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A SOLVING ALGORITHM FOR SYSTEMS OF NONLINEAR ALGEBRAIC EQUATIONS

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A Solving Algorithm for Systems of Nonlinear Algebraic Equations

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PREFACE

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An exact solution algorithm for various systems of nonlinear algebraic equations is presented. Several applications of the method are given, and a discussion of the results in the context of nonlinear programming problems is included.

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A SOLVING ALGORITHM FOR SYSTEMS OF NONLINEAR ALGEBRAIC EQUATIONS

I. INTRODUCTION

The purpose of this report is to draw the attention of prospective users to the possiblity of solving exactly systems of algebraic equations that occur in various contexts.

The first system to be considered consists of one quadratic equation in n unknowns $(x_1, x_2, \ldots, x_n)$ and n-1 linear equations in the same variables:

$$\sum_{i,j=1}^{n} q_{ij}\, x_i x_j = b$$

$$\sum_{k=1}^{n} a_{mk} x_k = b_m \; ; \; m = 1, \ldots, n-1 \tag{1}$$

Though the following results are valid (except where noted otherwise) when q, a, and b are complex numbers, the results are most useful for the real case.

The system of equations (1) can be interpreted in several ways. First, one can look upon it as arising from the purely geometric problem of determining the set of points of simultaneous intersection of an n-dimensional quadric surface with n-1 hyperplanes. It will be seen later that, for arbitrary coefficients, a solution does not always exist. Moreover, because of the nonlinearity of the system, we can expect the existence of more than one solution. This is clearly evident for the case n = 2 involving the real points of intersection of a conic section (ellipse, parabola, etc.) with a single straight line.

Second, a system of type (1) with three unknowns arises in a famous problem of classical mechanics (discussed by Euler and bearing his name) concerning the motion of a rigid body about a fixed point. Golubev[1] presents a solution for this case by a method attributed to Hess[2] and later modified by Shiff.[3]

Third, still another geometrical interpretation of system (1) is possible. Assume that, in an n-dimensional space endowed with an inner product of the form

$$(U, V) = \sum_{i,j=1}^{n} q_{ij} U_i V_j . \tag{2}$$

it is required to find a vector $x = (x_1, \ldots, x_n)$ of given norm (re: length) b and given projections b_m on n-1 <u>fixed</u> vectors $c_m = (c_{m1}, \ldots, c_{mn})$, where $m = 1, \ldots, n-1$. The resulting system of equations is then precisely that in (1), where the vectors $a_m = (a_{m1}, \ldots, a_{mn})$ are related to the vectors c_m by

$$a_m = c_m Q . \tag{3}$$

in which Q represents the matrix (q_{ij}) appearing in the definition of the inner product (2).

Finally, system (1) might be relevant to problems of nonlinear programming[4] where the quadratic form will represent a cost, or objective, function while the linear equations will represent constraints, or side conditions.

In section II we describe the method of solution for system (1) and discuss its possible solutions in detail. In section III applications of this algorithm to several simple examples are presented. In section IV extensions of the method to various systems — some of which consist of equations with nonlinearities of higher degree (possibly transcendental) — are presented, and the associated problem of extremizing a given function subject to linear and nonlinear constraints is discussed. In the final section, the solving algorithm is investigated from the viewpoint of implementation, and arguments concerning uniqueness of solutions are presented. In additon, it is shown that the solving algorithm is relevant to the nonlinear programming problem.

II. METHOD OF SOLUTION

The solutions presented below are obtained by a straightforward method, in marked contrast to that described by Golubev,[1] which seems needlessly complicated by the use of Grammians and partial differentiation.

We start by writing the equations of system (1) in more compact form:

$$x Q \tilde{x} = b$$

$$a_m \tilde{x} = b_m \; ; \; m = 1, \ldots, n-1 . \tag{4}$$

where the tilde ~ denotes the transpose (hence $\tilde{x}$ is a column vector).

To go on, we assume that the vectors a_m, $m = 1, \ldots, n-1$, are linearly independent. The opposite case will be discussed later. This linear independence of the a_m's implies that there exists at least one minor determinant based on n-1 columns of the rectangular matrix (a_{mk}; $m=1, \ldots, n-1$; $k=1, \ldots, n$) that does not vanish identically. This being the case, we may assume, without loss of generality, that the variables x_k have been already labeled in such a way that the first n-1 columns of the rectangular matrix yield a nonzero minor. Similarly, no generality is lost if the matrix Q is assumed to be symmetric, since this can always be accomplished by a suitable redefinition of the coefficients.

In view of these assumptions, let A denote the (n-1) x (n-1) matrix consisting of the first n-1 rows and first (n-1) columns of the rectangular matrix. Then system (4) can be put in the form

$$x Q \tilde{x} = b$$

$$A \tilde{x}^* = \tilde{b}^* - x_n \tilde{a}^* . \tag{5}$$

where $x^* = (x_1, \ldots, x_{n-1})$; $b^* = (b_1, \ldots, b_{n-1})$; and $a^* = (a_{1,n}, a_{2,n}, \ldots, a_{n-1,n})$.

By virtue of our assumption of linear independence, the matrix A is invertible, so that it is possible to solve for x^* from the linear equations in (5):

$$\tilde{x}^* = A^{-1}(\tilde{b}^* - x_n \tilde{a}^*) . \tag{6}$$

By transposition, we have also

$$x^* = (b^* - x_n a^*)\widetilde{A^{-1}} . \qquad (7)$$

If we partition the quadratic form to conform to the definition of x^*, we can write for the first equation in (5):

$$x^* Q^* \tilde{x}^* + 2(q^* \tilde{x}^*) x_n + q_{nn} x_n^2 = b . \qquad (8)$$

where $Q^* = (q_{ij};\ i = 1, \ldots, n-1;\ j = 1, \ldots, n-1)$ and $q^* = (q_{1n}, q_{2n}, \ldots, q_{n-1n})$. Substitution of expressions (6) and (7) for x^* and x^* into (8) yields

$$(b^* - x_n a^*)\widetilde{A^{-1}} Q^* A^{-1}(\tilde{b}^* - x_n \tilde{a}^*) + 2q^* A^{-1}(\tilde{b}^* - x_n \tilde{a}^*) x_n + q_{nn} x_n^2 = b . \qquad (9)$$

In both (8) and (9), the assumed symmetry of Q has been used. Moreover, since Q^* is also symmetric, (9) can be rewritten as the following quadratic in x_n:

$$\alpha x_n^2 + 2\beta x_n + \gamma = 0 . \qquad (10)$$

where

$$\alpha = q_{nn} - 2q^* A^{-1}\tilde{a}^* + a^* \widetilde{A^{-1}} Q^* A^{-1} \tilde{a}^* \qquad (11)$$

$$\beta = q^* A^{-1} \tilde{b}^* - a^* \widetilde{A^{-1}} Q^* A^{-1} \tilde{b}^* \qquad (12)$$

$$\gamma = b^{*}\widetilde{A^{-1}}Q^{*}A^{-1}\tilde{b}^{*} - b. \tag{13}$$

The two possible solutions of (10) are then

$$x_n^{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha}. \tag{14}$$

From (14), we are led to the following two possible cases:

1. $\underline{\alpha \neq 0}$

At most, two distinct solutions for x_n are possible. This implies the existence of, at most, two distinct solutions x^{*} for the original problem (as dictated by (6) or (7)). Exactly <u>two</u> distinct solutions will be obtained if and only if $\beta^2 - \alpha\gamma \neq 0$. In the opposite case of $\beta^2 - \alpha\gamma = 0$, there is only one solution.

2. $\underline{\alpha = 0}$

From (10), we obtain

$$2\beta x_n + \gamma = 0. \tag{15}$$

This implies that a <u>unique</u> solution will be obtained in this case if and only if $\beta \neq 0$.

The opposite case, $\beta = 0$, leads to two possibilities:

a. For $\gamma = 0$, it is clear that <u>any</u> x_n will satisfy the original equation (10). This, in turn, implies that there are an <u>infinite</u> number of solutions depending on one parameter x_n (or that there is a ∞^1-manifold of solutions). The geometric interpretation of this result is that the quadric surface and the hyperplanes intersect in a single "straight" line.

b. For $Y \neq 0$, (15) will be inconsistent. Hence, in this subcase no solution exists for x_n, and mutatis mutandis, no solution is possible for the original problem. This result raises no question if the rectangular $(n-1)xn$ matrix of the a-coefficients has only one nonvanishing $(n-1)x(n-1)$ - minor. On the other hand, if there are other nonzero minors, one can ask whether all of them will yield the same negative result. Though it is difficult to prove in general that this is indeed the case, we expect, on intuitive grounds, that the various nonzero minors will produce identical results.

From the discussion presented above, it is clear that the existence of a real solution (when all the entities entering the original system are real) requires satisfaction of the condition

$$\beta^2 - \alpha\gamma \geqq 0 . \tag{16}$$

In the opposite case, a real solution does not exist.

III. APPLICATIONS

In this section, the algorithm presented above is applied to several simple examples to obtain some well known results.

A vector $x = (x_1, \ldots, x_n)$ is to be found whose norm (length) is ℓ. In addition, x must be orthogonal to $n-1$ fixed real vectors a_m, $m = 1, \ldots, n-1$. These constraints imply that, in the terminology of the previous sections,

$$b = \ell^2 ; \quad b_m = 0 , \; m = 1, \ldots, n-1 . \tag{17}$$

For the norm, we assume the Euclidean length

$$(x, x) = \sum_{i=1}^{n} x_i^2 . \tag{18}$$

This implies that Q and Q^* are square unit matrices of n and $n-1$ dimensions, respectively, while $q_{nn} = 1$ and $q^* = 0$.

Substitution of these values in (11), (12), and (13) yields

$$\alpha = 1 + a^* \tilde{A}^{-1} A^{-1} \tilde{a}^* \tag{19}$$

$$\beta = 0 \tag{20}$$

$$\gamma = -\ell^2 . \tag{21}$$

The form of α reveals that it cannot vanish since, to vanish, the quadratic form in a^* of (19) must be equal to -1, which is impossible because the determinant of this quadratic is seen by inspection to be positive. This means that the quadratic form has a single minimum at $a^* = (0, \ldots, 0)$ and is positive at all other points. By assumption, $a^* \neq 0$, the form is positive definite, and $\alpha > 0$.

Use of these results gives the solutions

$$x_n^{\pm} = \pm \ell / \sqrt{\alpha}$$

$$x_k^{\pm} = \pm (\ell / \sqrt{\alpha}) \sum_{m=1}^{n-1} A_{km}^{-1} a_{mn} ; k = 1, \ldots, n-1 . \tag{22}$$

The existence of these two distinct real solutions, as pointed out above, is a basic feature of the solution. The following special cases of these results illustrate this multiplicity of solutions.

First, we specialize to the 2-dimensional space, the fixed vector being $a = (u_1, u_2)$. Then, assuming $u_1 \neq 0$, we have

$$A = u_1 \quad ; \quad A^{-1} = 1/u_1 \quad ; \quad a^* = u_2$$

$$\alpha = 1 + u_2^2 / u_1^2 . \tag{23}$$

Then

$$x_2^{\pm} = \pm \ell u_1 / \sqrt{u_1^2 + u_2^2}$$

$$x_1^{\pm} = \pm \ell u_2 / \sqrt{u_1^2 + u_2^2} . \tag{24}$$

The pictorial representation of this result is shown in figure 1. The solution vectors $x^{\pm}$ are of length ℓ and are perpendicular to the fixed vector a.

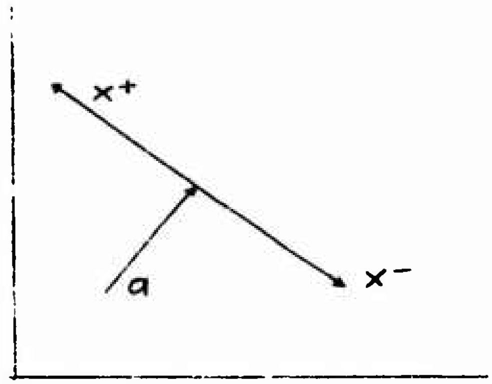

Figure 1

Next, we specialize to the 3-dimensional case where there are two fixed vectors $a_1 = (u_1, u_2, u_3)$ and $a_2 = (v_1, v_2, v_3)$. Assuming that

$$\Delta \equiv u_1 v_2 - u_2 v_1 \neq 0, \tag{25}$$

we get

$$A = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}; \quad A^{-1} = \frac{1}{\Delta}\begin{pmatrix} v_2 & -u_2 \\ -v_1 & u_1 \end{pmatrix}; \quad a^{*} = (u_3, v_3) \tag{26}$$

and

$$\begin{aligned} \alpha &= 1 + \frac{1}{\Delta^2}(u_3, v_3)\begin{pmatrix} v_2 & -v_1 \\ -u_2 & u_1 \end{pmatrix}\begin{pmatrix} v_2 & -u_2 \\ -v_1 & u_1 \end{pmatrix}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \\ &= \left[(u_1 v_2 - u_2 v_1)^2 + (u_2 v_3 - u_3 v_2)^2 \right. \\ &\qquad \left. + (v_1 u_3 - u_1 v_3)^2 \right] / \Delta^2 . \end{aligned} \tag{27}$$

The two solutions are then

$$x_3^{\pm} = \pm \ell/\sqrt{\alpha}$$
$$x_1^{\pm} = \pm \ell(v_2 u_3 - u_2 v_3)/\Delta\sqrt{\alpha}$$
$$x_2^{\pm} = \pm \ell(u_1 v_3 - v_1 u_3)/\Delta\sqrt{\alpha}\,. \tag{28}$$

Let us choose $\ell = \Delta\sqrt{\alpha}$. The solutions of (28) are now

$$x^{+} = \left([u_2 v_3 - u_3 v_2],\ [u_3 v_1 - u_1 v_3],\ [u_1 v_2 - u_2 v_1]\right)$$
$$x^{-} = -x^{+}. \tag{29}$$

We recognize, in the expression for the vector x^{+}, the usual definition of the cross product of two vectors a_1, a_2 given by the symbolic formula

$$\vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{30}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors in a Cartesian system of coordinates.

The geometric representation of this case is shown in figure 2. The two solutions are perpendicular to the plane defined by the two vectors a_1 and a_2.

It should be noted that the choice of x^{+} as the cross product of a_1 and a_2 corresponds to a right-handed system of coordinates, while x^{-} corresponds to a left-handed system.

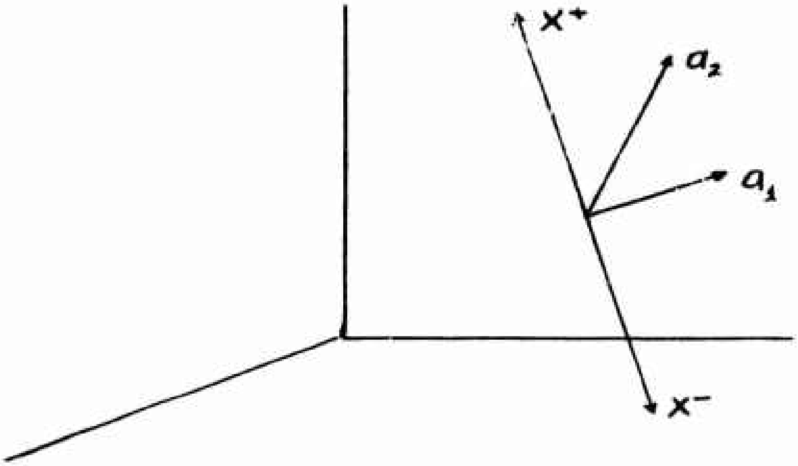

Figure 2

We can apply the previous formalism to the problem of finding the extrema (if existent) of the quadratic form $xQ\tilde{x}$ subject to the constraints

$$a_m \tilde{x} = b_m \quad ; m = 1, \ldots, n-1. \tag{31}$$

We substitute the results of section II (with the same assumptions) given in (6) and (7) in the function

$$f(x_1, \ldots, x_n) \equiv xQ\tilde{x} \quad , \tag{32}$$

which then becomes a function of the single variable x_n, given by

$$f(x_1, \ldots, x_n) = F(x_n) = \alpha x_n^2 + 2\beta x_n + \bar{\gamma}. \tag{33}$$

where α and β are defined in (11) and (12), respectively, and $\bar{\gamma}$ is defined by

$$\bar{\gamma} = b^* \widetilde{A^{-1}} Q^* A^{-1} \tilde{b}^* . \tag{34}$$

A discussion similar to that following (14) yields the two results below:

1. $\alpha \neq 0$

The extremum of $F(x_n)$ is at the point given by $F'(x_n) = 0$:

$$x_n = -\beta/\alpha . \tag{35}$$

This solution will be finite if $\alpha \neq 0$, which is the case considered here. This value of x_n will yield a minimum if $\alpha > 0$ and a maximum if $\alpha < 0$. Therefore, the unique extremum for $f(x_1, x_2, \ldots, x_n)$ in the present case will occur at

$$x = (x^*, -\beta/\alpha), \tag{36}$$

where

$$x^* = [b^* + (\beta/\alpha)a^*]\tilde{A}^{-1}. \tag{37}$$

2. $\alpha = 0$

In this case, $F(x_n)$ takes the simpler form

$$F(x_n) = 2\beta x_n + \bar{\gamma}. \tag{38}$$

If $\beta \neq 0 \neq \bar{\gamma}$, then (38) is the equation of a straight line, and it is obvious that no extremum exists in such a case.

If $\beta = 0$, the function $F(x_n)$ has the constant value $\bar{\gamma}$ throughout the region of space described by the side conditons (31), and again an extremum does not exist.

IV. GENERALIZATIONS

In this section, we present generalizations of the previous results in several directions.

Linear Independence of Vectors a_m

In the first instance, we retain the assumption of linear independence of the vectors a_m introduced in section II.

1. We start with an extension of our results to the following system:

$$p\tilde{x} + xQ\tilde{x} = b$$
$$a_m \tilde{x} = b_m \quad ; \; m = 1, \ldots, n-1 \; . \tag{39}$$

In (39), $p = (p_1, p_2, \ldots, p_n)$ where again the p_k values are complex numbers.

Under the assumption of linear independence of the a_m vectors, we can use (6) and (7) in the first equation of system (39). The resulting equation is a quadratic in x_n:

$$\alpha x_n^2 + 2\beta x_n + \gamma = 0 \, , \tag{40}$$

where α, β, and γ are defined below as

$$\alpha = q_{nn} - 2q^* A^{-1}\widetilde{a^*} + a^* \widetilde{A^{-1}} Q^* A^{-1} \widetilde{a^*} \tag{41}$$

$$\beta = \tfrac{1}{2} p_n + q^* A^{-1} \widetilde{b^*} - a^* \widetilde{A^{-1}} Q^* A^{-1} \widetilde{b^*} - \tfrac{1}{2} p^* A^{-1} \widetilde{a^*} \tag{42}$$

$$\gamma = b^* \widetilde{A^{-1}} Q^* A^{-1} \widetilde{b^*} + p^* A^{-1} \widetilde{b^*} - b \tag{43}$$

and

$$p^* = (p_1, p_2, \ldots, p_{n-1}) . \tag{44}$$

The existence and uniqueness of solutions to the system (39) follow the pattern established for system (1) and hence need not be repeated. As a concluding note to this extension, we remark that one might be tempted to try a reduction of system (39) to system (1) by a transformation of coordinates that will eliminate the linear term $p\widetilde{x}$. This indeed can be accomplished, but only in the special case where Q is invertible (i.e., $|Q| \neq 0$). For this special case, the appropriate transformation is the translation

$$x = y + \xi \tag{45}$$

where

$$\xi = -\frac{1}{2} p Q^{-1} . \tag{46}$$

It should be clear that the approach of (45) and (46) is more restrictive.

2. A second generalization is concerned with the system

$$\begin{aligned} f(x_1, \ldots, x_n) &= b \\ a_m \widetilde{x} &= b_m \; ; \; m = 1, \ldots, n-1 . \end{aligned} \tag{47}$$

Use of (7) in the first equation of system (47) yields an equation in one variable x_n:

$$F(x_n) = b . \tag{48}$$

The solutions of system (47) will depend on the nature of the solutions of (48).

In particular, if the function $f(x_1, \ldots, x_n)$ is of the form

$$f(x_1, \ldots, x_n) = g(c + p\widetilde{x} + xQ\widetilde{x}), \tag{49}$$

in which c is a constant scalar and p, Q, x have the same significance as in (39), then the equation for x_n will be

$$g(\alpha x_n^2 + 2\beta x_n + \gamma) = b, \tag{50}$$

where α, β, and γ can be easily written in terms of c, p, Q, A, a*, and b*. Moreover, if g(z) has an inverse, then the solutions of (50) will coincide with the solutions of

$$\alpha x_n^2 + 2\beta x_n + \gamma = g^{-1}(b). \tag{51}$$

In the evaluation of $g^{-1}(b)$, attention must be paid to the branch chosen for g^{-1}, since the function $g^{-1}(z)$ can be multivalued. Thus, we see that it is no more difficult to solve a system such as (49) than to solve system (39). This result can be of importance in problems for which the function $f(x_1, \ldots, x_n)$ may be approximated by a function of type (49). A truncated Taylor series for f is a special case of this form.

3. A third extension deals with the system

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= \beta_1 \\ f_2(x_1, \ldots, x_n) &= \beta_2 \\ a_m \tilde{x} &= b_m \; ; \quad m = 1, \ldots, n-2 \end{aligned} \tag{52}$$

where β_1, β_2, and b_m are complex numbers and the vectors a_m are again assumed to be linearly independent. This assumption implies that there exists at least one nonzero (n-2)x(n-2) minor of the a-coefficients. Again assuming a labeling such that the first n-2 columns yield this minor, we can write

$$\tilde{x}^* = A^{-1}(\tilde{b}^* - x_{n-1}\tilde{c}_1^* - x_n \tilde{c}_2^*), \tag{53}$$

where

$$X^* = (x_1, \dots, x_{n-2})$$
$$b^* = (b_1, \dots, b_{n-2})$$
$$c_1^* = (a_{1,n-1}, a_{2,n-1}, \dots, a_{n-2,n-1})$$
$$c_2^* = (a_{1,n}, a_{2,n}, \dots, a_{n-2,n}) \tag{54}$$

and A is the matrix of the nonzero (n-2) x (n-2) minor.

Substitution of (53) into the first two equations of system (52) results in the two simultaneous equations for x_{n-1}, x_n:

$$F_1(x_{n-1}, x_n) = \beta_1$$
$$F_2(x_{n-1}, x_n) = \beta_2 \, . \tag{55}$$

The points of intersection of these two planar curves will yield, through use of (53), all possible solutions of the original system (52). As in the simpler cases treated previously, the nature of these solutions will depend on the form of the functions f_1, f_2 as well as on the side conditions. For instance, if both f_1 and f_2 are quadratic forms, then (55) reduces to a single equation (in either x_{n-1} or x_n) of the fourth order. Hence, if no pathological cases occur, there are, at most, four distinct solutions to the original problem.

4. The preceding discussion points the way to a more general extension of the same type. The system to be considered now is of the form

$$f_k(x_1, \dots, x_n) = \beta_k \quad ; \quad k = 1, \dots, r$$
$$a_m \tilde{X} = b_m \quad ; \quad m = 1, \dots, n-r, \tag{56}$$

in which β_k and b_m are complex numbers and the vectors a_m are still assumed to be linearly independent. Use of the linear equations in (56) to eliminate, say, the first n−r x_j values reduces system (56) to a system of r equations

$$F_k(x_{n-r+1}, \ldots, x_n) = \beta_k; \quad k = 1, \ldots, r. \tag{57}$$

The points of intersection of these r hypersurfaces will then yield the possible solutions of (56). We remark that continuous manifolds of solutions will be obtained if the functions F_k are not linearly independent.

For the above extensions, one can formulate corresponding problems where extrema of functions, subject to linear and nonlinear constraints, are required. Suppose, for example, that instead of system (55), we must determine the extrema of $F_1(x_{n-1}, x_n)$ subject to the single constraint

$$F_2(x_{n-1}, x_n) = \beta_2 . \tag{58}$$

This can be done either by straightforward elimination of x_{n-1} from (57) or by the use of Lagrange multipliers.

<u>Linear Dependence of Vectors a_m</u>

We shall deal now with the extension of the previous results to the case of linear dependence of the vectors a_m appearing in system (1). The linear dependence of these vectors implies that one or more of the linear equations in system (1) are expressible as linear combinations of the rest. If this is the case, the redundant equations can be deleted and the remaining, say, s equations will be linearly independent. We shall immediately formulate the problem so as to include the more general system

$$\begin{aligned} f_k(x_1, \ldots, x_n) &= \beta_k; \quad k = 1, \ldots, r \\ a_m \tilde{X} &= b_m; \quad m = 1, \ldots, s, \quad s < n-r . \end{aligned} \tag{59}$$

As compared with system (56), the present system (59) consists of fewer equations; i.e., the number of unknowns exceeds the number of equations. The vectors a_m appearing in (59) are again linearly independent, and by a similar procedure the first s x_j values can be eliminated. The resulting system will then be

$$F_k(x_{s+1}, \ldots, x_n) = \beta_k \; ; \; k = 1, \ldots, r. \tag{60}$$

The original assumption $s < n-r$ implies also that $r < n-s$. Therefore, the possible solutions of (60) will depend on n-s-r parameters, and the solution of the original system (59) may include continuous manifolds of n-s-r dimensions.

The simplest system that can be solved explicitly is

$$\begin{aligned} p\tilde{x} + xQ\tilde{x} &= b \\ a_m \tilde{x} &= b_m \; ; \; m = 1, \ldots, n-2, \end{aligned} \tag{61}$$

where p is as defined in system (39).

Assuming that we already have the correct labeling, we can write

$$x^* = (b^* - x_{n-1}c_1^* - x_n c_2^*)\widetilde{A^{-1}}, \tag{62}$$

where all the entities are as defined in (54).

By substitution of (62) and its transpose into the first equation of (61), we obtain

$$p^*A^{-1}(\tilde{b}^* - x_{n-1}\tilde{c}_1^* - x_n\tilde{c}_2^*) + p_{n-1}x_{n-1} + p_n x_n$$
$$+ (b^* - x_{n-1}c_1^* - x_n c_2^*)\widetilde{A^{-1}}\,Q^*A^{-1}(\tilde{b}^* - x_{n-1}\tilde{c}_1^* - x_n\tilde{c}_2^*)$$
$$+ 2q_1^*A^{-1}(\tilde{b}^* - x_{n-1}\tilde{c}_1^* - x_n\tilde{c}_2^*)x_{n-1}$$
$$+ 2q_2^*A^{-1}(b^* - x_{n-1}\tilde{c}_1^* - x_n\tilde{c}_2^*)x_n$$
$$+ x^{(2)}Q^{(2)}\widetilde{x^{(2)}} = b. \tag{63}$$

All the starred quantities and A are n-2-dimensional, with definitions similar to those exhibited in (54), while q_1^*, q_2^* and $x^{(2)}$, $Q^{(2)}$ are defined by

$$q_1^* = (q_{1,n-1}, q_{2,n-1}, \ldots, q_{n-2,n-1})$$
$$q_2^* = (q_{1,n}, q_{2,n}, \ldots, q_{n-2,n})$$
$$x^{(2)} = (x_{n-1}, x_n)\;;\quad Q^{(2)} = \begin{pmatrix} q_{n-1,n-1} & q_{n-1,n} \\ q_{n,n-1} & q_{n,n} \end{pmatrix}.$$

The form of (63) reflects both the special partitioning of the variables into x^* and $x^{(2)}$ and the assumed symmetry of the matrix Q of the quadratic form in system (61).

Since Q^* is also symmetric, we can write (63) as

$$\alpha x_{n-1}^2 - 2\beta x_{n-1}x_n + \gamma x_n^2 + \sigma x_{n-1} + \tau x_n + \mu = 0, \tag{64}$$

where the coefficients are given by

$$
\begin{aligned}
\alpha &= q_{n-1\,n-1} - 2q_1^* A^{-1}\tilde{c}_1^* + c_1^* \tilde{A}^{-1} Q^* A^{-1} \tilde{c}_1^* \\
\beta &= c_1^* \tilde{A}^{-1} Q^* A^{-1} \tilde{c}_2^* + q_1^* A^{-1} \tilde{c}_2^* + q_2^* A^{-1} \tilde{c}_1^* - q_{n-1\,n} \\
\gamma &= q_{nn} - 2q_2^* A^{-1} \tilde{c}_2^* + c_2^* \tilde{A}^{-1} Q^* A^{-1} \tilde{c}_2^* \\
\sigma &= p_{n-1} - p^* A^{-1} \tilde{c}_1^* - 2q_1^* A^{-1} \tilde{b}^* - 2b^* \tilde{A}^{-1} Q^* A^{-1} \tilde{c}_1^* \\
\tau &= p_n - p^* A^{-1} \tilde{c}_2^* - 2q_2^* A^{-1} \tilde{b}^* - 2b^* \tilde{A}^{-1} Q^* A^{-1} \tilde{c}_2^* \\
\mu &= p^* A^{-1} \tilde{b}^* + b^* \tilde{A}^{-1} Q^* A^{-1} \tilde{b}^* - b .
\end{aligned}
\tag{65}
$$

If we solve for x_{n-1} from (64) we will get, at most, two distinct solutions which depend parametrically on x_n. The original problem has therefore, at most, two distinct one-parameter manifolds of solutions (if no pathological cases arise).

The above results show us how to proceed in finding the extrema of the function

$$f(x_1, \ldots, x_n) \equiv p\tilde{x} + x Q \tilde{x} \tag{66}$$

subject to the constraints

$$a_m \tilde{x} = b_m \quad ; \quad m = 1, \ldots, n-2 . \tag{67}$$

This problem is equivalent to finding the extrema of the function

$$F(x_{n-1}, x_n) \equiv \alpha x_{n-1}^2 - 2\beta x_{n-1} x_n + \gamma x_n^2 + \sigma x_{n-1} + \tau x_n + \mu \tag{68}$$

where $\bar{\mu}$ is equal to the μ defined in (65) with $b = 0$.

If we assume that no pathological cases arise, there is a single extremum at values of x_{n-1}, x_n obtained as a solution to the system of the following linear equations:

$$\frac{\partial F}{\partial x_{n-1}} \equiv 2\alpha x_{n-1} - 2\beta x_n + \sigma = 0$$

$$\frac{\partial F}{\partial x_n} \equiv -2\beta x_{n-1} + 2\gamma x_n + \tau = 0 \; ; \qquad (69)$$

namely,

$$x_{n-1} = \frac{\gamma\sigma + \beta\tau}{2(\beta^2 - \alpha\gamma)} \qquad ; \quad \beta^2 - \alpha\gamma \neq 0 .$$

$$x_n = \frac{\alpha\tau + \beta\sigma}{2(\beta^2 - \alpha\gamma)} \qquad (70)$$

Substitution of these values into (62) gives the value taken by the vector x^*, and thus the solution of the original problem has been found.

V. CONCLUSION

In the preceding sections we have presented exact algorithms of solution for algebraic systems consisting of mixed linear and nonlinear equations. These algorithms involve basically the evaluation of various determinants as well as a finite number of products and summations. This in itself does not constitute an obstacle to implementation because of the widespread availability of computing facilities. One question that can be raised in this context is whether these algorithms are the most rapid possible. The author cannot answer this equestion since alternative exact treatments of these same problems do not seem to exist, and therefore there is no basis for comparison.

At this point a second question must be raised. When discussing the uniqueness of the various solutions, we have assumed that a nonzero minor could be found. Since such a minor is not given from the start, we must first find it before applying the algorithm. For example, to find such a minor for the system (59), we must examine at most $\binom{n}{s}$ minors. Obviously, the actual number examined may be considerably smaller, especially if several such nonzero minors exist. This brings into prominence the second question: Does the application of the algorithm lead to identical solutions when various nonzero minors are used? By "identical" we mean that a change in labeling will bring into coincidence solutions based on different minors.

The answer to this seems to be in the affirmative; i.e., no new solutions are introduced by using different nonzero minors to start the algorithm. This can be verified directly for systems with few unknowns, but a proof for a general system (in particular for a system consisting of at least one highly nonlinear equation) is not known to the author.

Finally, we point out that the above results should be relevant in the context of nonlinear programming,[4] where one deals with systems of the type (59), ordinarily with $r = 1$. The variables $x = (x_1, \ldots, x_n)$ include the so-called slack variables, which arise from the transformation of a system of inequalities to a system of equations. The given single function $f(x_1, \ldots, x_n)$ must be extremized subject to the conditions $a_m \widetilde{x} = b_m$; $m = 1, \ldots, s$; and $s < n-1$. Although a variety of methods of attacking this problem have been in use for years, it is important to realize that these methods yield only approximate solutions. By contrast, the direct approach described here will always yield an exact function of $n-s-1$ variables to be extremized. If the points of maxima and minima of this function can be found in the <u>entire</u> $n-s-1$-dimensional space, then the programming problem will be completely solved if use is made of the residual constraints of non-negativity of the x_k, where $k = 1, \ldots, n$.

It should be clear from this discussion that, for these nonlinear programming problems, the actual choice of a method will be dictated by the capacity of the computing facility available, the computing time required, and the penalties involved in the use of approximate extremal points.

VI. REFERENCES

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