PAPER P-1182

ON THE LULEJIAN-I COMBAT MODEL

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INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER P-1182	2 GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On the Lulejian-I Combat Model		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER P-1182
7. AUTHOR(s) Alan F. Karr		8. CONTRACT OR GRANT NUMBER(s) IDA Independent Research
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Defense Analyses Program Analysis Division 400 Army-Navy Drive, Arlington, Va. 22202		10. PROGRAM ELEMENT PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE August 1976
		13. NUMBER OF PAGES 50
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE NA

16. DISTRIBUTION STATEMENT (of this Report)

This document is unclassified and suitable for public release.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lulejian-I Combat Model
Combat Simulation Model
Theater-Level Combat Simulation
Attrition Computation
Lanchester Attrition Process

FEBA Movement Optimization
Sequential Optimization

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper contains a review and summary of the Lulejian-I combat model together with comments and criticisms concerning the model. Emphases are on discussion of attrition equations in terms of underlying families of assumptions, on the optimization aspects of the model, and on the iterative model used in the model to jointly compute FEBA movement and ground combat casualties. Also treated are geographical,

20. structural, and organizational aspects of the model.

CONTENTS

1. INTRODUCTION

The Lulejian-I combat model is an aggregated, theater-level, computerized combat simulation model overlaid with a two-sided optimization structure which seeks to generate approximately optimal campaign-long allocations of major theater resources. In seeking to optimize resource entry into a logistics network, allocation of tactical aircraft to missions, and possible initial massing of the attacking side's resources, the model tries to represent in a game-theoretic context the adversary nature of the decision-making processes of the two sides.

Let us outline briefly the overall optimization structure of the model. The time period to be simulated is broken into a number of moves, each representing one or more days of combat; at the beginning of each move the two sides may change their allocations of entering supplies and tactical aircraft. For each of the possible choices of actions the evaluative portion of the model computes, from the resources present at the beginning of the move and those designated to enter during the move, the numbers of resources left at the end of the move and the territory gained. This evaluation is performed using a tactical air submodel, a ground combat assessment submodel, a logistics/interdiction submodel, and a ground force allocation submodel, in that order, for each day comprising the move.

Thus the structure is that of a two-person sequential game, as outlined in [4]. In principle, it is necessary to run the combat simulation once for each possible set of sequentially chosen actions by both sides. This is prohibitively time-

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consuming, so the model uses certain approximations to the
objective function of territorial gain in order to compute only
a very small set of outcomes; in the given context this is a
necessary simplification, but we don't know of any bounds on the
potential error committed by using it. More details on the
optimization procedure appear in Section 2.

This paper is a summary of the structure of the model,
together with some criticisms and praises of its various
aspects. The Lulejian and Associates report [8] was the source
of information on the Lulejian-I model used in preparation of
this paper. Attrition computations in the Lulejian-I model are
discussed and evaluated at length in Sections 5 and 6. Our
purpose therein is to promote reasoned discussion and com-
parison of various methods for calculating attrition in highly
aggregated combat models. By considering such factors as under-
lying assumptions, the rigor of derivations, and mathematical
consistency, we hope to add deduction and inference to the set
of useful criteria for comparison and evaluation of combat
models. That set of criteria also contains plausibility of
results and verisimilitude to historical data. By deduction
we mean that if various assumptions underlying each attrition
model are known and carefully stated (for some models there may
be more than one set of mathematical assumptions leading to the
same result or more than one physical situation compatible with
the mathematical assumptions) and if one believes that a set of
assumptions is satisfied, then he must necessarily accept
attrition calculated using an equation rigorously derived from
those assumptions.

The emphasis in this paper on attrition computations and
in particular on underlying assumptions and mathematical ques-
tions is a reflection of the biases and competence of the author.
It should not be construed to mean that other aspects of a combat
model, such as numbers and types of resources modeled, are not
important.

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2.   EMPHASES OF THE LULEJIAN-I MODEL

The model report [8] states that the main intended use of
the model is the "...assessment of alternative force levels,
compositions, and deployments in Central Europe...." Within
this context one main emphasis is determination of "approxi-
mately optimal" allocations of major theater resources. The
objective function of the optimization is (net) area gained.
All conclusions obtained from the model must be interpreted in
light of the overall optimization structure; for example, the
effects of different numbers of resources or different weapon
system capabilities may be evaluated differently by a model
which does not contain this particular optimization algorithm
and in particular by a model with no optimization structure at
all, such as IDAGAM I [1,2].

As noted in Section 1, the optimization structure of the
Lulejian-I model is that of a two-person sequential game, in
which the evaluative portion of the model (i.e., the combat
assessment) serves to compute intermediate and terminal payoffs
for the game solution algorithm. The type of forward sweep/
reverse sweep algorithm used to solve the game is known in other,
similar (indeed, simpler) contexts not to produce global optima.
Another ground for skepticism is the treatment of mixed strategies
and especially the meaning of the sequence of allocations used by
the two sides which is produced as a model output. In the lan-
guage of [4] we can interpret this sequence only as a *realization*
of a pair of "nearly optimal" strategies. The strategies them-
selves are (so far as we can determine) not computed by the model.
As a consequence, the results of the model have no prescriptive
content for choosing actions. The realization of actions

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computed could as well have arisen from nonoptimal strategies. And, moreover, the realized value of the game (that is, the realized FEBA movement) need bear no necessary relation to the game-theoretic value of the sequential game, which is an expectation with respect to optimal mixed strategies. For these reasons, the decision-making implications of the optimization structure of the model are of doubtful usefulness. Conclusions obtained concerning relative effects of force structures and strengths are tainted, but not entirely useless.

The model contains three processes in which optimization takes place. First, the allocation of tactical aircraft to missions is optimized, subject to constraints involving relative effectiveness of the different types of aircraft in different missions. While this is indeed an allocation problem of considerable interest and import, it should be noted that it is in precisely this context, cf. [3], that an algorithm similar to that in the Lulejian-I model is known to yield neither optimal strategies nor the correct game value. The second optimized process is the choice of proportions of supplies (general supplies, engineering supplies, and transport vehicles) entering the logistics network described in Section 4. Finally, the choice of sectors of initial attack and initial massing of forces may be optimized. The latter problem is clearly important and not well-treated in other, similar models. It is a positive aspect of this model to have included (albeit not necessarily entirely accurately) the phenomena of initial attack conditions.

A second emphasis of the Lulejian-I model is upon the effects of supply movement and supply shortages on theater-level combat results. To this end the model includes a notional logistics network on each side, through which supplies flow from a "port" to forward supply depots and to combat sectors; this network is discussed in more detail in Section 4. The effect of not enough supplies is a linear degradation of the effectiveness of combat resources requiring the supplies in question. That is, if there are A_0 resources present, each

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of which requires (on a particular day) r units of supplies,
and if S units of supplies are available that day, the effective
number of resources (as used in attrition computations, for
example) is

$$A = A_0 \min\{1, \frac{S}{rA_0}\} \ .$$

This makes sense provided one interprets the situation as being
that A of the resources participate in the combat at full effec-
tiveness, while the remaining A_0 - A resources are able neither
to inflict nor to receive attrition. Indeed, "grounded" re-
sources cannot be part of any interactions. For resources such
as aircraft this is plausible, but for "grounded" ground combat
resources (which move along with their companions) the plausi-
bility is less. Provided it be interpreted as we did above,
the linear form of the degradation seems reasonable. Other
models, however, treat the problem differently.

A third (after the optimization algorithm and supply repre-
sentation) emphasis of the Lulejian-I model is on an iterative
method for joint computation of FEBA movement and attrition to
ground combat maneuver forces, whose stated purpose is to take
explicit account of the tradeoff between position and casualty
rates which occurs in a combat situation. The mathematics of
this procedure will be discussed in Section 7; the following
comments are relevant at this point. First, the inclusion of
this procedure represents a *behavioral* assumption about combat
situations and, more specifically, about commanding officers'
decision-making processes. The assumption is that combat
decision makers do trade off casualty rates and position,
ordinarily by yielding position when casualty rates exceed cer-
tain acceptable levels and (but to a lesser extent) seeking
to advance and cause enemy casualties when their own side's
casualty rates are sufficiently low. Thus, the defending side
will cease to hold a position (while the attacker will cease to

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advance) when casualty rates exceed the thresholds. This
assumption is praiseworthy on two grounds: there is too little
attention devoted in combat modeling to this obviously important
problem of behavioral assumptions and the representation of the
behavior of combat decision makers; and this particular behav-
ioral assumption appears to be eminently reasonable. On the
negative side, the procedure increases the computational com-
plexity of the model as well as the number of computations
required, and is hence possibly one reason for the uniformly
simple attrition structure of the model, which is discussed in
detail in Sections 5 and 6. As seen in Section 7, there are
also difficulties with both the computational iteration scheme
in the model and the interpretation of that scheme as actually
representing the tradeoff it is alleged to represent. The
tradeoff also fails to take into account the asymmetry of the
attacker's and defender's goals. While it is plausible that
an attacker will continue to advance if casuality rates do not
exceed threshold values, the defender may not seek to push
back the attacker if his casualty rates are sufficiently low.
That is, the defender may simply hold his position but at
the cost of fewer casualties. It does not appear to us that
the computational procedures of the model account for this
phenomenon.

3. GEOGRAPHY

The geographic structure of the Lulejian-I model is conventional, as represented by the figure below.
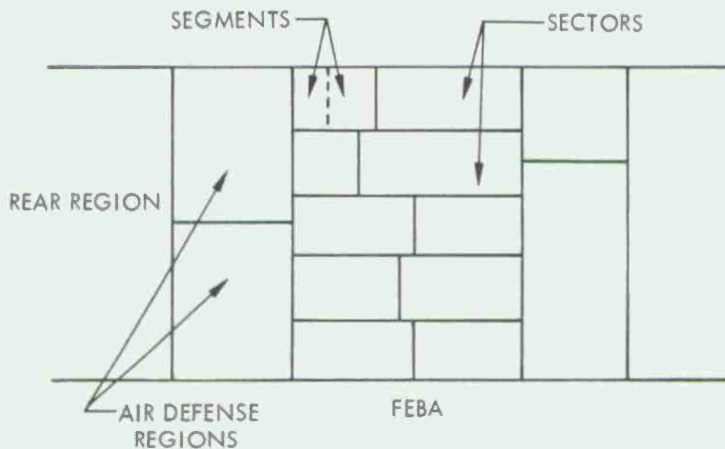
There may be up to ten sectors in which close combat occurs and in which the two sides are separated by a FEBA line; each sector may be composed of at most fifteen segments whose boundaries are parallel to the FEBA and perpendicular to the boundaries between sectors. These segments serve mainly to distinguish different kinds of terrain, of which three are permitted in the model. Sector boundaries are necessarily the same on both sides of the FEBA; the principal attrition and FEBA movement computations in the model are carried out individually for each sector.

Each side may have two air defense regions whose shared boundary evidently need not be at symmetric locations on the two sides. The depths of the air defense regions may also differ from side to side, but apparently not within one side.

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Contained in each air defense region are an air base and related supply stockage system. The logistics network discussed in Section 4 runs from the rear region to the combat sectors. In general, the geographical representations in the Lulejian-I model are less detailed than those in comparable models such as IDAGAM I [1,2].

4. RESOURCES AND RESOURCE ALLOCATIONS

The following broad classes of resources are modeled in
Lulejian-I.

A. Maneuver forces: 3 battalion types
   3 types of tanks
   3 types of armored personnel carriers
   3 types of antitank weapons
   3 types of mortars
   1 type of hand-held weapon.

B. Artillery forces: 6 types

C. Attack helicopter forces: 2 types

D. Tactical air forces
   5 types of tactical aircraft
   Aircraft shelters
   Airlift aircraft.

Each side may have up to six national participants whose
battalions may be structurally different. Usually, but not
necessarily, the three battalion types are infantry, mechanized
infantry, and armored battalions. Although bookkeeping in the
model is done at the battalion level, the engagements it seeks
to represent are of division- or corps-level magnitude.

Tactical aircraft may perform air defense, air base at-
tack, supply interdiction, close air support, air defense sup-
pression, and escort missions. Relative capabilities of air-
craft types in the various missions are provided as inputs by
the model user; daily allocations of aircraft to missions are

computed using the two-sided optimization procedure discussed in Section 2.

The Lulejian-I model includes the following kinds of supplies, which must pass from a "port" to the forward combat area through the supply network (discussed in the next paragraph).

- 4 types of battalion-sized combat units
  (usually infantry, mechanized infantry, armored and SAM battalions)
- 4 types of individual combat units
  (usually individual infantry, APC, tank and SAM replacements)
- General supplies
- Bridges to restore or increase network capacity
- Empty logistics vehicles.

The supply network is linear, as shown in the diagram below.
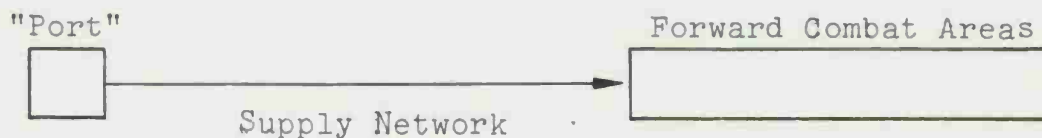
The term "port" is used here only in a generic sense and represents simply the entry point of the supply network. Port capacity is limited.

Time required for vehicles to transit from the port to the forward combat area is a function of the length and capacity of the network. The latter can be increased by the addition (notionally) of bridges and decreased as a result of interdiction by enemy aircraft. In addition, enemy aircraft can destroy vehicles and supplies traveling through the network.

It is assumed that the battalion-sized units themselves possess enough transportation vehicles to reach the forward areas; these resources enter the supply network immediately upon arrival at the port. The remaining resources are stockpiled in the port pending availability of transport vehicles,

and arrive at the ~~port~~ *FRONT?* in proportions determined by the optimization algorithm. However, so far as airlift capacity permits, individual infantry replacements are airlifted to the front.

Ground forces emerging from the network enter ground force reserve pools (battalion-sized units) and replacement pools (individual units) and thereafter are allocated in a manner described later in this section. Supplies are allocated to the two region depots and two air base depots on the basis of relative requirements that day. Empty transport vehicles re-enter the network and return to the port.

The equations used to compute the amounts of supplies entering the network, attrition to vehicles in the network, network capacity and output as a function of capacity and usage, network damage, and interdiction are all exponential equations of the type discussed in Section 5. Rather than present these equations here we refer the reader to Section 5 for details. The comments made there concerning exponential equations are, of course, equally pertinent here and perhaps even more so to nonattrition equations of exponential form. For those there is no underlying and rigorously derived equation.

User-specified or internally fixed and user-parameterized decision rules are used to control all aspects of ground force allocation other than the initial massing and initial choice of sectors of attack (discussed in Section 2); namely, selection of offensive/defensive postures in each sector, allocation of reserve battalions and replacement to sectors, and allocations of battalions into and out of a pool of "fought-out" units which have been withdrawn from on-line duty and cannot yet be recommitted to battle. After a period of time in the pool fought-out battalions enter reserve status, at which time they may re-enter combat. Replacement troops may be identified with a particular national participant and their assignment may be restricted to battalions of that participant.

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Ground force allocation computations are made last for each simulated day of combat and are made in the following order:

1. Determination of "fought-out" battalions and their transfer to pool;
2. Allocation of individual replacements to on-line, fought-out, and reserve battalions;
3. Determination of fought-out battalions to be returned to reserve status;
4. Selection of postures for next day;
5. Allocation of reserve battalions to sectors.

One of the four following replacement policies is chosen by user-prescribed input:

1. Pure unit replacement--replacements are allowed only to reserve battalions;
2. Modified unit replacement--replacements are sent first to reserve battalions and, if possible, then to fought-out battalions;
3. Modified individual replacement--individual replacements are sent to on-line, reserve, and fought-out battalions, respectively, in order of decreasing priority;
4. Pure individual replacement--individual replacements are allowed only to on-line battalions.

The exact equations used for the various computations will not be reproduced here; the reader is referred to [8].

Posture determination is based on a complicated set of rules involving the preceding day's posture, exposed flanks, success of the preceding day's attack, the number and length of exposed flanks on both sides, and FEBA movement in adjacent sectors on the preceding day. The model represents three basic postures: attack, hold (a hasty defense), and delay. We can find no evidence of a capability to model prepared defenses. The daily situation in a sector may thus be one of the following: an attack against a hold posture, an attack against a delay posture, or both sides holding (no attack). A

flank is exposed if the adjacent portion of the next sector is occupied by the opposition and critically exposed if its length exceeds a user-prescribed value. Withdrawal always occurs to reduce the length of a critically exposed flank. The precise rules for posture determination may be found in [8].

5. THE EXPONENTIAL ATTRITION EQUATION

Virtually all attrition computations in Lulejian-I employ
a particular exponential attrition equation, so it seems worth-
while to consider that equation in some detail before discussing
its application in the model. The generic attrition equation
used in Lulejian-I is of the form

$$(1) \qquad\qquad \Delta T = T\left(1 - e^{-p\frac{S}{T}}\right) ,$$

where

T = number of targets,

ΔT = attrition to targets,

S = number of shooters,

and where p is a "kill potential per shooter," which may depend
(in the model) on exogenous factors such as shooter and target
type or terrain, but must be independent of S and, especially,
of T. This seems to make physical sense only if T is so large
relative to S that detection is essentially no problem and
attrition thus depends only on the killing capabilities of
each shooter. S must also be small relative to T in order
that no target be attacked by more than one shooter.

It seems useful to seek, in the spirit of [6], a set of
carefully stated assumptions that leads to (1) or--more pre-
cisely--to an equation of binomial form to which (1) is an
approximation. Additional comments concerning the various
attrition equations discussed here can be found in [2] and
[6]. Here is one such set of assumptions that confirms the
heuristic remarks of the preceding paragraph.

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1) At a fixed time all T targets become vulnerable to detection and attack by all S searchers;

2) The probability that the ith searcher detects one or more targets is d for all $i = 1, \ldots, S$, where d is a constant in $[0,1]$. We emphasize that d is a function of neither S nor T;

3) Every target is equally likely to belong to the set of targets detected by the ith searcher. Of the targets detected by the ith searcher, he chooses exactly one to attack, according to a uniform distribution over those targets he has detected and independently of the detection process;

4) The conditional probability that a searcher kills a target, given that he attacks it, is a constant k;

5) No searcher may attack more than one target;

6) Detection and attack processes of the different searchers are mutually independent.

Consider now the jth target. The probability that this target is attacked by the ith searcher can be expressed as

$$P\{S_i \text{ attacks } T_j\} = P\{S_i \text{ makes a detection}\}$$
$$\cdot P\{S_i \text{ attacks } T_j | S_i \text{ makes a detection}\}$$
$$= d \cdot \frac{1}{T} ,$$

since targets are indistinguishable. Consequently the probability that T_j is killed is

$$P\{T_j \text{ killed}\} = 1 - \left(1 - \frac{kd}{T}\right)^S ,$$

so we have the following result.

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PROPOSITION. Subject to assumptions 1) to 6) above,

$$(2) \qquad E[\text{Targets killed}] = T\left[1-\left(1 - \frac{kd}{T}\right)^S\right].$$

If in (2) we approximate $\left(1-\frac{kd}{T}\right)^S$ by $\exp\left(-\frac{kdS}{T}\right)$ and put $p = kd$ (which now has the interpretation as the probability that a particular searcher kills *some* target), then we obtain (1). If, on the other hand, we approximate $\left(1-\frac{kd}{T}\right)^S$ by $1 - \frac{kdS}{T}$, i.e., the first two terms of the binomial expansion of $\left(1-\frac{kd}{T}\right)^S$, then (2) becomes

$$(3) \qquad \Delta T = kdS,$$

which is an attrition equation of Lanchester square form. Note that the approximation leading to (3) requires that $\frac{kd}{T} S$ be small, the physical interpretation of which is that there are so many targets, all searchers can simultaneously bring their capability to bear on (different) targets. This interpretation is consistent with both that of Lanchester and that of the work of Karr concerning stochastic analogues of Lanchester processes [5].

An interesting comparison arises between (2) and the binomial attrition equation derived in [6]. In [6], subject to a one-on-one detection hypothesis, one obtains the equation

$$(4) \qquad E[\text{Targets killed}] = T\{1-(1 - \frac{k}{T}[1-(1-d_1)^T])^S\},$$

where d_1 is the probability that a given searcher detects a *given* target. In this case the probability that a given searcher detects *some* target (i.e., one or more targets) is

(5)
$$d(T) = 1 - (1-d_1)^T \, ,$$

which is a function of T rather than a constant, as in (2).
Using (5) we can rewrite (4) as

$$E[\text{Targets killed}] = T\left[1-\left(1 - \frac{kd(T)}{T}\right)^S\right] \, ,$$

which illuminates both the similarities and the differences
between (2) and (4).

If in (4) we first make the approximation

$$(1-d_1)^T \sim 1 - d_1 T \, ,$$

and then the approximation

$$(1-kd_1)^S \sim 1 - dk_1 S \, ,$$

then (4) becomes

(6)
$$E[\text{Targets killed}] = kd_1 ST \, ,$$

which is of Lanchester linear form. Thus the linear-law version
of the attrition equation to which the prototype Lulejian-I
equation (1) is an approximation is the so-called "Kent equation"
(4). If it were desired to modify Lulejian-I to include linear-
law attrition for situations in which the one-on-one detection
hypotheses seem more plausible than the hypotheses of (2), then
(4), or some exponential approximation thereto, would seem to
be the proper equation to use.

Incidentally, the observation that (2) represents a form
of square-law attrition and (4) a form of linear-law attrition
buttresses the square-law/linear-law distinction made in [5]
and [7], as opposed to other, previous distinctions. In the

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context of both the continuous time stochastic processes discussed in [5] and the instantaneous processes discussed here, it is the linear-law process which involves one-on-one detections and the square-law process which has engagement rates (detection probabilities) independent of the size of the opposing force.

It was noted in [6] that there are formidable computational difficulties, although no theoretical ones, involved in generalizing (4) to the case of several types of searchers and targets, with detection and kill probabilities dependent on the type of target and type of searcher. The difficulties (cf. p. 25 of [6]) arise mainly from the fact that each shooter can attack at most one target. Because in the context leading to (2) the probability of detecting some target does not depend on the number of targets, derivation of the heterogeneous analogue is not so difficult.

Indeed, consider the following set of assumptions:

1) There are M types of searchers, $S(i)$ searchers of type i $(i=1,\ldots,M)$, N types of targets, and $T(j)$ targets of type j $(j=1,\ldots,N)$;

2) At a fixed time all targets become simultaneously vulnerable to detection and attack by all searchers;

3) The probability that a particular searcher of type i detects some target (of any type) is d_i, where d_i does not depend on the numbers of targets present;

4) All targets are equally likely to belong to the set of those detected by a particular searcher and of this set he chooses exactly one to attack, according to a uniform distribution and independently of the detection process;

5) The probability that a searcher of type i kills a target of type j, given that he attacks it, is k_{ij}.

6) No searcher may attack more than one target;

7) Detection and attack processes of all searchers are mutually independent.

One can then derive the following result.

PROPOSITION. Subject to assumptions 1) to 7) above, for each j

(7) $E[\text{type } j \text{ targets killed}] = T(j)\left[1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ij}d_i}{T}\right)^{S(i)}\right]$,

where

$$T = T(1) + \ldots + T(N)$$

is the total number of targets.

Making the same exponential approximation in (7) that yields (1) from (2) gives

(8) $E[\text{type } j \text{ targets killed}] \sim T(j)\left[1-\exp\left(-\frac{1}{T}\sum_{i=1}^{M}k_{ij}d_iS(i)\right)\right]$,

while making the approximations involved in putting

$$\prod_{i=1}^{M}\left(1-\frac{k_{ij}d_i}{T}\right)^{S(i)} \sim 1 - \sum_{i=1}^{M}\frac{k_{ij}d_iS(i)}{T}$$

yields

(9) $E[\text{type } j \text{ targets killed}] \sim \frac{T(j)}{T}\sum_{i=1}^{M}k_{ij}d_iS(i)$.

Now equation (9) is precisely analogous to the heterogeneous stochastic Lanchester square-law attrition process S3a of [5], so (7) is in fact a heterogeneous Lanchester square attrition equation. Moreover, the fact that (9) is analogous to process S3a of [5], rather than process S2 there, furthers the assertion made in [5] that S3a, and not S2, is the appropriate heterogeneous analogue of process S1 of [5].

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We now consider the attrition computations in the Lulejian-I model in more detail. First of all, equations (7) and (8) appear nowhere in the model; instead the model aggregates targets into a single class with kill probability, by type i searchers, of

(10)
$$\overline{k}_i = \frac{1}{T} \sum_{j=1}^{N} k_{ij} T(j) .$$

Attrition is computed and then allocated among target types on the basis of numbers of targets present, rather than on the same basis used in (10). Consequently,

$$E[\text{type } j \text{ targets killed}] = T(j) \left[1 - \exp\left(-\frac{1}{T} \sum_{i=1}^{M} \overline{k}_i d_i S(i) \right) \right] .$$

In several instances, the model represents detection and kill separately. For example (in the homogeneous case, for simplicity), the expected number of targets detected would be given by

(11a)
$$T_d = T(1 - e^{-dS/T}) ,$$

where d has the interpretation of a detection probability independent of the number of targets. Attrition would then be

(11b)
$$\Delta T = T_d \left(1 - e^{-kS/T_d} \right) ,$$

where k is a probability of kill. Substitution of (11a) into (11b) yields

$$\Delta T = T(1 - e^{-dS/T}) \left(1 - \exp\left[-kS/T(1 - e^{-dS/T}) \right] \right) ;$$

if one approximates, in the third factor, $(1 - e^{-dS/T})$ by dS/T, this last equation simplifies to

(12)
$$\Delta T = T(1 - e^{-dS/T})(1 - e^{-k/d}) ,$$

which is qualitatively the same as (1) insofar as dependence on

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S and T. Hence the preceding comments concerning equations (1) to (8) are germane to the attrition computations in Lulejian-I, even when detection and attrition are computed using separate equations.

Incidentally, it seems inappropriate to us that S should appear in (11b) rather than dS, which is the expected number of searchers that have detected some target, unless the intention is that all searchers can attempt to kill detected targets, whether or not they have themselves made a detection, which seems implausible to us. With this modification, (12) becomes

$$\Delta T = T(1-e^{-dS/T})(1-e^{-k})$$

$$\sim kT(1-e^{-dS/T}) \; ;$$

our previous comments remain relevant.

Use of exponential approximations of the form (1) or (8) to binomial equations such as (2) or (7), respectively, has both good and bad aspects. The main favorable point is computational efficiency: the time required to perform an exponentiation is approximately half that required (in (2), for example) to raise the quantity (1-kd/T) to the power S, especially when S is not an integer. When an equation is used many times in a model (there are even iterative calculations of close combat casualties, as described in Section 7), the reduction in computer time is significant. On the other hand, if the quantity kdS/T is ever very large (i.e., the approximation of (2) by (1) is not accurate), the error incurred using exponential approximations will then propagate (perhaps compounding itself) through subsequent calculations. There appears to be no way of ensuring, *a priori*, that such situations will not arise. Possibly less error would arise (especially for moderately large values of S) if in (2) one simply replaced S by its integer part (or rounded to the nearest integer). With careful

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computation of integer powers of (1-kd/T) the time required
should not greatly exceed that needed for the exponential
equation (1).

We wish to point out that computational efficiency is
not the usual justification for the use of (1) as opposed to
(2). In the first place, it is not clear that (2) was known
previously; the equation (1) may have been chosen simply on
an *ad hoc* basis, being justified--as is often the case--on the
basis of monotonicity and limit properties. Other, similar
justifications advanced in the report [8] are that exponential
equations account for "overlap" and decreasing marginal returns.

To summarize, we have the following objections to the
overall attrition methodology in Lulejian-I, aside from con-
siderations of whether the underlying assumptions are believed
satisfied in certain combat situations:

1. There is no representation of linear-law processes in
   which one-on-one detections are crucial. This seems
   especially questionable in the context of tactical
   air combat.

2. An aggregated equation is used instead of the correct
   (at least in terms of the underlying assumptions) equa-
   tions, namely (17).

3. The use of potentials obscures important physical dis-
   tinctions and computation of them may involve assump-
   tions inconsistent with those underlying the attrition
   model itself.

The next section of this paper contains more detailed
descriptions of the attrition calculations in Lulejian-I.
Inasmuch as essentially only one attrition equation is used,
the descriptions concern largely the computation of potentials
and the order of interactions, and also determination of the
numbers of weapons entering in various interactions.

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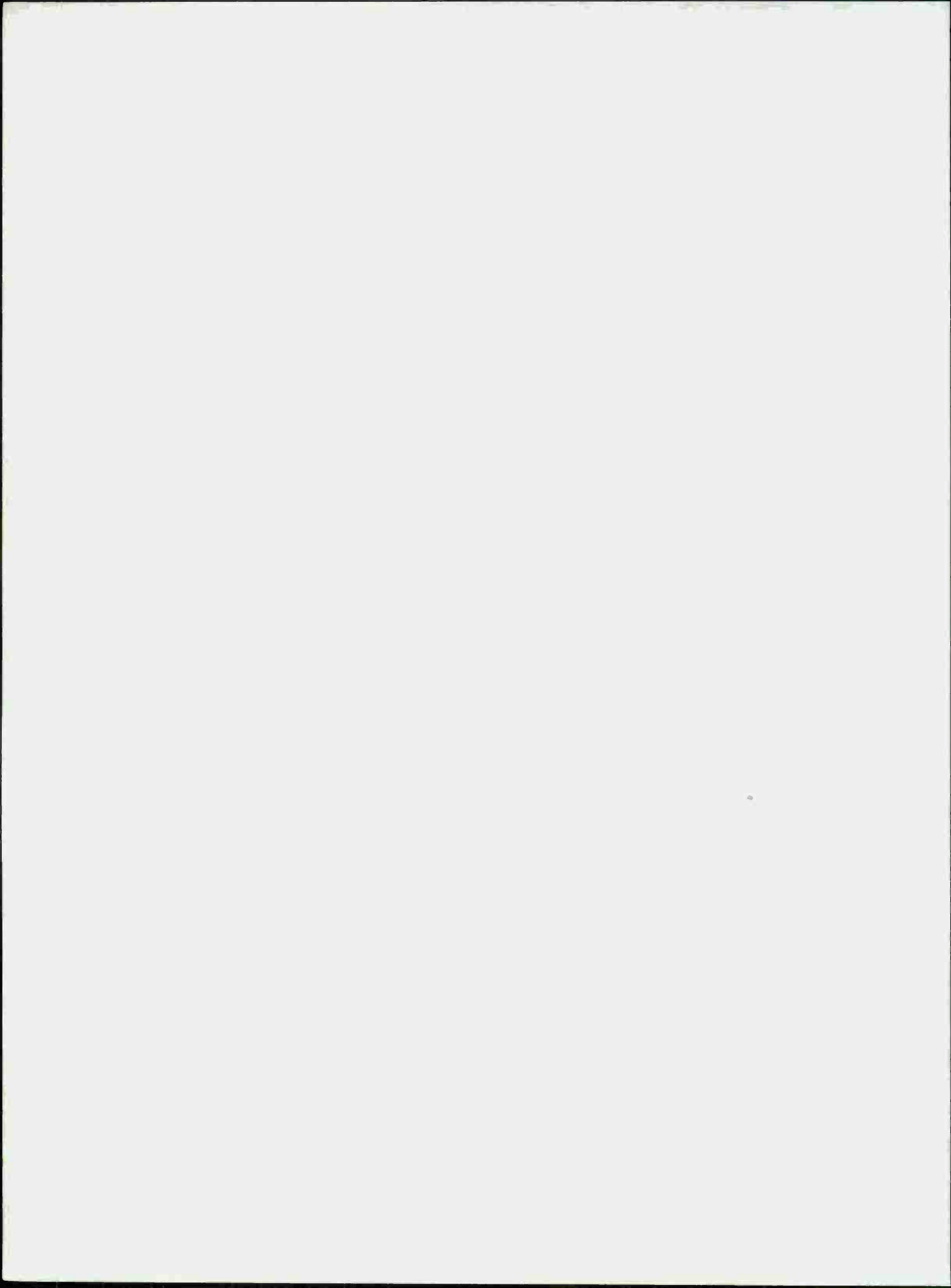
6. ATTRITION COMPUTATIONS

Tactical Air Combat Model

The Lulejian-I model represents weapon system performance degradation due to insufficient supplies by means of capability factors of the form

$$c = \frac{a}{r} \text{ ,}$$

where

c = capability factor,
a = allocated supplies,
r = required supplies.

One should best envision the fraction c of the weapon under consideration as receiving a full allocation of supplies with the remaining force "grounded." Thus the number of tactical aircraft allocated to mission type m is

(13) $A(m) = A \cdot f(m) \cdot s(m) \cdot c,$

where

A = total number of available aircraft,
f(m) = fraction of aircraft allocated to mission m,
s(m) = sortie rate on mission m,

and c is as computed above. The allocation fraction f(m) is computed by the optimization algorithm discussed in Section 2. Aircraft are further allocated to CAS and interdiction missions.

Suppression aircraft associated with attack aircraft assigned to interdiction targets may suppress area-deployed air defense artillery (ADA) on a fly-by basis. This results in no kills of

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ADA but only in reduced effectiveness, according to the equation

$$(14) \qquad f = 1 - \exp[-\frac{1}{\alpha} \sum_i A(i)\beta(i)] \, ,$$

where

i = suppression aircraft type,

$A(i)$ = number of type i suppression aircraft,

$\beta(i)$ = area suppressed by one type i aircraft,

α = area over which ADA is deployed,

and f is the fraction of ADA sites suppressed.

Attacking aircraft are then subject to attrition by area-deployed ADA, first by SAMs and then by AAA, because SAMs have greater range. Kill potentials are computed using the equations

$$(15) \qquad k(s) = n(s) \cdot c \cdot p_1 \cdot p_2$$
$$k(a) = n(a) \cdot c \cdot p \, ,$$

where

$n(s)[n(a)]$ = number of unsuppressed SAM[AAA] sites,

c = capability factor,

p_1 = tracking probability for SAM,

p_2 = probability of kill given tracking, for SAM,

p = kill probability for AAA.

As discussed in Section 2, the probabilities p_1 and p must refer to "some attacking aircraft," not "a particular attacking aircraft" and must be independent of the number of attacking aircraft. Attrition to aircraft by SAMs is then

$$(16a) \qquad \Delta A(s) = A(1 - e^{-k(s)/A})$$

and that by AAA is

$$(16b) \qquad \Delta A(a) = (A - \Delta A(s))(1 - e^{-k(a)/(A-\Delta A(s))}) \, .$$

As mentioned in Section 5, losses are allocated on the basis of

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numbers of aircraft of various types present, whereas p_1, p_2, p are computed by weighting type-dependent probabilities. Consumptions of SAMs and AAA ammunition are also computed.

The air battle proceeds in the following manner: escorts of attack aircraft seek to engage aircraft of the defender. Unengaged defenders, together with a user-input fraction of engaged, but surviving, defenders (with suitably degraded effectiveness) then engage the attack aircraft. All engaged attackers abort their mission. We now give a more detailed description.

First the defense must detect the attacking force and vector aircraft to engage it. The number of attack aircraft detected is

$$(17) \qquad A_2 = \min\left\{M, \ A_1\left(1 - e^{-2M/A_1}\right)\right\},$$

where

M = maximum number of aircraft the defense can detect,

A_1 = A − ΔA(s) − ΔA(a) = number of attacking aircraft which have survived area-deployed ADA.

The remaining A_1 − A_2 attacking aircraft are not detected and proceed unmolested to their targets. We can find no rigorous derivation of (17); some of the comments in Section 5 are applicable to it. Note that M is independent of the sizes of defending and attacking forces and so presumably represents a single, centralized detection system such as AWACS or ground-based radar.

Engagement potential of type i escorts against type j defenders is then computed to be

$$(18) \qquad p(i,j) = E(i) \ \frac{D(j)}{\sum\limits_{k} D(k)} \ r(i,j) \ ,$$

where

E(i) = number of type i escort aircraft,

D(j) = number of type j defender aircraft,

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$r(i,j)$ = engagement potential of one type i escort against *only* type j defenders,

and where $p(i,j)$ is the potential of type i escorts to engage type j defenders. Note (once more) that the $r(i,j)$ are independent of the numbers of defenders present and are presumably in units of "type j defenders per type i escort." The number of engaged defenders of type j is thus

$$(19) \qquad D_1(j) = D(j)\left(1 - \exp\left[\frac{1}{D(j)} \sum_i p(i,j)\right]\right) \ .$$

The "derivation" of this equation given in the report [8] involving Poisson distributions is incorrect. It would imply that the engagement potential of escorts is always fully realized.

As a consequence of (19) the number of type j defenders engaged by type i escorts is computed to be

$$(20) \qquad D_1(i,j) = D_1(j) \frac{p(i,j)}{\sum_k p(i,k)} \ .$$

The number of type j defenders killed by type i escorts is therefore

$$(21) \qquad \Delta D_1(i,j) = D_1(i,j)\left(1 - \exp\left[-p(i,j)k_1(j,i)/D_1(i,j)\right]\right) \ ,$$

where k_1 is the probability of kill given engagement, while the number of type i escorts killed by type j defenders is

$$(22) \qquad \Delta E(i,j) = p(i,j)\left(1 - \exp\left[-p(i,j)k_2(j,i)/D_1(i,j)\right]\right) \ ,$$

where the $k_2(j,i)$ are probabilities of kill given engagement and $p(i,j)$ is given by (18). The logic behind (22) is that each type i escort is involved in $p(i,j)/E(i)$ engagements with type j defenders and vulnerable in each of them.

The interaction between defending interceptors and attack aircraft is treated analogously, so we omit a detailed description.

A certain fraction f_j of type j defenders that are engaged but
not killed by escorts are allowed to engage the attack aircraft.
These are treated as different types of defenders in order to
reflect differential effectiveness.

No other interactions in the model are represented in such
detail; as previously noted most attrition calculations are
performed with weighted parameters and aggregated forces. The
objective is to obtain numbers of kills as a function of the
type of killing weapon. In general this appears to be an ex-
tremely difficult theoretical problem, although it can be handled
fairly easily within the context of the continuous time stochastic
Lanchester processes discussed in [5] by enlarging the state
space; the computational difficulties are always severe. A
difficulty in attempting to allocate kills on the basis of the
killing weapon, within the context of (7) or (8), is that some
target may have been killed more than once; i.e., by more than
one searcher. Such "overkills" cannot be sensibly allocated.
In equation (7)

$$1 - \prod_{i=1}^{M} (1 - k_{ij} d_i/T)^{S(i)}$$

is the probability that a given target of type j is killed (by
one or more searchers). The conditional probability that it is
killed by one or more type j searchers, given that it is killed,
is therefore

$$(23) \qquad q(i,j) = \frac{1 - (1 - k_{ij} d_i/T)^{S(i)}}{1 - \prod_{\ell=1}^{M} (1 - k_{\ell j} d_\ell/T)^{S(\ell)}} .$$

The "overkill" alluded to above is reflected in the fact that
$\sum_i q(i,j)$ may exceed 1. If, however, it is sufficiently close
to 1 (or if for each j the $q(i,j)$ were linearly normalized
to sum to 1) then the expected number of type j targets killed
by type i searchers becomes.

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$$\Delta T(i,j) = \Delta T(j) \frac{[1 - (1 - k_{ij}d_i/T)^{S(i)}]}{\sum\limits_{\ell} [1 - (1 - k_{\ell j}d_\ell/T)^{S(\ell)}]} .$$

Note that this is not the same as the equations used in Lulejian-I.

Interdiction aircraft are allocated proportionately to target types, and CAS aircraft to sectors, on the basis of numbers of battalions present; ADA suppression aircraft assigned to interdiction aircraft are allocated in the same proportions.

In the first interaction, suppression aircraft are vulnerable to point target defenses—first to SAMs and then to AAA in the same manner as described by equations (15) and (16). The calculations are performed by using weighted parameters and aggregated numbers, but are disaggregated on the same weighted basis, thus preventing a possible inconsistency alluded to in Section 5.

Suppression aircraft surviving point target ADA then attack these air defense sites, some of which are destroyed permanently, while others are suppressed for one day only. Potential of suppression aircraft against air defense sites is given by

(24) $$p = \sum_i A(i)p(i) ,$$

where

   $A(i)$ = number of suppression aircraft of type i,
   $p(i)$ = fraction of one ADA site suppressed by one type i aircraft, per sortie.

It is assumed that not more than one ADA site can be suppressed on one sortie. The number of ADA sites suppressed is then

(25) $$\tilde{N} = N(1 - e^{-p/N}) ,$$

where

   N = number of ADA sites;

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these sites cannot interact with attacking aircraft. The number
of ADA sites destroyed is

(26) $$\Delta N = q\tilde{N} \ ,$$

where

q = probability an ADA site is killed, given it is suppressed.

Suppressed, but not killed, sites return to full effectiveness in
the next day's combat. Attacked but unsuppressed ADA battalions
have decreased effectiveness against attack aircraft, losses of
which are then computed in an entirely analogous manner.

There is present in each combat sector additional ADA to
defend against enemy aircraft on the CAS mission in that sector.
As with the interdiction mission, the attack aircraft have ADA
suppressor aircraft associated with them. The number of sup-
pressor aircraft engaged by SAM sites is given by

(27) $$A_1 = A(1 - \exp[- 2nrc/w]),$$

where

    A = number of ADA suppressor aircraft attacking in the
        sector,

    n = number of ADA sites in the sector,

    r = site acquisition radius,

    c = capability factor based on general supplies,

    w = sector width.

In order that (27) be plausible, the ratio r/w must be small.
The number of suppressor aircraft destroyed by SAMs is there-
fore

(28) $$\Delta A = A_1(1 - \exp[- nmcc'q_1q_2/A_1]) \ ,$$

where c, n, A_1 are as in (27) and where

    m = number of SAMs fired per site,

    c' = capability factor based on ammunition supplies,

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q_1 = tracking probability,

q_2 = conditional probability of kill, given tracking.

As discussed in connection with air-to-air attrition, a plausible interpretation of (27) and (28) requires that detection be carried out by a system physically distinct from the firing system and that there be some communication of detection information (as opposed to each site being able to attack only those aircraft it detects). Note that while detection information is communicated, attacks are uncoordinated in the sense of being probabilistically independent; this seems implausible.

The remaining A-ΔA suppression aircraft are then vulnerable to AAA sites and attrition is calculated entirely analogously to the preceding computations (using a two-step process). ADA suppression is then computed in the same manner as described in equations (14) or (25).

Aircraft losses to, and suppression of, ADA sites associated with logistics targets and air bases are computed analogously. The same is true for calculations involving attack aircraft assigned to the CAS mission.

We next discuss attrition of aircraft on the ground at air bases. Of the airlift aircraft at an air base, a user-input fraction are at risk. The number of tactical aircraft at risk is determined by means of the equation

$$(29) \qquad B = \sum_i \big(B(i) - \Delta B(i)\big)\big(1 - s(i)cd(i)\big) ,$$

where

i = mission type,

$B(i)$ = number of tactical aircraft assigned to mission i,

$\Delta B(i)$ = aircraft losses on mission i,

$s(i)$ = sortie rate on mission i,

c = capability factor based on supplies,

$d(i)$ = fraction of day required for one sortie on mission i;

and where B is the number of tactical aircraft at risk. Tactical aircraft are sheltered, without priority as to type, to the extent possible. Thus, of the B tactical aircraft at risk,

$$(30) \qquad\qquad B_s = \min\{B, S\}$$

are sheltered, where S is the number of available shelters, and the remaining $B_u = B - B_s$ are parked in the open. Airlift aircraft are never sheltered, even if shelters remain unoccupied by tactical aircraft.

The Lulejian-I model treats aircraft in the open as "point targets" and aircraft in shelters as "area targets." Kill potential of attacking aircraft against unsheltered aircraft is

$$(31) \qquad\qquad p_u = \sum_j A_j k_j \ ,$$

where

> j = type of attacking aircraft,
> A_j = number of type j attacking aircraft not engaged by any defenses,
> k_j = potential of one type j attacker, in units of expected number of unsheltered aircraft destroyed.

Potential against sheltered aircraft is

$$(32) \qquad\qquad p_s = \sum_j A_j \ell_j \ ,$$

where ℓ_j is the kill potential of one type j aircraft against shelters, in units of area. Since (as will be made explicit in (33) and (34) below) no allocation of attacking aircraft to different targets is made, the potentials k_j and ℓ_j must account for some sort of allocation of fire on the part of attacking aircraft. Attrition to aircraft in the open is

$$(33) \qquad \Delta B_u = (\hat{B} + B_u)\left(1 - \exp[-\, p_u/(\hat{B} + B_u)]\right) \ ,$$

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where $\hat{B}$ is the number of airlift aircraft at risk. Attrition to shelters is

$$\Delta S = S(1 - \exp[- p_s/a]) ,$$

where

a = area *initially* occupied by shelters.

Use of the initial area implies that destroyed shelters are not discernible to attackers.

The number of sheltered aircraft destroyed is then

$$(34) \qquad \Delta B_s = \begin{cases} \Delta S \quad , \quad \text{if } B_s = S \\[2ex] \Delta S \dfrac{B}{S} \quad , \quad \text{if } B_s = B . \end{cases}$$

As usual, losses to different types of aircraft are allocated proportionately.

Ground Combat Model

As is done for most air combat computations, all weapons within each weapon class (tanks, APCs, artillery, CAS aircraft, helicopters, mortars, and antitank weapons), are aggregated into one weapon type with averaged parameters for the purpose of computing ground combat losses. Losses are allocated on the basis of numerical proportions.

To begin, for each sector (calculations are done individually for each sector) the number of on-line defending battalions of each type (infantry, mechanized infantry, and tank battalions) is computed by means of the equation

$$(35) \quad B_1(i) = B(i)[f_0 + (f_m - f_0)(1 - e^{-1})^{-1}(1 - e^{-\Delta F/F_m})] ,$$

where

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i = battalion type (1 = infantry, 2 = mechanized infantry, 3 = tank),

$B(i)$ = number of battalions in sector,

f_0 = fraction of battalions on line when no FEBA movement is occurring,

f_m = fraction of battalions on line when maximum FEBA movement is occurring,

F_m = maximum possible FEBA movement,

ΔF = present day's FEBA movement,

and where $B_1(i)$ is the number of type i defending battalions on line. The iterative computation scheme discussed in Section 7 in fact begins at this point.

The assumptions leading to (35) are not easily discerned. When $\Delta F = 0$, (35) reduces to

$$B_1(i) = B(i)[f_0] \; ;$$

that is, when there is no FEBA movement the defender, through a combination (presumably) of choice and necessity, maintains a fraction f_0 of his battalions on line. On the other hand, if $\Delta F = F_m$ then (35) becomes

$$B_1(i) = B(i)[f_m] \; ,$$

since in this case

$$\left(1 - e^{-\Delta F/F_m}\right) = \left(1 - e^{-1}\right) \; .$$

Thus (35) is just an interpolation:

$$B_1(i) = B(i)\left[f_0\left(\frac{e^{-\Delta F/F_m} - e^{-1}}{1 - e^{-1}}\right) + f_m\left(\frac{1 - e^{-\Delta F/F_m}}{1 - e^{-1}}\right)\right] \; .$$

Why an interpolation is appropriate is not clear. Moreover, use of this form has implications concerning positioning of forces within a sector, the rate at which reserve forces may be committed to try to stem an advance, and the rate at which

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reserve forces become "on-line" as the FEBA moves toward them.
None of these implications is explained in the report [8].

The next state of interaction represented is contacting
of on-line defending battalions by attacking battalions. The
number of type i defending battalions contacted by attacking
battalions is given by

$$(36) \qquad B_2(i) = B_1(i)\left(1 - \exp\left[- \frac{1}{F_m w} \sum_j w_j s_j R(j)\right]\right) ,$$

where

$\qquad j$ = attacking battalion type,

$\qquad R(j)$ = number of type j attacking battalions,

$\qquad w$ = sector width,

$\qquad w_j$ = search width of type j battalion,

$\qquad s_j$ = advance rate (maximum) of type j battalions.

Here again, the exponential equation seems to be used on a rather
ad hoc basis. Note that $B_2(i)/B_1(i)$ is independent of i; that is,
the fraction of defending battalions contacted is the same for all
battalion types. This assumption that all battalion types are
equally easy to contact seems implausible. An analogous equation
gives numbers $R_2(j)$ of type j attacking battalions contacted by
the defense; contact is thus neither mutual nor symmetric.

The next interaction is "location of opposing forces at
the squad level," which should not be construed to mean that
the Lulejian-I model operates at this level of detail. On the
contrary, it is more aggregated than IDAGAM I [1,2], at least
in terms of attrition calculations. The purpose of this parti-
cular set of computations is to take into account, in a fairly
explicit fashion, the effect of separation distances of oppos-
ing forces. More detailed discussions of the rationale for
treating distance appear in Sections 2 and 7, which (briefly
stated) is that each side can reduce its casualties to an

acceptable level by increasing the separation distance between
the two sides. The defender accomplishes this by retreating,
the attacker by breaking off and withdrawing. FEBA movement
and ground combat casualties are, therefore, computed jointly
using an iterative technique discussed in more detail in the
next section. Thus "location at squad level" is but a surro-
gate used to account for the effect of separation distances on
ground combat attrition.

The model keeps a record of nine separation distances (be-
tween each kind of defending battalion and each kind of attacking
battalion). The number of located type i defending battalions is
computed using the equation

$$(37) \qquad B_3(i) = B_2(i)\left(1 - \exp\left[- \sum_{j=1}^{3} p_{ij}/r_{ij}\right]\right),$$

where

$B_3(i)$ = number of located defending battalions of type i,

p_{ij} = "potential of type j attacking battalions to
locate type i defending battalions,"

r_{ij} = separation distance between type i defending
battalions and type j attacking battalions.

The location potentials p_{ij}, so far as we can determine from
[8], depend on neither the numbers of defending battalions nor
the numbers of attacking battalions. These potentials must
also, evidently, be valid at unit separation distance. In
order that (37) have the proper dimensions, the p_{ij} should be
in units of distance, which we cannot interpret in a plausible
manner. Note that (37) also contains an explicit quantification,
but without a physically plausible and rigorously stated under-
lying assumption, of the effect of separation distance on the
ability to locate opposing battalions. Again, the exponential
equation seems to have been chosen on an *ad hoc* basis.

Similar equations give the numbers $R_3(j)$ of accurately located attacking battalions. Located numbers of battalions are transformed to located numbers $I_D(I_A)$ of defending (attacking) infantry, numbers $T_D(T_A)$ of defending (attacking) tanks, and numbers $P_D(P_A)$ of defending (attacking) armored personnel carriers (APCs) in the obvious manner. Similar notations with "~" denote contacted numbers.

All *contacted* infantry and armor are vulnerable to support fire from the opposing side (arising from support battalions, artillery, aircraft, and helicopters). The number of defending infantry killed by support fire is

$$(38) \qquad \Delta \tilde{I}_D^{(1)} = \tilde{I}_D \left(1 - e^{-fd/\tilde{I}_D} \right),$$

where d is an effective density of defending infantry and f is the antipersonnel potential (APP) of support fire by the attacker (in units of area per day). The computation of f involves summing over the different types of support fire weapons; the contribution from each weapon type depends linearly on the number of weapons of that type present.

Armor losses to fire from CAS aircraft and helicopters in support roles are computed in the following manner. The number of defending tanks acquired for fire is

$$(39) \qquad T_D^{(1)} = \tilde{T}_D \left(1 - \exp\left[\left(- \frac{T_D}{\tilde{T}_D} \frac{\tilde{T}_D}{\tilde{T}_D + \tilde{P}_D} \right) f \right] \right),$$

where

$T_D^{(1)}$ = number of defending tanks acquired,

T_D, $\tilde{T}_D$, $\tilde{P}_D$ are as above,

and where f is the antiarmor potential of support fire units of the attacker, in units of search potential per day, and is obtained by summing contributions from aircraft and helicopters.

The factor $T_D/\tilde{T}_D$ is the fraction of contacted tanks located and
is asserted in [8] to account for information received from
maneuver units. We do not understand how it does so. In order
that (39) be consistent with the other equations in the model it
seems necessary to include an additional factor of $1/\tilde{T}$ in the
exponential term. Acquisition of defending APCs and attacking
armor is entirely analogous. Antiarmor attrition potential p is
obtained by summing contributions from CAS aircraft, helicopters,
and artillery and is a linear combination of the numbers of wea-
pons present. Note that artillery are involved in attrition but
not acquisition, implying further communication of information
among various weapon systems. The number of defending tanks
destroyed by support fire is thus

$$(40) \qquad \Delta T_D^{(1)} = T_D^{(1)} \left(1 - \exp\left[- p \Big/ \left(T_D^{(1)} + P_D^{(1)} \right) \right] \right) .$$

Losses of APCs and attacking armor are computed in the same way.

Next we discuss computation of infantry losses due to oppos-
ing maneuver units; in some sense this and similar equations for
armor losses are the main attrition equations of the model. Anti-
personnel potential of attacking maneuver unit weapons is

$$(41) \qquad p = \alpha(\tilde{I}_A - (1/2)\Delta I_A^{(1)})$$
$$+ \beta(\tilde{T}_A - (1/2)\Delta T_A^{(1)})$$
$$+ \gamma(\tilde{P}_A - (1/2)\Delta P_A^{(1)}) ,$$

where α, β, γ are per weapon antipersonnel potentials and are of
the form $c + d/R$ where c, d are constants (which depend on the
weapon type) and R is the appropriate separation distance; cf.
(37). Here also $\tilde{I}_A$, $\tilde{T}_A$, $\tilde{P}_A$ are the numbers of *contacted* attacking
infantry, tanks, and APCs, respectively, and $I_A^{(1)}$, $T_A^{(1)}$, $P_A^{(1)}$ are
losses due to support fire of the defending side. The adjustments
for losses represent an effort to account for the fact that all

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support fire-induced casualties do not occur at one time. This
is one of the few attempts we have seen to deal explicitly with
the problem of discrete computation of losses occurring over
time; in most cases, however, fractional losses are sufficiently
low that this adjustment does not cause a perceptible increase
in the accuracy or realism of the model. The number of defen-
ding infantry killed by attacking maneuver unit weapons is then
computed using the equation

$$(42) \qquad \Delta I_D^{(2)} = \left[I_D \left(1 - \frac{\Delta I_D^{(1)}}{\tilde{I}_D} \right) \right] \left(1 - e^{-\frac{p}{I_D}} \right),$$

where

p = APP computed in (41),

I_D = number of located infantry, from (37),

and other quantities are as previously computed. The factor

$\left(1 - \dfrac{I_D^{(1)}}{\tilde{I}_D} \right)$ is the fraction of contacted infantry which survive

supporting fire. It is assumed that the same fraction of located
infantry survive and that only these are vulnerable to maneuver
unit weapons. Hence for purposes of computing each side's losses
all support fire casualties are assumed to occur at once at the
beginning of the day, while for computing potential to destroy
elements on the other side a different assumption is made. More-
over, the exponential term in (42) should contain the same ad-
justment as does the other factor; this is an inconsistency in
(42). Losses of tanks and APCs to maneuver unit weapons are
computed using analogous equations, but taking into account the
relative indistinguishability of tanks and APCs. Thus, the
number of defending tanks destroyed by maneuver units is

(43) $\quad \Delta T_D^{(2)} = \left[T_D \left(1 - \dfrac{\Delta T_D^{(1)}}{\tilde{T}_D} \right) \right]$

$$\times \left(1 - \exp \left\{ - p \left[T_D \left(1 - \frac{T_D^{(1)}}{\tilde{T}_D} \right) + P_D \left(1 - \frac{P_D^{(1)}}{\tilde{P}_D} \right) \right]^{-1} \right\} \right) ,$$

where p is now the antiarmor potential of maneuver unit weapons of the attacker. Similarly, the number of defending APCs destroyed by maneuver units is

(44) $\quad \Delta P_D^{(2)} = \left[P_D \left(1 - \dfrac{\Delta P_D^{(1)}}{\tilde{P}_D} \right) \right]$

$$\times \left(1 - \exp \left\{ - p \left[T_D \left(1 - \frac{T_D^{(1)}}{\tilde{T}_D} \right) + P_D \left(1 - \frac{P_D^{(1)}}{\tilde{P}_D} \right) \right]^{-1} \right\} \right) ,$$

where p is again the antiarmor potential. Note that the second inconsistency of (42) does not occur in (43) or (44).

Hence the main ground combat attrition equation in Lulejian-I contains two inconsistencies, neither of which is likely (for small loss rates) to cause any practical effect on the results of the model, but each of which could do so in certain situations and each of which can rather easily be corrected.

Attack helicopters are assumed to carry out grouped attacks so as to attempt to saturate AAA defenses. After much argument the attrition equation derived in the report turns out to be

(45) $$\Delta H = cH ,$$

where H is the number of attacking helicopters and c is the maximum "acceptable" attrition rate for helicopters. The helicopters are assumed to choose a standoff distance r such that (45) holds.

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In this case, subject to the usual exponential attrition equation, one can compute that

$$(46) \qquad r = - \frac{\alpha}{\log(1-c)} \frac{N}{H} ,$$

where

N = number of AAA guns,

α = kill potential per AAA gun at unit range.

Thus kill potential of AAA guns against helicopters is inversely proportional to range.

The support fire contribution of helicopters is also inversely proportional to range and is given by

$$(47) \qquad f_H = H \frac{p}{r} ,$$

where

p = support fire potential at unit range,

and r is given by (46).

The attrition equations described here are used in an iterative joint calculation of FEBA movement and losses, described in further detail in the next section.

7. FEBA MOVEMENT CALCULATION

A unique feature of Lulejian-I is its iterative method for joint calculation of attrition to ground combat maneuver forces and FEBA movement. As previously discussed, use of this methodology is justified by the behavioral assumption that combat commanders trade off casualties and separation distance, generally by withdrawing when casualties exceed acceptable levels. We now give a more detailed description of the mathematical representation of this process in Lulejian-I. For an alternative treatment of the problem we refer the reader to IDAGAM I [1,2].

We introduce the following notations, which are essentially, but not precisely, those of Section 6.

$I_D[I_A]$ = total number of defending [attacking] infantry committed to battle,

$\Delta I_D[\Delta I_A]$ = total casualties to defending [attacking] infantry,

$T_D[T_A]$ = total number of defending [attacking] tanks committed to battle,

$\Delta T_D[\Delta T_A]$ = total losses of defending [attacking] tanks,

$P_D[P_A]$ = total number of defending [attacking] APCs committed to battle,

$\Delta P_D[\Delta P_A]$ = total losses of defending [attacking] APCs,

$R(U,V)$ = mean separation distance between type U defending battalions and type V attacking battalions (I - infantry battalions, P - mechanized infantry battalions, T - armored battalions),

$c_D(U)[c_A(U)]$ = maximum acceptable attrition rates for defending [attacking] battalions of type U.

All losses are functions of the separation distances R.

The model assumes that movement of each type of weapon is directly proportional to the ratio of actual attrition to maximum acceptable attrition. Thus the forward movement of attacking the infantry is

$$(48a) \qquad M_A(I,R) = S\left(1 - \frac{\Delta I_A(R)}{c_A(I)I_A}\right) ,$$

where S is the maximum rate of movement (achieved only if no casualties are suffered; i.e., during unopposed advance). The movement is negative if and only if actual casualties ΔI_A exceed maximum acceptable casualties $C_A(I)I_A$. Equations analogous to (48a) are used to compute the following quantities:

$$M_A(T,R) = \text{movement of attacking tanks} ,$$
$$M_A(P,R) = \text{movement of attacking APCs} .$$

Movement of defending infantry (with positive movement still in the direction of the attacker) is given by

$$(48b) \qquad M_D(I,R) = S\left(\frac{\Delta I_D(R)}{C_D(I)I_D} - 1\right) .$$

The corresponding movements of defending tanks and armored personnel carriers are computed analogously.

For purposes of consistency the model requires that all six movements $M_A(I,R)$, ..., $M_D(P,R)$ be equal (so that there is a well-defined "FEBA movement," namely, the common value of the six movements). Thus one wants R to be such that

$$(49) \quad M_A(I,R) = M_A(T,R) = M_A(P,R) = M_D(I,R) = M_D(T,R) = M_D(P,R) .$$

There are five independent separation distances (the remaining four can be computed from these) and (49) contains five equations.

Let us consider for a moment the meaning of the FEBA movement computation embodied in (48) and (49). If the common value of the six movements is positive then

1) The attacker receives fewer than the maximum acceptable casualties to all maneuver unit types;

2) The defender receives more than the maximum acceptable casualties to each type of maneuver unit;

3) The attacker advances.

If, on the other hand, the common value is negative, then

1) The attacker receives more than the maximum acceptable casualties to each type of maneuver unit;

2) The defender receives fewer casualties than the maximum acceptable number;

3) The attacker retreats.

Only in the seemingly unlikely event that every maneuver unit type on both sides receives exactly its maximum acceptable casualties does the FEBA fail to move. This seems implausible to us; a more realistic view would be that over a rather large set of force levels and capabilities no FEBA movement takes place.

Observe, in all cases, that all types of maneuver units on a particular side incur the same ratio of actual to acceptable casualties, that (except if no FEBA movement occurs) exactly one side exceeds its casualty thresholds, and that the amount by which the advancing side is below its thresholds is related to the amount by which the retreating side exceeds its thresholds. None of these mathematical assumptions is even physically motivated, let alone plausible or desirable.

In order to avoid difficulties with S as the maximum FEBA movement, one must assume that

(50)
$$\frac{\Delta I_A(R)}{C_A(I)I_A} \leq 2 \ ,$$

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and similarly for other casualty ratios. For this implicit assumption that actual casualties cannot exceed acceptable casualties by a factor of more than two we know of no physical justification, nor even of a plausibility argument.

Thus the purpose of the iterative computation is to select values of R for the given day such that (49) is satisfied, in which case that day's losses and FEBA movement can then be computed as described in Section 6 and (48), respectively. The iteration process is thus the following:

1. Choose initial estimates of R and ΔS (the common value of the movements when (49) is satisfied);

2. Use the separation distances to calculate, by the attrition methodology described above (namely equations (35) through (47)), losses and, by (48), movements;

3. Check whether the equations (49) are satisfied. If so, the computation is done. If not, estimate new separation distances R and FEBA movement ΔS and return to Step 2.

The technique used to produce new estimates for R and ΔS is essentially a gradient method (i.e., Newton's method). When the iteration scheme converges (*if* it does; there is no proof in the report that it will, but experience seems to indicate that it does) the day's losses and FEBA movement have then been computed.

More basic still are questions of existence and uniqueness of solutions to the system (49). Neither of these problems is addressed in the report [8] nor is there any indication of their having been dealt with in another document. The system (49) consists of five nonlinear equations in five unknowns; four of the nine equations presented in the report are identities and need not be considered when discussing existence, uniqueness and computation of solutions. In the absence of linearity or a

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surrogate such as convexity, there are few explicit results concerning existence and uniqueness of solutions to such systems of equations.

To illustrate the difficulties involved, we consider a simplified situation in which there is one type of maneuver force on the defending side and two types of maneuver forces on the attacking side. Suppose, for concreteness, that the defending side possesses only tanks and the attacking side possesses tanks and APCs. The relevant numbers of resources are T_D, T_A and P_A, respectively; the respective attrition thresholds are $c_D(T)$, $c_A(T)$ and $c_A(P)$. $R(T,T)$ is the separation distance between defending and attacking tanks; $R(T,P)$ is the separation distance between defending tanks and attacking APCs. Without loss of generality, we assume that the maximum FEBA movement S is equal to one. We then wish to solve the equations

$$
\text{(51a)} \qquad \frac{\Delta T_A}{c_A(T)T_A} = \frac{\Delta P_A}{c_A(P)P_A}
$$

$$
\text{(51b)} \qquad \frac{\Delta T_D}{c_D(T)T_D} - 1 = 1 - \frac{\Delta T_A}{c_A(T)T_A} \ .
$$

Based on Equations (35) through (47) we simplify the loss equations relating ΔT_D, ΔT_A and ΔP_A to the separation distances to become

$$\Delta T_A = T_A \left[1 - \exp\left(-\frac{k_1 T_D}{T_A R(T,T)} \right) \right]$$

(52)
$$\Delta P_A = P_A \left[1 - \exp\left(-\frac{k_2 T_D}{P_A R(T,P)} \right) \right]$$

$$\Delta T_D = T_D \left[1 - \exp\left(-\frac{\ell_1 T_A}{T_D R(T,T)} \right) \right]$$

where k_1, k_2, ℓ_1 are constants representing the combined effect of various potentials discussed in Section 6. Note that defending tanks are invulnerable to attacking APCs.

The goal is to find values of $R(T,T)$ and $R(T,P)$ such that (51) is satisfied, where ΔT_A, ΔP_A and ΔT_D are computed using (52). We first show that there is a unique value of $R(T,T)$ such that

$$\frac{\Delta T_D}{c_D(T) T_D} - 1 = 1 - \frac{\Delta T_A}{c_A(T) T_A} \; ;$$

i.e., there is a unique solution to equation (51b). Using (52) this equation may be transformed to

(53)
$$2 - \frac{1}{c_A(T)} - \frac{1}{c_D(T)} + \frac{1}{c_A(T)} e^{-k_1 T_D/T_A \; R(T,T)}$$
$$+ \frac{1}{c_D(T)} e^{-k_2 T_A/T_D \; R(T,T)} = 0 \; .$$

Let us consider the left-hand side of (53) as a function f of $a = e^{-1/R(T,T)}$. If there is a unique root of the equation

$$f(a) = 0 \; , \quad a \in [0,1] \; ,$$

then there exists a unique value of $R(T,T)$ satisfying (53). We observe from (53), however, that

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(54) $$f(0) = 2 - \frac{1}{c_A(T)} - \frac{1}{c_D(T)} \; ,$$

which is strictly negative unless $c_A(T) = c_D(T) = 1$ and zero in this case, that

(55) $$f(1) = 2 \; ,$$

and that

(56) $$f'(a) = \frac{k_1 T_D}{c_A(T) T_A} a^{(k_1 T_D/T_A)-1} + \frac{k_2 T_A}{c_D(T) T_D} a^{(k_2 T_A/T_D)-1} \; ,$$

which is strictly positive on $(0,1]$. It is immediate from (54), (55), and (56) that there exists a unique point $a \in [0,1]$ such that $f(a) = 0$.

The preceding development fixes a unique value $R(T,T)$ such that (51b) holds. With this $R(T,T)$ fixed, it remains to solve (51a) for $R(T,P)$. After obvious simplifications one can write (51a) as

(57) $$\frac{1}{c_A(T)}\left(1-e^{-k_1 T_D/T_A R(T,T)}\right) = \frac{1}{c_A(P)}\left(1-e^{-\ell T_D/P_A R(T,P)}\right) \; .$$

Recalling that $R(T,T)$ is fixed, let us write $a = e^{-1/R(T,P)}$, whereupon (57) becomes

(58) $$\frac{1}{c_A(T)}\left(1-e^{-k_1 T_D/T_A R(T,T)}\right) - \frac{1}{c_A(P)} + \frac{1}{c_A(P)}a^{\ell T_D/P_A} = 0 \; .$$

Consider the left-hand side of (58) as a function g of $a \in [0,1]$; we wish to consider existence and uniqueness of solutions to the equation $g(a) = 0$. Proceeding as in (54) to (56), we see that

$$g(0) = \frac{1}{c_A(T)}\left(1-e^{-k_1 T_D/T_A R(T,T)}\right) - \frac{1}{c_A(P)} \; ,$$

$$g(1) = \frac{1}{c_A(T)}\left(1-e^{-k_1 T_D/T_A R(T,T)}\right) \; ,$$

which is positive, and that

$$g'(a) = \frac{\ell T_D}{c_A(P)P_A} a^{(\ell T_D/P_A)-1} \; ,$$

which is positive for $a \in [0,1]$. If $g(0) \leq 0$, then there exists a unique $a \in [0,1]$ such that $g(a) = 0$, and hence a unique solution $R(T,T)$, $R(T,P)$ to (51).

If, on the other hand $g(0) > 0$ then there exists no $a \in [0,1]$ with $g(a) = 0$, and hence no solution to (51). One can easily find combinations of parameters such that $g(0)$ is positive; one such set is

$$T_D = T_A$$
$$k_1 = 1$$
$$R(T,T) = 1$$
$$c_A(T) = 1/4$$
$$c_A(P) = 1/2 \; .$$

In this case

$$\frac{1}{c_A(T)}\left(1-e^{-k_1 T_D/T_A R(T,T)}\right) - \frac{1}{c_A(P)} = (1/4)^{-1}(1-e^{-1}) - (1/2)^{-1}$$
$$\simeq 4(.63) - 2$$
$$= .52 \; .$$

Consequently, equations of the sort used in Lulejian-I may fail to admit a solution at all.

A second example further illustrates the difficulties that can arise. Suppose that each type of maneuver battalion is vulnerable only to opposing battalions of the same type. In this case the attrition equations can be written for armored battalions as

$$(59) \quad \begin{aligned} \Delta T_A &= T_A\left(1-e^{-k_1 T_D/T_A R(T,T)}\right) \\ \Delta T_D &= T_D\left(1-e^{-k_2 T_A/T_D R(T,T)}\right), \end{aligned}$$

for mechanized infantry battalions as

$$(60) \quad \begin{aligned} \Delta P_A &= P_A\left(1-e^{-\ell_1 P_D/P_A R(P,P)}\right) \\ \Delta P_D &= P_D\left(1-e^{-\ell_2 P_A/P_D R(P,P)}\right), \end{aligned}$$

and for infantry battalions

$$(61) \quad \begin{aligned} \Delta I_A &= I_A\left(1-e^{-m_1 I_D/I_A R(I,I)}\right) \\ \Delta I_D &= I_D\left(1-e^{-m_2 I_A/I_D R(I,I)}\right). \end{aligned}$$

Here k_1, k_2, ℓ_1, ℓ_2, m_1, m_2 represent combined potentials.

The analysis used in the first example shows that there is a unique value of $R(T,T)$ such that

$$1 - \frac{\Delta T_A}{c_A(T)T_A} = \frac{\Delta T_D}{c_D(T)T_D} - 1,$$

where ΔT_A, ΔT_D are computed using (59); let us denote by b_1 the common value of both sides of (62). Similarly, there is a unique value of $R(P,P)$ such that, subject to (60),

$$1 - \frac{\Delta P_A}{c_A(P)P_A} = \frac{\Delta T_D}{c_D(P)P_D} - 1 \equiv b_2,$$

and there is a unique value of $R(I,I)$ such that, subject to (61),

$$1 - \frac{\Delta I_A}{c_A(I)I_A} = \frac{\Delta I_D}{c_D(I)I_D} - 1 \equiv b_3 \ .$$

Now one of two cases must obtain. The first is that b_1, b_2, b_3 are not all equal; in this case there is no solution to (49). The second is that $b_1 = b_2 = b_3$ in which case there is a two-dimensional continuum of solutions to (49), for no constraints are placed on the values of $R(T,P)$ and $R(I,P)$. Either case is unacceptable to the user of the model.

The examples above do not demonstrate that the system (49) always fails to have a solution--only that in some cases a solution may fail to exist. In the sense that they represent situations with special structure and are based on equations not exactly identical with those in the model, the examples are not fully general and the question of whether, when every maneuver battalion is present and vulnerable to all types of opposing battalions, solutions exist and are unique, is still open. Our examples do demonstrate that there exist difficulties with both existence and uniqueness in certain cases; consequently such difficulties must be presumed to exist in all cases in which a proof to the contrary is lacking. That is, until a proof is constructed that verifies existence and uniqueness of solutions to (49) in the exact form of the equations appearing in the model, the user who is scientific and rational must view the entire computation with suspicion. When difficulties arise with special cases of a computational procedure, it is the responsibility of the proponents of that procedure to rigorously identify those cases in which the difficulties do not obtain.
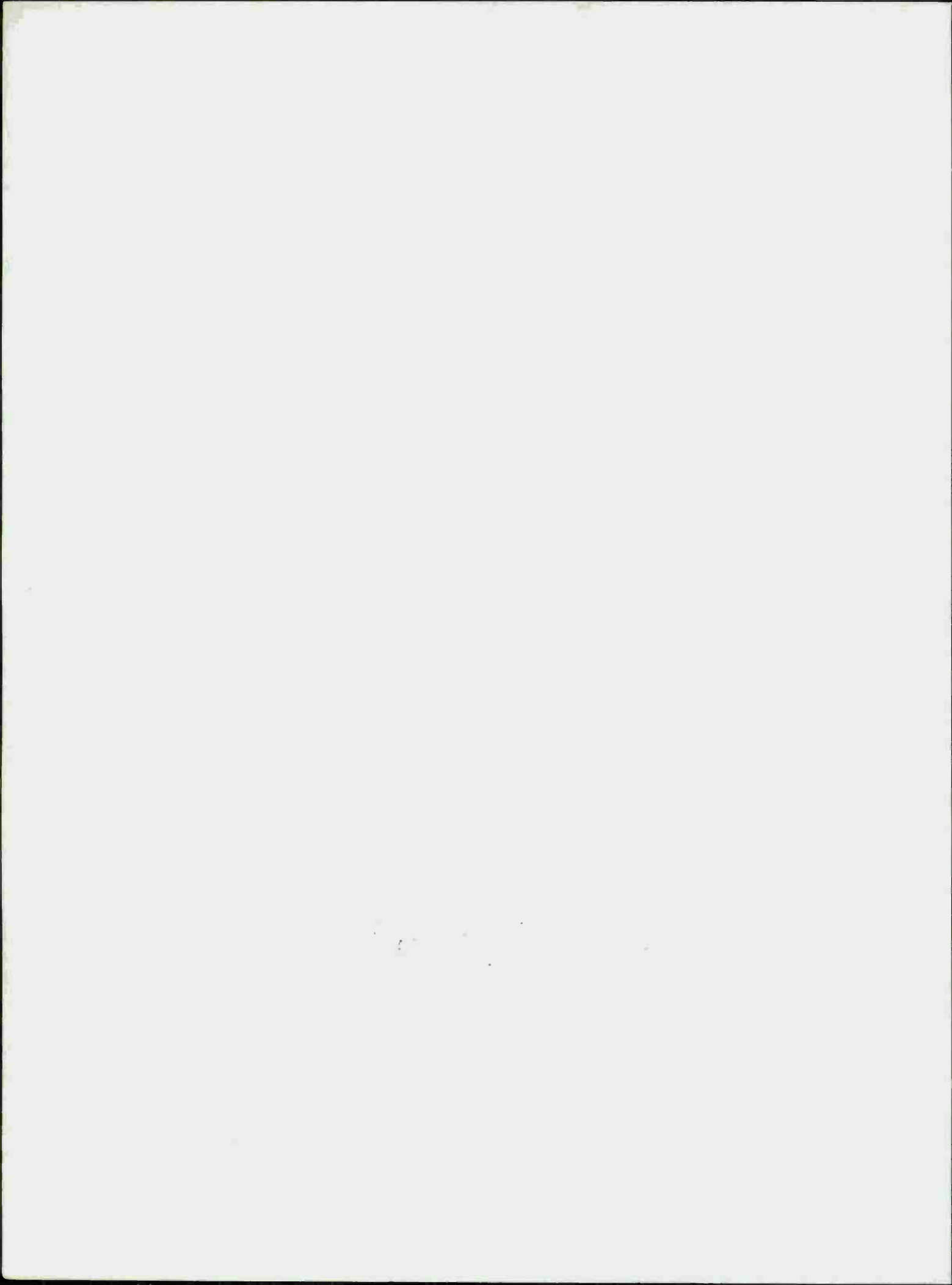
In the absence of a rigorous mathematical proof (which, in view of the complexity of the model, we feel would be most difficult to construct) empirical evidence may be useful if treated with the proper skepticism. Convergence of the computational algorithm--if it is properly programmed and contains no hidden

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devices that ensure "convergence"--is valid empirical evidence for the existence of solutions, but provides no information concerning uniqueness. The same is true of observed smoothness of the dependence of solutions on parameter values. Any extrapolation based on empirical testing, especially if qualitative (i.e., to different equations) should be regarded as unreliable.

To conclude, we believe that (and it should be noted that the point is both philosophical and practical) the entire model must be viewed with suspicion, if not downright distrust, until and unless the existence and uniqueness question is settled. Our examples demonstrate that it is very much an open question at this time.

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