

AD-A026 533

RIA-76-U381

WVT-TR-76017

AD

USADACS Technical Library



5 0712 01010892 5

PROPAGATION OF A CIRCULAR CRACK IN TORSION

**TECHNICAL
LIBRARY**

May 1976



**BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189**

TECHNICAL REPORT

AMCMS No. 611102.11.H4500.30

19970930 059

Pron No. EJ-6-Y0012-EJ-M7

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DTIC QUALITY INSPECTED 1

DISPOSITION

Destroy this report when it is no longer needed. Do not return it to the originator.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER WVT-TR-76017	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PROPAGATION OF A CIRCULAR CRACK IN TORSION		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert L. Ryan		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Benet Weapons Laboratory Watervliet Arsenal, Watervliet, N.Y. 12189 SARWV-RT		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS AMCMS No. 611102.11.H4500.30 Pron No. EJ-6-Y0012-EJ-M7
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Armament Command Rock Island, Illinois 61201		12. REPORT DATE May 1976
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 26
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE		
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Fracture (Mechanics) Integral Equations Torsion Crack Propagation		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The propagation of a circular crack under dynamic torsional loading conditions is considered. The crack perimeter speed is taken to be an arbitrary monotonic increasing function of time. A general solution is obtained by means of integral transforms, together with Kostrov's technique; particular results are extracted for an initial value problem and for a suddenly appearing crack problem. The dynamic stresses near the periphery of the crack are found to have an inverse square root singularity.		

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

WVT-TR-76017

AD

PROPAGATION OF A CIRCULAR CRACK IN TORSION

Robert L. Ryan

May 1976



**BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189**

TECHNICAL REPORT

AMCMS No. 611102.11.H4500.30

Pron No. EJ-6-Y0012-EJ-M7

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

TABLE OF CONTENTS

	<u>Page</u>
DD Form 1473	
INTRODUCTION	1
PROBLEM FORMULATION AND SOLUTION	2
CASE 1: INITIAL VALUE PROBLEM	15
CASE 2: SUDDENLY APPEARING CRACK	17
DISCUSSION	20
REFERENCES	21
APPENDIX	22

LIST OF ILLUSTRATIONS

1. Solution Region: Drawn for the Case $t < R$, $r > r^*(t)$	6
2. Solution Region: Drawn for the Case $t > R$, $r > r^*(t)$	7
3. Diagram for Crack Plane Stress Solution	8
4. Diagram for Crack Plane Displacement Solution	14

INTRODUCTION

This report deals with the mathematical aspects of a problem of importance in the failure of structures, namely, the sudden spreading of a crack in a stressed medium. A previous study on the dynamic problem of torsion of a circular crack was made by Sih and Embley [1]. Their work, however, is restricted to the case of a crack which opens instantaneously to a fixed radius and thereafter remains stationary. Most work on crack propagation under other loading conditions is based on the mathematically expedient supposition that propagation occurs at constant speed. It is more realistic to consider the speed of propagation as a function of time based on some physical hypothesis. Recently, Kostrov [2] has proposed an analytical technique for treating a variable speed crack and has obtained results for the case of anti-plane loading conditions. In this report Kostrov's technique is applied to the case of torsional loading conditions.

Kostrov's method has the distinct advantage over other methods, notably the Weiner-Hopf method and the dual integral equation method, in that mixed problems with time dependent boundaries can be treated. Moreover, the near-field solution can be deduced rather easily, at least for torsion and anti-plane strain conditions. An inherent drawback of Kostrov's method is that for a finite length crack the

¹G. C. SIH and G. T. EMBLEY (1972) Journal of Applied Mechanics, 395-400. Sudden Twisting of a Penny-Shaped Crack.

²B. V. KOSTROV (1966) Journal of Applied Mathematics and Mechanics, 30(6), 1241-1248. Unsteady Propagation of Longitudinal Shear Cracks.

analytical results are limited to the time interval for which waves emanating from opposite points on the crack border have not yet come into contact; it appears that numerical integration is required for the time in which wave interaction occurs. On the basis of the results of the study by Sih and Embley [1], however, the dynamic stress intensity factor reaches its maximum value before wave interaction occurs. Thus it is reasonable to suppose that the results obtained in this report on the dynamic stress intensity factor represent the maximum value that can be expected.

PROBLEM FORMULATION AND SOLUTION

Consider an infinite elastic solid with shear modulus μ and density ρ referred to cylindrical coordinates r, θ, z . At time $t=0$ a circular crack of radius a occupies the plane $z=0$; for time $t>0$ the crack radius is $r^*(t) = a + R(t)$, where $R(t)$ is a monotonic increasing function, with $R(0)=0$. It is assumed that $\frac{dR}{dt} < c$, where $c=(\mu/\rho)^{1/2}$ is the torsional wave speed of the solid. Suppose that the solid is loaded by tractions of equal magnitude applied to the adjacent edges of the crack surface, and that these tractions are independent of θ . Then the displacement $v(r, z, t)$ will be anti-symmetric in z , so that it is sufficient to consider only the region $z \geq 0$. For this problem the stress components are

$$\sigma_{\theta z} = \mu v_{,z} \quad , \quad \sigma_{\theta r} = \mu \left(v_{,r} - \frac{1}{r} v \right) \quad , \quad (1)$$

¹ G. C. SIH and G. T. EMBLEY (1972) Journal of Applied Mechanics, 395-400. Sudden Twisting of a Penny-Shaped Crack.

and the displacement satisfies the equation

$$v_{,rr} + \frac{1}{r} v_{,r} - \frac{1}{r^2} v + v_{,zz} = v_{,tt} , \quad (2)$$

wherein for convenience ct has been replaced by t . The boundary conditions on $z=0$ may be written as

$$\sigma_{\theta z} = f(r, t) , \quad r < r^*(t) , \quad (3)$$

$$v = 0 , \quad r > r^*(t) , \quad (4)$$

where $f(r, t)$ is a continuous function. Since by superposition any initial displacements may be separated, then without loss in generality let

$$v = 0, v_{,t} = 0 , \quad t = 0 . \quad (5)$$

The problem as formulated by equations (1)-(5) constitutes a mixed boundary value problem. In order to apply Kostrov's method it is first necessary to obtain a relationship between the displacement and the stress $\sigma_{\theta z}$ on the plane $z=0$. To this end let

$$v(\xi, z, p) = \int_0^\infty e^{-pt} dt \int_0^\infty r J_1(\xi r) v(r, z, t) dr , \quad (6)$$

$$\Sigma(\xi, z, p) = \int_0^\infty e^{-pt} dt \int_0^\infty r J_1(\xi r) \sigma(r, z, t) dr , \quad (7)$$

denote Hankel-Laplace transforms; in (7) σ denotes $\sigma_{\theta z}$. Applying (6) to equation (2) and using (5) gives

$$v_{,zz} - \alpha^2 v = 0 , \quad z \geq 0 , \quad (8)$$

where $\alpha^2 = \xi^2 + p^2$. The appropriate solution of (8) is

$$v(\xi, z, p) = v(\xi, p) e^{-\alpha z} , \quad (9)$$

provided that $\text{Re}\{\alpha\} \geq 0$. Likewise, applying (7) to (1) and using (9) gives

$$\Sigma(\xi, z, p) = \Sigma(\xi, p) e^{-\alpha z}, \quad (10)$$

and

$$\Sigma(\xi, p) = -\mu \alpha v(\xi, p). \quad (11)$$

Formula (11) gives the relationship between the transformed stress and displacement components on $z=0$. Now substituting (11) into (9) and applying the Hankel-Laplace inversion formula gives

$$\mu v(r, z, t) = -\frac{1}{2\pi i} \int_0^\infty \xi J_1(\xi r) d\xi \int_B e^{pt} \frac{\Sigma(\xi, p)}{\alpha} e^{-\alpha z} dp, \quad (12)$$

where B denotes the well-known Bromwich contour. Using the result

$$\frac{1}{2\pi i} \int_B e^{pt} \frac{e^{-\alpha z}}{\alpha} dp = \begin{cases} 0, & 0 < t < z, \\ J_0(\xi u), & z < t < \infty, \end{cases} \quad (13)$$

where $u = (t^2 - z^2)^{1/2}$, together with the convolution theorem gives

$$\mu v(r, z, t) = - \int_0^{t-z} dt' \int_0^\infty r' \sigma(r', t') dr' \int_0^\infty \xi J_1(r\xi) J_1(r'\xi) J_0(s\xi) d\xi, \quad (14)$$

wherein $s = [(t-t')^2 - z^2]^{1/2}$. The inner integral may be expressed as [3]

$$\int_0^\infty \xi J_1(r\xi) J_1(r'\xi) J_0(s\xi) d\xi = \begin{cases} 0, & 0 < r' < |s-r|, \\ \frac{1}{\pi r r'} \cot \phi, & |s-r| < r' < s+r, \\ 0, & r+s < r', \end{cases} \quad (15)$$

³I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

where

$$\cot\phi = (r^2+r'^2-s^2) \{[s^2-(r-r')^2] [(r+r')^2-s^2]\}^{-1/2} . \quad (16)$$

Hence (12) becomes

$$\mu v(r,z,t) = - \frac{1}{\pi r} \int_R \int \sigma(r',t') \cot\phi \, dr' dt' . \quad (17)$$

Region R is depicted in figures (1) and (2) for the situations in which $0 < t < (r^2+z^2)^{1/2}$ and $(r^2+z^2)^{1/2} < t$, respectively. In figure (2) region R is bounded by the line $t'=0$ and the two hyperbolas $s=r+r'$ and $s=|r-r'|$. The dashed lines represent the limiting boundary of region R as $z \rightarrow 0$, and in the sequel will be referred to as region R_0 .

Formula (17) may be used in conjunction with Kostrov's method to determine the stress ahead of the crack perimeter. Thus putting $z=0$ in (17) and invoking conditions (3) and (4) gives

$$\int_{R_{02}} \int \sigma(r',t') \cot\phi \, dr' dt' = - \int_{R_{01}} \int f(r',t') \cot\phi \, dr' dt', \quad r > r^*(t) \quad (18)$$

where $\phi = \phi|_{z=0}$, and where, as shown in figure (3) regions R_{01} and R_{02} denote portions of region R_0 to the left and right, respectively, of the crack perimeter $r' = r^*(t')$. It is clear that $\sigma(r',t')=0$ in region R_{03} .

Equation (18) is now recast in terms of the coordinates $\xi' = r'-t'$ and $\eta' = r'+t'$. Thus using the notation $\sigma(r',t') = \tau(\xi',\eta')$, $f(r',t') = g(\xi',\eta')$ gives

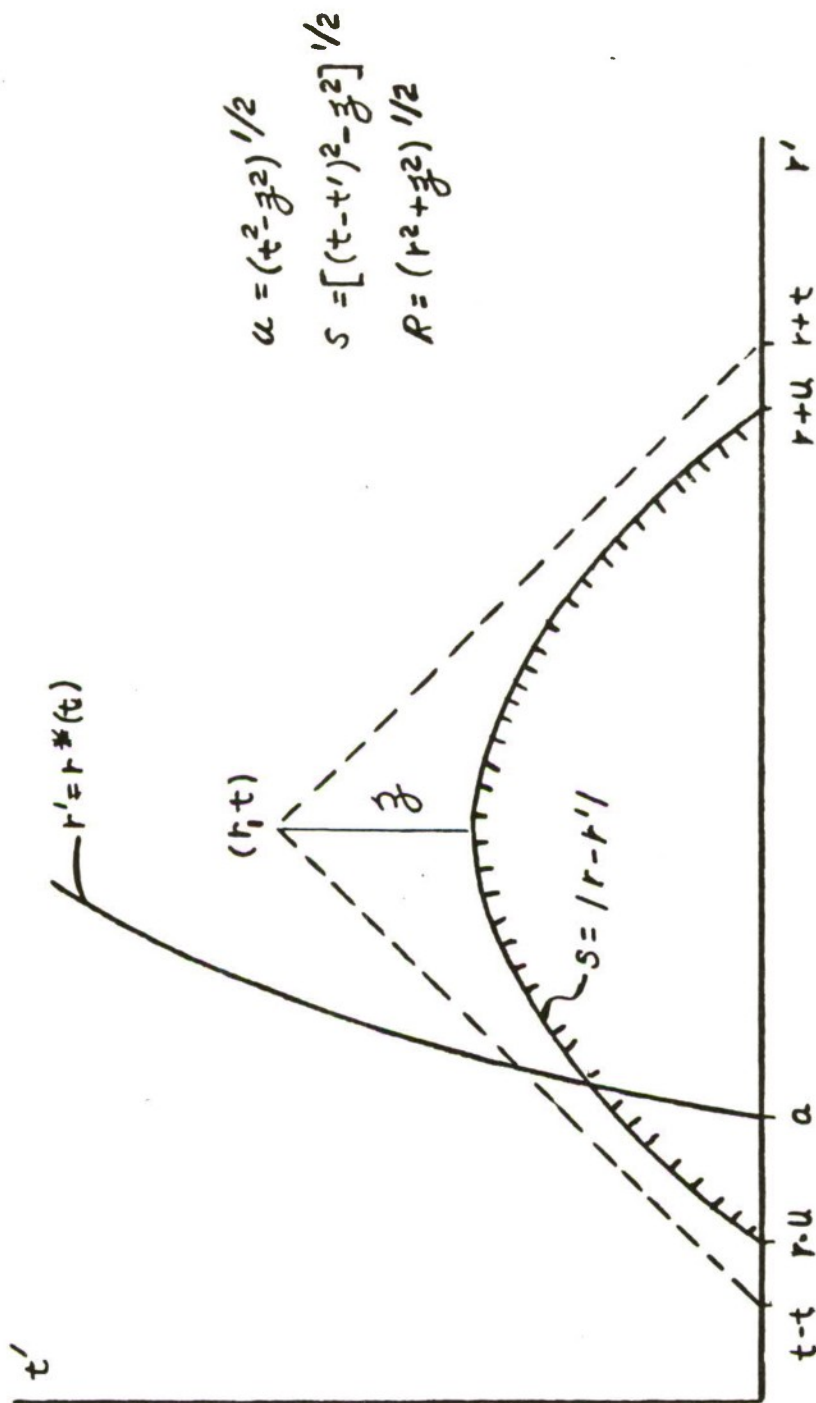


Figure 1. Solution Region: Drawn for the Case $t < R$, $r > r^*(t)$

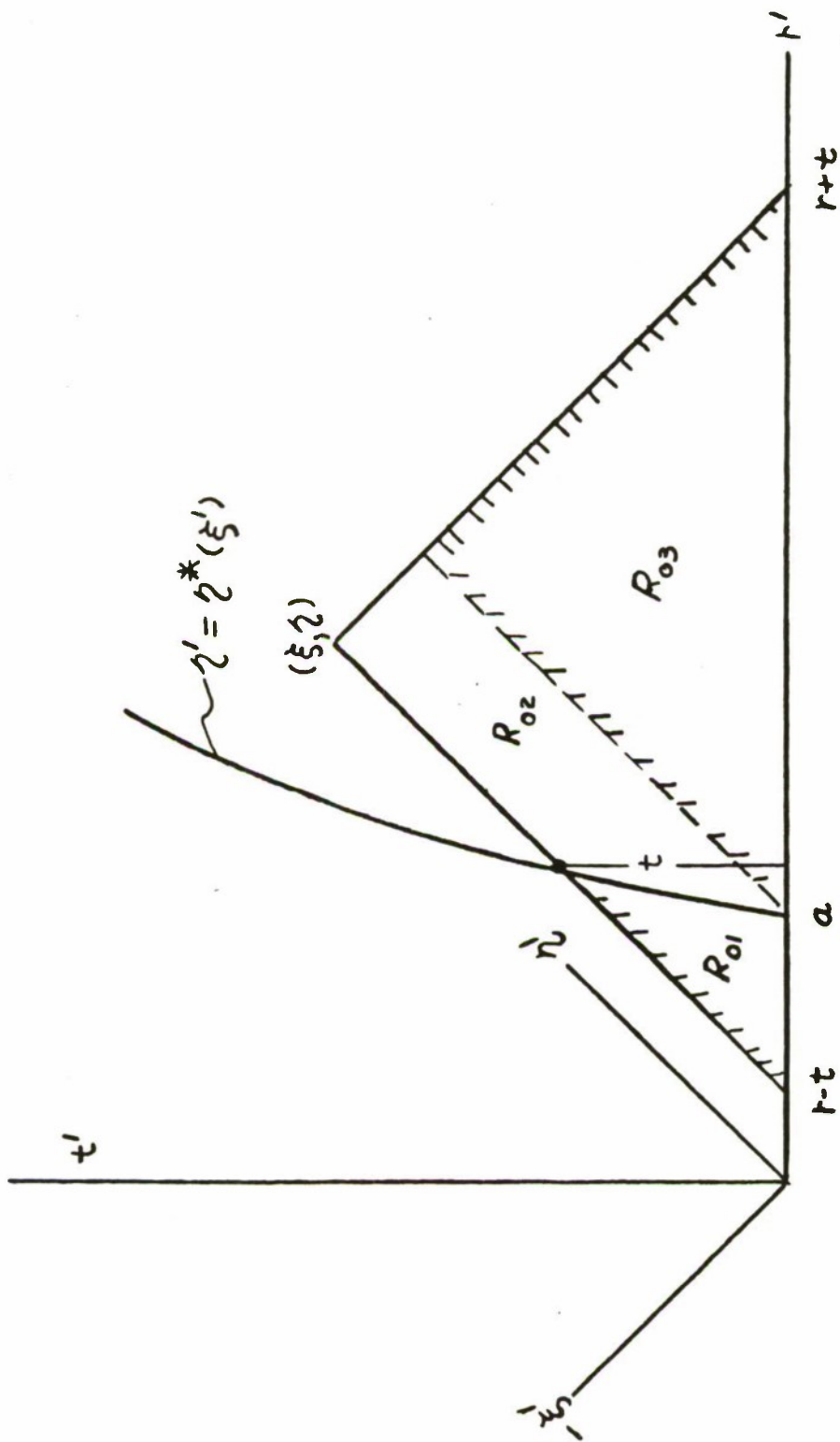


Figure 3. Diagram for Crack Plane Stress Solution

$$\int_{\xi}^a \frac{d\xi'}{(\xi' - \xi)^{1/2} (\eta + \xi')^{1/2}} \left\{ \int_{\eta^*(\xi')}^{\eta} \tau(\xi', \eta') \frac{N(\xi', \xi, \eta', \eta)}{(\eta - \eta')^{1/2} (\xi + \xi')^{1/2}} d\eta' + F(\xi', \xi, \eta) \right\} = 0, \quad (19)$$

where

$$F(\xi', \xi, \eta) = \int_{\xi'}^{\eta^*(\xi')} g(\xi', \eta') \frac{N(\xi', \xi, \eta', \eta)}{(\eta - \eta')^{1/2} (\xi + \eta')^{1/2}} d\eta', \quad (20)$$

$$N(\xi', \xi, \eta', \eta) = 2 (\xi' \eta' + \xi \eta) + (\eta' - \xi') (\eta - \xi), \quad (21)$$

and $\eta' = \eta^*(\xi')$ denotes the crack perimeter. Note that since $r' = r^*(t')$ then $t' = r^{*-1}(r') = t^*(r')$, so that $\eta^*(\xi')$ is given implicitly by the formula

$$\eta^* = \xi' + 2 t^* \left[\frac{1}{2} (\eta^* + \xi') \right]. \quad (22)$$

Now equation (19) is satisfied if the expression within braces vanishes.

Moreover substituting

$$\tau(\xi', \eta') = (\xi + \eta')^{1/2} \psi(\xi', \xi, \eta'), \quad (23)$$

leads to an Abel type integral equation for the stress ahead of the crack perimeter of the form

$$\int_{\eta^*(\xi')}^{\eta} \psi(\xi', \xi, \eta') \left[(\eta - \eta')^{1/2} (2\xi - \xi' + \eta') + \frac{(\xi + \eta')(\xi' + \eta')}{(\eta - \eta')^{1/2}} \right] d\eta' = -F(\xi', \xi, \eta). \quad (24)$$

The solution of (24) may be obtained in a straightforward way by use of Laplace transforms; the result is written as

$$\frac{d}{d\eta} [(\xi' + \eta)^2 (\xi + \eta)^{1/2} \psi(\xi', \xi, \eta)] = -\frac{1}{\pi} \frac{\eta + \xi'}{(\eta + \xi)^{1/2}} \frac{d^2}{d\eta^2} \int_{\eta^*(\xi')}^{\eta} \frac{F(\xi', \xi, \eta')}{(\eta - \eta')^{1/2}} d\eta'. \quad (25)$$

To get the solution at the point (ξ, η) let $\xi' = \xi$ and substitute from (23). This gives

$$\frac{d}{d\eta} [(\eta + \xi)^2 \tau(\xi, \eta)] = - \frac{1}{\pi} (\eta + \xi)^{1/2} \frac{d^2}{d\eta^2} \int_{\eta^*(\xi)}^{\eta} \frac{F(\xi, \xi, \eta')}{(\eta - \eta')^{1/2}} d\eta'. \quad (26)$$

Now since $F(\xi, \xi, \eta') = (\xi + \eta') E(\xi, \eta')$, where

$$E(\xi, \eta') = \int_{\xi}^{\eta^*(\xi)} g(\xi, v) \frac{(\xi + v)^{1/2}}{(\eta' - v)^{1/2}} dv, \quad (27)$$

then (26) may be written as

$$\frac{d}{d\eta} [(\eta + \xi)^2 \tau(\xi, \eta)] = - \frac{1}{\pi} \frac{d}{d\eta} [(\eta + \xi)^{3/2} \frac{d}{d\eta} \int_{\eta^*(\xi)}^{\eta} \frac{E(\xi, \eta')}{(\eta - \eta')^{1/2}} d\eta']. \quad (28)$$

Finally, substituting from (27) and performing one integration yields the result

$$\tau(\xi, \eta) = - \frac{1}{\pi(\eta + \xi)^{1/2} [\eta - \eta^*(\xi)]^{1/2}} \int_{\xi}^{\eta^*(\xi)} g(\xi, \eta') \frac{(\xi + \eta')(\eta^* - \eta')^{1/2}}{\eta - \eta'} d\eta', \quad \eta > \eta^*(\xi). \quad (29)$$

In terms of the original variables (29) becomes

$$\sigma(r, t) = - \frac{1}{\pi r^{1/2} [r - r^*(\tau)]^{1/2}} \int_{|r-t|}^{r^*(\tau)} \frac{f(r', t - r + r')}{|r-t|} \cdot \frac{r'^{1/2} [r^*(\tau) - r']^{1/2}}{r - r'} dr', \quad r > r^*(t). \quad (30)$$

In formula (30) the quantity τ , (not to be confused with the stress $\tau(\xi, \eta)$), is given implicitly by the relationship $t - \tau = r - r^*(\tau)$, as may be seen from figure (3). The quantity τ represents the retarded time associated with the moving crack border, that is, a point r experiences

at time t the disturbance, in the form of a wave, caused by the moving crack border when the border location was at $r^*(\tau)$.

The result (30) is valid for the time interval $r-a < t < r+a$. This interval corresponds to the time when the disturbance from the moving crack perimeter point $(r^*(t), \theta+\pi)$ has not yet reached the crack position $(r^*(t), \theta)$. To determine $\sigma(r, t)$ for larger values of time it appears necessary to replace $f(r', t')$ by $-\sigma(r', t')$ on those portions of the range of integration on which the stress is unknown. This procedure corresponds to repeated diffraction of the waves at the crack perimeter. It does not appear feasible to carry out this procedure analytically since multiple integrals are involved.

Since it is clear from figure (3) that the condition $r > r^*(t)$ is equivalent to the condition $r > r^*(\tau)$, it follows that formula (30) defines the stress ahead of the actual crack tip. Moreover, formula (30) gives a singularity of the order $1/2$ at the actual crack tip location. That this is so follows from a Lagrange expansion of the quantity $r - r^*(\tau)$, which is of the form [4]

$$r - r^*(\tau) = r - r^*(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial t}\right)^{n-1} \dot{r}^*(t) \rho^n(t), \quad (31)$$

⁴E. T. WHITTAKER and G. N. WATSON (1927) Modern Analysis, Fourth Edition, Cambridge University Press.

where

$$\dot{r}^*(t) = \frac{dr^*}{dt}, \text{ and } \rho(t) = \{[r-r^*(t)]^2 + z^2\}^{1/2}.$$

On $z=0$ (31) gives

$$r-r^*(\tau) = \frac{r-r^*(t)}{1-\dot{r}^*(t)} + \text{terms of order } \rho^2. \quad (32)$$

Hence as $r \rightarrow r^*(t)$ the term $[r-r^*(\tau)]^{-1/2}$ in formula (30) may be re-

placed by the term $\left[\frac{r-r^*(t)}{1-\dot{r}^*(t)} \right]^{-1/2}$, so that in the neighborhood of

the crack perimeter

$$\sigma(r,t) \sim \frac{k}{\pi \sqrt{r[r-r^*(t)]}}, \quad (33)$$

where k is the coefficient of stress intensity

$$k = \frac{\sqrt{1-\dot{r}^*(t)}}{\sqrt{r^*(t)}} \int_{|r^*(t)-t|}^{r^*(t)} \frac{f(r', t-r^*(t)+r')}{\sqrt{r^*(t)-r'}} \frac{\sqrt{r'}}{\sqrt{r^*(t)-r'}} dr'. \quad (34)$$

With the crack plane stress completely determined (at least for the time interval previously noted), it is now a relatively simple matter to obtain the crack displacement. Thus reverting to the coordinates $\xi=r-t$, $\eta=r+t$, letting $w(\xi, \eta) = v(r, t)$, and employing the previous designation for the stress component in formula (17) gives for $z=0$

$$\begin{aligned}
-2\pi(\xi+\eta) w(\xi,\eta) = & \int_{\xi}^{\xi^*(\eta)} \frac{d\xi'}{(\xi'-\xi)^{1/2}(\eta+\xi')^{1/2}} G(\xi',\xi,\eta) \\
& + \int_{\xi^*(\eta)}^a \frac{d\xi'}{(\xi'-\xi)^{1/2}(\eta+\xi')^{1/2}} \left[F(\xi',\xi,\eta) \right. \\
& \left. + \int_{\eta^*}^{\eta} \tau(\xi',\eta') \frac{N(\xi',\xi,\eta',\eta) d\eta'}{(\eta-\eta')^{1/2}(\xi+\eta')^{1/2}} \right]
\end{aligned} \tag{35}$$

where

$$G(\xi',\xi,\eta) = \int_{\xi'}^{\eta} g(\xi',\eta') \frac{N(\xi',\xi,\eta',\eta)}{(\eta-\eta')^{1/2}(\xi+\eta')^{1/2}} d\eta' . \tag{36}$$

In (35) $\xi^*(\eta)$ denotes the ξ' co-ordinate of the intersection of the line $\eta'=\eta$ and the crack locus $\eta'=\eta^*(\xi')$, i.e. $\xi^*(\eta) = \eta_{(\eta)}^{*-1}$. Now from (19) the bracketed quantity within the second integral vanishes.

Hence rewriting (35) in terms of the original variables yields

$$\mu v(r,t) = - \frac{1}{\pi r} \int_{R_1} \int f(r',t') \cot \phi_0 dr' dt' , \quad r < r^*(t), \tag{37}$$

where region R_1 is shown in figure (4). Formula (37) represents the displacement for general dynamic loading of a circular crack of initial radius a . It is of interest to note that the solution is independent of the stress ahead of the crack perimeter. Expressions (30) and (37) will now be utilized to study results for two particular cases.

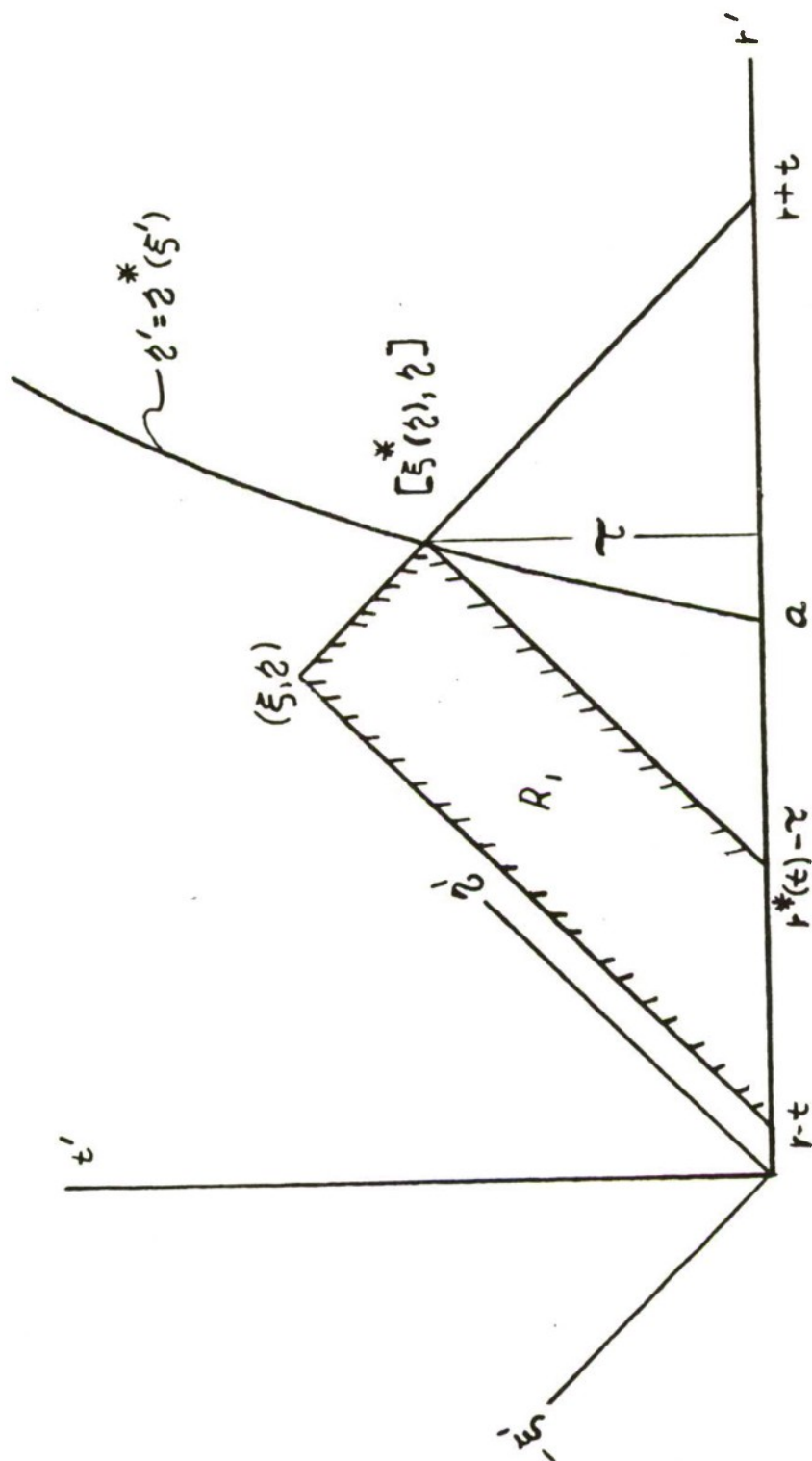


Figure 4. Diagram for Crack Plane Displacement Solution

Case 1. Initial Value Problem.

In formula (30) put

$$\begin{aligned} f(r,t) &= 0, \quad 0 < r < a \\ &= -\sigma^{(0)}(r), \quad a < r < r^*(t), \end{aligned} \quad (38)$$

where $\sigma^{(0)}(r)$ is some initial state of stress. The resulting expression represents the perturbed stress arising from the motion of the crack perimeter. As an illustration let

$$\sigma^{(0)}(r) = \frac{S}{2r\sqrt{r^2-a^2}} (2r^2-a^2) H(r-a). \quad (39)$$

Expression (39) is the result given by Sneddon [5] for the stress on the crack plane $z=0$ under conditions of uniform stress S at infinity. Introducing (39) into (30) yields

$$\sigma(r,t) = \frac{S}{2\pi\sqrt{r[r-r^*(\tau)]}} \int_a^{r^*(\tau)} \frac{2r'^2-a^2}{\sqrt{r'(r'^2-a^2)}} \frac{\sqrt{r^*(\tau)-r'}}{r-r'} dr', \quad r > r^*(t). \quad (40)$$

Since the initial stress on the freshly formed portion of the crack surface has been annulled, then, by superposition, (40) combined with (39) gives the actual stress ahead of the crack perimeter for the case of uniform stress S at infinity. The integral may be expressed in terms of known elliptic integrals; these expressions are given in the appendix.

In the limit as $r \rightarrow r^*(t)$ it is found that the stress intensity

⁵ I. N. SNEDDON and M. LOWENGRUB (1969) Crack Problems in the Classical Theory of Elasticity, Wiley and Sons.

factor for this case is

$$k = -\frac{5}{2} \left[\frac{1-r^*(t)}{r^*(t)} \right]^{1/2} \int_a^{r^*(t)} \frac{2r'^2 - a^2}{\sqrt{r'(r'^2 - a^2)} [r^*(t) - r']} dr' \quad (41)$$

Explicit expressions for the integral term may be obtained from those given in the appendix by putting $r=r^*(\tau) = r^*(t)$.

The perturbed crack displacement for this case follows by substituting (38) into (37). Reference to figure (4) shows that for $0 < r-t < a$ the result may be expressed as

$$\mu v(r, t) = \frac{1}{\pi r} \int_a^{r^*(\tau)} \sigma^{(0)}(r') dr' \int_{\tau-r^*(\tau)+r'}^{t+r-r'} \cot \phi dt' \quad (42)$$

wherein $\sigma^{(0)}(r')$ is given by (39) and τ is given implicitly by $t-\tau = r^*(\tau)-r$. With the aid of tables [3] it is found that (42) may be written as

$$\mu v(r, t) = \frac{1}{\pi r} \int_a^{r^*(\tau)} \sigma^{(0)}(r') dr' I(r', r, t) \quad (43)$$

where

$$I(r', r, t) = \frac{r^2 + r'^2}{r + r'} F(\kappa, q) - (r + r') E(\kappa, q) + \frac{1}{w} \{ [(r + r')^2 - w^2] [w^2 - (r - r')^2] \}^{1/2} ,$$

in which $F(\kappa, q)$ and $E(\kappa, q)$ are elliptic integrals of the first and second kind, respectively, with

$$\kappa = \sin^{-1} \left\{ \frac{r+r'}{w} \left[\frac{w^2 - (r-r')^2}{4rr'} \right]^{1/2} \right\} , \quad q = \frac{(4rr')^{1/2}}{r+r'}$$

³ I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

and $w = t - \tau + r^*(\tau) - r'$. For $0 < t - r < a$ identical results are found, hence (43) is valid for $0 < |t - r| < a$.

Case 2: Suddenly Appearing Crack

For this case let $f(r, t) = -S$. Then formula (30) gives

$$\sigma(r, t) = \frac{S}{\pi \sqrt{r[r - r^*(\tau)]}} \int_{|r-t|}^{r^*(\tau)} \frac{\sqrt{r'[r^*(\tau) - r']}}{r - r'} dr' \quad r > r^*(t). \quad (44)$$

The result (44), superposed with a constant stress S , gives the solution for a suddenly appearing crack under uniform stress S at infinity. Here it is assumed that the crack opens instantaneously to radius a and thereafter is located at $r = r^*(t)$. For this case the integral may be evaluated explicitly; the result is

$$\begin{aligned} \sigma(r, t) = & \frac{S}{\pi \sqrt{r[r - r^*(\tau)]}} \sqrt{|t - r| [r^*(\tau) - |t - r|]} \\ & + \frac{1}{2} [2r - r^*(\tau)] \sin^{-1} \left[\frac{2}{r^*(\tau)} \sqrt{|t - r| [r^*(\tau) - |t - r|]} \right] \\ & - \sqrt{r[r - r^*(\tau)]} \sin^{-1} \left[\frac{2 \sqrt{r|t - r| [r^*(\tau) - |t - r|] [r - r^*(\tau)]}}{r^*(\tau) (r - |t - r|)} \right] \end{aligned} \quad (45)$$

As an example, if the crack speed is a linear function of time, say $r^*(t) = a + \beta t$, with $\beta < 1$, then since $t - \tau = r - r^*(\tau)$ it follows that

$$\tau = \frac{t - r + a}{1 - \beta}. \quad \text{Putting this result in (45) gives for } r - r^*(\tau) < t < r$$

$$\begin{aligned}
\sigma(r,t) = & \frac{S}{\pi\sqrt{r(r-a-\beta t)}} \left\{ \sqrt{(r-t)(t-r+a)} + \frac{1}{2} \frac{(2-\beta)r-a-\beta t}{\sqrt{1-\beta}} \right. \\
& \sin^{-1} \left[\frac{2\sqrt{1-\beta}}{a+\beta(t-r)} \sqrt{(r-t)(t-r+a)} \right] - \frac{\sqrt{r(r-a-\beta t)}}{\sqrt{1-\beta}} \\
& \left. \sin^{-1} \left[\frac{2}{t[a+\beta(t-r)]} \sqrt{r(r-t)(t-r+a)(r-a-\beta t)} \right] \right\} \quad (46)
\end{aligned}$$

The crack surface displacement may likewise be obtained explicitly. Thus substituting into (37) and referring to figure (4) gives for $0 < r-t < a$

$$\mu v(r,t) = \frac{S}{\pi r} \int_0^t dt' \int_{r+t'-t}^{r^*(\tau)-|t'-\tau|} \cot \phi \, dr' \quad , \quad r < r^*(t) \quad . \quad (47)$$

Then using tables [3] gives

$$\mu v(r,t) = \frac{S}{\pi r} \int_0^t J(r,s,t) \, ds \quad , \quad r < r^*(t) \quad , \quad (48)$$

where

$$J(r,s,t) = (r-s)F(\kappa,q) + (r+s)E(\kappa,q) - \frac{1}{\lambda} \sqrt{[(r+s)^2 - \lambda^2][\lambda^2 - (r-s)^2]} \quad ,$$

in which

$$\kappa = \sin^{-1} \frac{r+s}{\lambda} \frac{\sqrt{\lambda^2 - (r-s)^2}}{\sqrt{4rs}} \quad , \quad q = \frac{\sqrt{4rs}}{r+s}$$

$$\text{and } \lambda = r^*(\tau) - |t-s-\tau| \quad .$$

It is of interest to note that if $r-a < t < r+a$ the results simplify

³
I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

even further, since for this time interval a point on the crack surface has not yet experienced the effect of the waves from the crack perimeter. For this case formula (48) is replaced by

$$\mu v(r,t) = \frac{S}{\pi r} \int_0^t ds \int_{r-s}^{r+s} \cot \phi \, dr' \quad , \quad (49)$$

which, upon evaluation of the inner integral, gives

$$\mu v(r,t) = \frac{S}{\pi r} \int_0^t [(r-s)K(\frac{\sqrt{4rs}}{r+s}) + (r+s)E(\frac{\sqrt{4rs}}{r+s})] \, ds, \quad (50)$$

where K and E are complete elliptic integrals. Now substituting $s=rw$ and applying the transformation formulae

$$K(\frac{2\sqrt{w}}{1+w}) = (1+w)K(w)$$

$$E(\frac{2\sqrt{w}}{1+w}) = \frac{2}{1+w} E(w) - (1-w)K(w)$$

gives

$$v(r,t) = \frac{2Sr}{\pi} \int_0^{t/r} E(w) \, dw \quad , \quad r-a < t < r+a \quad . \quad (51)$$

The final result follows by using

$$\int_0^{t/r} E(w) \, dw = \frac{\pi}{2} \frac{t}{r} \left[1 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2 (t/r)^{2n}}{(2n+1) 2^{4n} (n!)^4} \right] \quad (52)$$

The resulting expression is equivalent to that which would be obtained on the surface of a half-space loaded by a step function load of magnitude S.

DISCUSSION

The analytical results contained herein may be used as a basis for studying the physical aspects of the dynamic torsional fracturing process. As demonstrated by Kostrov [2] in the case of anti-plane strain, the crack propagation speed may be made definite by applying the physical hypothesis that the work done in the rupture process depends only on the crack speed. This hypothesis, together with Barenblatt's cohesion model leads to a differential equation governing the crack perimeter locus.

Another useful extension of the results of this report is to incorporate the effects of a plastic zone near the crack border. This may be done by using Dugdale's model as shown by Achenbach [6] for the case of anti-plane strain. These considerations, it is felt, are more appropriately left as subjects of future investigations.

² B. V. KOSTROV (1966) Journal of Applied Mathematics and Mechanics, 30(6), 1241-1248. Unsteady Propagation of Longitudinal Shear Cracks.

⁶ J. D. ACHENBACH (1970) International Journal of Engineering Science, Vol 8, 947-966. Brittle and Ductile Extension of a Finite Crack by a Horizontally Polarized Shear Wave.

REFERENCES

1. G. C. SIH and G. T. EMBLEY (1972) Journal of Applied Mechanics, 395-400. Sudden Twisting of a Penny-Shaped Crack.
2. B. V. KOSTROV (1966) Journal of Applied Mathematics and Mechanics, 30(6), 1241-1248. Unsteady Propagation of Longitudinal Shear Cracks.
3. I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.
4. E. T. WHITTAKER and G. N. WATSON (1927) Modern Analysis, Fourth Edition, Cambridge University Press.
5. I. N. SNEDDON and M. LOWENGRUB (1969) Crack Problems in the Classical Theory of Elasticity, Wiley and Sons.
6. J. D. ACHENBACH (1970) International Journal of Engineering Science, Vol 8, 947-966. Brittle and Ductile Extension of a Finite Crack by a Horizontally Polarized Shear Wave.

APPENDIX

The integral term in equation (40) may be reduced as follows:

Let

$$A = \int_a^b \frac{(2x^2 - a^2)}{\sqrt{x(x^2 - a^2)}} \frac{\sqrt{b-x}}{r-x} dx, \quad r > b, \quad (A1)$$

where $b = r^*(\tau)$. Changing the integration variable to $u = r-x$ leads to

$$A = -a^2 I + 2J, \quad (A2)$$

where

$$I = I_1 - \beta I_2, \quad (A3)$$

$$J = J_1 - \beta J_2, \quad (A4)$$

$$I_1 = \int_{\beta}^{\alpha} \frac{du}{\sqrt{(r-u)(\alpha-u)(u-\beta)(c-u)}} \quad (A5)$$

$$I_2 = \int_{\beta}^{\alpha} \frac{du}{u\sqrt{(r-u)(-u)(u-)(c-u)}} \quad (A6)$$

$$J_1 = \int_{\beta}^{\alpha} \frac{\sqrt{r-u^3} du}{\sqrt{(\alpha-u)(c-u)(u-\beta)}} \quad (A7)$$

$$J_2 = \int_{\beta}^{\alpha} \frac{\sqrt{r-u^3} du}{u\sqrt{(\alpha-u)(c-u)(u-\beta)}} \quad (A8)$$

In (A3) - (A8), $\beta=r-a$, $\alpha=r-b$, $c=r+a$, so that $c>r>\alpha>\beta$. Now from tables [3] it is found that

³ I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

$$I_1 = \frac{2}{\sqrt{(c-\alpha)(r-\beta)}} K\left(\frac{\sqrt{(c-r)(\alpha-\beta)}}{\sqrt{(c-\alpha)(r-\beta)}}\right), \quad (A9)$$

where K is the complete elliptic integral of the 1st kind. Hence replacing the original variables gives

$$I_1 = \frac{2}{\sqrt{2ar^*(\tau)}} K\left(\frac{\sqrt{r^*(\tau)-a}}{\sqrt{2r^*(\tau)}}\right). \quad (A10)$$

$$I_2 = \frac{2[a+r^*(\tau)]}{(r+a)[r-r^*(\tau)]\sqrt{2ar^*(\tau)}} II\left(\pi/2, \frac{(r+a)[a-r^*(\tau)]}{2a[r-r^*(\tau)]}, \frac{\sqrt{r^*(\tau)-a}}{2r^*(\tau)}\right), \quad (A11)$$

when II is the elliptic integral of the third kind. To evaluate J , make the further change of integration variable $z = \frac{u-\beta}{\alpha-\beta}$, so that

(A7) becomes

$$J_1 = \frac{\sqrt{(r-\beta)^3}}{\sqrt{c-\beta}} \int_0^1 \frac{\sqrt{(1-\delta z)^3}}{\sqrt{z(1-z)(1-rz)}} dz, \quad (A12)$$

in which $\delta = \frac{\alpha-\beta}{r-\beta}$, and $\gamma = \frac{\alpha-\beta}{c-\beta}$. Formula (A12), as given in tables [3], may be expressed as

$$J_1 = \frac{\sqrt{[r^*(\tau)]^3}}{\sqrt{a+r^*(\tau)}} \pi F_1\left(\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, 1; \frac{r^*(\tau)-a}{r^*(\tau)}, \frac{r^*(\tau)-a}{r^*(\tau)+a}\right), \quad (A13)$$

where F_1 is a hypergeometric function. Finally, integral J_2 is evaluated by writing

$$J_2 = r J_{21} - J_{22}. \quad (A14)$$

³

I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

In equation (A14)

$$J_{21} = r I_2 - I_1, \quad (A15)$$

in which I_2 and I_1 are given by (A10) and (A11), and

$$J_{22} = \int_{\beta}^{\alpha} \frac{\sqrt{r-u} \, du}{\sqrt{(\alpha-u)(c-u)(u-\beta)}}. \quad (A16)$$

The integral J_{22} is given directly by tables [3]; the result in terms of the original variables is

$$J_{22} = \frac{\sqrt{2a}}{\sqrt{r^*(\tau)}} \, \Pi\left(\pi/2, \frac{r^*(\tau)-a}{r^*(\tau)}, \frac{\sqrt{r^*(\tau)-a}}{\sqrt{2r^*(\tau)}}\right). \quad (A17)$$

Substituting (A9)-(A17) into (A2)-(A4) gives the result for (A1).

3

I. S. GRADSHTEYN and I. M. RYZHIK (1965) Tables of Integrals, Series, and Products, Academic Press.

WATERVLIET ARSENAL INTERNAL DISTRIBUTION LIST

May 1976

	<u>No. of Copies</u>
COMMANDER	1
DIRECTOR, BENET WEAPONS LABORATORY	1
DIRECTOR, DEVELOPMENT ENGINEERING DIRECTORATE	1
ATTN: RD-AT	1
RD-MR	1
RD-PE	1
RD-RM	1
RD-SE	1
RD-SP	1
DIRECTOR, ENGINEERING SUPPORT DIRECTORATE	1
DIRECTOR, RESEARCH DIRECTORATE	2
ATTN: RR-AM	1
RR-C	1
RR-ME	1
RR-PS	1
TECHNICAL LIBRARY	5
TECHNICAL PUBLICATIONS & EDITING BRANCH	2
DIRECTOR, OPERATIONS DIRECTORATE	1
DIRECTOR, PROCUREMENT DIRECTORATE	1
DIRECTOR, PRODUCT ASSURANCE DIRECTORATE	1
PATENT ADVISORS	1

EXTERNAL DISTRIBUTION LIST

May 1976

1 copy to each

CDR
US ARMY MAT & DEV READ. COMD
ATTN: DRCRD
DRCRD-TC
DRCRD-W
5001 EISENHOWER AVE
ALEXANDRIA, VA 22304

OFC OF THE DIR. OF DEFENSE R&E
ATTN: ASST DIRECTOR MATERIALS
THE PENTAGON
WASHINGTON, D.C. 20315

CDR
US ARMY TANK-AUTMV COMD
ATTN: AMDTA-UL
AMSTA-RKM MAT LAB
WARREN, MICHIGAN 48090

CDR
PICATINNY ARSENAL
ATTN: SARPA-TS-S
SARPA-VP3 (PLASTICS
TECH EVAL CEN)
DOVER, NJ 07801

CDR
FRANKFORD ARSENAL
ATTN: SARFA
PHILADELPHIA, PA 19137

DIRECTOR
US ARMY BALLISTIC RSCH LABS
ATTN: AMXBR-LB
ABERDEEN PROVING GROUND
MARYLAND 21005

CDR
US ARMY RSCH OFC (DURHAM)
BOX CM, DUKE STATION
ATTN: RDRD-IPL
DURHAM, NC 27706

CDR
WEST POINT MIL ACADEMY
ATTN: CHMN, MECH ENGR DEPT
WEST POINT, NY 10996

CDR
US ARMY ARMT COMD
ATTN: AMSAR-PPW-IR
AMSAR-RD
AMSAR-RDG
ROCK ISLAND, IL 61201

CDR
US ARMY ARMT COMD
FLD SVC DIV
ARMCOM ARMT SYS OFC
ATTN: AMSAR-ASF
ROCK ISLAND, IL 61201

CDR
US ARMY ELCT COMD
FT MONMOUTH, NJ 07703

CDR
REDSTONE ARSENAL
ATTN: AMSMI-RRS
AMSMI-RSM
ALABAMA 35809

CDR
ROCK ISLAND ARSENAL
ATTN: SARRI-RDD
ROCK ISLAND, IL 61202

CDR
US ARMY FGN SCIENCE & TECH CEN
ATTN: AMXST-SD
220 7TH STREET N.E.
CHARLOTTESVILLE, VA 22901

DIRECTOR
US ARMY PDN EQ. AGENCY
ATTN: AMXPE-MT
ROCK ISLAND, IL 61201

CDR
HQ, US ARMY AVN SCH
ATTN: OFC OF THE LIBRARIAN
FT RUCKER, ALABAMA 36362

EXTERNAL DISTRIBUTION LIST (Cont)

1 copy to each

CDR
US NAVAL WPNS LAB
CHIEF, MAT SCIENCE DIV
ATTN: MR. D. MALYEVAC
DAHLGREN, VA 22448

DIRECTOR
NAVAL RSCH LAB
ATTN: DIR. MECH DIV
WASHINGTON, D.C. 20375

DIRECTOR
NAVAL RSCH LAB
CODE 26-27 (DOCU LIB.)
WASHINGTON, D.C. 20375

NASA SCIENTIFIC & TECH INFO FAC
PO BOX 8757, ATTN: ACQ BR
BALTIMORE/WASHINGTON INTL AIRPORT
MARYLAND 21240

DEFENSE METALS INFO CEN
BATTELLE INSTITUTE
505 KING AVE
COLUMBUS, OHIO 43201

MANUEL E. PRADO / G. STISSER
LAWRENCE LIVERMORE LAB
PO BOX 808
LIVERMORE, CA 94550

DR. ROBERT QUATTRONE
CHIEF, MAT BR
US ARMY R&S GROUP, EUR
BOX 65, FPO N.Y. 09510

2 copies to each

CDR
US ARMY MOB EQUIP RSCH & DEV COMD
ATTN: TECH DOCU CEN
FT BELVOIR, VA 22060

CDR
US ARMY MAT RSCH AGCY
ATTN: AMXMR - TECH INFO CEN
WATERTOWN, MASS 02172

CDR
WRIGHT-PATTERSON AFB
ATTN: AFML/MXA
OHIO 45433

CDR
REDSTONE ARSENAL
ATTN: DOCU & TECH INFO BR
ALABAMA 35809

12 copies

CDR
DEFENSE DOCU CEN
ATTN: DDC-TCA
CAMERON STATION
ALEXANDRIA, VA 22314

NOTE: PLEASE NOTIFY CDR, WATERVLIET ARSENAL, ATTN: SARWV-RT-TP,
WATERVLIET, N.Y. 12189, IF ANY CHANGE IS REQUIRED TO THE ABOVE.