \max_{r^1 \varepsilon R^1} \min_{r^2 \varepsilon R^2} \tilde{O}(r^1,r^2) \ .$$

Therefore

$$\max_{\rho^1 \varepsilon P^1} \min_{\rho^2 \varepsilon P^2} \hat{O}(\rho^1,\rho^2) \geq \max_{r^1 \varepsilon R^1} \min_{r^2 \varepsilon R^2} \tilde{O}(r^1,r^2) \ .$$

Assertion b) is proven in an entirely analogous manner.

PROPOSITION: If GN has a pure strategy saddlepoint, so does GA. In this case the values of the two games are equal.

PROOF: This follows trivially from inequalities (a) and (b) of the Lemma.

PROPOSITION: Let (r^1,r^2) be a pure strategy saddlepoint for the game GN, where

$$r^1 = s_1^1 s_2^1 \ldots s_M^1, \quad r^2 = s_1^2 s_2^2 \ldots s_M^2 \ .$$

Let

$$\rho^1 = f_1^1 f_2^1 \ldots f_M^1, \quad \rho^2 = f_1^2 f_2^2 \ldots f_M^2$$

where

$$f_1^1 = s_1, \ f_1^2 = s_1^2 \ ,$$

$$f_j^1(\cdot) = s_j^1 \quad \text{for all arguments, } j=2\ldots M$$

$$f_j^2(\cdot) = s_j^2 \quad \text{for all arguments, } j=2\ldots M \ .$$

Then (ρ^1,ρ^2) is a saddlepoint for the game GA.

PROOF: Clearly $\hat{O}(\rho^1,\rho^2) = \tilde{O}(r^1,r^2)$ = value of GM = value of GA.

We have yet to show that (ρ^1,ρ^2) is a saddlepoint. Suppose there is some $\underline{\rho}^1$ such that $\hat{O}(\underline{\rho}^1,\rho^2) > \hat{O}(\rho^1,\rho^2)$.

Now, $\underline{\rho}^1$ represents the sequence

$$\underline{f}_1^1\underline{f}_2^1\dots\underline{f}_M^1 \ .$$

Compute

$$\underline{s}_1^1 = \underline{f}_1^1$$

$$\underline{s}_2^1 = \underline{f}_2^1(s_1^2)$$

$$\underline{s}_3^1 = \underline{f}_3^1(s_1^2 s_2^2)$$

and so on until s s ...s have been computed.

Therefore

$$\hat{O}(\underline{\rho}^1,\rho^2) = O(\underline{s}_1^1\underline{s}_2^2\dots\underline{s}_M^1;\ s_1^2 s_2^2\dots s_M^2) > \hat{O}(\rho^1,\rho^2) \text{ (by assumption)}$$

$$= \tilde{O}(r^1,r^2) = O(s_1^1 s_2^1\dots s_M^1;\ s_1^2 s_2^2\dots s_M^2)$$

So that there is some $\underline{r}^1 = \underline{s}_1^1\underline{s}_2^1\dots\underline{s}_M^1$

$$\tilde{O}(\underline{r}^1,r^2) > \tilde{O}(r^1,r^2) \ .$$

This contradicts the assertion that (r^1,r^2) was a saddlepoint for GN.

In a similar way one proves that there is no $\underline{\rho}^2$ such that

$$\hat{O}(\rho^1,\underline{\rho}^2) < \hat{O}(\rho^1,\rho^2) \ .$$

What we have shown so far is that whenever the nonadaptive game has a pure strategy saddlepoint, then so does the adaptive game. Moreover, when this occurs, at least one of the adaptive saddlepoints will be the "image" of a nonadaptive saddlepoint, hence it suffices to look at the nonadaptive game. It is important to realize that knowing that the adaptive game has a pure strategy saddlepoint yields no information about the non-adaptive game *except* in the following circumstance:

PROPOSITION: If GA has a pure strategy saddlepoint (ρ^1,ρ^2) where

$$\rho^1 = f_1^1 f_2^1 \ldots f_M^1$$

$$\rho^2 = f_1^2 f_2^2 \ldots f_M^2$$

and the f_j^i have the property that

$$f_1^1 = \text{some } s_1^1 \qquad f_1^2 = \text{some } s_1^2$$

and

$$f_j^i(\cdot) = \text{some } s_j^i \quad \text{for } all \text{ arguments}$$

$$i=1,2; \qquad j=2,\ldots,M$$

then if $r^1 = s_1^1 s_2^1 \ldots s_M^1 \quad r^2 = s_1^2 s_2^2 \ldots s_M^2$, (r^1,r^2) is a saddlepoint of the game GN.

PROOF: Since (ρ^1,ρ^2) is a pure strategy saddlepoint of GA

$$\hat{O}(\rho^1,\rho^2) = \min_{\underline{\rho}^2 \varepsilon P^2} \hat{O}(\rho^1,\underline{\rho}^2) .$$

But

$$\hat{O}(\rho^1,\rho^2) \quad O(s_1^1 s_2^1 \ldots s_M^1,\ s_1^2 s_2^2 \ldots s_M^2) .$$

Therefore

$$O(s_1^1 s_2^1 \ldots s_M^1;\ s_1^2 s_2^2 \ldots s_M^2) = \min_{\underline{s}_j^2 \varepsilon S_j^2} O(s_1^1 s_2^1 \ldots s_M^1;\ \underline{s}_1^2 \underline{s}_2^2 \ldots \underline{s}_M^2)$$

or equivalently

$$\tilde{O}(r^1,r^2) = \min_{\underline{r}^2 \varepsilon R^2} \tilde{O}(r^1,\underline{r}^2) .$$

In a similar manner, it is possible to show

$$\tilde{O}(r^1,r^2) = \max_{\underline{r}^1 \varepsilon R^1} \tilde{O}(\underline{r}^1,r^2) .$$

Hence $\tilde{O}(r^1,r^2)$ is the minimum of its row, the maximum of its column so that (r^1,r^2) is a pure strategy saddlepoint.

EXAMPLES: The following examples are the two-stage game with

$$S_1^1 = \{a,b\} \qquad S_2^1 = \{a,b\}$$

$$S_1^2 = \{p,q\} \qquad S_2^2 = \{p,q\} .$$

The outcomes can be found in the table for the nonadaptive case of each example. In all cases $\varepsilon,\gamma > 0$

Example 1 shows a game in which the nonadaptive game has a saddlepoint, so that the adaptive game also has one. Moreover, if one looked only at the adaptive game, one would notice that one of the saddlepoints (in the upper left hand corner) corresponds to a nonadaptive strategy pair which is a saddlepoint of the nonadaptive game. This illustrates the final Proposition.

Example 2 is a case where the nonadaptive game has no saddlepoint, although the adaptive game has one.

In Example 3, neither game has a saddlepoint.

Example 1

	pp	pq	qp	qq
aa	0^*	γ	γ	0
ab	$-\varepsilon$	γ	0	$-\varepsilon$
ba	$-\varepsilon$	$-\varepsilon$	0	γ
bb	0	γ	$-\varepsilon$	γ

MaxMin = 0
MinMax = 0
* saddlepoint

Nonadaptive Case

	$p\ {}^{p}/_{p}$	$p\ {}^{p}/_{q}$	$p\ {}^{q}/_{p}$	$p\ {}^{q}/_{q}$	$q\ {}^{p}/_{p}$	$q\ {}^{p}/_{q}$	$q\ {}^{q}/_{p}$	$q\ {}^{q}/_{q}$
$a\ {}^{a}/_{a}$	0^*	0	γ	γ	γ	γ	0^*	0
$a\ {}^{a}/_{b}$	0	0	γ	γ	0	0	$-\varepsilon$	$-\varepsilon$
$a\ {}^{b}/_{a}$	$-\varepsilon$	$-\varepsilon$	γ	γ	γ	γ	0	0
$a\ {}^{b}/_{b}$	$-\varepsilon$	$-\varepsilon$	γ	γ	0	0	$-\varepsilon$	$-\varepsilon$
$b\ {}^{a}/_{a}$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	0	γ	0	γ
$b\ {}^{a}/_{b}$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	γ	$-\varepsilon$	γ
$b\ {}^{b}/_{a}$	0^*	γ	0	γ	0	γ	0^*	γ
$b\ {}^{b}/_{b}$	0	γ	0	γ	$-\varepsilon$	γ	$-\varepsilon$	γ

MaxMin = 0 MinMax = 0 * saddlepoint

Adaptive Case

Example 2

	pp	pq	qp	qq
aa	0	γ	$-\varepsilon$	0
ab	$-\varepsilon$	0	γ	γ
ba	$-\varepsilon$	$-\varepsilon$	γ	$-\varepsilon$
bb	γ	$-\varepsilon$	γ	0

MaxMin = $-\varepsilon$

MinMax = γ

Nonadaptive Case

	$p\ {}^{p}/_{p}$	$p\ {}^{p}/_{q}$	$p\ {}^{q}/_{p}$	$p\ {}^{q}/_{q}$	$q\ {}^{p}/_{p}$	$q\ {}^{p}/_{q}$	$q\ {}^{q}/_{p}$	$q\ {}^{q}/_{q}$
$a\ {}^{a}/_{a}$	0	0	γ	γ	$-\varepsilon$	$-\varepsilon$	0	0
$a\ {}^{a}/_{b}$	0	0^{*}	γ	γ	γ	γ	γ	γ
$a\ {}^{b}/_{a}$	$-\varepsilon$	$-\varepsilon$	0	0	$-\varepsilon$	$-\varepsilon$	0	0
$a\ {}^{b}/_{b}$	$-\varepsilon$	$-\varepsilon$	0	0	γ	γ	γ	γ
$b\ {}^{a}/_{a}$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	γ	$-\varepsilon$	γ	$-\gamma$
$b\ {}^{a}/_{b}$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	$-\varepsilon$	γ	0	γ	0
$b\ {}^{b}/_{a}$	γ	$-\varepsilon$	γ	$-\varepsilon$	γ	$-\varepsilon$	γ	$-\varepsilon$
$b\ {}^{b}/_{b}$	γ	$-\varepsilon$	γ	$-\varepsilon$	γ	0	γ	0

MaxMin = 0 MinMax = 0 * saddlepoint

Adaptive Case

Example 3

	pp	pq	qp	qq
aa	0	$-\varepsilon$	γ	$-\varepsilon$
ab	$-\varepsilon$	γ	0	γ
ba	γ	0	γ	$-\varepsilon$
bb	$-\varepsilon$	γ	$-\varepsilon$	0

MaxMin = $-\varepsilon$

MinMax = γ

Nonadaptive Case

	$p\ {}^{p}/_{p}$	$p\ {}^{p}/_{q}$	$p\ {}^{q}/_{p}$	$p\ {}^{q}/_{q}$	$q\ {}^{p}/_{p}$	$q\ {}^{p}/_{q}$	$q\ {}^{q}/_{p}$	$q\ {}^{q}/_{q}$
$a\ {}^{a}/_{a}$	0	0	$-\varepsilon$	$-\varepsilon$	γ	γ	$-\varepsilon$	$-\varepsilon$
$a\ {}^{a}/_{b}$	0	0	$-\varepsilon$	$-\varepsilon$	0	0	γ	γ
$a\ {}^{b}/_{a}$	$-\varepsilon$	$-\varepsilon$	γ	γ	γ	γ	$-\varepsilon$	$-\varepsilon$
$a\ {}^{b}/_{b}$	$-\varepsilon$	$-\varepsilon$	γ	γ	0	0	γ	γ
$b\ {}^{a}/_{a}$	γ	0	γ	0	γ	$-\varepsilon$	γ	$-\varepsilon$
$b\ {}^{a}/_{b}$	γ	0	γ	0	$-\varepsilon$	0	$-\varepsilon$	0
$b\ {}^{b}/_{a}$	$-\varepsilon$	γ	$-\varepsilon$	γ	γ	$-\varepsilon$	γ	$-\varepsilon$
$b\ {}^{b}/_{b}$	$-\varepsilon$	γ	$-\varepsilon$	γ	$-\varepsilon$	0	$-\varepsilon$	0

MaxMin = $-\varepsilon$ MinMax = γ

Adaptive Case

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APPENDIX B

SADDLEPOINTS OF ADAPTIVE GAMES

by

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APPENDIX B

SADDLEPOINTS OF ADAPTIVE GAMES

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In this appendix we examine the relationship between saddlepoints of adaptive games and saddlepoints for their associated nonadaptive games. In particular, we show that the MaxMin and MinMax values of the nonadaptive game bound the same quantities for the adaptive game. It follows that the adaptive game will have a saddlepoint if the nonadaptive game has one. An example is given which shows that the opposite is not true, i.e., the adaptive game may have a saddlepoint while the nonadaptive game does not.

Notation:

Let n (number of periods) be given. For each n, let:

$A^i(x^i,y^i,p^i,q^i)$ = payoff in period i when in state (p^i,q^i) and actions x^i and y^i are chosen by the players,

$P^i(x^i,y^i,p^i,q^i)$ = state p^{i+1} resulting from state (p^i,q^i) when actions x^i and y^i are chosen,

$Q^i(x^i,y^i,p^i,q^i)$ = state q^{i+1} resulting from state (p^i,q^i) when actions x^i and y^i are chosen,

X^i,Y^i = constraint sets on decision variables x^i and y^i in period i,

(p^1,q^1) = given initial state.

Then

$$A(x,y) = \sum_{i=1}^{n} A^i(x^i,y^i,p^i,q^i)$$

where

$$
\begin{aligned}
x &= (x^1, \dots, x^n) \\
y &= (y^1, \dots, y^n) \\
p^{i+1} &= P^i(x^i, y^i, p^i, q^i) \\
q^{i+1} &= Q^i(x^i, y^i, p^i, q^i)
\end{aligned}
$$

is the cumulative payoff function whose value is uniquely determined when the players choose vectors x and y.

In order to define the game G, we first define the types of strategies available to the players I and II. An *adaptive* strategy for player I is a sequence of (vector-valued) functions

$$x^1(\cdot), \dots, x^n(\cdot)$$

where $x^i(\cdot) : (p^i, q^i) \to X^i$. Thus an adaptive strategy for player I prescribes an action or decision for each state (p^i, q^i) that can be realized in each period i. A similar definition holds for player II.

A *nonadaptive* strategy for player I is a sequence of vectors

$$x^i, \dots, x^n$$

where $x^i \in X^i$. Thus a nonadaptive strategy for player I also prescribes an action or decision for each period i, but there is no state dependence.

Let

S = set of all nonadaptive strategies for I,

$\mathcal{S}$ = set of all adaptive strategies for I,

and

T = set of all nonadaptive strategies for II,

$\mathcal{T}$ = set of all adaptive strategies for II.

Note $S \subset \mathcal{S}$, $T \subset \mathcal{T}$.

Player I's adaptive problem is

$$\max_{s \in \mathcal{S}} \min_{t \in \mathcal{T}} A(s,t) \doteq \underline{V}$$

and his nonadaptive problem is

$$\max_{s \varepsilon S} \min_{t \varepsilon T} A(s,t) \doteq \underline{v} \ .$$

Player II's adaptive problem is

$$\min_{t \varepsilon \mathcal{T}} \max_{s \varepsilon \mathcal{S}} A(s,t) \doteq \overline{V}$$

and his nonadaptive problem is

$$\min_{t \varepsilon T} \max_{s \varepsilon S} A(s,t) \doteq \overline{v} \ .$$

We have

$$\underline{V} \leq \overline{V}$$

and

$$\underline{v} \leq \overline{v} \ .$$

We will show that

$$\underline{v} \leq \underline{V} \leq \overline{V} \leq \overline{v} \ .$$

<u>Results</u>:

First, we prove the following result:

<u>Theorem</u>. With the definitions given above, if $s^0 = (x^1(\cdot), \ldots, x^n(\cdot))$ is any adaptive strategy for I, then

$$\min_{t \varepsilon \mathcal{T}} A(s^0,t) = \min_{t \varepsilon T} A(s^0,t) \ .$$

<u>Proof</u>. Since $T \subset \mathcal{T}$, we have

$$\min_{t \varepsilon \mathcal{T}} A(s^0,t) \leq \min_{t \varepsilon T} A(s^0,t) \ .$$

Let $t^0 = (y^1(\cdot), \ldots, y^n(\cdot))$ solve[1] the left of these two problems,

[1] Without proper assumptions on the constituent functions, the problem may have no solution. We shall assume that the proper assumptions do hold.

and set

$$\bar{x}^1 = x^1(p^1,q^1) \qquad \bar{y}^1 = y^1(p^1,q^1)$$

and

$$p^2 = P^1(\bar{x}^1,\bar{y}^1,p^1,q^1)$$

$$q^2 = Q^1(\bar{x}^1,\bar{y}^1,p^1,q^1) .$$

Continuing, set

$$\bar{x}^2 = x^2(p^2,q^2) \qquad \bar{y}^2 = y^2(p^2,q^2)$$

and

$$p^3 = P^2(\bar{x}^2,\bar{y}^2,p^2,q^2)$$

$$q^3 = Q^2(\bar{x}^2,\bar{y}^2,p^2,q^2)$$

etc., to produce two sequences of decisions

$$\bar{x}^1,\ldots,\bar{x}^n$$

and

$$\bar{y}^1,\ldots,\bar{y}^n$$

for the players. (This pair of sequences is often termed a *play* of the game G whose payoff is A when strategies s and t are employed.)

Note that

$$\bar{t} \doteq (\bar{y}^1,\ldots,\bar{y}^n) \ \varepsilon\ T .$$

But

$$A(s^0,t^0) = A(s^0,\bar{t})$$

since the same play of the game is produced by either of the pairs (s^0,t^0) or $(s^0,\bar{t})$. Thus

$$\begin{aligned}\min_{t \in \mathcal{T}} A(s^0,t) = A(s^0,t^0) = A(s^0,\bar{t}) \\ \geq \min_{t \in T} A(s^0,t) \\ \geq \min_{t \in \mathcal{T}} A(s^0,t)\end{aligned}$$

and the proof is complete.

The theorem implies that player II can restrict himself to his nonadaptive strategies if player I announces his intent to employ a given adaptive strategy. This is analogous to the situation in matrix games, wherein a player may restrict himself to his pure strategies if his opponent announces his intent to employ a certain mixed strategy.

Corollary. With the above definitions

$$\max_{s \in S} \min_{t \in T} A(s,t) \leq \max_{s \in \mathcal{S}} \min_{t \in \mathcal{T}} A(s,t)$$

i.e.,

$$\underline{v} \leq \underline{V}$$

Proof. Let s denote any nonadaptive strategy for I. Then

$$\min_{t \in T} A(s,t) = \min_{t \in \mathcal{T}} A(s,t)$$

so

$$\begin{aligned}\underline{v} = \max_{s \in S} \min_{t \in T} A(s,t) &= \max_{s \in S} \min_{t \in \mathcal{T}} A(s,t) \\ &\leq \max_{s \in \mathcal{S}} \min_{t \in \mathcal{T}} A(s,t) = \underline{V}\end{aligned}$$

since $S \subset \mathcal{S}$.

This corollary implies that the first ("outside") player generally does better with an adaptive strategy, but the second ("inside") player cannot improve his payoff by employing an adaptive strategy.

Let G = (A,S,T) denote the *nonadaptive game* with payoff A and strategy sets S and T. This *adaptive* game is denoted by *G* = (A,*S*,*T*).

<u>Corollary</u>. If G has a saddlepoint, then so does *G*.

<u>Proof</u>. From the above corollary

$$\underline{v} \le \underline{V} \le \overline{V} \le \overline{v} \; .$$

If G has a saddlepoint, the $\underline{v} = \overline{v}$, so $\underline{V} = \overline{V}$ which implies that *G* has a saddlepoint.

This corollary was also established by Berkovitz[1] in the context of differential games.

The following example shows that *G* may possess a saddlepoint while G does not. In Figure 1, the extensive form of *G* is given with terminal payoffs as indicated. The normal form of the game is given in Table 1, where we employ the notation

(R|L,R)

to stand for the strategy:

> go down and to the right when in period 1;
> go down and to the left when in period 2 if
> your opponent went left in period 1, otherwise,
> go down and to the right in period 2.

[1] Berkowitz, L., Lectures on Differential Games, in Kuhn, H. and Szego, G. (editors) *Differential Games and Related Topics*, North-Holland, 1971.

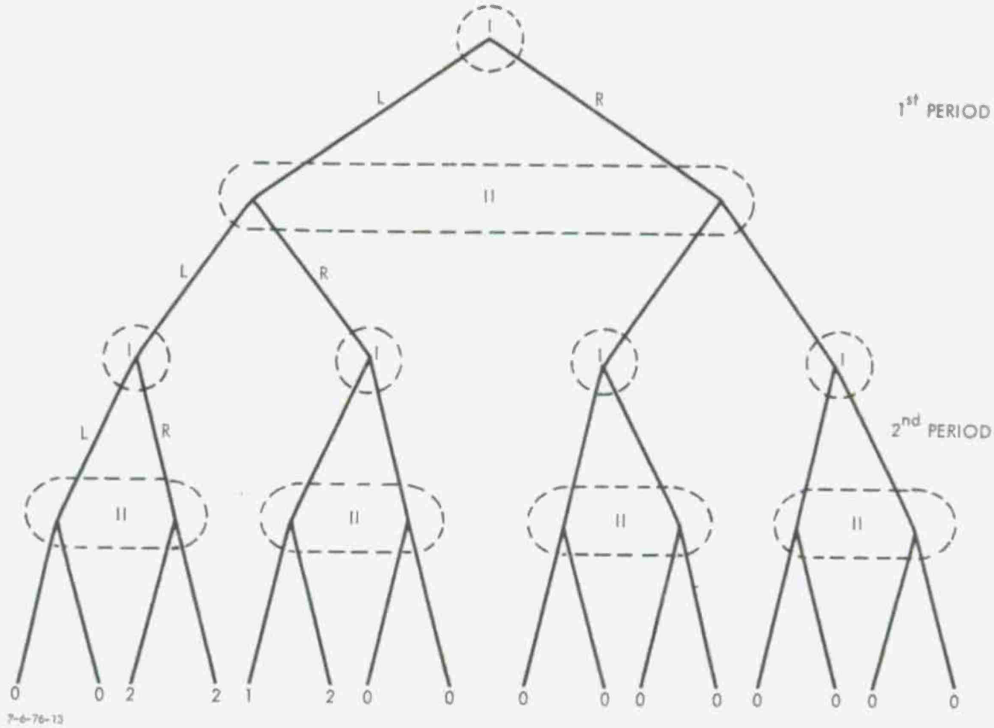

Figure 1. THE ADAPTIVE GAME
(EXTENSIVE FORM)

Table 1. THE ADAPTIVE GAME--NORMAL FORM

	(L|L,L)	(L|L,R)	(L|R,L)	(L|R,R)	(R|L,L)	(R|L,R)	(R|R,L)	(R|R,R)
(L|L,L)	0	0	0	0	1	1	2	2
(L|L,R)	0	0	0	0	0	0	0	0
(L|R,L)	2	2	2	2	1	1	2	2
(L|R,R)	2	2	2	2	0	0	0	0
(R|L,L)	0	0	0	0	0	0	0	0
(R|L,R)	0	0	0	0	0	0	0	0
(R|R,L)	0	0	0	0	0	0	0	0
(R|R,R)	0	0	0	0	0	0	0	0

There are 2 saddlepoints (circled).

In Figure 2, the extensive form of the nonadaptive game is displayed. The corresponding normal form is given in Table 2. Note that there is no saddlepoint.

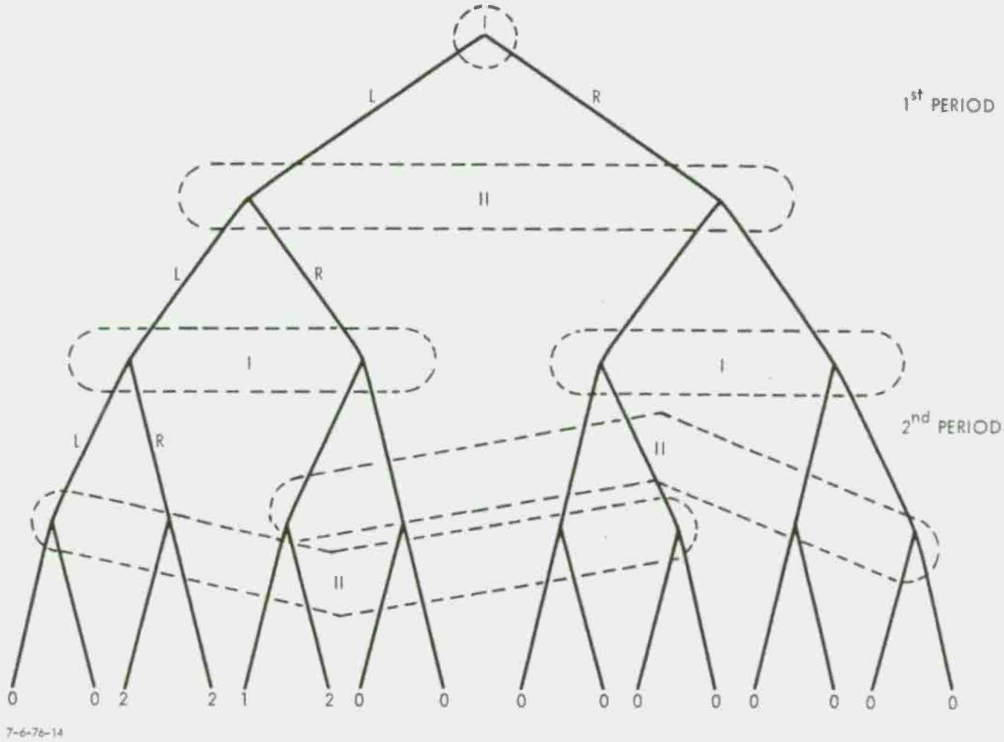

Figure 2. THE NONADAPTIVE GAME (EXTENSIVE FORM)

Table 2. THE NONADAPTIVE GAME--NORMAL FORM

	LL	LR	RL	RR
LL	0	0	1	2
LR	2	2	0	0
RL	0	0	0	0
RR	0	0	0	0

In these examples we have $\underline{v} = 0$, $\underline{V} = \overline{v} = \underline{V} = \overline{V} = 1$. Note that $\overline{v} = \overline{V}$, a reflection of the fact that one of player II's MinMax adaptive strategies is actually a nonadaptive strategy (R|L,L).

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