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# A CONFRONTATION OF MATHEMATICAL MODELS FOR FATIGUE LIFE WITH ACTUAL SERVICE DATA

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

From a collection of over a thousand different groups of qualified aluminum fatigue data, an empiric distribution is obtained which can serve as a standard for comparison of various mathematical models for fatigue. On this basis several commonly used parametric models, as well as some mathematical suppositions, can be discarded if good predictions are desired of the earliest fatigue failures. One of the models not disqualified, which contains sufficient

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) flexibility to fit the actual data, is a distribution of the log-life with three parameters, namely, location, scale and flexure. Because maximum likelihood estimation of all three parameters cannot be accomplished simultaneously, the location parameter is posited as being calculated on the basis of a cumulative damage rule known *a priori* while the scale parameter is determined from the material and the flexure parameter describes the presence of flaws and/or the time to crack initiation. These relations with the physics of materials provide a design basis for calculation of reliability.

# PREFACE

The research reported herein was conducted under Air Force Contract F 33615-73-C-4016 at Washington State University. The work was initiated under project 7071. The technical monitor of the contract was Dr. H. Leon Harter, Aerospace Research Laboratories (until 30 June 1975) and Air Force Flight Dynamics Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio. The principal investigator was Professor Sam C. Saunders.

The author would like to acknowledge the use of data from reports previously sponsored by the Air Force Materials Laboratory. These investigations were conducted, in part, by the Fatigue Research Group of the Boeing Commerical Airplane Company. Specifically conversations with Messrs. Ian Whittaker, Rao Varanosi and Joseph P. Butler of that group, were helpful.

The final report covers the work conducted during the period from September 15, 1974 until September 15, 1975. The manuscript was submitted for publication in March 1976.

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## SECTION I

## INTRODUCTION

The increasing acceptability of statistical considerations in airframe design criteria has led to some well-known, beneficial results through the application of life-length distributions to the calculation of reliability, see [1], [2]. On the other hand statistical methods are often blamed, since the grouping together of failures under the category "statistical" is detrimental to the improvement of service life, when the causes of failure are not identified and corrective action not taken (presumably "random" failures have no "assignable" cause). Because of the dichotomy of failure modes inherent in this interpretation, it is sometimes believed that the use of statistical methods precludes the idea, long associated with deterministic models, that each failure should be fixed and all other similar components in use should be repaired and altered so that the reoccurrence of that failure mode is avoided.

This false conclusion may have been fostered by the adoption of statistical models from other fields so that the parameters governing the distributional law are without physical interpretation. This state of affairs is deemed desirable by some statistical protagonists under the supposition that lack of such specific physical meaning entails a universal statistical applicability.

Some of the recent results in the study of life distributions, which have been specialized so that they take into account the known physical and material nature of the fatigue process, will be presented and the implication of these distributional assumptions to cost effective analysis of inspection and maintenance schedules in fleet reliability will be made. The utility of these distributions will be argued both for metals and for composite materials.

## SECTION II

## THE COLLECTED DATA

As a result of the extreme variabliity of the observed fatigue life of nominally identical specimens under nominally identical loads, it has been the practice in engineering studies to record the logarithm of life in either units of time or number of cycles. Because of this historical fact, denote the random variable which shall be recorded from the observed life as $X = \ln T$ where for mathematical convenience the natural, rather than the common, logarithm is taken.

To be specific, the confrontation with a large collection of data gathered over a score of years by a single air frame manufacturer, and reported in [3], which contains a large number of similarly qualified groups of data, will be made. These data were qualified in the sense that they satisfy the following restrictions:

1. All data were from components fashioned from the same aluminum alloy. Only geometries typical of aircraft structural components were permitted.
2. The imposed stress was always low enough that every fatigue failure exceeds 100 cycles, but high enough that no failure exceeds $3 \times 10^6$ cycles.
3. There were no rotating - bending tests, no unnotched specimens, no bonded lap-joints or hand forgings included.
4. Only actual service conditions were imposed upon the components.

However, even with these qualifications the data consisted of a large number of small groups, each one of which was too small to exhibit statistical regularity and moreover the groups themselves were so inhomogeneous that statistical analysis among the groups was quite difficult without some unifying assumtions.

In order to reduce the data to reckoning the same plausible assumptions as in [5] were made.

1° Each group contains a number of nominally identical specimens exposed to the same service load.

2° The mean (characteristic) log-life is determined by the maximum stress imposed.

3° The variance of the log-life is the same for every group.

It is known that the variance of the life itself increases as the mean life increases. What is assumed in 3° is that the variance is proportional to the square of the mean, i.e. the coefficient of variation is the same, so that 3° is not in disagreement with fact.

Assume there are m groups of data and each group has a different number of observations. In the jth group the ith observation is

$$x_{ij} \quad \text{for} \quad i = 1, \ldots, n_j \quad \text{and} \quad j = 1, \ldots, m \quad .$$

The above assumptions imply that $x_{ij}$ are observations of a random variable $X_{ij}$ for which

$$EX_{ij} = \mu_j \qquad \operatorname{var}(X_{ij}) = \sigma^2 \quad .$$

Let $\underline{x}^\dagger = (x_1, \ldots, x_m)$ denote a group of n such qualified observations, written as the transform of a column vector.

Let $A = (a_{ij})$ be an m×n symmetric matrix such that

$$\sum_{j=1}^{n} a_{ij} = 0 \ , \ \sum_{j=1}^{n} a_{ij}^2 = 1 \qquad \text{for each} \quad i = 1, \ldots, n \quad .$$

Now consider the set of observations transformed by $A$, namely $\underline{y} = A\underline{x}$ or

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for } i = 1, \ldots, n .$$

If $\underline{X}$ denotes a random, column vector of independent identically distributed variates with mean $\mu$ and variance $\sigma^2$, then the vector $\underline{Y} = A\,\underline{X}$ has

$$E\,\underline{Y} = \underline{0} \quad \text{and} \quad \text{var}(\underline{Y}) = \sigma^2 AA' .$$

Since the matrix $AA'$ has diagonal elements of unity then each variate in $\underline{Y}$ has a marginal distribution with mean zero and variance $\sigma^2$, however they may be dependent even if they were originally assumed to be normal.

Since the value of $n$ changes from group to group there must be a specification of $A$ for each different value of $n$. The choice made for $A$ was

$$a_{ij} = \sqrt{\frac{n}{n-1}} \left(\delta_{ij} - \frac{1}{n}\right)$$

where $\delta_{ij}$ is the Kronecker delta.

This means that the transformed data are

$$y_{ij} = \sqrt{\frac{n_j}{n_j-1}} \left(x_{ij} - \bar{x}_{\cdot j}\right) \quad \text{for} \quad \begin{matrix} i = 1, \ldots, n_j \\ j = 1, \ldots, m \end{matrix}$$

with

$$\bar{x}_{\cdot j} = \left(\sum_{i=1}^{n_j} x_{ij}/n_j\right) .$$

It follows that

$$E\,Y_{ij} = 0 , \qquad \text{var}(Y_{ij}) = \sigma^2 .$$

In this case $S_j^2 = \frac{1}{n_j} \sum_{i=1}^{n_j} Y_{ij}^2$

is an unbiased estimate of $\sigma^2$. Moreover the estimate

$$\hat{\sigma}^2 = \frac{1}{m} \sum_{j=1}^{m} S_j^2$$

is a consistent, unbiased estimate for $\sigma^2$. However, in case fatigue life is actually log-normally distributed then

$$\tilde{\sigma}^2 = \left[ \sum^{m} n_j S_j^2 \Big/ \sum^{m} (n_j - 1) \right]$$

is the minimum variance unbiased estimate of $\sigma^2$.

See the discussion of the treatment of fatigue data in [6].

If all the data from the $m$ groups of transformed data namely,

$$y_{ij} \quad \text{for} \quad i = 1, \ldots, n_j \quad \text{and} \quad j = 1, \ldots, m ,$$

are pooled together to form an empiric cumulative what will it look like? Denote the random empiric cumulative by $\mathfrak{F}$ defined by

$$\mathfrak{F}(x) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left[ c(y_{ij}, x) \Big/ \sum_{j=1}^{m} n_j \right]$$

where $c(y,x) = 1$ if $y \leq x$ and $0$ otherwise. Clearly

$$E\mathfrak{F}(x) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left[ P[Y_{ij} \leq x] \Big/ \sum_{j=1}^{m} n_j \right] .$$

Since from the assumptions made, for given $j$ the variate $Y_{ij}$ has the same marginal distribution for each $i = 1, \ldots, n_j$, say $H(\cdot\, ; n_j)$, we have

$$E\mathfrak{F}(x) = \sum_{j=1}^{m} \left( \frac{n_j}{\sum_1^m n_j} \right) H(x\, ; n_j) \quad . \tag{1}$$

Thus in all cases the distribution will be close to a mixture of distributions which are determined by the sample sizes. The actual distribution is very difficult to determine analytically unless the assumption of log-normality of fatigue life is true. In that case $H(\cdot\ ; n_j)$ would be a normal distribution of mean zero and variance $\sigma^2$ independently of $n_j$. Hence, the pooled data would tend toward a straight line with slope determined by $\sigma$ when plotted on log-normal probability paper.

An illustration of this data is taken from Ref[5] and presented here as Figure 1.

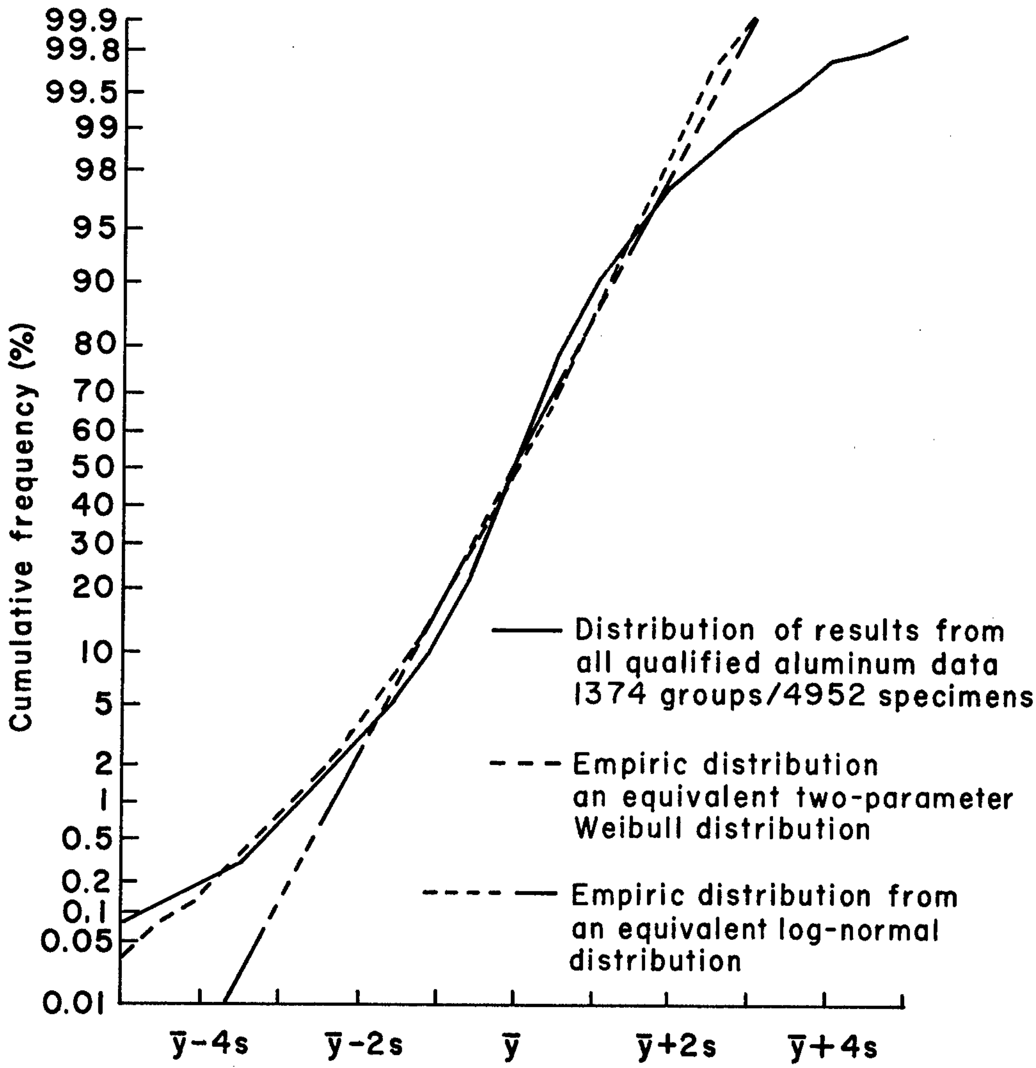


Figure 1. All Qualified Aluminum Alloy Data (Compared to Various Distribution Models)

## SECTION III

## THE MECHANISM OF FATIGUE CRACK GROWTH

It has been postulated by some investigators, see [4] and the citations there, that a three-phase growth model is necessary to describe in sufficient detail the behavior of a fatigue crack with enough exactitude to determine a statistical distribution which can be used to predict fatigue life.

In order to be self-contained and to provide a basis for comparison, a summary of the argument given in [4] will be presented here. The three stages of fatigue crack growth will be described in terms of the time necessary for certain successive transition events to occur. The interoccurence times of these events will be called $T_0$, $T_1$, $T_2$ where

$T_0$ is the time from the beginning of service until crack initiation takes place.

$T_1$ is the time following initiation, during which the fatigue crack grows in a stable fashion as a linear function of the number of load applications, until a critical size is reached.

$T_2$ is the time following critical size until fracture, or containment, during which the crack grows at an increasing rate governed by the stress intensity factor.

The total time from beginning of service until fracture is then given by the sum $T = T_0 + T_1 + T_2$ .

In order to describe this with more mathematical detail let $s(t)$ denote the size of the crack at any time $t > 0$ ; then in accord with the various stages, the behavior can be described as follows:

1' $s(t) = 0$ for $0 \le t < t_0$

2' $s' = a_1$ for $t_0 \le t < t_0 + t_1$

$$\therefore\ s(t) = s_0 + a_1(t-t_0)\ .$$

We now adopt two laws from fracture mechanics which were assumed, p.4 in [7]:

(i)' The stress intensity factor is proportional to the square root of crack length.

(ii)' The logarithm of growth rate is an increasing linear function of the logarithm of stress intensity.

From these assumptions it has been shown in [7] there follows the differential equation

3' $s' = a_2(s/s_1)^{b+1}$ for $t > t_0 + t_1$

where $a_2 > 0$ and $b > -1$ are disposable constants.

If we define the propagation function $p$ for $t > 0$ by

$$p(t) = \begin{cases} (1 - bt)^{-1/b} & \text{if } b \neq 0 \\ e^t & \text{if } b = 0 \ , \end{cases} \tag{2}$$

then during all stages we have

$$s(t) = \begin{cases} 0 & 0 \le t < t_0 \\ a_1(t-t_0) + s_0 & t_1 \le t < t_1 \\ s_1\ p[a_2(t-t_1)/s_1] & t \ge t_1\ . \end{cases}$$

Thus for $a_2 > a_1$ we can write the total time $t$ until fracture as

$$t = t_0 + \frac{s_1-s_0}{a_1} + \frac{s_1}{a_2}\ p^{-1}(s_2/s_1) \tag{3}$$

where $s_0$, $s_1$, $s_2$ are the critical crack sizes at the times of transition of phase. The behavior of this differential system of expected crack growth has been plotted in [4] . It is the behavior of the total fatigue life as defined in equation (3) as a stochastic variable which is to be examined here.

In order to account for the observed statistical variability in the life length measurements certain of the disposable constants appearing in equation (3) must be assumed to be random variables across the population of components which will be subjected to life determination. The questions to be answered by comparison of the mathematical model with the data are: Which variables should be designated as random? Which variables should be thought of as constants exactly determinable from engineering measurements? It is to these questions that we turn in the next section.

## SECTION IV

## THE CONVOLUTION OF INDEPENDENT STAGES

Suppose that the critical crack sizes $s_0, s_1, s_2$ can be determined by analysis from the geometry of the specimen and the constants determining the exact propagation function $p$, given in equation Eq(2), are known. This leaves $t_0, a_1, a_2$ as candidates for stochastic variables.

Under this presumption the time until fracture is

$$T = T_0 + T_1 + T_2$$

where

$$T_1 = \frac{s_1 - s_0}{A_1} \quad , \qquad T_2 = \frac{s_1}{A_2} p^{-1}\left(\frac{s_2}{s_1}\right) \quad .$$

Here $A_1$ and $A_2$ denote crack propagation rates which vary stochastically over the population of material specimens i.e. the rates of crack propagation through the metal in stages 1 and 2, say $a_1, a_2$ respectively, will be different for each metallic specimen. Much experimental evidence supports this contention, see [7] and the references given there. It is presumed here, as a mathematical convenience, that $A_1, A_2$ have $\xi$-normal distributions as defined in [8]. Since these variates possess the reciprocal property this is equivalent to assuming the times of transition are random. This point was discussed in [4].

Thus the total time until fracture has been represented as the sum of three separate distinct periods which are respectively the time to initiation, the time of linear crack growth per cycle and the time of crack propagation from critical size by fracture mechanical principles, until complete fracture or arrest.

The question presents itself: Is it possible to convolve three independent random variables representing time spent in each stage so as to obtain from them a distribution which will, if mixed in the right proportion, match that empiric distribution from the data which has been exhibited in Figure (1)? This question, also asked in [4], was answered through extensive mathematical comparison of the observed distribution of fatigue life in Figure (1) with the sum

$$\sum_{i=1}^{3} \beta_i \Psi(\alpha_i Z_i)$$

where $Z_i$ for $i = 1, 2, 3$ are independent standard normal variates and

$$\Psi(x) = \ln(x + \sqrt{x^2 + 1}) \ .$$

The answer appears to be no. The evidence, in summary, is that if the variates are independent almost any convolution of three such smooth unimodal distributions is very close to normality.

Because of the analytical difficulties with a three-fold convolution the actual method used to study the distribution of the sum was mathematical simulation. This method was as follows:

At the $j$th replication, generate by machine the standard normal variates $z_{ij}$ $i = 1, 2, 3$ from which with preassigned values $a_{ij}\beta_i$ for $i = 1, 2, 3$ compute the $j$th value

$$t_j = \sum_{i=1}^{3} \beta_i \Psi(\alpha_i z_{ij}) \qquad \text{for } j = 1, \ldots, \qquad .$$

The empiric cumulative distribution was formed from the set of observations

$t_1, \ldots, t_M$ for $M = 1{,}000$ and plotted on the appropriate probability scale by plotter on the machine. Some results of this simulation are in Figure 2.

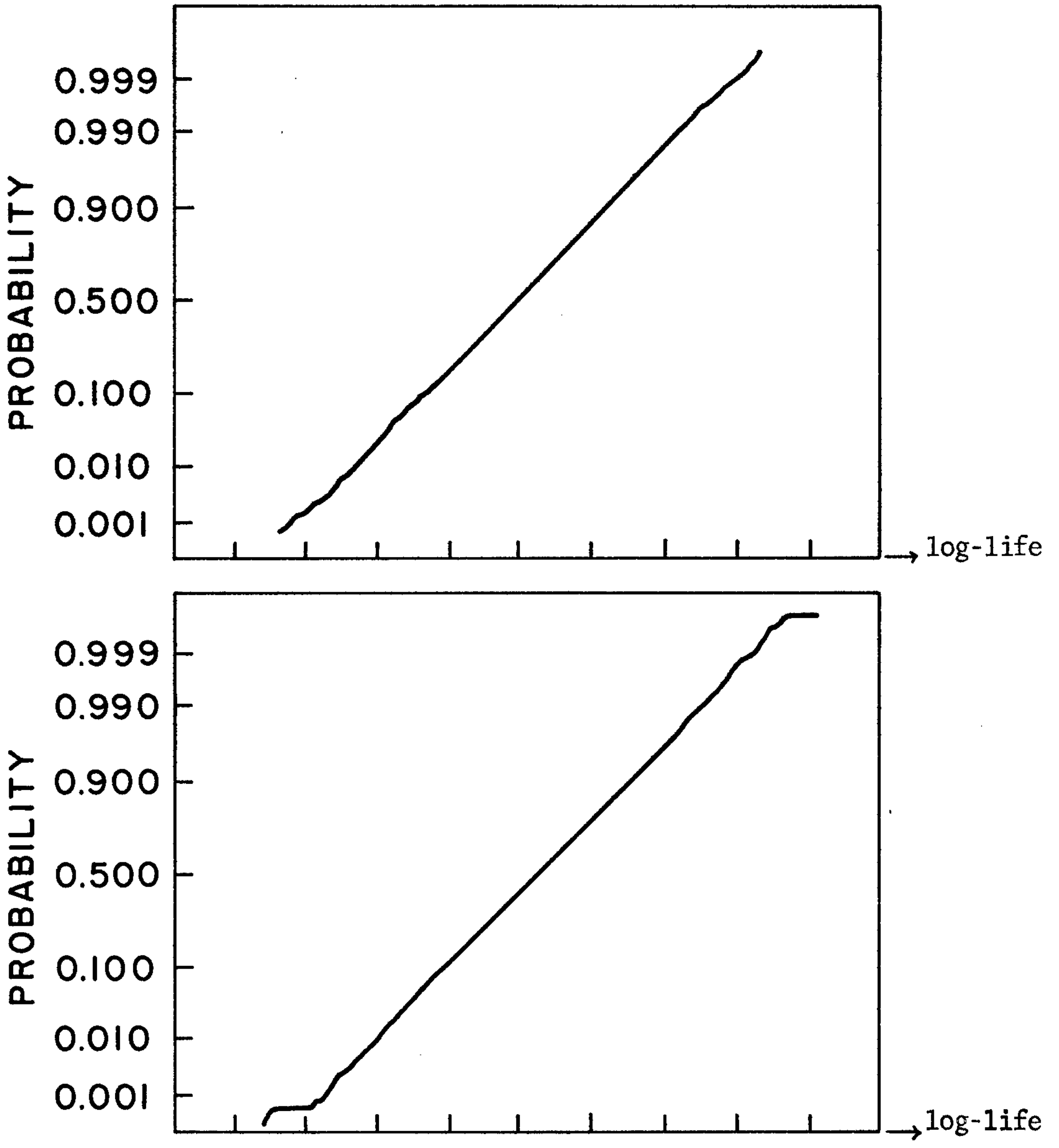


Figure 2. Empiric Distribution of 3-fold Convolution

It is seen that the convolution of these three variates, when transformed by scale and location to standard form, yields nearly a straight line. This indicates that only a log-normal variate can be obtained by this method. The initial and terminal behavior of the curve obtained is far from that actually observed. Is it possible that three separate stages are too many? Perhaps there is no initial random period of latency of the crack and the first stage should be eliminated.

If the crack is always initiated at time zero then $T_0 = 0$ and only two stages of crack growth remain, namely, the low cycle fatigue and fracture mechanical propagation. These would correspond to the linear crack growth and accelerating crack growth phases of the time to failure.

The next question is whether the behavior exhibited in the empiric distribution can be obtained by simulation of a two-stage model, namely

$$\sum_{1}^{2} \beta_i \, \Psi(\alpha_i Z_i) \quad .$$

Again the answer appears to be no; see Figure 3.

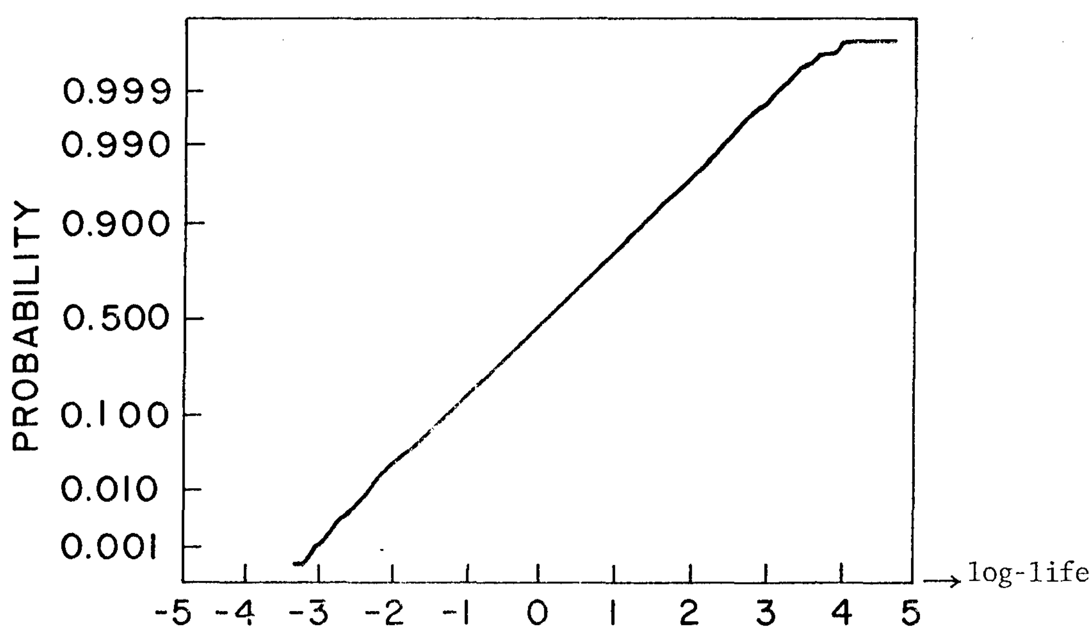


Figure 3. Empiric Distribution of Convolution

## SECTION V

## THE MODEL:
## A REVISED DERIVATION

Let us consider a coordinate axis with a logarithmic abscissa and an ordinate which is the inverse of a standard normal distribution in order to facilitate comparison with the empiric cumulative obtained from the data. On such a scale the log-normal distribution would plot as a straight line.

On this coordiante axis let us construct a rectangle formed from the sides of parallel horizontal and vertical asymptotes of distributions which will now be considered.

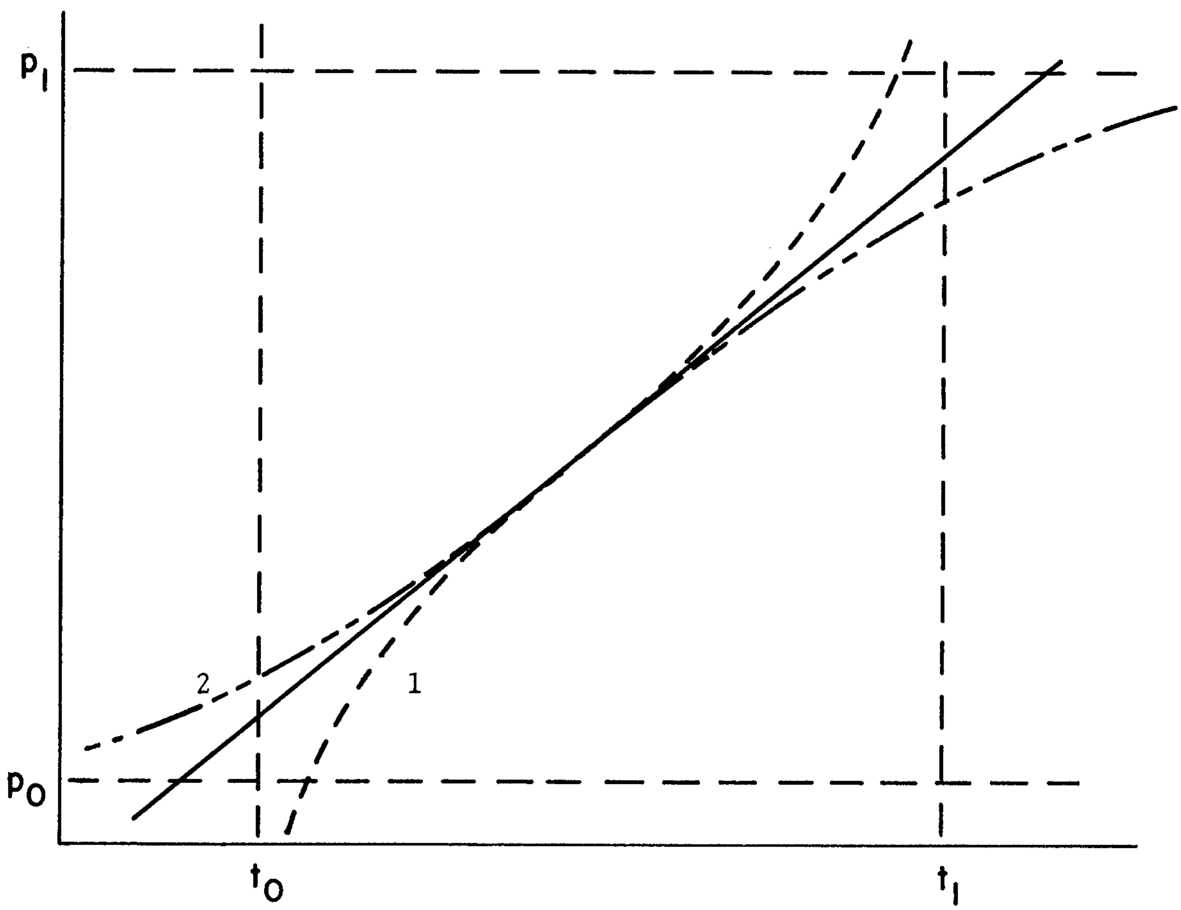


Figure 4. Flexure of Distribution with Finite Support

Suppose that a distribution has support only between $t_0$ and $t_1$; then for a component with this distribution the interpretation can be made:

$t_0$ = the time before which no failures can occur .

$t_1$ = the time before which all failures must occur.

Such a distribution would have the appearance of 1 in figure 4 . Suppose that a distribution is contained between horizontal asymptotes $p_0$ and $p_1$ . Then for a component with a life distribution like 2 in figure 4 an interpretation can also be made.

$p_0$ = the probability of being failed at time zero

$1-p_1$ = the probability of never failing.

The actual data, although compounded from being the mixture of observations transformed by scale and location, appears to exhibit an S-shaped symmetry somewhat like distribution 2 in Figure 4 . That is to say, the data indicates a tendency toward having a probability of being failed at time zero, as well as a probability of never failing. Let us now try to explain the observed behavior of the data in terms of simple mathematical models.

It has been argued in [8] that if a crack must progress a fixed distance $\omega$ , with incremental steps of varying (random) length and with each step having the same distribution with mean $\mu$ and variance $\sigma^2$ then the distribution of time (number of steps) for the crack to progress the given distance $\omega$ , call it $T_0$ , is given, at least for small enough step sizes, by

$$\Phi\left[\frac{1}{\alpha}\,\xi_0(t/\beta)\right] \qquad \text{for} \quad t > 0 \; .$$

Here $\Phi$ is the standard normal distribution and

$$\beta = \omega/\mu \qquad\qquad \alpha = \sigma/\sqrt{\mu\omega}$$

with $\xi_0$ defined by

$$\xi_0(x) = \sqrt{x} - \frac{1}{\sqrt{x}} \qquad\qquad 0 < x < \infty \quad .$$

If we suppose that the initial size $s_0$ of the crack at time zero is a random variable, then the distance, $W = s_1 - s_0$ , the crack must progress to attain the critical size $s_1$ is a random variable, which is assumed to be normal with mean $\omega$ and variance $\rho^2$ , in order to make the integration easy. From these assumptions it can be shown, as in [4], that the distribution of time for the crack to exceed $s_1$ , say $T_1$ , is given by

$$\Phi\left[\frac{1}{\alpha}\,\xi_1(t/\beta)\right] \qquad\qquad t > 0$$

where

$$\xi_1(t) = \xi(t)/\left(1+\frac{\epsilon}{t}\right)^{\frac{1}{2}} \qquad\qquad \text{and} \qquad \epsilon = \rho^2/\beta\sigma^2 \quad .$$

Clearly as $t \to \infty$ , $\xi_1(t) \to -\epsilon^{-\frac{1}{2}}$ , and $\Phi\left(\frac{-1}{\sqrt{\epsilon}}\right)$ is the probability the structure is broken at time zero.

Thus it plainly appears that the introduction of a random initial crack size will push the toe of the distribtuion toward the left and upwards in a direction more in consonance with the data. The fact that within this particular family of distributions a positive probability of having the structure failed at time zero is always obtained is a consequence of the assumption that the initial flaw was normally distributed and, hence, there is some positive probability of exceeding the terminal crack size $s_2$ .

On the other hand, if the crack progresses from a critical size $s_1$ to terminal size at a rate which is decreasing as a function of crack length the resulting distribution would still have its tail pushed down from the log-normal line toward the right. Again the form would be more nearly that obtained empirically from the data. To effect this, a differential equation model, alternative to the one given previously in section IV and discussed earlier in [7], is now postulated. The substitution of assumption (i)' for (i) accomplishes this, where

(i)' the stress intensity factor is proportional to a power of the crack length.

If the power of the proportionality were negative one-half, this assumption would be appropriate for cracks radiating from both sides of a hole in a panel which is under tension, as well as in other structures in which the initial stages of crack growth cause relaxation of the stress intensity. This is, in fact, the law which has been ostensibly verified in such investigations.

If we replace (i) by (i)' the only difference is that the resulting differential equation of 3° need not satisfy the restriction $b > -1$ for the second disposable constant. With this restriction dropped the propagation function in equation (1) is exactly the same but $b$ may be any real number. And, in case $b \leq -1$, the rate of propagation decreases for $t > t_1$.

From (i)' is obtained, if we let $K(s)$ be the stress intensity at crack length $s$, the relation for some real $\epsilon$, positive $\nu$

$$K(s) = \nu s^{\epsilon} \qquad s \geq 0 .$$

From (ii) is obtained, for some real $\delta$ and positive $\eta$, the relation

$$\ln s' = \delta + \eta \ln K(s) \qquad \text{for} \quad s \geq 0 \; .$$

From these two equations is obtained a differential equation the solution of which is given in 3° , but $\eta$ can now be any real number and consequently $b$ can be any real number. The difference in the behavior of the crack length as a function of time is illustrated in Figure (5).

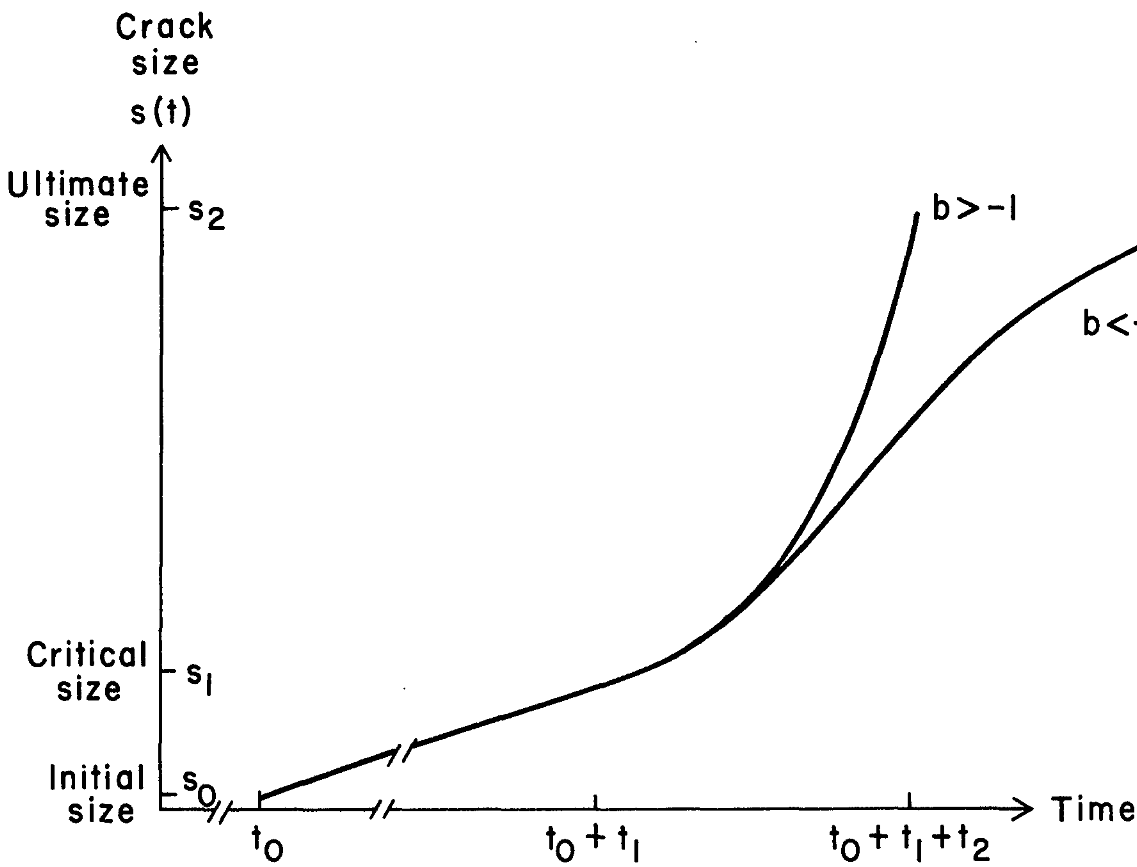


Figure 5. Crack Growth as a Function of Time

## SECTION VI

## SIMILARITY BETWEEN THE FATIGUE LIFE OF METALLIC AND COMPOSITE MATERIALS

A mathematical model which is sufficiently general to encompass virtually all the observed variational behavior of the fatigue process and which follows mathematically from two laws of fracture mechanics has been described. The summarization of this complex mathematical description into a probability law which is sufficiently general that it cannot be disqualified by the data, but is sufficiently simple that it can be easily utilized, is the goal of this discussion.

The law of distribution which is proposed here as a candidate is the same one which was studied in [4] as regards the estimation of parameters. If $T$ denotes the fatigue life of a component, let $X = \ln T$ be the life recorded in logarithmic units. Assume there exists a known transformation $\omega \in \Omega$ where

$$\omega \in \Omega \quad \text{iff} \quad \omega' \geq 0 \quad , \quad \omega' \geq \omega'' \quad \text{and} \quad \omega \quad \text{is odd} .$$

Here it is presumed that $\omega$ is determined by the type of material and the manufacturing process as well as the geometry of the component.

With $\omega$ given assume there exist $\alpha, \nu > 0$ , $-\infty < \mu < \infty$ such that

$$X \sim G(\cdot \; ; \alpha, \nu, \mu)$$

where

$$G(x:\alpha, \nu, \mu) = \Phi\left[\frac{1}{\alpha}\,\omega\left(\frac{x-\mu}{\nu}\right)\right] \qquad -\infty < x < \infty . \tag{4}$$

In this formulation $\mu$ is the location *parameter*, $\nu$ is the *scale* parameter and $\alpha$ is the *flexure* parameter. The flexure and scale together control the shape of the distribution.

It is thought that the correspondence between the physics of materials and the parameter space is as follows:

the location parameter $\mu$ is controlled by the maximum imposed stress.

the scale parameter $\gamma$ is determined by the variation in the material quality and the distribution of stress.

the flexure parameter $\alpha$ is governed by the distribution of initial flaws and the probability with which the flaws exceed the initial crack size.

The Weibull distribution has received some attention recently as an appropriate model for the distribution of life of composite structures, see [10], [11], [12]. This has been based on an extrapolation of the success of the Weibull distribution as a prediction of fatigue life from all types of metallic structures including both high and low strength steels, various aluminum alloys and titanium as well, see [5]. The values of the estimates of the shape parameter for the composite materials studied were near unity, that is to say the fitted Weibull distribution was close to an exponential distribution.

This low estimate of the Weibull shape parameter indicates that the corresponding value of the scale parameter for the log-life should be high since one is almost the reciprocal of the other. However, because there are, as yet, only sufficient data to determine the central portion of the distribution any extrapolation to the tails must be hazardous. Note that the conclusion drawn from this estimated life distribution being near exponential is that composite structures "don't wear out". This does not mean they don't fail, only that if a component has not failed, it is probably not worthwhile to replace it with a new one no matter how old.

As data accumulated for metallic components, the fit of the Weibull distribution became less certain since the hypotheses could be rejected in many instances. This rejection was principally due to the existence of long lived components which were incompatible with the assumption of increasing failure rate. This could be explained, as was done by Weibull himself, by postulating that the data was a mixture of two different Weibull distributions with different characteristic lives. This assumption, as we know, will explain the flattening of the tail of the distribution as demonstrated in Figure (6).

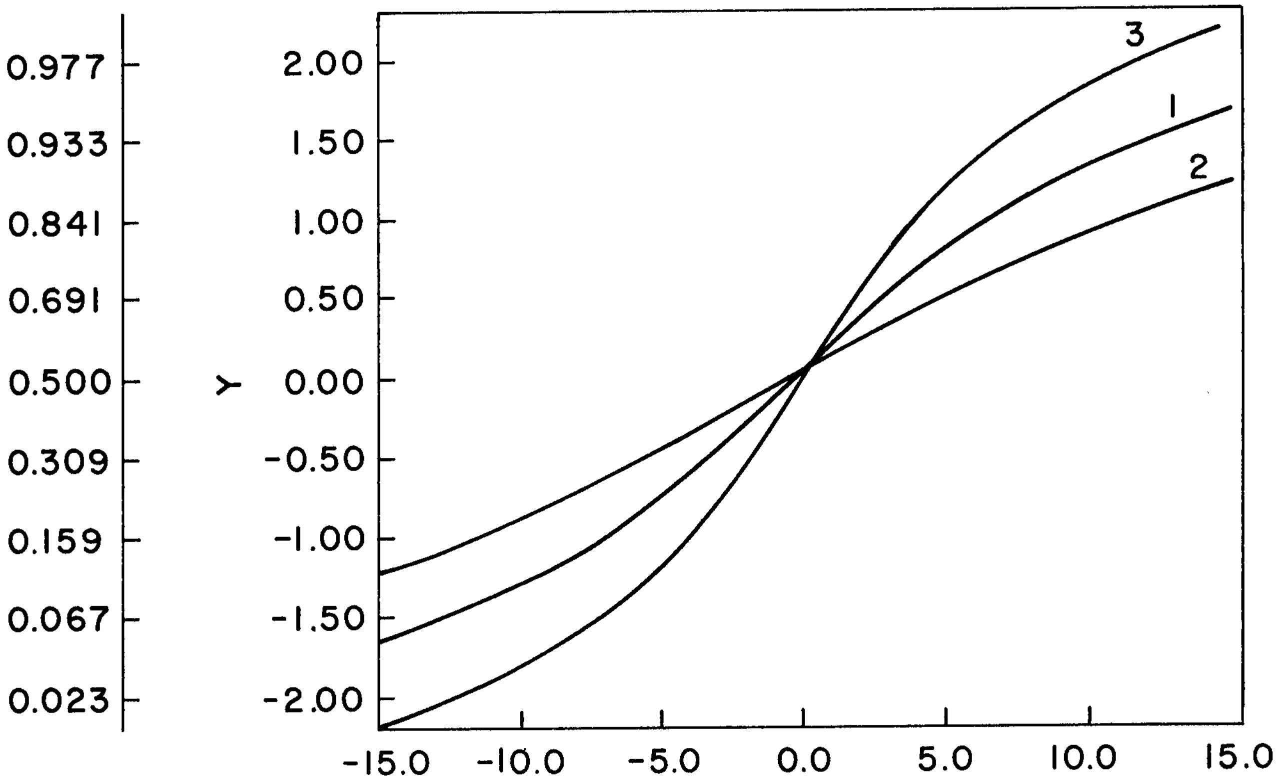


Figure 6. Graph of Distribution with Various Flexures and Fixed Location and Scale

But whether or not the estimation of the four or five parameters necessary to determine the correct mixture of Weibull distributions will be easier than the estimation of the three parameters for the distribution as defined in equation (4) has yet to be demonstrated.

It is our supposition that the extensive flexibility of the family of distributions given in (4), with its presently existing computerized estimation techniques from truncated data, will be very useful in the estimation of inspection periods necessary to maintain the reliability of composite structures against failure from fatigue.

## SECTION VII

## CONCLUSIONS

In preceding studies, Ref [4], the fatigue process has been analysed during separate stages of its development. The question that was addressed was how these separate phases of crack growth, with the stochastic variation across the population of components and service environments, should be combined to generate a mathematical model of the total time to failure distribution which would agree with the data on fatigue life of metallic components that had been collected from actual airframe structure in service.

The first supposition was that the convolution of independent variates representing the times of transition from stage to stage could be accomplished so that the final distribution agreed with the standardized data which had been assembled. Dozens of trials indicated that this was probably not so.

The next suggestion was that two such stages, without the first-stage time-to-crack initiation variate, could be convolved to obtain a distribution agreeing with the data. Unfortunately the same negative results were obtained.

In an attempt to provide an explanation for what has been observed, certain functions, which could be easily integrated in closed form, were selected and the integrated results investigated. The effect of a random initial flaw size on the toe of the distribution of total life was found to be in agreement with the observed data.

Further investigation revealed that if the final stages of crack growth exhibit crack deceleration,or a probability of crack arrest, this alters the tail of the distribution of total life in a way which agrees with the observed data. This necessitated a reinterpretation of the original fracture mechanical law under which the propagation factor was derived with a consequent extension of its applicability.

These studies resulted in the adoption of a three-parameter distribtuion of fatigue life which reflect, to a degree, the main features in each distinct phase of behavior. These three parameters can be expressed as the scale, location and flexure of the distribution of the logarithm of life.

There are distinct advantages to the generality which is afforded by the introduction of a third, or flexure, parameter. The principal one is that it affords a degree of control over the tails of the distribution and it avoids the "safe life" concept during the initial period of use of a new component when in fact the evidence is that the phenomenon of incipient early failures, called infant mortality, is often observed.

These were shortcomings noted in earlier studies of this family of distributions, namely, that when all three parameters were unknown the usual method of maximizing the likelihood could not be used to estimate all three parameters simultaneously.

The practical consequence was that a complete determination of the parameters governing the fatigue life must be accomplished by calculations relying in part upon knowledge from other disciplines. Some of the disposable constants which appear in the model must be related by theory to physical measurements determined from the material or physical geometry of the specimen. This effort has been undertaken previously for the Weibull distribution of fatigue life for metallic components. It is now being attempted for different types of woods in the construction of glue-laminated beams for use in the building industry.

It is the hope that this three-parameter model coupled with the selection of the appropriate $\omega$-transformation of the log-life will provide as much flexibility in the selection of a distribution of life as can be found by mixing two Weibull distributions each having two or three parameters. Moreover the problem of parameter estimation from limited or truncated samples for the prediction of life in service will be easier for this $\omega$-transformed distribution of log-life than from the mixture of Weibull distributions.

Because of its construction in accord with fracture mechanical principles, which should be virtually the same in any material, it is expected that this derived distribution should have significant applicability and success in the calculations of inspection periods to prevent fatigue failure in structures of either metallic alloy or composite materials. Such results must await a study of the type performed in Reference [13].

## REFERENCES


[1] Butler, J.P., "Reliability Analysis in the Estimation of Transport-type Aircraft Fatigue Performance" in International Conference on Structural Safety and Reliability, Ed. by A. M. Freudenthal, Pergamon Press, pp. 181-212, 1972.

[2] Probabilistic Aspects of Fatigue, Ed. by R.A. Heller, American Society for Testing and Materials, STP 511, 1972.

[3] Whittaker, I.C., and Besuner, P.C., "A Reliability Analysis Approach to Fatigue Life Variability of Aircraft Structures", Air Force Materials Laboratory Technical Report, AFML-TR-69-65, 1969.

[4] Saunders, Sam C., "A Synthesis of Crack Growth Theories in Fatigue into Some Statistical Distributions of Fatigue Life", Aerospace Research Laboratories Technical Report, ARL-TR-75-0178, 1975.

[5] Whittaker, I.C., "Development of Titanium and Steel Fatigue Variability Model for Application of Reliability Analysis Approach to Aircraft Structures", Air Force Materials Laboratory Technical Report, AFML-TR-72-236, Appendix I, 1972.

[6] Saunders, Sam C., "The Treatment of Data in Fatigue Analysis" in Proceedings of the Colloquium on Structural Reliability, Ed. by J.L. Swedlow, T.A. Cruse and T.C. Halpin, Carnegie-Mellon Univ., pp. 136-15, 1972.

[7] Whittaker, I.C. and Saunders, S.C., "Application of Reliability Analysis to Aircraft Structures Subject to Fatigue Crack Growth and Periodic Structural Inspection", Air Force Material Laboratories Technical Report, AFML-TR-73-92, 1973.

[8] Saunders, S.C., "A Family of Random Variables Closed under Reciprocation", J. Amer. Statist. Assoc., 69, 533-539, 1974.

[9] Birnbaum, Z.W. and Saunders, S.C., "A New Family of Life Distributions", J. Appl. Probab., 6, 319-327, 1969.

[10] Robinson, E.Y., "Estimating Weibull Parameters for Composite Materials", in Proceedings of the Colloquium on Structured Reliability, Ed. by J.L. Swedlow, T.A. Cruse, and J.C. Halpin, Carnegie-Mellon Univ., pp. 463-526, 1972.

[11] Jerina, K.J. and Halpin, J.C., "Statistical Stress Concentration Effects in Composites", loc. cit., pp. 308-322.

[12] Eisenmann, J.R., Kaminski, B.E., Reed, K.L., and Wilkins, D.J. "Toward Reliable Composites: An Examination of Design Methodology" loc. cit., pp 359-381.

[13] Whittaker, I.C. and Saunders, S.C., "Exploratory Development on Application of Reliability Analysis to Aircraft Structures Considering Interaction of Cumulative Fatigue Damage and Ultimate Strength", Air Force Materials Laboratory Technical Report, AFML-TR-72-283, 1973.



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