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BY

RICHARD W. COTTLE and JONG-SHI PANG

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ABSTRACT

Recently, Mangasarian [18], [19] has discussed the idea of solving certain classes of linear complementarity problems as linear programs. The present paper (1) demonstrates how these complementarity problems are related to the theory of polyhedral sets having least elements and (2) discusses the question of whether the linear programming approach can be recommended for solving them.

1. INTRODUCTION

It is a fairly well-known fact that if a linear complementarity problem has a solution, then it has a solution which is an extreme point of its "feasible set." This means that if an appropriate linear form were known, i.e., one whose minimum over the feasible set would necessarily occur at a complementarity solution, then the linear complementarity problem could be solved as a linear program. Typically, one does not know an appropriate linear form in advance and can not rapidly find one. But there are exceptional cases, some of which have been noted in the literature. (See [7], [9], [18], [19], [26].) It is our contention that these linear complementarity problems solvable as linear programs are related to the theory of polyhedral sets with least elements. Examples of this relationship are made explicit by Cottle and Veinott [9] and by Tamir [26]. Some numerical experience based on this observation is reported in Cottle, Golub and Sacher [7].

More recently, Mangasarian [18], [19] has produced several additional examples of linear complementarity problems whose solutions can be obtained via linear programming (which incidentally is not intended here to imply the use of the simplex method or any of its derivatives). Mangasarian's results in this area are not explicitly based on least element arguments, but rather on a key lemma having to do with optimal dual variables. Our primary purpose in this paper is to demonstrate that Mangasarian's theory can be interpreted in terms of least elements of polyhedral sets. For the most part, our methods are matrix-theoretic. In the course of our investigation, we uncovered a few results of this type; they are included here because we believe them to be new and of independent interest.

The possibility suggested by the linear programming formulation of a linear complementarity problem raises the question of whether this approach can be recommended in practice. Hence our secondary purpose in this paper is to give at least a tentative answer by reporting the computational experience we have gathered in solving some linear programs--of the type that could arise from linear complementarity problems--by an iterative (relaxation) procedure rather than by the simplex method or any of its variants. Motivation for using an iterative method can be found in the size and structure of the matrices one might expect to encounter in some potential applications of the linear complementarity problem.

The plan of the paper is the following. In section 2, we cover a bit of background material. The section has two parts. The first part fixes our notation and gives some characterizations of matrices in terms of the linear complementarity problem. The second part is a synopsis of the main results Mangasarian obtained in [18] and [19]. In section 3, we develop our least-element interpretation of the subject and present some incidental matrix-theoretic results. In the fourth and final section, we discuss our somewhat preliminary computational experience with solving linear complementarity problems as linear programs by relaxation methods.

2. BACKGROUND

2.1 Miscellaneous preliminaries

Throughout this paper, R_+^n will denote the nonnegative orthant of the Euclidean n-space R^n and $R^{n \times m}$ will denote the class of real $n \times m$ matrices. We denote the i-th column (row) of a matrix $A \in R^{n \times m}$ by $A_{.i}$ ($A_{i.}$). A real matrix $A \in R^{n \times n}$ is said to be a Z-matrix (P-matrix) if it has non-positive off diagonal entries (positive principal minors). We shall call a matrix $A \in R^{n \times n}$ a K-matrix (or a Minkowski matrix) if it is both a P- and Z-matrix simultaneously. The classes of all real Z-, P- and K-matrices will be denoted by Z, P and K respectively. They are treated extensively by Fiedler and Pták [14].

For a vector $q \in R^n$ and a matrix $M \in R^{n \times n}$, the linear complementarity problem, denoted by (q,M) is that of finding $x \in R^n$ such that

$$(2.1) \qquad q + Mx \geq 0, \quad x \geq 0 \quad \text{and} \quad x^T(q + Mx) = 0 .$$

By the feasible set for (q,M) we mean the polyhedral set

$$X(q,M) = \{x \in R^n : q + Mx \geq 0, \; x \geq 0\} .$$

We say that the problem (q,M) is feasible if $X(q,M)$ is nonempty. A subset S of R^n is said to be bounded below if there is a vector $x' \in R^n$ such that $x \geq x'$ for all $x \in S$. The vector $\bar{x} \in S$ is the least element of S if $\bar{x} \leq x$ for all $x \in S$. It is clear that the least element, if it exists, must be unique.

Minkowski matrices as well as P- and Z-matrices play very important roles in the linear complementarity problem. It is well-known (see Samelson et al. [25]) that the problem (q,M) has a unique solution for every $q \in R^n$ if and only if $M \in P$. Tamir [26] characterized Z-matrices in the following way.

Theorem. The matrix $M \in R^{n \times n}$ is a Z-matrix if and only if for each vector $q \in R^n$ for which the feasible set $X(q,M)$ is nonempty, there exists a least element $\bar{x}$ in $X(q,M)$ satisfying $x^T(q + Mx) = 0$.

Cottle and Veinott [9] proved the following characterization of K-matrices.

Theorem. The matrix $M \in R^{n \times n}$ is Minkowski if and only if for each $q \in R^n$, the feasible set $X(q,M)$ has a least element $\bar{x}$ which is the only vector in $X(q,M)$ satisfying $x^T(q + Mx) = 0$.

Note that the characterizations of Z- and K-matrices are in terms of least elements of the feasible set $X(q,M)$. This feature is of fundamental importance in the present work.

Various methods for solving the linear complementarity problem (q,M) in the important special case where M is a Z-matrix have been considered intensively by a number of authors [3], [7], [8], [11], [23], [24]. While Mangasarian's proposal [18], [19] to solve linear complementarity problems (q,M) as linear programs is not entirely new, his results definitely appear to enlarge the class of problems to which this solution strategy is applicable. Specifically, he proved that for certain classes of matrices M, it is possible to find a vector p such that each solution of the linear program

$$\text{(2.2)} \qquad \text{minimize } p^T x \text{ subject to } q + Mx \geq 0,\ x \geq 0$$

solves the problem (q,M). We denote the linear program (2.2) by the triple (p,q,M). Its dual is equivalent to

$$\text{(2.3)} \qquad \text{minimize } q^T y \text{ subject to } p - M^T y \geq 0,\ y \geq 0\ ,$$

which is just $(q,p,-M^T)$. We say that a linear complementarity problem (q,M) is <u>LP-solvable</u> if we can find a vector $p \in R^n$ such that each solution of the linear program (p,q,M) solves (q,M).

Recognizing that most LP-solvable linear complementarity problems arise from the discretization of (partial) differential equations (see [7], [8], [11], [23]) and that the properties of the matrices so obtained (e.g. Z-matrices) are not so conducive to efficient solution of the linear programs by the simplex method, Mangasarian proposed the use of relaxation methods ([1], [2], [13], [20]) for solving inequality systems. In particular, solving the linear program (2.2) is equivalent in a logical sense to solving the linear inequalities:

$$(2.4) \qquad \begin{pmatrix} M & 0 \\ 0 & -M^T \\ I & 0 \\ 0 & I \\ -p^T & -q^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} -q \\ -p \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which consist of primal and dual feasibilities and the reverse of the weak duality of the linear program (2.2). Presumably, the computational advantage offered by relaxation methods is their capacity for preserving matrix sparsity.

2.2 Mangasarian's results

Our purpose here is to summarize the principal results obtained by Mangasarian in the aforementioned papers. The fundamental theorem is the following:

Theorem 2.1. Let the feasible set $X(q,M)$ be nonempty, and let M satisfy

$$(2.5) \qquad MX = Y$$

$$(2.6) \qquad r^T X + s^T Y \geq 0 \quad \text{for some} \quad r, s \in R^n_+ ,$$

where $X, Y \in Z$. Then the linear complementarity problem (q,M) can be solved by solving the linear program (p,q,M) with $p = r + M^T s$.

The proof of the theorem depends heavily on the key lemma below.

Lemma 2.2. If $\bar{x}$ solves the linear program (p,q,M) and there exists an optimal solution $\bar{y}$ of $(q,p,-M^T)$ such that $\bar{y} + p - M^T\bar{y} > 0$, then $\bar{x}$ solves the problem (q,M).

The following corollary identifies some classes of matrices satisfying conditions (2.5) and (2.6) in Theorem 2.1.

Corollary 2.3. Let the feasible set $X(q,M)$ be nonempty and let $e \in R^n$ be any positive vector. Then for each of the cases when

(a) $M = YX^{-1}$, $X \in K$, $Y \in Z$ $(p = r \geq 0,\ r^TX > 0)$

(b) $M = YX^{-1}$, $X \in Z$, $Y \in K$ $(p = M^Ts,\ s \geq 0,\ s^TY > 0)$

(c) $M \in Z$ $(p = e)$

(d) $M^{-1} \in Z$ $(p = M^Te)$

(e) $-M \in K$ $(p = -e$ or $p = M^Te)$

(f) $-M^{-1} \in K$ $(p = -M^Te$ or $p = e)$,

the linear complementarity problem (q,M) has a solution which can be obtained by solving the linear program (p,q,M) with the indicated p.

The results above are drawn from the first of the two papers. In the second paper, Mangasarian extends the class of LP-solvable linear complementarity problems by establishing the following remarkable theorem.

Theorem 2.4. Let the feasible set $X(q,M)$ be nonempty, and suppose there exist $X, Y \in R^{n\times n}$, $A \in R^{m\times m}$, $B, H \in R^{n\times m}$, $G \in R^{m\times n}$, $p \in R^n_+$ and $p_0 \in R^m_+$ such that

$$(2.7) \quad MX = Y + BG; \quad MH \geq BA; \quad X, Y, A \in Z; \quad G, H \geq 0$$

$$(2.8) \qquad (p^T, p_0^T) \begin{bmatrix} X & -H \\ -G & A \end{bmatrix} > 0 .$$

Then the linear complementarity problem (q,M) has a solution which can be obtained by solving the linear program (p,q,M).

By specializing Theorem 2.4, Mangasarian produced the following table.

Matrix M of (1)	Conditions on M	Vector p of (2)	Conditions on p
1. $M = YX^{-1}$	$X \in K,\ Y \in Z$	p	$p \geq 0,\ p^T X > 0$
2. $M = YX^{-1}$	$X \in Z,\ Y \in K$	$p = M^T s$	$s \geq 0,\ s^T Y > 0$
3. M	$M \in Z$	p	$p > 0$
4. M	$M^{-1} \in Z$	$p = M^T e$	$e > 0$
5. $M = Y + ab^T$	$Y \in K,\ a \geq 0,\ b > 0$	$p = b$	$b > 0$
6. $M = 2X - Y$	$X \in Z,\ Y \in K$ $X \geq Y$ (componentwise)	$p = M^T p_0$	$p_0 > 0,\ p_0^T Y > 0$
7. $M \geq 0$ (component-wise)	$m_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n} m_{ij}$ $\forall\ j = 1, \ldots, n$	$p = M^T e$	$e^T = (1, \ldots, 1) \in R^n$
8. $M \geq 0$ (component-wise)	$m_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} m_{ij}$ $\forall\ i = 1, \ldots, n$	$p = M^T p_0$	$p_0 > 0$ $p_0^T(-M + 2\,\text{diag}\ M) > 0$

TABLE 1

3. CONNECTION WITH LEAST ELEMENTS

In this section we develop our least element interpretation of Mangasarian's theory. The cornerstone of our approach is a strengthening of Theorem 2.1. The new result (Theorem 3.9) makes it possible to invoke the theory of polyhedral sets with least elements. The desired relationship between the two theories is made explicit in Theorem 3.11. Except for the matrix-theoretic results mentioned earlier, the rest of the section is concerned with showing how Theorem 2.4 and the special cases enumerated in Table 1 can be related to Theorem 3.9 and thereby to the least element theory. We begin by reviewing a few more pertinent facts.

Definition 3.1. Let M be a square matrix. By a principal rearrangement of M, we mean a matrix $\tilde{M} = P^T MP$ where P is a permutation matrix.

The following facts are well-known (see e.g. [4]):

(i) The classes of Z-, P- and K-matrices are invariant under principal rearrangements.

(ii) The inverse of a P-matrix is a P-matrix.

(iii) The property of a matrix belonging to any one of the three classes Z, P and K is inherited by each of its principal submatrices.

In [14], Fiedler and Pták gave a list of thirteen equivalent conditions for a Z-matrix to be a K-matrix. Here we quote three which this paper needs later.

Proposition 3.2. Let $A \in Z$. Then the conditions below are equivalent to each other:

(iv) there exists a vector $x \geq 0$ such that $Ax > 0$;

(v) the inverse A^{-1} exists and $A^{-1} \geq 0$;

(vi) the principal minors of A are positive.

Definition 3.3. Let A be a nonsingular principal submatrix of a square matrix M. Let $\ddot{M}$ be a principal rearrangement of M such that $\tilde{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then the *Schur complement of A in M*, denoted by (M/A), is the matrix $D - CA^{-1}B$.

Properties and applications of the Schur complements have been surveyed in Cottle [5]. A proof of the following proposition can be found in Crabtree [10].

Proposition 3.4. Let A be a nonsingular principal submatrix of the Minkowski matrix M. Then (M/A) is itself a Minkowski matrix.

Definition 3.5. The matrix $A \in R^{n \times m}$ is said to be *Leontief* if it has exactly one positive element in each column and there is a vector $x \in R^m$ such that $x \geq 0$ and $Ax > 0$.

Proposition 3.6. Let $A \in R^{n \times m}$ be a Leontief matrix. Then there exists a submatrix $B \in R^{n \times n}$ such that B is itself Leontief. Furthermore, B^{-1} exists and is nonnegative.

A proof of the preceding proposition is given in Dantzig [12]. It is based on an application of the simplex method of linear programming.

We are now ready to present our results. We first state an alternative proposition which is an immediate consequence of the well-known theorem of Kuhn-Fourier [15] on the solvability of a system of linear relations.

Proposition 3.7. Let $X, Y \in R^{n \times n}$. Then the following two conditions are equivalent to each other:

$$r^T X + s^T Y > 0 \quad \text{for some} \quad r, s \in R^n_+ \tag{3.1}$$

and

$$\left.\begin{array}{r} Xu \leq 0 \\ Yu \leq 0 \\ u \geq 0 \end{array}\right\} \Rightarrow u = 0 \,. \tag{3.1'}$$

The lemma below provides necessary and sufficient conditions for two Z-matrices X and Y to satisfy condition (3.1).

Lemma 3.8. Let $X, Y \in R^{n \times n}$ be Z-matrices. Then (3.1) holds if and only if there exist a principal rearrangement with permutation matrix P and a partitioning of X and Y such that

$$P^T XP = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad P^T YP = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \tag{3.2}$$

$$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in K \tag{3.3}$$

Proof: "Sufficiency". We shall show that condition (3.1)' holds. Let u satisfy $Xu \leq 0$, $Yu \leq 0$ and $u \geq 0$. We then have

$$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} u \leq 0$$

which implies $u \leq 0$ (and thus $u = 0$) because

$$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix}^{-1} \geq 0 \ .$$

"Necessity". Condition (3.1) can be rewritten as

(3.4) $$(X^T, Y^T) \binom{r}{s} > 0 \quad \text{with} \quad \binom{r}{s} \geq 0 \ .$$

The matrix $A = (X^T, Y^T) \in R^{n \times 2n}$ has at most one positive element in each column. We may assume, without loss of generality, that it has exactly one in each column: otherwise we can always delete those columns of A consisting entirely of non-positive entries and delete at the same time the corresponding components of the vector $(r^T, s^T)^T$, then we are left with a smaller system satisfying (3.4) in which the matrix has exactly one positive element in each column and we can work with this smaller matrix. Now the matrix A is Leontief; hence by Proposition 3.6, there exists a submatrix $B \in R^{n \times n}$ such that B is itself Leontief. B^{-1} exists and is nonnegative. Note that the same column of X^T and Y^T cannot both simultaneously appear in B

because B has exactly one positive element in each column. Hence by permuting the columns of B, if necessary, we may assume that the i-th column of B is either the i-th column of X^T or the i-th column of Y^T. Then B is a complementary submatrix of (X^T, Y^T). This suggests the permutation and the partitioning, thus proving (3.2) and (3.3).

Using this lemma, we give necessary and sufficient conditions for a matrix M to satisfy conditions (2.5) and (2.6) of Theorem 2.1.

<u>Theorem 3.9.</u> Let M, X and Y be $n \times n$ matrices with X, Y in Z. Then

(3.5) $$MX = Y$$

(3.6) $$r^T X + s^T Y > 0 \quad \text{for some} \quad r, s \geq 0$$

if and only if there is a principal rearrangement and partitioning of M, X and Y such that

(3.7) $$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

(3.8) $$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ is nonsingular .}$$

(3.9) $$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in K .$$

Proof: We first note that if P is a permutation matrix, then (3.5) holds if and only if $(P^T MP)(P^T XP) = P^T YP$. Therefore (3.5) holds if and only if it holds for every principal rearrangement of M, X and Y. Now, the sufficiency part of the theorem follows immediately from Lemma 3.8 and the observation above. For the necessity part, it remains to verify condition (3.8). Condition (3.9) implies that X_{11} is nonsingular. Solving for M_{21} in the equation

$$Y_{21} = M_{21}X_{11} + M_{22}X_{21} .$$

we obtain

$$M_{21} = (Y_{21} - M_{22}X_{21})X_{11}^{-1} .$$

Hence

$$\begin{aligned} Y_{22} &= M_{21}X_{12} + M_{22}X_{22} \\ &= (Y_{21} - M_{22}X_{21})X_{11}^{-1}X_{12} + M_{22}X_{22} , \end{aligned}$$

or,

$$Y_{22} - Y_{21}X_{11}^{-1}X_{12} = M_{22}(X_{22} - X_{21}X_{11}^{-1}X_{12}) .$$

Note that the matrix on the left side is just the Schur complement of Y_{22} in the Minkowski matrix

$$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} .$$

By Proposition 3.4, it is nonsingular. Therefore so is the matrix $X_{22} - X_{21}X_{11}^{-1}X_{12}$. Now (3.8) follows from Schur's determinental formula [5]

$$\det\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \det X_{11} \det(X_{22} - X_{21}X_{11}^{-1}X_{12}) .$$

This completes the proof of the theorem.

Notation. Let $\mathcal{C}$ denote the class of square matrices satisfying conditions (3.5) and (3.6).

It is clear that, using Theorem 3.9, we can readily construct matrices belonging to the class $\mathcal{C}$ starting with any Minkowski matrix. In the sequel, we shall focus our discussion on this class $\mathcal{C}$. Our purpose is to establish a relationship between the class of linear complementarity problems (q,M) with $M \in \mathcal{C}$ and the theory of polyhedral sets having least elements. In order to achieve this, we state and prove the following lemma.

Lemma 3.10. Let $X, Y \in R^{n\times n}$ be Z-matrices and let $(s,q) \in R^n \times R^n$. Suppose that the polyhedral set

$$V = \{v \in R^n : q + Yv \geq 0,\ s + Xv \geq 0\}$$

is nonempty and bounded below. Then there exists a least element $\bar{v} \in V$ satisfying $(q + Yv)^T(s + Xv) = 0$. Furthermore this least element can be obtained by solving the linear program

(3.10) minimize r^Tv subject to $q + Yv \geq 0$, $s + Xv \geq 0$

for any positive vector $r \in R^n$.

Proof: Consider the linear program (3.10) where $r \in R^n$ is positive. Since the constraint set V is bounded below and obviously closed, problem (3.10) has a solution, say $\bar{v}$. We want to prove that $\bar{v}$ is the least element of V. (This will imply that $\bar{v}$ solves the linear program (3.10) for any other choices of the positive vector r.) So let $v \in V$ and $v' = (v_i')$ be the vector with $v_i' = \min(v_i, \bar{v}_i)$ for each i. Consider index k. We may assume, without loss of generality, that $v_k' = v_k$. Then we have

$$(q + Yv')_k = q_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} Y_{k\ell} v_\ell' + Y_{kk} v_k'$$

$$\geq q_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} Y_{k\ell} v_\ell + Y_{kk} v_k$$

$$\geq 0 .$$

Similarly, we can deduce $(s + Xv')_k \geq 0$. These inequalities hold for $k = 1, \ldots, n$. Therefore $v' \in V$. By the definition of $\bar{v}$, it follows that

$$r^T \bar{v} \leq r^T v' \leq r^T \bar{v} .$$

Hence $\bar{v} = v' \leq v$. This shows that $\bar{v}$ is indeed the least element of V. It remains to verify that $\bar{v}$ satisfies the complementarity property. Clearly we have $(q + Y\bar{v})^T (s + X\bar{v}) \geq 0$ because $\bar{v} \in V$. Suppose $(q + Y\bar{v})_i > 0$ and $(s + X\bar{v})_i > 0$ for some index i. Let $\epsilon > 0$ and consider the vector $v = \bar{v} - \epsilon e^i$ where e^i is the i-th unit vector. We have

$$q + Yv = q + Y\bar{v} - \epsilon Ye^i .$$

Therefore

$$(q + Yv)_j \geq 0 \quad \text{for every } j \neq i ,$$

and

$$(q + Yv)_i = q_i + (Y\bar{v})_i - \epsilon Y_{ii} .$$

Similarly, we have

$$(s + Xv)_j \geq 0 \quad \text{for every } j \neq i$$

and,

$$(s + Xv)_i = (s + X\bar{v})_i - \epsilon X_{ii} .$$

Clearly, $Y_{ii} \leq 0$ $(X_{ii} \leq 0)$ implies $(q + Yv)_i \geq 0$ $((s + Xv)_i \geq 0)$. If $Y_{ii} > 0$ and $X_{ii} > 0$, then choose $\epsilon > 0$ such that

$$0 < \epsilon < \min((q + Y\bar{v})_i/Y_{ii}, (s + X\bar{v})_i/X_{ii}) .$$

With this choice of ϵ, we see that $(q + Yv)_i \geq 0$ and $(s + Xv)_i \geq 0$. Hence it follows that $v \in V$. But $r^T v < r^T \bar{v}$, contradicting the fact that $\bar{v}$ solves (3.10).

Consider the problem (q,M) with $M \in \mathcal{C}$. Then the inequalities

$$q + YX^{-1}x \geq 0 , \qquad x \geq 0$$

define the feasible set for (q,M). With $x = Xv$, the inequalities above can be expressed in the form

$$(3.11) \qquad \binom{X}{Y} v \geq \binom{0}{-q} .$$

Let V be the set of all solutions of (3.11). Then it is clear that (q,M) is feasible if and only if $V \neq \emptyset$. Partitioning the vector

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

according to Theorem 3.9, we see that

$$v \in V \text{ implies that } \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} v \geq \begin{pmatrix} 0 \\ -q_2 \end{pmatrix}$$

Since the matrix

$$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

is Minkowski by condition (3.9) of Theorem 3.9, its inverse is non-negative by Proposition 3.2. Therefore we have

$$v \in V \text{ implies } v \geq \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -q_2 \end{pmatrix}$$

i.e., the set V is bounded below. Hence if $V \neq \emptyset$, it follows from Lemma 3.10 that V has a least element $\bar{v}$ satisfying $(q + Yv)^T(Xv) = 0$ and $\bar{v}$ can be obtained by solving the linear program

$$\text{minimize } r^T v \text{ subject to } q + Yv \geq 0,\ Xv \geq 0$$

for any positive vector $r \in R^n$. Letting $\bar{x} = X\bar{v}$, we see that $\bar{x}$ is a solution of (q,M) and it can be obtained by solving the linear program

$$(3.12) \qquad \text{minimize } (X^{-T}r)^T x \quad \text{subject to} \quad q + Mx \geq 0,\ x \geq 0$$

which is just $(X^{-T}r, q, M)$. Summing this up, we have proved

<u>Theorem 3.11</u>. Let $M \in C$ and suppose that the problem (q,M) is feasible. Then there exists a bijective, linear map $L: R^n \to R^n$ such that the feasible set $X(q,M)$ is mapped onto a polyhedral set V having a least element $\bar{v}$ whose pre-image $\bar{x} = L^{-1}(\bar{v})$ solves the problem (q,M).

It is this theorem which provides the desired relationship between the linear complementarity problems with matrices in class C and the theory of polyhedral sets having least elements. Together with Lemma 3.10, the theorem also provides an interpretation for the conditions imposed on the vector p chosen in the objective function of the corresponding linear programs.

<u>Remark 1</u>. When using the linear program (3.12) to obtain a solution to the linear complementarity problem (q,M), one must, first of all, solve the system of linear equations $p^T X = r^T$ where $r^T > 0$ is chosen arbitrarily, to get the vector p used in the objective function of the linear program. For large n, this problem of finding p is not an insignificant task.

<u>Remark 2</u>. For $M \in \mathcal{C}$, Mangasarian shows in Theorem 2.4 that the problem (q,M) can be solved via the linear program (p,q,M) for any p belonging to the class

$$\rho_1 = \{p \in R^n : p^T = r^T + s^T M \text{ for some } (r,s) \geq 0 \text{ such that } r^T X + s^T Y > 0\}.$$

Our least element argument above shows that such a vector p can be chosen arbitrarily in the class

$$\rho_2 = \{p \in R^n : p^T = r^T X^{-1} \text{ for some } r > 0\} .$$

Here we demonstrate that $\rho_1 = \rho_2$. Note that

$$p \in \rho_1 \iff p^T = r^T + s^T M = (r^T X + s^T Y) X^{-1}$$

for some $r, s \geq 0$ and $r^T X + s^T Y > 0$. Hence it suffices to show that for any $t \in R^n$, $t > 0$, there exist $r, s \geq 0$ such that $(X^T, Y^T)\binom{r}{s} = t$. We mentioned in the proof of Lemma 3.8 that we may assume without loss of generality that the matrix (X^T, Y^T) is Leontief. We also showed that there exists a complementary submatrix B of (X^T, Y^T) which has a nonnegative inverse. Clearly, $B^{-1}t > 0$. Now we define the vector (r^T, s^T) as follows:

$$r_i = \begin{cases} (B^{-1}t)_i & \text{if } B_{.i} = (X^T)_{.i} \\ 0 & \text{otherwise ,} \end{cases}$$

and

$$
s_i = \begin{cases} (B^{-1}t)_i & \text{if } B_{.i} = (Y^T)_{.i} \\ 0 & \text{otherwise ,} \end{cases}
$$

for each $i = 1,\ldots,n$. Then it can readily be verified that (r^T, s^T) is the desired vector.

Remark 3. The proof of Theorem 3.11 shows that condition (3.6) implies that the polyhedral set V is bounded below. The converse is also true and is an immediate consequence of the duality theorem of linear programming.

Remark 4. For $M \in C$ and $q \in R^n$, the solution $\bar{x}$ of the problem (q,M) obtained in Theorem 3.11 need not be the least element of the feasible set $X(q,M)$ under the usual ordering of R^n, as the following example shows. However, it will be demonstrated in Pang [22] that $\bar{x}$ is always the least element under the partial ordering induced by the polyhedral cone

$$
D = \{q \in R^n : X^{-1}q \geq 0\}
$$

where $M = YX^{-1}$ with X and Y satisfying conditions (3.5) and (3.6).

Example.

$$M = \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & -3 \\ -\frac{1}{2} & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} \frac{1}{2} & -2 \\ -\frac{1}{2} & 3 \end{pmatrix}$$

$$q^T = (-1, 6) \; .$$

$$M = YX^{-1} \; ; \qquad\qquad Y \in K \; .$$

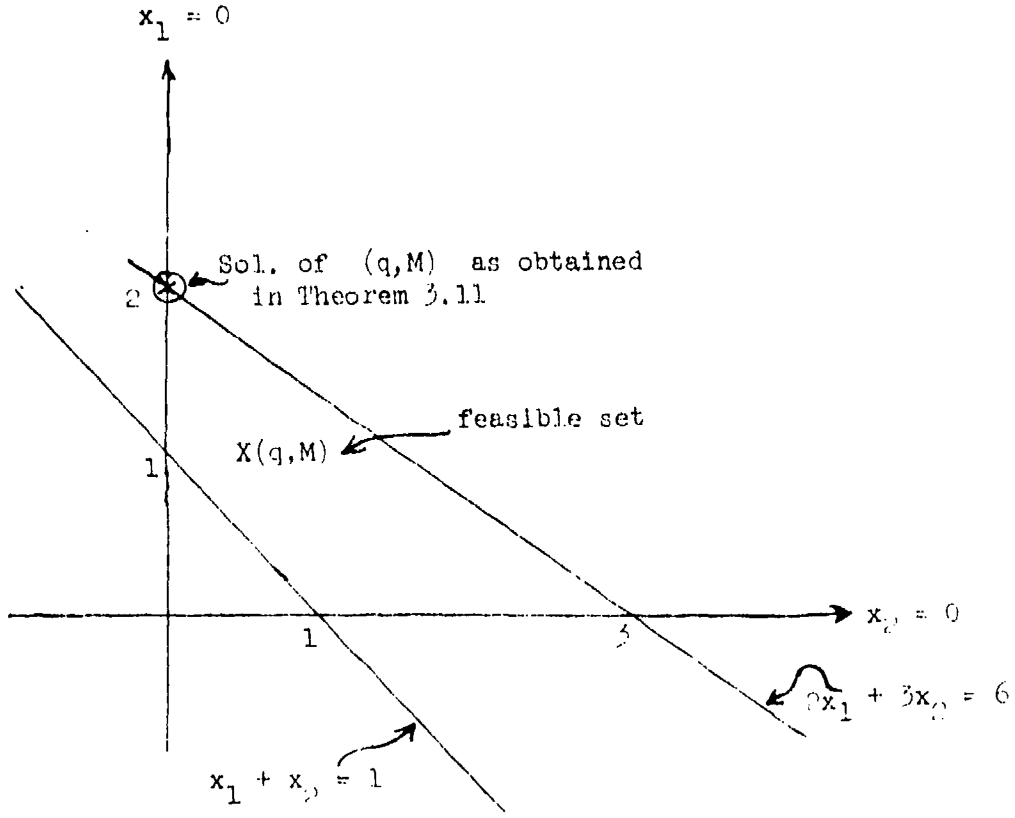
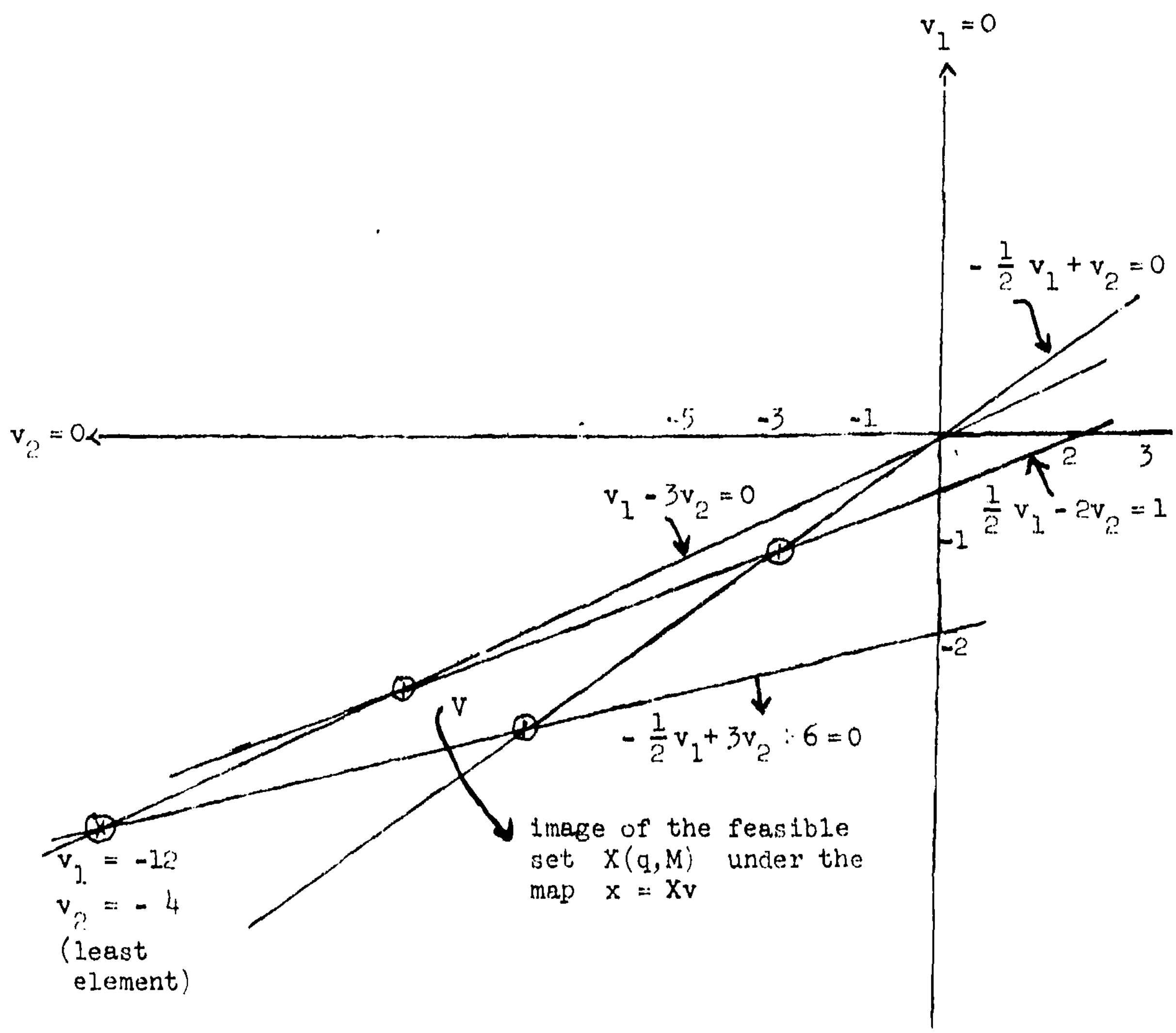


FIGURE 1



$v_1 = 0$
$-\frac{1}{2} v_1 + v_2 = 0$
$v_2 = 0$
-5
-3
-1
2
3
$v_1 - 3v_2 = 0$
$\frac{1}{2} v_1 - 2v_2 = 1$
-1
-2
V
$-\frac{1}{2} v_1 + 3v_2 + 6 = 0$
image of the feasible
set X(q,M) under the
map x = Xv
$v_1 = -12$
$v_2 = -4$
(least
element)

FIGURE 2

Having established the desired relationship mentioned above, we proceed to investigate the classes of matrices introduced by Mangasarian. It is clear that all the classes of matrices in Corollary 2.3 are subclasses of $\mathcal{C}$.

In [19], Mangasarian introduced the "slack linear complementarity problem" of finding $x, y \in R_+^n$ such that

$$(3.13) \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix} + \begin{pmatrix} M & B \\ 0 & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq 0$$

and

$$u^T x = v^T y = 0$$

in order to extend the class of LP-solvable linear complementarity problems. It is clear that if A satisfies the condition that $x^T A x \neq 0$ for all $0 \neq x \geq 0$, and if (3.13) has a solution $(\bar{x}, \bar{y})$, it necessarily follows that $\bar{y} = 0$. So $\bar{x}$ solves (q,M). Mangasarian then shows that if M satisfies conditions (2.7) and (2.8), the matrix $\begin{bmatrix} M & B \\ 0 & I \end{bmatrix}$ satisfies conditions (2.5) and (2.6). He then invokes Theorem 2.1 and the above observation to obtain Theorem 2.4.

Here we intend to derive Theorem 2.4 using a different approach. We want to employ the above established relationship. Our eventual aim is to show that a matrix satisfying conditions (2.7) and (2.8) belongs to the class $\mathcal{C}$.

It has long been an open problem in the theory of linear complementarity problems to characterize the class $\mathcal{K}$ of matrices for which the feasibility of the linear complementarity problems implies their solvability. Although this class $\mathcal{K}$ is still a mystery, various subclasses have been explored and studied intensively. (See e.g. Lemke [17].) Theorem 3.11 shows that the class $\mathcal{C}$ is a subclass of $\mathcal{K}$. The following example shows that it is a <u>proper</u> subclass of $\mathcal{K}$.

<u>Example</u>. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then it can easily be shown that the problem (q,M) always has a solution for every $q \in R^2$. We claim that $M \notin \mathcal{C}$. Suppose not, then there exist $X, Y, \in R^{2\times 2}$, both Z-matrices satisfying conditions (3.5) and (3.6). We have

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

which implies

$$x_{11} + x_{21} = y_{11}$$

and

$$x_{11} + x_{21} = y_{21} .$$

Thus $y_{11} = y_{21} \leq 0$. Similarly, $y_{22} = y_{12} \leq 0$. Lemma 3.8 would then imply $X \in K$. In particular, $\det X = x_{11}x_{22} - x_{12}x_{21} > 0$. However,

$$0 < x_{11} = y_{21} - x_{21} \leq -x_{21}$$

and

$$0 < x_{22} = y_{22} - x_{12} \leq -x_{12} .$$

Therefore, $x_{11}x_{22} \leq x_{12}x_{21}$ which is a contradiction. This proves that $M \notin \mathcal{C}$.

As the last four subclasses of matrices in Table 1 are all defined so explicitly in terms of well-known classes of matrices, it is natural to ask whether any of these will perhaps add to our knowledge about the class K. In the sequel, we shall study these special classes separately and show they all belong to P which is of course already well-known in the theory of linear complementarity problems. (See [4], [6], [25].)

Proposition 3.12. Let $Y \in K \cap R^{n \times n}$ and $a, b \in R^n_+$. Then the matrix $Y + ab^T \in P$.

Proof: The hypothesis is inherited by principal submatrices, hence it suffices to show $\det(Y + ab^T) > 0$. Clearly,

$$\det(Y + ab^T) = (\det Y)\ (\det(I + (Y^{-1}a)b^T)) .$$

Using the formula

$$\det(xy^T - \lambda I) = (-1)^n \lambda^{n-1}(\lambda - x^T y)$$

which holds for all $x, y \in R^n$ and $\lambda \in R$, we obviously have (substituting $\lambda = -1$, $x = Y^{-1}a$ and $y = b$)

$$\det(Y + ab^T) = (\det Y)(1 + (Y^{-1}a)^T b) \geq \det Y > 0$$

because $Y^{-1} \geq 0$ and $a, b \geq 0$. Therefore $Y + ab^T \in P$.

Proposition 3.13. If $M = Y + ab^T$ where $Y \in K$ and $a, b \geq 0$, then there exists a matrix $X \in K$ such that $M = YX^{-1}$.

Proof. Since $Y \in K$ and $a \geq 0$, it follows that $\bar{a} \equiv Y^{-1}a \geq 0$. We may write $M = Y(I + \bar{a}b^T)$. Let $X \equiv (I + \bar{a}b^T)^{-1}$. Then $M = YX^{-1}$. It remains to show that $X \in K$. An easy calculation shows that

$$X = I - \frac{1}{1 + b^T\bar{a}} \bar{a}b^T \in Z.$$

It follows from Proposition 3.12 that $X^{-1} = I + \bar{a}b^T \in P$; hence $X \in P$ by fact (ii) mentioned earlier. Therefore $X \in K$. This completes the proof.

Corollary 3.14. If $M = Y + ab^T$ where $Y \in K$ and $a, b \geq 0$, then $M \in C$.

Thus we have shown that the class of matrices

$$\{M \in R^{n \times n}: M = Y + ab^T,\ Y \in K \text{ and } a, b \in R^n_+\}$$

is a subclass of C. Furthermore, the matrices X and Y in the factorization $M = YX^{-1}$ are Minkowski.

Remark. In [19], Mangasarian treated the above class of matrices with the assumption that the vector b is strictly positive. Here, we have relaxed this assumption slightly and merely required b to be nonnegative.

We next proceed to another class. The proof of the following lemma can be found in Fiedler and Pták [14].

Lemma 3.15. Let $A \in Z$, $B \in K$ and $A \geq B$, then

(vii) $A \in K$

(viii) $A^{-1}B \in K$, $BA^{-1} \in K$

(ix) $A^{-1}B \leq I$, $BA^{-1} \leq I$.

Lemma 3.16. Let $A \in Z$, $B \in K$, $I \geq B$ and $A \geq B$. Then the matrix $I + A - B \in P$.

Proof: The hypothesis are inherited by principal submatrices, hence it suffices to show $\det(I + A - B) > 0$. Let $C = A - B \geq 0$. Then $B = A - C$ belongs to K by assumption. We have

$$(I + C)(A - C) = A - C + CB = A - C(I - B) \leq A .$$

Hence $(I + C)(A - C) \in Z$. Since $I + C \geq I$ and $A - C \in K$, it follows readily from condition (iv) in Proposition 3.2 that $(I + C)(A - C) \in K$. Hence

$$\det(I + A - B) = \det(I + C)$$
$$= \det(I + C)(A - C)/\det(A - C) > 0 .$$

This completes the proof.

Lemma 3.17. Let $Z_1, Z_2, Z_3 \in Z$. $Z_1 \geq Z_2 \geq Z_3$ and $Z_3 \in K$. Then the matrix $Z_1 + Z_2 - Z_3 \in P$.

Proof: Again, it suffices to show $\det(Z_1 + Z_2 - Z_3) > 0$. We write

$$Z_1 + Z_2 - Z_3 = Z_1(I + Z_1^{-1}Z_2 - Z_1^{-1}Z_3) .$$

Lemma 3.15 implies the assumptions of Lemma 3.16 are satisfied with $A = Z_1^{-1}Z_2$ and $B = Z_1^{-1}Z_3$. Therefore $\det(I + Z_1^{-1}Z_2 - Z_1^{-1}Z_3) > 0$. Lemma 3.15 also implies that $\det Z_1 > 0$. Hence $\det(Z_1 + Z_2 - Z_3) > 0$.

Corollary 3.18. Let $A \in Z$, $B \in K$ and $A \geq B$. Then the matrix $2A - B \in P$.

Remark. As a matter of fact, following the proof of Lemma 3.17, we may deduce that if A, B satisfy the assumptions in Corollary 3.18, then the matrix $\lambda A - B \in P$ for all $\lambda \geq 2$.

Corollary 3.19. If $A \in Z$, $B \in K$ and $A \geq B$, then $\det(2A - B) \geq \det B$.

Proof: According to Theorem 1 of Ostrowski [21], if $Y \in Z$ has positive diagonals and M is a matrix such that $|m_{ii}| \geq y_{ii}$ and $|m_{ij}| \leq -y_{ij}$ for all $i \neq j$, then $|\det M| \geq \det Y$. So choosing $Y = B$, $M = 2A - B$, we have $|\det M| = \det M$ and $|m_{ii}| = m_{ii} = 2a_{ii} - b_{ii} \geq b_{ii}$ and $0 \geq 2a_{ij} \geq 2b_{ij}$ for all $i \neq j$. Hence $-b_{ij} \geq 2a_{ij} - b_{ij} \geq b_{ij}$, i.e., $|2a_{ij} - b_{ij}| \leq -b_{ij}$. Ostrowski's theorem applies and the proof is complete.

Proposition 3.20. Let $M = 2A - B$ where $A \in Z$, $B \in K$ and $A \geq B$. Then there exist $Y \in Z$ and $X \in K$ such that $M = YX^{-1}$.

Proof. Lemma 3.15 implies A^{-1} exists. We may write

$$M = 2A - B = (2B - BA^{-1}B)(B^{-1}A)$$
$$= (2B - BA^{-1}B)(A^{-1}B)^{-1}$$

Let $X = A^{-1}B$ and $Y = 2B - BA^{-1}B$. Then Lemma 3.15 implies that $X \in K$ and $X \leq I$. We have

$$Y = 2B - BA^{-1}B = A - (A - B)(I - A^{-1}B) \leq A.$$

Hence $Y \in Z$ and the proof is complete.

Corollary 3.21. Let $M = 2A - B$ with $A \in Z$, $B \in K$ and $A \geq B$. Then $M \in C$.

Corollary 3.22. Let M satisfy either of the following conditions:

$$m_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} |m_{ij}| , \qquad i = 1, \ldots , n, \tag{3.14}$$

$$m_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n} |m_{ij}| , \qquad j = 1, \ldots , n . \tag{3.15}$$

then $M \in C$.

Proof: It suffices to show that M satisfies the assumptions in Corollary 3.21. Define the matrix $A = (a_{ij})$ as follows:

$$a_{ij} = \begin{cases} m_{ii} & \text{if } j = i \\ m_{ij} & \text{if } j \neq i \text{ and } m_{ij} \leq 0 \\ 0 & \text{if } j \neq i \text{ and } m_{ij} > 0 . \end{cases}$$

Let $B = 2A - M$; clearly, $M = 2A - B$ and $A \geq B$. We obviously have $A \in Z$, thus $B \in Z$. It remains to show that $B \in K$. If $B = (b_{ij})$ then by its definition,

$$b_{ij} = \begin{cases} m_{ii} & \text{if } j = i \\ m_{ij} & \text{if } j \neq i,\ m_{ij} \leq 0 \\ -m_{ij} & \text{if } j \neq i,\ m_{ij} > 0 , \end{cases}$$

so that letting $e^T = (1, \ldots , 1) \in R^n$, we have for $i = 1, \ldots , n$,

$$\begin{aligned} (Be)_i &= \sum_{j=1}^{n} b_{ij} \\ &= m_{ii} + \sum_{\substack{j=1 \\ j\neq i,\, m_{ij} \leq 0}}^{n} m_{ij} + \sum_{\substack{j=1 \\ j\neq i,\, m_{ij} > 0}}^{n} (-m_{ij}) . \end{aligned}$$

Thus, $Be > 0$ if condition (3.14) is satisfied. Similarly, we may deduce that $B^Te > 0$ if condition (3.15) is satisfied. Therefore, in either case, it follows that $B \in K$. This completes the proof.

<u>Remark 1</u>. The proof above and Lemma 3.20 show that a matrix satisfying conditions (3.14) and (3.15) belongs to the class P.

Remark 2. In [19], Mangasarian treated the classes of matrices M satisfying conditions (3.14) and (3.15) with the additional assumption that M is nonnegative. The corollary above enlarges these classes by omitting the nonnegativity assumption and shows that the corresponding linear complementarity problems are still related to polyhedral sets with least elements.

Up to this point, we have shown that all matrices in Mangasarian's Table 1 belong to the class C. Furthermore, we have established that the last four subclasses belong to P, thus to the class K mentioned earlier. Therefore they do not extend our knowledge about the class K.

Remark. If M belongs to any of these four subclasses, the problem (q,M) has a solution for every $q \in R$ because M must necessarily be a P-matrix. Thus the feasibility assumption can be removed.

As all matrices in Table 1 are obtained by specializing the conditions (2.7) and (2.8) in Theorem 2.4, it is natural to ask whether a matrix satisfying these two conditions alone belongs to C. We answer this question by establishing the following theorem which summarizes some of the previous results.

Theorem 3.23. Let $M \in R^{n \times n}$. Suppose there exist $X, Y \in R^{n \times n}$, $A \in R^{m \times m}$, $B, H \in R^{n \times m}$, $G \in R^{m \times n}$, $p \in R^n_+$ and $p_0 \in R^m_+$ satisfying

(3.16) $MX = Y + BG$; $MH \geq BG$; $X, Y, A \in Z$; and $G, H \geq 0$;

(3.17) $$(p^T, p_0^T) \begin{bmatrix} X & -H \\ -G & A \end{bmatrix} > 0 \, .$$

Then

(3.18) M has the representation

$$M = \bar{Y}\bar{X}^{-1}$$

where

$$\bar{Y} \equiv Y - \bar{H}A^{-1}G, \quad \bar{X} \equiv X - HA^{-1}G, \quad \text{and} \quad \bar{H} \equiv MH - BA \, ;$$

(3.19) furthermore, $\bar{Y} \in Z$ and $\bar{X} \in K$.

<u>Proof</u>: The matrix $\begin{bmatrix} X & -H \\ -G & A \end{bmatrix}$ belongs to class Z, thus condition (3.17) implies that it belongs to K. In particular A^{-1} exists. We have

$$\bar{Y} \equiv Y - \bar{H}A^{-1}G = Y - (MH - BA)A^{-1}G$$
$$= Y + BG - M(HA^{-1}G)$$
$$= M(X - HA^{-1}G) = M\bar{X} \, .$$

The matrix $\bar{X} \equiv X - HA^{-1}G$ is the Schur complement of A in the Minkowski matrix $\begin{bmatrix} X & -H \\ -G & A \end{bmatrix}$, hence $\bar{X}$ is itself Minkowski by Proposition 3.4 and in particular, nonsingular. Thus $M = \bar{Y}\bar{X}^{-1}$, establishing (3.18). It remains to verify that $\bar{Y} \in Z$. We have $\bar{Y} \equiv Y - \bar{H}A^{-1}G$ where $\bar{H} = MH - BA \geq 0$ and $G \geq 0$ by assumption; moreover, $A^{-1} \geq 0$ because $A \in K$. Therefore $\bar{Y} \in Z$. This completes the proof of the theorem.

Corollary 3.24. Suppose $M \in \mathbb{R}^{n \times n}$ satisfies conditions (3.16) and (3.17). Then $M \in \mathcal{C}$.

Remark. In the factorization $M = \tilde{Y}\tilde{X}^{-1}$ above, the matrix $\tilde{Y} \equiv Y - \tilde{H}A^{-1}G$ contains the matrix $\tilde{H}$ which is defined in terms of M, i.e. the factorization involves M itself implicitly. This somewhat awkward situation can be remedied by solving for M using condition (3.16) to obtain $M = (Y + BG)X^{-1}$. Substituting into the definition of $\tilde{H}$, we see that M is no longer involved in the factors.

4. COMPUTATIONAL EXPERIENCE

Methods for solving linear complementarity problems for various classes of matrices have been proposed and investigated intensively. Among these are the principal pivoting method [4], [5] and Lemke's almost complementarity pivoting algorithm [6], [16]. These methods (and some others) work rather satisfactorily for matrices of reasonable size. But in many applications of the linear complementarity problems to partial differential equations, the matrices are often large, sparse and specially structured. See [21] for example. Methods like those mentioned above seem to be inefficient when applied to these problems. For one thing, most of the nice properties (especially the sparsity which is a very important factor for efficiency) that the matrices originally possess will be destroyed when the problems are being processed. Recognizing this disadvantage, one would like to use iterative (relaxation) procedures which, presumably, have the computational advantage of preserving matrix sparsity.

In [18], Mangasarian proposed formulating linear complementarity problems as linear programs and solving them by applying relaxation methods to the linear inequality system (2.4). In this section, we discuss our somewhat preliminary computational experience using this solution strategy and attempt to answer the question of whether this approach can be recommended in practice.

For the sake of clarity, we first review part of the theory of relaxation methods for solving linear inequality systems [1], [2], [13], [20].

We want to find a vector $z \in R^n$ satisfying the system of linear inequalities: $Az \leq b$ where $A \in R^{m \times n}$ and $b \in R^m$. Let $A_i \equiv A_{i\cdot}$ be the i-th row of the matrix A. The relaxation method (due to Eremin [13]) constructs a sequence $\{z^k\}$ in the following manner:

(i) Choose $z^0 \in R^n$ arbitrarily. Let $k = 0$.

(ii) If $Az^k \leq b$, the procedure terminates.

If not, then some linear inequality is violated. Let i be the smallest index of the most violated constraints:

$$A_i z^k - b_i = \max_{1 \leq j \leq n} (A_j z^k - b_j) .$$

(iii) Define

$$z^{k+1} = z^k - \lambda^k \left(\frac{A_i z^k - b_i}{\|A_i\|^2} \right) A_i^T . \qquad (4.1)$$

Theorem (Eremin [13]).

The sequence $\{z^k\}$ defined by the relaxation method under the assumptions

(i) $\lambda^k \in (0,2]$, $k = 1,2,\ldots$

(ii) $\inf \lambda^k > 0$.

converges to one of the solutions of the linear inequality system $Az \leq b$ if the latter is consistent.

We now apply the method to solve the linear program (p,q,M), or equivalently, to find vectors $x, y \in R^n$ such that the following system of linear inequalities is satisfied.

$$(4.2) \qquad \begin{pmatrix} -M & 0 \\ 0 & M^T \\ -I & 0 \\ 0 & -I \\ p^T & q^T \end{pmatrix} \begin{pmatrix} x \\ \\ y \end{pmatrix} \leq \begin{pmatrix} q \\ p \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Remark. The vectors x and y are primal and dual variables, respectively.

We now present an algorithm based on the relaxation method for solving (4.2). Let $\varepsilon > 0$ be some preassigned positive tolerance. The algorithm starts by choosing

$$z^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}$$

arbitrarily. Initially $\ell = 0$.

Step 1. Let $z^\ell = \begin{pmatrix} x^\ell \\ y^\ell \end{pmatrix}$ and define

$$(4.3) \qquad d(z^\ell) = \max\Big(\max_{1 \leq i \leq n} \big(-\sum_{j=1}^{n} m_{ij} x_j^\ell - q_i\big), \max_{1 \leq j \leq n} \big(\sum_{k=1}^{n} m_{kj} y_k^\ell - p_j\big),$$

$$\max_{1 \leq i \leq n} (-x_i^\ell), \max_{1 \leq j \leq n} (-y_j^\ell), \sum_{i=1}^{n} p_i x_i^\ell + \sum_{j=1}^{n} q_j y_j^\ell \Big) .$$

Step 2. If $d(z^{\ell}) \leq \epsilon$, stop.

Otherwise, construct $z^{\ell+1} = \begin{pmatrix} x^{\ell+1} \\ y^{\ell+1} \end{pmatrix}$ according to (4.1) and return to step 1 with $\ell+1$ replacing ℓ.

The main work involved in each iteration cycle of the algorithm is the computation of the quantity $d(z^{\ell})$ and the updating of the new iterates. When actually programmed, $d(z^{\ell})$ is computed as in (4.3). While in constructing the new approximate solution $z^{\ell+1}$, an index is set to check where the maximizing term comes from, in order to avoid computing the same components repeatedly. For instance, if the maximizing term is in $\{-\sum_{j=1}^{n} m_{ij}x_j^{\ell} - q_i\}_{i=1}^{n}$, then the vector y^{ℓ} need not be updated; on the other hand, if the maximizing term comes from $\{-x_i^{\ell}\}_{i=1}^{n}$, then only one component of x^{ℓ} will be changed while the other components of z^{ℓ} will remain unchanged in the next iteration. These features of the system are important in reducing the computational effort of the algorithm.

There is a variation of the algorithm that one might want to consider when actually coding it. In (4.2) where the new iterate $z^{\ell+1}$ is computed, it is necessary to divide by the 2-norm of the gradient of the most violated constraint. If the number of iterations is large compared to $4n+1$ which is the number of constraints in (4.1), it would be more economical to reduce these divisions if possible. A way to achieve this is to work with the "normalized system" which is obtained from (4.1) by normalizing each constraint separately. The division steps in (4.2) are then no longer needed. An obvious disadvantage of this "normalized system" is the need for more storage space for the whole matrix and also for the constant vectors. But if the problem itself were of moderate size so that no storage problem would occur, then one might try to use this "normalized system" instead of the original system (4.1).

We performed several experiments to solve the problem (q,M) where $q \in R^n$ was randomly chosen and

$$M = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

We chose $p^T = (1, \ldots, 1) \in R^n$. The relaxation parameters $\{\lambda^k\}$ were chosen to be the same in each iteration. The tolerance ϵ was generously chosen to be 10^{-4}. The algorithm was then applied to the inequality system (4.1). All the computation was done on the IBM 370/168 using FORTRAN H with Opt = 2. The results are summarized in the following tables.

Inputs: $n = 5$. $q^T = (2, -1, -4, 6, -5)$;

starting iterate $x^0 = y^0 = 0$; original system.

Solution of (q,M): $x^T = (0, 2, 3, 0, 2.5)$.

Value of λ	No. of Iterations	Execution Time (sec)
.7	1737	.58
.8	1380	.50
.9	1108	.45
1.1	149	.38
1.2	649	.37
1.3	505	.33
1.4	432	.33
1.5	358	.31
1.6	176	.27
1.7	949	.42
1.8	668	.37

TABLE 2

Inputs: $n = 5$. $q^T = (2, -1, -4, 6, -5)$

starting iterate $x^0 = y^0 = 0$; normalized system.

Solution of (q,M): $x^T = (0, 2, 3, 0, 2.5)$

Value of λ	No. of Iterations	Execution time (sec)
1.0	669	.38
1.5	150	.28
1.6	887	.41
1.8	965	.43

TABLE 3

Inputs: $n = 5$. $q^T = (2, -1, -3, 4, -5)$;

starting iterate: $x^0 = (3, 3, 3, 3, 3)^T$,

$y^0 = (2, \ldots, 2)^T$; normalized system.

Solution of (q,M): $x^T = (0, 2, 3, 1, 5)$

Value of λ	No. of iterations	Execution time (sec)
.8	17364	3.12
.9	14263	2.63
1.0	11664	2.19
1.1	9663	1.88
1.2	7926	1.57
1.3	6571	1.36
1.4	5626	1.19
1.5	4381	1.21
1.6	3352	.81
1.7	2353	.65
1.8	1490	.50
1.9	664	.37

TABLE 4

Inputs: $n = 7$. $q^T = (2, -1, -4, 6, -5, 3, -2)$

starting iterate $x^0 = y^0 = 0$; original system.

Solution of problem (q,M): $x^T = (0, 2, 3, 0, 2.75, .5, 1.25)$

Value of λ	No. of iterations	Execution time (sec)
1.95	2042	.74
1.8	9820	2.63
1.7	14831	3.87
1.6	19935*	4.60

TABLE 5

*20000 is the maximum number of iterations that we allow.

Inputs: $n = 7$. $q^T = (2, -1, -4, 6, -5, 3, -2)$

starting iterate: $x^0 = (2, \ldots, 2)^T$, $y^0 = (1, \ldots, 1)$

original system; solution of (q,M) given in Table 5.

Value of λ	No. of iterations	Execution time (sec)
1.95	1362	.57
1.85	5910	1.68
1.75	10597	2.67
1.65	14313	3.65
1.55	19219	4.95

TABLE 6

The problems solved in these experiments are far from the size and difficulty one would encounter in practice. Nevertheless, they suggest that the relaxation method described above lacks the efficiency required for solving more realistic problems. It is conceivable that by some clever modification the method could be made more attractive.

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Recently, Mangasarian [13], [19] has discussed the idea of solving certain classes of linear complementarity problems as linear programs. was discussed The present paper (1) demonstrates how these complementarity problems are related to the theory of polyhedral sets having least elements and (2) discusses the question of whether the linear programming approach can be recommended for solving them.

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