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ANTAGONISTIC GAMES

Lowell Bruce Anderson

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This paper defines two subsets of noncooperative, finite, two-person (bimatrix) games. The first subset, called *antagonistic games*, consists of bimatrix games in which if one player does better, the other player necessarily must do worse (where better and worse are determined by comparing the payoffs that result from playing two alternative pairs of actions). Zero-sum games form a proper

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Results and counterexamples are given that relate some standard theorems on zero-sum games to the corresponding conjectures on antagonistic and strength-ratios games. Alternative utilities for strength ratios that convert strength ratios into zero-sum games are given. Applications of considering payoffs based on strength ratios occur, for example, in defense analyses, where net assessments of forces and potential losses are usually measured in terms of ratios, not differences, in strengths.

ABSTRACT

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CONTENTS

I. INTRODUCTION

Two-person noncooperative games are generally classified as to whether or not they are equivalent to zero-sum games, with usually no distinction being made among the various types of games which are not equivalent to zero-sum games. The purpose of this paper is (a) to discuss another subset of two-person games, which we call antagonistic games; and (b) to present a method for converting some antagonistic games into zero-sum games. All games considered in this paper are assumed to be noncooperative, finite, two-person (bimatrix) games.

A. DEFINITION OF ANTAGONISTIC GAMES

Let Blue and Red denote the two players and let the payoffs to them be denoted by $P_b(x,y)$ and $P_r(x,y)$, respectively, when Blue plays (possibly mixed) strategy x and Red plays (possibly mixed) strategy y. For simplicity, we will not distinguish an action from the pure strategy that plays that action with probability one, and we write $P_b(i,j)$ in place of $P_b(x,y)$ when Blue plays the pure strategy that selects action i and Red plays the pure strategy that selects action j.

(1) DEFINITION. A two-person game with payoff functions P_b and P_r is called an antagonistic game if P_b and P_r have the property that

(2) $$P_b(i,j) > P_b(k,\ell) \text{ if and only if } P_r(i,j) < P_r(k,\ell)$$

for all pure Blue strategies i and k and all pure Red strategies j and ℓ.

Condition (2) defines a very natural subset of two-person games; namely, those games in which if one player does better, the other player necessarily must do worse (where better and worse are determined by comparing the payoffs that result from playing two alternative pairs of actions). Zero-sum games clearly form a proper subset of antagonistic games; the distinction is that for zero-sum games, if one player does better the other must do worse *by the same amount*. Accordingly, it is reasonable to ask how many of the results concerning zero-sum games extend to all antagonistic games. Roughly speaking, the answer is that results involving mixed strategies do not extend to all antagonistic games, but results limited to pure strategies generally can be extended. Specific theorems and examples are given in Section II.

It should be noted that some translations of Russian publications on game theory define the term "antagonistic game" equivalently with the term "zero-sum game." Since "zero-sum game" is the standard terminology outside the Russian literature, we make the above, more general definition for "antagonistic game."

B. DEFINITION OF STRENGTH-RATIO GAMES

A subset of antagonistic games that is essentially disjoint from the set of zero-sum games can be defined as follows:[1]

(3) DEFINITION. A two-person game with payoff functions P_b and P_r is called a strength-ratio game if

$$P_b(i,j) = [P_r(i,j)]^{-1} > 0$$

for all pure Blue strategies i and all pure Red strategies j.

[1]The game whose payoff is 1.0 to each player for all possible strategies is both a strength-ratio game and a constant-sum game (which is equivalent to a zero-sum game). With this exception, the set of strength-ratio games is clearly disjoint from the set of constant-sum games.

We call these games "strength-ratio games," rather than just "ratio games," in order to distinguish them from ratio games as defined by Schroeder (1970) [6].

C. BACKGROUND

Measures of effectiveness used in defense analyses are frequently in the form of ratios of strengths of the two opposing sides (where "strength" is usually stated in terms either of resources or of capability to inflict losses). One exception to the use of ratios has been in the development and use of game theoretical models of combat. In these models, the measures are assumed to be differences in strengths of the two sides in order to make the resulting games zero-sum games. Clearly, different results can be obtained depending on whether the measure used is the difference or the ratio of these strengths. Pugh and Mayberry (1973) [5] use this observation to motivate a discussion that applies to all nonzero-sum games. In Section III, we show that this observation need not limit an analysis to those results that apply all nonzero-sum games. In that section, we give a procedure which produces a zero-sum game that is based on strength ratios, but is not equivalent to the corresponding strength-ratio (and hence nonzero-sum) game.

II. ANTAGONISTIC GAMES

In this section we consider the following four questions:

1. Do all games satisfying condition (2) satisfy the corresponding condition for mixed strategies, i.e., do antagonistic games satisfy

(4) $$P_b(x,y) > P_b(z,w) \text{ if and only if } P_r(x,y) < P_r(z,w)$$

for all (possibly mixed) Blue strategies x and z and Red strategies y and w? (Zero-sum games clearly satisfy this condition.)

2. Do all pairs of equilibrium strategies of antagonistic games give the same payoff to Blue (and, symmetrically, do they all give the payoff to Red)? If not, can anything be said about different equilibrium payoffs?

3. Is there any relationship between the equilibrium strategies of the zero-sum game whose payoffs to Blue are given by P_b (or the zero-sum game whose payoffs to Red are given by P_r) and the antagonistic game whose payoffs are given by (P_b, P_r)?

4. "Threats" cannot be made in two-person zero-sum games, but they can be made in two-person nonzero-sum games. Can threats be made in antagonistic games?

The answer to question 1 is no. Counterexamples can easily be constructed from the examples given in answering the other questions below.

Questions 2 and 3 involve pairs of equilibrium strategies, which are defined as follows:

(5) DEFINITION. A pair of strategies (x^*, y^*) is called a pair of equilibrium strategies for the two-person game with payoff functions P_b for Blue and P_r for Red if

$$P_b(x,y^*) \leq P_b(x^*,y^*)$$

for all Blue strategies x and

$$P_r(x^*,y) \leq P_r(x^*,y^*)$$

for all Red strategies y.[1]

The answer to question 2 above is given by the following

(6) THEOREM. All pairs of pure equilibrium strategies of an antagonistic game give the same payoff to Blue and the same payoff to Red. Further, the payoff to either side from a pair of pure equilibrium strategies can be less than, but cannot be greater than, the payoff from a pair of equilibrium strategies that are not pure.

PROOF. Suppose (i,j) and (k,ℓ) are both pairs of pure equilibrium strategies. Then

$$P_b(k,j) \leq P_b(i,j) ,$$

and so

$$P_r(k,j) \geq P_r(i,j) , \tag{7}$$

by the condition for antagonistic games. Since (k,ℓ) is also a pair of equilibrium strategies,

$$P_r(k,j) \leq P_r(k,\ell) . \tag{8}$$

Combining (7) and (8) gives that

$$P_r(i,j) \leq P_r(k,\ell) .$$

Thus, by symmetry, all pairs of pure equilibrium strategies for

[1]A well known theorem is that there exists at least one pair of equilibrium strategies for any finite two-person game—see, for example, Owen (1968) [4].

antagonistic games have the same payoff to Blue and the same payoff to Red.

Now suppose (i,j) is a pair of pure equilibrium strategies and (x,y) is a pair of mixed equilibrium strategies, and let x_k be the probability that Blue plays action k under strategy x. Then

$$P_b(k,j) \leq P_b(i,j)$$

for all k, so by the condition for antagonistic games

$$P_r(k,j) \geq P_r(i,j)$$

for all k, and so

$$P_r(x,j) = \sum_k x_k P_r(k,j) \geq P_r(i,j) \ .$$

Since (x,y) is a pair of equilibrium strategies,

$$P_r(x,j) \leq P_r(x,y) \ ,$$

and so

$$P_r(i,j) \leq P_r(x,y) \ .$$

Thus, the payoff to Red from any pair of pure equilibrium strategies must be less than or equal to the payoff to Red from any pair of mixed equilibrium strategies. Clearly, the same statement also holds for Blue.

All that remains is to give an example of an antagonistic game that has a pair of pure equilibrium strategies, and has a pair of mixed equilibrium strategies with a greater payoff to each side than the pure equilibrium strategies give. Such an example with three actions for each side is:

$$\begin{array}{cc} & \text{RED} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{BLUE} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (1, 1) & (2, 1/2) & (2, 1/2) \\ (1/2, 2) & (10, 1/10) & (1/10, 10) \\ (1/2, 2) & (1/10, 10) & (10, 1/10) \end{bmatrix} \end{array},$$

where the terms in parentheses correspond to $(P_b(i,j), P_r(i,j))$ for Blue playing action i and Red playing action j. The pair of strategies (1, 0, 0), (1, 0, 0) is in equilibrium with a payoff of 1.0 to each side; and the pair of strategies (0, 1/2, 1/2), (0, 1/2, 1/2) is in equilibrium with a payoff of 5.05 to each side. ▯

The answer to question 3 is given by the following

(9) THEOREM. Let P_b and P_r be the payoff functions for an antagonistic game. Then the following conditions are equivalent:

(a) the pair of pure strategies (i,j) is an equilibrium pair for the antagonistic game,

(b) the pair of pure strategies (i,j) gives a saddlepoint for the zero-sum game whose payoff function to Blue is P_b,

(c) the pair of pure strategies (i,j) gives a saddlepoint for the zero-sum game whose payoff function to Red is P_r.

In particular, the antagonistic game has a pair of pure equilibrium strategies if and only if the zero-sum game whose payoff function to Blue is P_b (or the zero-sum game whose payoff function to Red is P_r) has a saddlepoint.

PROOF. Let $P_r^0 = -P_b$ so that P_r^0 denotes the payoff function to Red for the zero-sum game whose payoff function to Blue is P_b, and suppose that (i,j) gives a saddlepoint for this zero-sum game. Then

(10) $$P_r^0(i,\ell) \le P_r^0(i,j) \quad \text{for all } \ell.$$

Now suppose that there is a (possibly mixed) Red strategy y such that

$$P_r(i,y) > P_r(i,j) .$$

Then there must be some pure Red strategy ℓ' such that

$$P_r(i,\ell') > P_r(i,j)$$

but then

$$P_b(i,\ell') < P_b(i,j)$$

by the condition for antagonistic games. This inequality gives that

$$P_r^0(i,\ell') > P_r^0(i,j)$$

which contradicts (10). Therefore

$$P_r(i,y) \le P_r(i,j)$$

for all Red strategies y. Applying the identical argument to Blue strategies gives that (i,j) is an equilibrium pair of pure strategies for the antagonistic game.

Now suppose that (i,j) is an equilibrium pair of pure strategies for the antagonistic game. Then

$$P_r(i,\ell) \le P_r(i,j)$$

for all pure Red strategies ℓ. Therefore, by the antagonistic game condition, .

$$P_b(i,\ell) \ge P_b(i,j)$$

and so

$$P_r^0(i,\ell) \le P_r^0(i,j)$$

for all pure Red strategies ℓ. By the equilibrium assumption,

$$P_b(k,j) \leq P_b(i,j)$$

for all pure Blue strategies k, and so (i,j) is a saddlepoint for the zero-sum game whose payoff function to Blue is P_b. Interchanging the roles of Red and Blue completes the proof. $\square$

The answer to question 4 depends on how one defines "threats." For the purpose of this paper we give the following

(11) DEFINITION. A threat to Red by Blue in a two-person game is said to exist if there exist three (possibly mixed) pairs of strategies (x_1,y_1), (x_2,y_2), (x_3,y_3) such that:

(a) x_1 maximizes $P_b(\cdot,y_1)$,

(b) y_2 maximizes $P_r(x_2,\cdot)$,

(c) $P_b(x_1,y_1) > P_b(x_2,y_2)$,

(d) $P_r(x_3,y_3) > P_r(x_1,y_1) > P_r(x_2,y_2)$.

In this case, we call y_1 Red's compliance strategy and call x_2 Blue's threat strategy.

The rationale behind this definition is as follows. Blue threatens to play x_2 unless Red plays y_1. If Red complies and agrees to play y_1, and if Blue can enforce this agreement, then Blue can maximize over his possible payoffs and (11a) follows. To exclude "idle" threats, suppose that if Red does not agree to comply, then Blue is required to play x_2. In this case Red can maximize over his possible payoffs and (11b) follows. Blue would not threaten Red into playing y_1 if Blue could do at least as well by himself with y_2 and so (11c) follows. Finally, Blue cannot threaten Red with x_2 to force Red to play y_1 if $P_r(x_2,y_2) \geq P_r(x_1,y_1)$; and Red is not threatened into playing y_1 if y_1 is the best Red can do--i.e., there must exist some strategy pair (x_3,y_3) such that $P_r(x_3,y_3) > P_r(x_1,y_1)$; and so (11d) follows.[1]

Given this definition, can an antagonistic game have a threat?[2] The following slight change to the example following Theorem (6) gives an obvious threat to Red by Blue

$$\begin{array}{cc} & \text{RED} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{BLUE} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (1, 1) & (2, 1/2) & (2, 1/2) \\ (1/2, 2) & (10, 1/10) & (1/100, 100) \\ (1/2, 2) & (1/10, 10) & (10, 1/10) \end{bmatrix}. \end{array}$$

Blue can threaten to play $x_2 = (1, 0, 0)$ unless Red complies and agrees to play, say, $y_2 = (0, 98/100, 2/100)$. This strategy is not the only possible Red compliance strategy, but all other compliance strategies must also be mixed strategies.

(12) THEOREM. If a threat to Red by Blue exists in an antagonistic game, then Red's compliance strategy (y_1) cannot be a pure strategy.

PROOF. Suppose that y_1 is a pure strategy, say j. Let i be a pure Blue strategy that maximizes $P_b(i,j)$, let x_2 be Blue's threat strategy, and let ℓ be a pure Red strategy that maximizes $P_r(x_2,\cdot)$. Then

(13) $$P_b(i,j) \geq P_b(k,j)$$

---

[1]Note that conditions (11a) through (11d) are not similar to Nash's conditions for resolving threats; these conditions serve only to define threats. Note also that the results that follow do not depend on (11d). So if one wishes to define "threat" in such a way that Br'er Rabbit is threatened by the briar patch, then condition (11d) can be deleted and these results still hold.

[2]Clearly it is possible for an antagonistic game not to have a threat, since zero-sum games are a subset of antagonistic games.

for all pure Blue strategies k, and

(14) $$P_r(x_2,\ell) \geq P_r(x_2,j) .$$

From (13)

$$P_r(k,j) \geq P_r(i,j)$$

for all k, and so

(15) $$P_r(x_2,j) \geq P_r(i,j) .$$

Combining (14) and (15) gives that

$$P_r(x_2,\ell) \geq P_r(i,j),$$

but this contradicts (11d) which says that

$$P_r(i,j) > P_r(x_2,\ell) .$$

Therefore y_1 cannot be a pure strategy. ▯

Theorem 12 is of interest for the following reason. One can conceive of several ways to enforce an agreement to play a pure strategy. For example, the resources that would allow a side to play any other strategy but that particular pure strategy could be altered or dismantled. However, it is considerably more difficult to conceive of realistic ways to enforce an agreement to play a mixed strategy if one of the actions in that mixed strategy produces a greater payoff to the "threatened" side. Theorem 12 states that all compliance strategies in antagonistic games are mixed strategies; and so each compliance strategy must play (with positive probability) at least one action that has a greater payoff to the threatened side than that compliance strategy gives. Thus, while threats can exist in antagonistic games, it may not be realistic to ignore questions concerning the enforcement of the strategies involved in those threats.

III. UTILITIES FOR STRENGTH RATIOS THAT YIELD ZERO-SUM GAMES

Since strength-ratio games are a subset of antagonistic games, the theorems of Section II also apply to strength-ratio games. Further, the examples in Section II are intended to indicate the limitations of attempting to extend additional results from zero-sum games to antagonistic games; and since both of these examples are also strength-ratio games, they also indicate that the same limitations apply even if one considers only strength-ratio games. Thus, we do not attempt to find results peculiar to strength-ratio games in this section. Instead, we give alternative utilities for strength ratios. These utilities are relatively plausible, and games based on these utilities are zero-sum games, not strength-ratio games.

Let S_b and S_r be two functions, each mapping the cross product of the Blue and Red pure strategies (actions) into the positive reals. We call S_b and S_r Blue's strength and Red's strength, respectively.[1]

It is reasonable to believe that each player might want to maximize the ratio of his strength to that of his opponent, as

[1]For defense analyses, strength ratios can be thought of as the ratio of quantities possessed by two opposing forces (i.e., force ratios). Examples are the number of warheads as a measure for strategic forces, the total available firepower as a measure for conventional ground and air forces, or the tons of ordnance that are delivered in support of ground forces as a measure for tactical air forces. Alternatively, another type of strength ratio useful in defense analyses is the ratio of the capability of the two opposing forces to inflict losses on each other (i.e., loss ratios). Examples are weapon-loss ratios or loss-rate ratios for conventional forces, and the ratio of the fractional value of each side's economy that would be lost for strategic forces.

opposed to maximizing the difference to their strengths. But, for example, it may not be reasonable to assume that Blue is indifferent between (a) a 1-to-1 strength ratio with probability 1, and (b) a 9-to-5 strength ratio in his favor with probability 1/2 and a 1-to-5 strength ratio against him with probability 1/2--which is what equating Blue's utility with the numerical value of the strength ratio would give. It seems generally more reasonable to assume that Blue is indifferent between (a) a 1-to-1 strength ratio with probability 1, and (b) a 5-to-1 strength ratio in his favor with probability 1/2 and a 1-to-5 strength ratio against him with probability 1/2.

One way to incorporate this second indifference into a two-person payoff structure is as follows. Let

$$Q_b(i,j) = \begin{cases} \dfrac{S_b(i,j)}{S_r(i,j)} - 1 & \text{if } \dfrac{S_b(i,j)}{S_r(i,j)} \geq 1 \\[2ex] 1 - \dfrac{S_r(i,j)}{S_b(i,j)} & \text{if } \dfrac{S_b(i,j)}{S_r(i,j)} < 1 \end{cases}$$

and

$$Q_r(i,j) = \begin{cases} \dfrac{S_r(i,j)}{S_b(i,j)} - 1 & \text{if } \dfrac{S_r(i,j)}{S_b(i,j)} \geq 1 \\[2ex] 1 - \dfrac{S_b(i,j)}{S_r(i,j)} & \text{if } \dfrac{S_r(i,j)}{S_b(i,j)} < 1 \end{cases}$$

for all pure Blue strategies i and pure Red strategies j, and let the payoffs to Blue and Red be given by

$$P_b(x,y) = \sum_{i,j} x_i Q_b(i,j) y_j$$

$$P_r(x,y) = \sum_{i,j} x_i Q_r(i,j) y_j$$

for all Blue strategies x and Red strategies y. Note that these payoff functions are defined symmetrically in terms of Blue and Red, and they have the property that a Blue favorable n-to-one strength ratio has the same additional utility to Blue over a one-to-one strength ratio as a one-to-one strength ratio has to Blue over a Red favorable one-to-n strength ratio.

If P_b and P_r are defined as above, the resulting game is a zero-sum game (it is easy to check that $Q_b(i,j) + Q_r(i,j) \equiv 0$). Accordingly, strength ratios can be considered in a reasonable way within the context of zero-sum games, which allows an analysis to consider strength ratios without restricting that analysis to only those results that hold for all nonzero-sum games.[1]

A more general definition of Q_b and Q_r is as follows. Let f be any odd function ($f(-x) = -f(x)$). Then if Q_b is defined by

$$(16)\quad Q_b(i,j) = \begin{cases} f\left(\dfrac{S_b(i,j)}{S_r(i,j)} - 1\right) & \text{if } \dfrac{S_b(i,j)}{S_r(i,j)} \geq 1 \\[2ex] f\left(1 - \dfrac{S_r(i,j)}{S_b(i,j)}\right) & \text{if } \dfrac{S_b(i,j)}{S_r(i,j)} < 1 \end{cases}$$

and Q_r is defined symmetrically, the resulting game has the characteristics described above (in particular, it is a zero-sum game). This more general definition can be used if utility does not increase linearly with strength ratio.

Finally, we remark that the payoff structure described above is implicitly contained in two existing game theoretical

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[1] If, for a particular problem, the utility functions for the two players are known and these known utilities result in a strength-ratio game as defined in Section I, then the above method cannot be applied. But if the strengths S_b and S_r represent physical quantities and an analyst must develop a payoff structure based on these quantities, then the above structure appears to be plausible and useful.

models, but the descriptions of these models do not explicitly point out that this structure is there. The two models are the OPTSA models (see Bracken, Falk, and Karr (1975) [2], and Anderson, Bracken, and Schwartz (1975) [1]) and the ATACM model (see Fish (1975) [3]). For both these models, S_b can be interpreted as Blue air firepower delivered in support of ground operations in a conventional war, and S_r is the same for Red. Both models consider $S_b - S_r$ as a possible payoff for Blue. Both models also consider "ground movement" as a payoff measure, where ground movement is a function, f, of strength ratios where the Blue strength consists of the Blue ground strength plus S_b, and the same for Red. By zeroing out the ground strengths and by interpreting f as a utility function as in (16), the structure described above is obtained. Thus, both these models as currently programmed can use ratios of air firepower as well as differences in air firepower as measures of effectiveness.

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REFERENCES

[1] Anderson, L.B., J. Bracken, and E. Schwartz. *Revised OPTSA Model*. 3 vols. Paper P-1111, Institute for Defense Analyses, Arlington, VA., September 1975.

[2] Bracken, Jerome, James E. Falk, and Alan F. Karr. "Two Models for Optimal Allocation of Aircraft Sorties." *Operations Research*, 23, No. 5 (1975).

[3] Fish, J.R. *ATACM: ACDA Tactical Air Campaign Model*. ACDA/PAB-249, Ketron, Inc., Arlington, VA., October 1975.

[4] Owen, G. *Game Theory*. W.B. Saunders Company, Philadelphia, PA., 1968.

[5] Pugh, George E., and John P. Mayberry. "Theory of Measures of Effectiveness for General Purpose Military Forces: Part I. A Zero-Sum Payoff Appropriate for Evaluating Combat Strategies." *Operations Research*, 21, No. 4 (1973).

[6] Schroeder, Roger G. "Linear Programming Solutions to Ratio Games." *Operations Research*, 18, No. 2 (1970).