BALLISTIC RESEARCH LABORATORY

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# REPORT

RÖGGLA'S EQUATION AND ITS APPLICATION TO INTERIOR BALLISTIC PROBLEMS

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by

R. H. KENT

ABERDEEN PROVING GROUND
ABERDEEN, MD.

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ADDENDUM TO BALLISTIC RESEARCH LABORATORY REPORT NO. 48 (REVISED)
by R. H. Kent

## Initial and Maximum Values of the Roggla-Kent Dimensionless Pressure.

In Ballistic Research Laboratory Report No. 48 Kent derives Eq. (9), viz.

$$\eta \frac{d^3\eta}{d\theta^3} = \left(\frac{d^2\eta}{d\theta^2}\right)r - \bar{\gamma}\,\frac{d\eta}{d\theta}\,\frac{d^2\eta}{d\theta^2},$$

where $\bar{\gamma}$ is a mean ratio of specific heats, r the exponent in the rate-of-burning law, $\eta$ the free volume expansion ratio, and $\theta$ a dimensionless time variable. For fixed values of r and $\bar{\gamma}$ this is a universal differential equation applicable to all guns, subject to the limitations of the Roggla theory. For a universal value of the initial second derivative

$$\left(\frac{d^2\eta}{d\theta^2}\right) \equiv a$$

it gives $\eta$ as a universal function of $\theta$, and thus $\frac{d^2\eta}{d\theta^2}$ as a universal function of $\eta$, viz $\varphi(\eta)$. Except for the ordinate scale Roggla's P $-\eta$ chart for his "zero degree of degressiveness" is a plot of $\varphi(\eta)$ vs $\eta$, so that $\frac{d^2\eta}{d\theta^2}$ may well be called the dimensionless pressure.

Together with the values r = 0.7 and $\bar{\gamma}$ = 1.2 Roggla used a value 1000/43.5 = 23.0 for the ratio of maximum pressure to initial pressure. Numerical integration of Eq. (9) led to the value 0.640 for $[\varphi(\eta)]_{max}$ for values of $a$ about equal to 0.028. The value found for $[\varphi(\eta)]_{max}$ was not sensitive to the value of $a$, being 0.640 for a = 0.03. Thus one can say that for a = 0.0278, $[\varphi(\eta)]_{max}$ has the value 0.640. The ratio of 0.640 to 0.0278, however, is just 23.0. We thus have the result that Roggla's "zero degressiveness" P $-\eta$ curve corresponds to the values:

$$\varphi(1) = a = 0.0278$$
$$[\varphi(\eta)]_{max} = 0.640$$

The numerical integration showed the maximum value of $\varphi(\eta)$ to occur at $\eta = 1.84$, Roggla's value 1.93 thus being in error by 5%.

J. P. Vinti

J. P. Vinti

MAY 25 1942

## RÖGGLA'S EQUATION AND ITS APPLICATION TO INTERIOR BALLISTIC PROBLEMS (REVISED)

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Some time ago Röggla published a series of interior
ballistic charts together with a pamphlet* explaining the
method of computing velocities and pressures. While these
charts have a useful field of application, the theoretical
treatment of Röggla may also be used advantageously in
calculating, without their aid, the performance of a new
weapon when interior ballistic data for a weapon differing
not too widely from it in caliber, muzzle velocity etc. are
known or in calculating the effect of a change in bullet
weight, web thickness etc. in a given weapon. For this
reason an exposition is given of the deduction of Röggla's
equation and a description of the method of its application
to interior ballistic problems.

Röggla's equation is the following:

$$p_1 = p \left[ \frac{(\overline{\gamma_1}-1)e_1\,\sigma_1 b_1}{(\overline{\gamma}-1)e\,\sigma b} \right]^{\frac{2}{3-2r}} \left( \frac{O_{o_1}}{O_o} \right)^{\frac{2}{3-2r}} \left( \frac{v_o}{v_{o_1}} \right)^{\frac{1}{3-2r}} \left( \frac{a_1}{a'} \right)^{\frac{1}{3-2r}} \left( \frac{a}{a_1} \right)^{\frac{2}{3-2r}} {**} . \quad (1)$$

In this equation;

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FIRING RECORD FILE.</span>

$p$ = pressure,

$\overline{\gamma}$ = a pseudo ratio of specific heats,  <span style="color:red">JUL 21 1941</span>

$e$ = the energy of the powder per unit weight,

* Neue Diagramme Für Die Angewandte Innere Ballistik, Theorie und Beispiele"
  Edmund Röggla, Pilsen, 1930
**Obtained from Röggla's (7) on page 6 of "Theorie und Beispiele."

$\sigma$ = the density of the powder,

b = the specific rate of burning,

r = exponent of pressure in the rate of burning law,*

$O_o$ = the initial surface of the charge,

$v_o$ = the initial free volume,

G' = the reduced weight of the projectile which takes account also of the weight of charge,

q = the cross sectional area of the bore.

The derivation of Röggla's equation depends fundamentally upon Résal's equation which may be written in the form:

$$\text{I} \qquad\qquad \text{II} \qquad \text{III}$$

$$e\,\sigma\,b\,O\,p^r\,dt = p\,dv + \frac{p\,dv + v\,dp}{\gamma - 1} \qquad . \qquad (2)$$

The term under I gives the amount of energy delivered by the powder charge in time dt on the assumption that the rate of burning is proportional to the rth power of the pressure. The term under II is the work done during the corresponding time. The term under III represents the increase in the internal energy of the powder gas. However, it holds only on the contrary to fact assumption that the ratio of the specific heats is constant.

The internal energy of a gas of mass, M, is

$$M \int_o^T c_v\,dT = M\,\overline{c}_v T,$$

where $c_v$ is the specific heat at constant volume and T the absolute temperature. In the case of most gases

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* While the experiments of Crow and Grimshaw seem to have shown that when a pure organic powder is burnt, the rate of burning is directly proportional to the pressure, other experiments e.g. those in the 240 mm Howitzer tend to show that when there is a considerable admixture of black powder the exponent is reduced possibly to as low as .7.

Further work on the influence of black powder on the rate of burning needs to be done.

$$c_p - c_v = R,$$

where $c_p$ is the specific heat at constant pressure and $R$ is the gas constant. Hence we may write the internal energy of the gas as

$$M \bar{c}_v T = \frac{M \bar{c}_v R T}{c_p - c_v} = \frac{M p(v-b)^*}{\dfrac{c_p}{c_v} - \dfrac{c_v}{c_v}} \quad .$$

If $c_v$ is a constant, then

$$\frac{c_p}{c_v} - 1 = \gamma - 1,$$

On the other hand, if $c_v$ is not a constant we may designate

$$\frac{R}{c_v} \text{ by } (\bar{\gamma} - 1)$$

from which it follows that

$$\bar{\gamma} = 1 + \frac{R}{c_v} \quad .$$

We call $\bar{\gamma}$ the pseudo ratio of specific heats.

If $c_v$ is a function of temperature, then in general $\bar{\gamma}$ is also a function of temperature. If the variation in the temperature of the powder gas is not too great, $\bar{\gamma}$ may be taken as approximately constant while the projectile is in the bore. In view of this the $\gamma$ in Résal's equation (2) should be replaced by $\bar{\gamma}$ .

In the term under III, v really represents the free volume back of the projectile whereas under II, v represents total volume. By free volume is meant the volume back of the projectile minus the volume of the unburnt powder and the co-volume of the burnt powder. However, on the assumption that the co-volume of the powder is equal to the volume

* b is the co-volume in this expression.

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of the equivalent mass of solid powder, the equation is
still valid in this respect. While this is not accurately
true, it is possibly not a serious error in view of the fact
that Résal's equation should be used for only approximate
computations.

We accordingly rewrite Résal's equation in place of
(2) as

$$e^{\sigma} b \, 0 \, p^r \, dt = p \, dv + \frac{p \, dv + v \, dp}{\gamma - 1} \tag{2a}$$

and v now represents the free volume both under II and III.

We next introduce the free space ratio, $\eta$ , which is the
ratio of free volume at any time to the original free volume.
We accordingly obtain

$$dv = d\eta \cdot v_o.$$

The assumption is made that the passive resistance is pro-
portional to the pressure. As a result of this hypothesis, the
work, $A = \int p dv$, is given by

$$A = \frac{G' c^2}{2g} ,$$

where G' is the reduced weight of the projectile, which takes
account of kinetic energy of the powder gas and also of the
work of the passive resistance and c is the velocity. It
may readily be shown that if G is the weight of the projectile,
L the weight of charge and if the pressure equivalent to the
forcing resistance is given by

$$\frac{\mu}{gq} \cdot \frac{Gdc}{dt} , \quad ^{*}$$

then the reduced weight of the projectile G' is given by

$$G' = G(1 + \mu + \frac{L}{3G} ).$$

We accordingly also obtain

$$dA = \frac{G'}{g} \, cdc = pdv. \tag{3}$$

Since the velocity is the derivative of the travel with respect
to the time the following relations are easily obtained:

* In other words, $\mu$ is the ratio of the pressure equivalent to the forcing
  resistance to the pressure effective for the acceleration of the projectile.

$$\frac{dc}{dt} = \frac{d^2 s}{dt^2} = \frac{1}{q}\, v_o\, \frac{d^2\eta}{dt^2} \qquad (4)$$

$$p\, dv = \frac{G'}{g}\, \frac{v_o^2}{q^2}\, \frac{d\eta}{dt}\, \frac{d^2\eta}{dt^2}\, dt \qquad (5)$$

$$p = \frac{G'}{g}\, \frac{v_o}{q^2}\, \frac{d^2\eta}{dt^2},\ dp = \frac{G'}{g}\, \frac{v_o}{q^2}\, \frac{d^3\eta}{dt^3}\, dt. \qquad (6)$$

By means of the substitution of these relations in (2a) one obtains

$$\frac{d^3\eta}{dt^3} = \dddot{\eta} = \frac{B\,\ddot{\eta}^r - \overline{\gamma}\dot\eta\ddot\eta}{\eta}\ ^{*}, \qquad (7)$$

where $B = (\overline{\gamma}-1)\, e\, \sigma\, b\, Q_o \left(\dfrac{g\, q^2}{G'v_c}\right)^{1-r} \cdot \dfrac{1}{v_o}$.

The above three paragraphs are practically equivalent to a translation of paragraph 2 of "Neue Diagramme für Die Angewandte Innere Ballistik, Theorie und Beispiele."

In our simplified treatment we assume that the total burning surface of the charge is constant, a condition which is approximately realized in the case of strip and tubular powder.

Thus Röggla's equation applies more precisely to grains with constant burning surface and not to degressively or to progressively burning grains. However, for the latter powders, it may be used as an approximation.

Let $\theta = jt$, so that

$$\dot\eta = j\, \frac{d\eta}{d\theta}$$

$$\ddot\eta = j^2\, \frac{d^2\eta}{d\theta^2}$$

$$\dddot\eta = j^3\, \frac{d^3\eta}{d\theta^3}.$$

* The superscript dots signify differentiation with respect to time.

Then (7) becomes

$$j^3 \frac{d^3\eta}{d\theta^3} = \frac{B \; j^{2r}\left(\frac{d^2\eta}{d\theta^2}\right)^r - \overline{\gamma} \; j^3 \frac{d\eta}{d\theta} \; \frac{d^2\eta}{d\theta^2}}{\eta} \qquad . \qquad (8)$$

If we take $j = B^{\frac{1}{3-2r}}$, (8) reduces to

$$\eta \; \frac{d^3\eta}{d\theta^3} = \left(\frac{d^2\eta}{d\theta^2}\right)^r - \overline{\gamma} \; \frac{d\eta}{d\theta} \cdot \frac{d^2\eta}{d\theta^2} \qquad . \qquad (9)$$

It is thus evident that on Röggla's assumptions there is one differential equation for all interior ballistic trajectories.

We now take initial conditions as follows:

$$\eta_o = 1$$

$$\left(\frac{d\eta}{d\theta}\right)_o = 0$$

$$\left(\frac{d^2\eta}{d\theta^2}\right)_o = a, \text{ a constant for all trajectories.}$$

The first of these conditions holds by definition, the second expresses the fact that the initial velocity is zero, while the third is undoubtedly contrary to fact but is very convenient and may not lead to excessive errors. There is only one solution, $\eta = f(\theta)$, of (9) satisfying these initial conditions and hence on the given assumptions there is only one interior ballistic trajectory, $\eta = f(\theta)$, for all combinations of gun, powder and projectile.

Equation (6) gives the pressure as a function of the time. Now

$$\ddot{\eta} = j^2 \frac{d^2\eta}{d\theta^2} = j^2 f''(\theta) = j^2 f''\left[f^{-1}(\eta)\right] = j^2 \varphi(\eta),$$

where the function $f^{-1}$ is the inverse of $f$ and $\varphi(\eta)$ is defined by

$$\varphi(\eta) \equiv f'' \left[ f^{-1}(\eta) \right] \quad .$$

Hence the pressure can be expressed as a function of the free space ratio, $\eta$, as follows:

$$p = \frac{G'}{g} \frac{v_o}{q^2} \ddot{\eta} = \frac{G'}{g} \frac{v_o}{q^2} j^2 \varphi(\eta) = \frac{G' v_o}{g q^2} \frac{2}{3-2r} \varphi(\eta)$$

$$= \left[ \frac{G' (\overline{\gamma}-1)^2 e^2 \sigma^2 b^2 p_o^2}{g q^2 v_o} \right]^{\frac{1}{3-2r}} \varphi(\eta) . \tag{10}$$

The value of $\eta$ at which $\varphi(\eta)$ is a maximum gives the position of maximum pressure.

If $p$ and $p_1$ are the pressures for two different weapons, and $\eta$ and $\eta_1$ their corresponding free space ratios, we have, letting $\dfrac{G'}{g} \dfrac{v_o}{q^2}$, the coefficient of $\ddot{\eta}$ in (5), be $k$

and the corresponding coefficient for the case of the second weapon be $k_1$,

$$p = k\ddot{\eta}$$

$$p_1 = k_1 \ddot{\eta}_1 .$$

From the above relations we obtain

$$p = k j^2 \frac{d^2\eta}{d\theta^2}$$

$$p_1 = k_1 j_1^2 \frac{d^2\eta_1}{d\theta_1^2} ,$$

where $\theta_1 = j_1 t$ and $j_1 = B_1^{\frac{1}{3-2r}}$ .

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and $\dfrac{\rho_1}{\rho} = \dfrac{k_1}{k}\left(\dfrac{B_1}{B}\right)^{\frac{2}{3-2r}} \dfrac{\dfrac{d^2\eta_1}{d\Theta_1^2}}{\dfrac{d^2\eta}{d\Theta^2}}$ .

For equal free space ratios, that is for $\eta_1 = \eta$ ,

$$\frac{\rho_1}{\rho} = \frac{k_1}{k}\left(\frac{B_1}{B}\right)^{\frac{2}{3-2r}} .$$

Substituting in this equation the values of k and B, we obtain Föggla's equation

$$\rho_1 = \rho\left[\left(\frac{(\overline{\gamma}_1-1)e_1\sigma_1 b_1}{(\overline{\gamma}-1)e\sigma b}\right)^2\left(\frac{\rho_{o_1}}{\rho_o}\right)^2\left(\frac{v_o}{v_{o_1}}\right)\left(\frac{G_1}{G^!}\right)\left(\frac{a}{a_1}\right)^2\right]^{\frac{1}{3-2r}} . \qquad (1) \quad *$$

Since for two given sets of conditions the ratio of $\rho_1$ to $\rho$ is constant, it follows that the ratio of the starting pressures is the same constant. This is therefore an additional assumption. It may be stated in this form: the ratio of the starting pressures is assumed to be equal to the ratio of the maximum pressures.

It may easily be shown that if the shape of the grain and the density are unchanged

$$\frac{\rho_{o_1}}{\rho_o} = \frac{C_1 w}{C w_1} . \qquad **$$

If the compositions of the powder are also identical, equation (1) becomes

* We distinguish between $\overline{\gamma}_1$ and $\overline{\gamma}$ in this expression is spite of our assumption that $\gamma_1 = \gamma$ because a relatively small change in $\overline{\gamma}$ will produce a relatively large change in $(\overline{\gamma} - 1)$.

** C is the weight of charge, w is the web thickness.

$$p_1 = p\left[\left(\frac{C_1 w}{C w_1}\right)^2 \cdot \left(\frac{v_o}{v_{o_1}}\right)\left(\frac{G_1'}{G'}\right)\left(\frac{q}{q_1}\right)^2\right]^{\frac{1}{3-2r}} . \tag{11}$$

Several interesting deductions may be made directly
from this equation; one is that if the web thickness de-
creases while the maximum pressure is kept the same the
position of the point of maximum pressure will be at a
greater distance from the breech than formerly.

It follows directly from equation (11) that if $w_1$ is
less than $w$ and the pressure remains unchanged, $v_{o_1}$
must be greater than $v_o$. As is shown on page 7 the pressure
is a function of the free space ratio. Hence if the initial
free volume is increased it follows that the travel for a
given free space ratio will be greater. Hence since the
position of the point of maximum pressure depends directly
on the free space ratio, it follows that for the quicker
powder the point of maximum pressure will be further
removed from the breech.

This equation may also be used for deriving the differ-
ential coefficients for the maximum pressure which are used
in correcting maximum pressures for weight of projectile,
chamber capacity, etc.

We write (See Hitchcock's "Differential Coefficients
for Interior Ballistics", File R-II-23)

$$\frac{dp}{p} = \alpha_1 \frac{dC}{C} + \beta_1 \frac{dV}{V} + n_1 \frac{dG'}{G'} + \kappa_1 \frac{dw}{w} ,$$

where $p$ represents the maximum pressure and $V$ represents the
chamber capacity.

If we now let

$$p_1 = p + dp,$$

$$C_1 = C + dC,$$

$$w_1 = w + dw, \text{ etc.,}$$

we obtain from (11)

$$1 + \frac{d\rho}{\rho} = 1 + \frac{2}{3-2r} \cdot \frac{dC}{C} - \frac{1}{3-2r} \cdot \frac{dv_o}{v_o} + \frac{1}{3-2r} \cdot \frac{dG'}{G'} - \frac{2}{3-2r} \cdot \frac{dw}{w}. \quad (12)$$

In case the chamber capacity V is kept constant, then

$$dv_o = -\frac{dC}{\sigma},$$

since

$$v_o = V - \frac{C}{\sigma}.$$

In case the charge is kept constant while the chamber volume varies,

$$dv_o = dV.$$

Hence

$$dv_o = dV - \frac{dC}{\sigma}$$

and

$$\frac{dv_o}{v_o} = \frac{V}{v_o} \cdot \frac{dV}{V} - \frac{C}{\sigma v_o} \cdot \frac{dC}{C}.$$

Equation (12) for the differential coefficients thus becomes

$$\frac{d\rho}{\rho} = \frac{1}{3-2r} \left( 2 + \frac{C}{\sigma v_o} \right) \frac{dC}{C} - \frac{1}{3-2r} \cdot \frac{V}{v_o} \cdot \frac{dV}{V} + \frac{1}{3-2r} \cdot \frac{dG'}{G'} - \frac{2}{3-2r} \frac{dw}{w}. \quad (13)$$

For purposes of comparison, the values of the differential coefficients have been computed for two guns from the above equation, taking r = .7, .8 and 1 and placed in a table together with the values given in Hitchcock's paper which are deduced from the Interior Ballistic Tables and also some experimental values.

In the Table the values computed by Röggla's equation are
headed by R, Hitchcock's values by H, and the experimental
values by E.

| Gun | V | Wt.of Proj. | Wt. of Chg. | Exp.* r | $\alpha_1$ | | | $\beta_1$ | | | $n_1$ | | $\kappa_1$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | R | H | E | R | H | E | R | H | R | H |
| 75 mm Mod. 1897 | 84 | 11.72 | 1.316 | .7 | 1.48 | | | -.85 | | | .625 | | -1.25 | |
| | | | | .8 | 1.69 | 1.43 | 2.01 | -.98 | -.75 | -.92 | .715 | .63 | -1.43 | -1.33 |
| | | | | 1.0 | 2.37 | | | -1.37 | | | 1.00 | | -2.00 | |
| 3"M1 1917 | 296 | 15 | 4.61 | .7 | 1.49 | | | -.85 | | | .625 | | -1.25 | |
| | | | | .8 | 1.70 | 1.57 | 1.67 | -.98 | -.77 | | .715 | .65 | -1.43 | -1.46 |
| | | | | 1.0 | 2.38 | | | -1.37 | | | 1.00 | | -2.00 | |

NOTE: The few experimental values are obtained by the use of coppers, the
coefficients might be appreciably different if they were determined
by the piezo-electric gauge.

* The coefficients have been computed as shown above for various values
of the exponent, r, of the pressure in the rate of burning law.

In conclusion we proceed to solve some problems by means of Röggla's equation and check them by the firing results.

## 1st Problem

Given

Chamber capacity - 84 cu. in.
Weight of charge - 1.316 lbs.
Density of powder - .057 lb/in$^3$
Weight of projectile - 11.72 lbs.
Maximum pressure - 23,350 lbs.

Required: to compute the maximum pressure in the same gun with the following conditions:

Chamber capacity - 84 cu. in.
Weight of charge - 1.551 lbs.
Weight of projectile - 15.96 lbs.
The powder is of the same lot.

The results obtained by the computations are the following:

For $r = 1$, $p_1 = 47,100$

For $r = .8$, $p_1 = 38,600$

For $r = .7$, $p_1 = 36,200$

Observed pressure, 35,200

## 2nd Problem

Given

$C = 20.2$ oz.
$V = 84$ cu. in.
$p = 30,370$ lb/in$^2$

Required: to compute $p_1$ if $C_1 = 18$ oz. and other variables are unchanged.

Result of computations:

$p_1 = 25,540$

$p_1$ observed $= 25,300$.

<u>3rd Problem</u>

     Given

$C = 1.28$ lbs.
$V = 34$ cu. in.
$p = 36,830$ lb/in$^2$

     Required: to compute $p_1$ for

$C_1 = .445$ lbs.

$V_1 = 35$ cu. in., and other variables unchanged.

The result of computation:

$p_1 = 16,000$ lb/in$^2$

$p_1$ observed = 16,150*.


     From the above examples, it may be seen that very satis-
factory results are obtained, using $r = .7$. Hence in spite
of the fact that we know that the rate of burning of the
pure organic powder in the closed chamber is unity, in
view of the more satisfactory results obtained by the ex-
ponent .7, such an exponent should be used in applying
Höggla's equation to ordinary problems. There is evidence
that the reduction of the exponent below unity serves to
compensate for the contrary to fact assumptions that the
starting pressure is proportional to the maximum pressure and
that the forcing resistance is proportional to the acceleration
of the projectile.

     I am indebted to Miss B. I. Hart for help in the revision.


                             R. H. Kent
                             R. H. Kent


* The last two examples were computed by Mr. Lane, using $r = .7$.