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METHODS FOR SOLVING THE VISCOELASTICITY EQUATIONS FOR CYLINDER SPHERE PROBLEMS

BY
G.C. Gaunaurd

22 MARCH 1976

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This report considers techniques used to solve the Navier field equations of viscoelasticity in the Kelvin-Voigt or Maxwell models, for cylindrical or spherical geometries. Introducing scalar and vector potentials into the viscoelasticity equations formulation, ultimately yields "telegraph-type" partial differential equations governing those potentials. For harmonic time-dependence, these reduce to scalar and vector Helmholtz's equations with complex propagation constants. These constants are shown to be related to the viscoelastic material-

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constants in a more or less complicated fashion depending on the viscoelastic model used. The stresses, strains and displacements are then found from these potentials for a dozen cases of interest in those two coordinate systems. The formulation resembles that of electrodynamics in a Coulomb gauge.

The above information is vital to set-up and solve various kinds of boundary-value-problems of dynamic viscoelasticity which appear when studying cases of acoustic scattering from sound-absorbing structures; problems we are now addressing. The analysis is summarized in two large Tables, set-up in a conveniently accessible form. Remarks on "complex-moduli" are examined in a final section under the light of the viscoelasticity "Correspondence Theorem" and a list of recommendations and conclusions is given at the end.

NSWC/WOL/TR-76-20 22 March 1976

METHODS FOR SOLVING THE VISCOELASTICITY EQUATIONS FOR CYLINDER AND SPHERE PROBLEMS

This report describes techniques used to solve the field equations of dynamic viscoelasticity in two geometries, two viscoelastic models, and a dozen different situations of interest. This information is needed to study acoustic scattering from sound-absorbing structures of cylindrical and spherical shapes, which are cases presently under study by the author. This report sets up most of the theoretical viscoelasticity foundations needed to deal with the other sound scattering problems under study.

This work is continuing and it was done as part of an NSWC project entitled "Acoustical Properties of Ordnance Materials", Task No. MAT-03L-000/ZR00-001-010, Problem 127, which deals with acoustic scattering from objects covered with viscoelastic materials. This is a progress report describing work done during FY 76.

J. R. DIXON
By direction

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CONTENTS

TABLES

I. INTRODUCTION The basic viscoelastic models.

The deformation of a viscoelastic solid under any kind of external loads is usually studied by means of viscoelastic models. Two basic models commonly used are associated with the names of Kelvin[1]-Voigt[2] and Maxwell[3]. In the Kelvin-Voigt model, the elastic and viscous properties of each material point (or particle) of the body, which can be respectively represented by a spring and a dashpot, are assumed connected in parallel. In the Maxwell model they are assumed connected in series. We repeat that this description applies at each point in the viscoelastic solid and it is as if the body contained a continuous distribution of damped oscillators, (viz, elementary mass-spring-dashpot systems) one at each one of its material points. As if it were not clear enough already, the entire "Kelvin-Voigt solid" can not be replaced by ONE spring connected in parallel with ONE dashpot. This simplistic view of a deformable solid as a single particle, may be useful in some other elementary context such as that used when one treats a body as a particle, but it is of no use in viscoelasticity. Otherwise viscoelasticity could not be viewed as a field-theory capable of describing stress and displacement fields at each point in a body, since by that oversimplification, the body has been reduced to a particle. We emphasize this rather trivial, but quickly forgotten point, because it is common to find authors who try to use, say, the "Kelvin model" as a single spring in parallel with a single dashpot, only to find that the model is "no-good" and that they must go to "more-degrees-of-freedom systems", such as three "Kelvin-models" in series, or other similar configurations, to obtain meaningful results. It is clear that this approach does not give the Kelvin model, as it truly is, even a chance to "work". These authors have replaced the continuous field-equations of viscoelasticity[4], by a set of three ordinary second-order differential equations of the simple

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[1] William Thomson (Lord Kelvin), b. at Belfast, [illegible]; d. near Glasgow, 1907. Professor at Glascow Univ, 1846-1889. Buried at Westminster Abbey, London (near Newton's tomb).

[2] Woldemur Voigt, b. at Leipzig 1850; d. at Göttingen 1919. Professor of Mechanics at Göttingen Univ, Germany.

[3] James C. Maxwell, b. Edinburgh, 1831; d. Cambridge, 1879. Professor of Physics at King's College and at Cambridge Univ, 1860-1879. Founder of the Cavendish Lab at Cambridge.

[4] ie, a set of three scalar partial differential equations hopelessly coupled and non-linear, governing the displacement-field in the body, which can be linearized for small deformations and small deformation-gradients, and the "linear" theory of viscoelasticity then results. These equations are called the Navier equations of viscoelasticity.

damped-oscillator type. This replacement is, in no way, equivalent to solving the field-equations of Navier.

The basic point of the continuum field-theory approach to viscoelasticity is that the model assumptions, that the spring and dashpot are connected in series or in parallel at each point in the body, are immediately reflected in the fact that the field-equations resulting from either one of those models turn out to be different. To fix the ideas we now give the linearized form of the field-equations for both these models.

i) Kelvin-Voigt Model (parallel):

$$\left[1+\frac{\mu_v}{\mu_e}\frac{\partial}{\partial t}\right]\nabla^2\vec{u}+\left(\frac{\lambda_e+\mu_e}{\mu_e}\right)\left[1+\frac{\lambda_v+\mu_v}{\lambda_e+\mu_e}\frac{\partial}{\partial t}\right]\vec{\nabla}(\vec{\nabla}\cdot\vec{u})=\frac{1}{C_s^2}\frac{\partial^2\vec{u}}{\partial t^2}$$

ii) Maxwell Model (series):

$$\left[1+\frac{1}{2\beta}\frac{\partial}{\partial t}\right]\left[\nabla^2\vec{u}+\left(\frac{\lambda_e+\mu_e}{\mu_e}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})-\frac{1}{C_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\right]+\frac{3\alpha}{2\beta}\left[\nabla^2\vec{u}+\frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})-\frac{1}{C_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\right]=\frac{1}{C_s^2}\frac{\partial^2\vec{u}}{\partial t^2}+\frac{3\alpha+2\beta}{C_s^2}\frac{\partial\vec{u}}{\partial t}\quad\left(C_s^2=\frac{\mu_e}{\rho}\right)$$

Here ρ is the material density, λ_e, μ_e are the elastic Lamé constants, $\vec{u}$ is the vector-displacement field, and λ_v, μ_v (or α, β in the Maxwell model) are the viscosity coefficients. As given above, in differential operator form, these equations hold in any coordinate system. These are the Navier equations one must solve in the viscoelastic body. It is possible to derive eqs. (i) and (ii) starting from the basic idea that the spring and dashpot at each material point are connected either in parallel or in series respectively. Once the displacement field components are found by solving (i) or (ii) with suitable boundary conditions, one can then find the stresses from them.

In the absence of viscosity (ie, $\lambda_v = 0$, $\mu_v = 0$ for the Kelvin model or $\alpha = 0$, $\beta = 0$ for the Maxwell model) both field equations (i) and (ii) given above reduce to,

iii) The Field equations of linear dynamic elasticity:

$$\nabla^2\vec{u}+\left(\frac{\lambda_e+\mu_e}{\mu_e}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})=\frac{1}{C_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\ .$$

since now there are only springs and no dashpots at each material point. The scalar components of this vector equation are the Navier equations of elasticity in the absence of body-forces as found by Navier[5] for the elastic body. Equations (i) and (ii) are also called the Navier equations (of viscoelasticity) by extension, since Navier never worked with viscoelastic solids.

In most materials of interest, the "constants" λ_e, μ_e and the "coefficients" λ_v, μ_v are really not constants, but frequency-dependent parameters. Thus, few materials are completely describable by the Kelvin or the Maxwell models in the continuum sense of eqs. (i) and (ii). The material behavior of viscoelastic substances tends to follow one or the other model in different regions of the parameters involved, say, frequency among others. This means that in general, these models of field-equations (i) and (ii) are quite "good". Rubbers at low frequencies are known to be well described by the Kelvin model (i). Pulse-tube measurements exploit this factual observation. Since these models are not perfect, researchers in this field have proposed more complicated models.

It is not hard to see that various Maxwell elements in series at each point in the body have the properties of a single Maxwell element with equivalent spring and dashpot constants given by $1/k_{eq} = \sum_{i=1}^{n} 1/k_i$ and $1/\eta_{eq} = \sum_{i=1}^{n} 1/\eta_i$ respectively. Various Kelvin elements in parallel at each point in the body have the properties of a single Kelvin element with $k_{eq} = \sum_{i=1}^{n} k_i$ and $\eta_{eq} = \sum_{i=1}^{n} \eta_i$.

On the other hand, Kelvin elements in series, or Maxwell elements in parallel, have more complicated properties. In their desire to generalize the basic models (i) and (ii), researchers have invented the so called "standard viscoelastic model". It consists of a Maxwell element in parallel with a Kelvin element at each point in the body. The linearized field equations which result in this model when two springs and two dashpots are connected as described above must be very complicated and rare, since I can not find one single reference to them. I was able to derive the particular subcase which results when the two dashpots are described by <u>one</u> single viscous constant η and both springs by the <u>same</u> two elastic constants λ_e and μ_e. The resulting field equations in this still very

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[5] C. L. Navier, (1827) Mémoire sur les lois de l'equilibre et du mouvement des corps solides élastiques. Mém. Acad. Sci. Paris 7.

general case are,

$$\text{iv)}\quad \mu_e\left[1-\delta_{t_1t_2}\right]\nabla^2\vec{u} + \left[\lambda_e+\mu_e-\frac{\mu_e}{3}\delta_{t_1t_2}\right]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) +$$

$$+ t_1\frac{\partial}{\partial t}\left\{(\lambda_e+\frac{2}{3}\mu_e)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \rho\ddot{\vec{u}}\right\} + \mu_e t_2\frac{\partial}{\partial t}\left\{\nabla^2\vec{u}+\frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})\right\} =$$

$$= \rho\ddot{\vec{u}}$$

where $\delta_{t_1t_2}$ is the Kronecker delta equal to one (zero) for $t_1 = t_2$ (or $t_1 \neq t_2$). Further t_1 (or t_2) equal η/μ_e. For $t_1 = 0$ and $t_2 = \eta/\mu_e$ (here $\eta = \mu_v$), eqs. (iv) reduce to the Kelvin model equations (i). When $t_1 = t_2 = \eta/\mu_e$ (which amounts to setting $2\beta = 1/\eta$ and $-3\alpha = 1/\eta$), equations (iv) reduce to the Maxwell model equations (ii) with $3\alpha + 2\beta = 0$. The quantities t_1 and t_2 are the retardation or relaxation times of the Kelvin and Maxwell models respectively. The constitutive (ie, stress-strain) relations of this viscoelastic model are also given in ref (6). I know of no viscoelastic boundary-value-problem that has ever been analytically solved using this model, which many agree is more realistic than (i) or (ii), since it can describe wider material behaviors and a wider variety of materials. Since the "standard viscoelastic model" is so hard to handle, we must realistically conclude that all analytical viscoelastic problems we are bound to see solved in the near future will be based on either the Kelvin or the Maxwell models of field equations (i) or (ii) respectively.

Occasionally we find a reference in the literature which contains a very complicated network such as a dozen "Maxwell elements" in parallel. Such a complicated network immediately implies that this is not a continuum field-theory approach, but rather that the body has been replaced by twelve coupled damped-oscillators. Work of this nature happens to be mostly experimental, chemical, and containing little mathematical analysis. These complicated networks are basically intended as pictorial descriptions without much physico-mathematical discussion of the response of the twelve coupled-oscillators, which, per se, is far from being a trivial problem. Sometimes one comes across an entire textbook dealing with various aspects of viscoelasticity without a single reference to the continuum approach, or the field-equations for the viscoelastic models. This old fashioned tendency is out-dated today. Mechanics of deformable media has become more highly mathematized now-a-days, than ever before.

II. SOLUTION OF THE FIELD EQUATIONS. (Tables 1 & 2)

We will now present a way to solve the field-equations of linear dynamic viscoelasticity (i) and (ii). The technique we will follow is common in other field-theories (ie, electro-dynamics) but we believe it is novel in viscoelasticity. It consists of introducing scalar and vector potentials ϕ and $\vec{\psi}$ such that $\vec{u} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{\psi}$ and $\vec{\nabla} \cdot \vec{\psi} = 0$. By splitting the displacement field $\vec{u}$ into irrotational and solenoidal parts in this fashion, it turns out that the field equations of each model (i) or (ii) are automatically satisfied provided that the scalar and vector potentials satisfy certain scalar and vector telegraph-type equations. The vector potential has three scalar components, but since the solenoidal gauge condition $\vec{\nabla} \cdot \vec{\psi} = 0$ must be satisfied, only two of the three scalar functions are independent. Those two (ψ and χ where ψ is <u>not</u> $|\vec{\psi}|$) together with the scalar potential ϕ, form the three independent scalar potentials that can be used to solve the problem. If these three independent scalar potentials satisfy three scalar telegraph-type equations then it can be shown that the field equations are automatically satisfied.

It is then possible to express all the stress and displacement components in terms of these independent potentials, which are determined first by solving the telegraph-type equations they must satisfy. In this fashion we can determine all the stresses and displacements in the body needed to completely solve the problem.

To discuss these solutions for the Kelvin-Voigt or Maxwell solids we have constructed two charts. (Tables 1 and 2). Table 1 deals with these viscoelastic models for the general case of arbitrary time-dependence. Table 2 covers the important case of harmonic time-dependence of the form exp $(-i\omega t)$. The coordinate systems covered in those tables are the cylindrical and the spherical. The cylindrical system is studied in general in columns B and C for the Kelvin or Maxwell models respectively. It is also studied in the (z-independent) <u>plane-strain</u> subcase, which is applicable to infinitely long cylinders, in columns D and E, for the Kelvin and Maxwell models, respectively. The spherical system is also presented in full generality in column F for the Kelvin model. It is also given in column G for the axi-symmetric case without azimuthal dependence ϕ, again for the Kelvin model. The Maxwell model is not covered for spherical coordinates in these tables. The basic result of these tables, particularly Table 2, for harmonic time-dependence, is that if we introduce independent scalar potentials which satisfy the Holmholtz's equations with complex propagation constants given in item (10), then the field equations given in items (7) are automatically satisfied for each of the cases considered. Furthermore, the displacement and stress-field components are found from the potentials by the relations in items (4) and (8) respectively. Notice that the complex propagation constants given in item (10) are related to the elastic and viscous constants of the viscoelastic material

TABLE 1 : EQUATIONS OF LINEAR DYNAMIC VISCOELASTICITY IN CYLINDRICAL AND SPHERICAL COORDINA

A) GENERAL FORMULAS VALID FOR ANY VISCOELASTIC MODEL IN ANY COORDINATE SYSTEM.	B) KELVIN VOIGT VISCOELASTIC MODEL (PARALLEL) FOR GENERAL 3 DIMENSIONAL CYLINDRICAL COORDINATES.
1) RELATION BETWEEN THE TRACES τ_{kk} AND ϵ_{kk}: $\tau_{kk} = \left(\frac{3P+2Q}{3R+2S}\right)\epsilon_{kk}$ τ_{kk} = HYDROSTATIC OR BULK STRESS , ϵ_{kk} = DILATATION.	1) RELATION BETWEEN THE TRACES τ_{kk} AND ϵ_{kk}: $\tau_{kk} = [(3\lambda_e+2\mu_e)+(3\lambda_v+2\mu_v)\frac{\partial}{\partial t}]\,\epsilon_{kk}$ λ_e, μ_e = ELASTIC (LAMÉ) CONSTANTS , λ_v, μ_v = VISCOUS CONSTANTS
2) CONSTITUTIVE RELATIONS: $R\delta_{ij}\tau_{kk} + 2S\tau_{ij} = P\delta_{ij}\epsilon_{kk} + 2Q\epsilon_{ij}$ OR, WHICH IS THE SAME, $\tau_{ij} = \frac{2(PS-RQ)}{2S(3R+2S)}\delta_{ij}\epsilon_{kk} + \frac{2Q}{2S}\epsilon_{ij}$, WHERE, $\epsilon_{kk} = \Delta = \vec{\nabla}\cdot\vec{u}$ & $R = \alpha + \sum_{k=1}^{r}\alpha_k\frac{\partial^k}{\partial t^k}$, $S = \beta + \sum_{k=1}^{r}\beta_k\frac{\partial^k}{\partial t^k}$, $P = \lambda + \sum_{k=1}^{r}\lambda_k\frac{\partial^k}{\partial t^k}$ & $Q = \mu + \sum_{k=1}^{r}\mu_k\frac{\partial^k}{\partial t^k}$.	2) CONSTITUTIVE RELATIONS: HERE, $R = 0$, $S = \frac{1}{2}$, $P = \lambda_e + \lambda_v\frac{\partial}{\partial t}$, $Q = \mu_e + \mu_v\frac{\partial}{\partial t}$ ∴ $\tau_{ij} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\delta_{ij}\epsilon_{kk} + 2(\mu_e + \mu_v\frac{\partial}{\partial t})\epsilon_{ij}$ CLEARLY, $\alpha = 0$, $\alpha_k = 0$ $\forall k = 1, 2, \ldots r$, $\beta = \frac{1}{2}$, $\beta_k = 0$ $\forall k = 1, 2, \ldots r$, $\lambda = \lambda_e$, $\mu = \mu_e$, $\lambda_1 = \lambda_v$, $\mu_1 = \mu_v$ $\lambda_k = 0$ $\forall k \neq 1$, $\mu_k = 0$ $\forall k \neq 1$.
3) STRAIN - DISPLACEMENT RELATIONS: $\epsilon_{ij} = u_{(i,j)} = \frac{1}{2}[u_{i,j} + u_{j,i}]$	3) STRAIN - DISPLACEMENT RELATIONS: $\epsilon_{rr} = \frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$, $\epsilon_{zz} = \frac{\partial u_z}{\partial z}$, $2\epsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}$
4) DISPLACEMENT - INDEPENDENT POTENTIAL RELATIONS: NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.	4) DISPLACEMENT - INDEPENDENT POTENTIAL RELATIONS: $\vec{u} = \vec{\nabla}(\varphi + \frac{\partial\psi}{\partial z}) - \hat{e}_z\nabla^2\psi + \hat{e}_z\times\vec{\nabla}(\nabla^2\chi) = \vec{\nabla}\varphi + \vec{\nabla}\times[(\vec{\nabla}\psi)\times\hat{e}_z$ OR IN COMPONENT FORM: $u_r = \frac{\partial}{\partial r}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)$, $u_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}[\varphi + \frac{\partial\psi}{\partial z}] + \partial$ HERE: $\varphi = \varphi(r,\theta,z,t)$, $\psi = \psi(r,\theta,z,t)$, $\chi = \chi(r,\theta,z,t)$, $\vec{\nabla} =$ NOTE: $\vec{u} = \vec{\nabla}\varphi + \vec{\nabla}\times\vec{\psi}$ & $\vec{\nabla}\cdot\vec{\psi} = 0$.
5) STRESS - DISPLACEMENT RELATIONS: $\tau_{ij} = \frac{2(PS-RQ)}{2S(3R+2S)}\delta_{ij}\Delta + \frac{2Q}{2S}\epsilon_{ij}$, WHERE $\Delta = u_{k,k} = \vec{\nabla}\cdot\vec{u}$ & $\epsilon_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}]$ AS IN (3).	5) STRESS - DISPLACEMENT RELATIONS. $\tau_{ij} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\delta_{ij}\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t})\epsilon_{ij}$ WHERE ϵ_{ij} ARE GIVEN IN 3) AND $\Delta = \vec{\nabla}\cdot\vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$.
6) STRAIN - INDEPENDENT POTENTIAL RELATIONS NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.	6) STRAIN INDEPENDENT POTENTIAL RELATIONS: $\epsilon_{rr} = \frac{\partial^2}{\partial r^2}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)] = \frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^3\psi}{\partial z\partial r^2} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\nabla^2\chi)$ $\epsilon_{\theta\theta} = [\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}][\varphi + \frac{\partial\psi}{\partial z}] + \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)] = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}$ $\epsilon_{zz} = \frac{\partial^2}{\partial z^2}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{\partial}{\partial z}(\nabla^2\psi) = \frac{\partial^2\varphi}{\partial z^2} - \frac{\partial}{\partial z}[\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r}$ $2\epsilon_{\theta z} = \frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi) + \frac{\partial^2}{\partial z\partial r}(\nabla^2\chi) = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z}$ $2\epsilon_{rz} = 2\frac{\partial^2}{\partial r\partial z}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{\partial}{\partial r}(\nabla^2\psi) - \frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi) = 2\frac{\partial^2\varphi}{\partial r\partial z} +$ $2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}\{\frac{1}{r}[\varphi + \frac{\partial\psi}{\partial z}]\} + [\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi) = 2\,\partial$
7) THE FIELD EQUATIONS: (FOR THE DISPLACEMENT FIELD) i): $[QR + 2S(P+Q)]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + Q(3R+2S)\nabla^2\vec{u} = 2S(3R+2S)\rho\ddot{\vec{u}}$ ii): $2[2QR + S(P+2Q)]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - Q(3R+2S)\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = 2S(3R+2S)\rho\ddot{\vec{u}}$ NOTE: $\nabla^2\vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{u})$ Q, R, S, P ARE OPERATORS DEPENDING ON THE MODEL.	7) FIELD EQUATIONS: i): $[\mu_e + \mu_v\frac{\partial}{\partial t}]\nabla^2\vec{u} + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) = \rho\frac{\partial^2\vec{u}}{\partial t^2}$ OR, IN COMPONENT FORM: $[\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}$ $[\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2}$ $[\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_z] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2}$ WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN 4) AND 5). ii): $[\lambda_e + 2\mu_e + (\lambda_v + 2\mu_v)\frac{\partial}{\partial t}]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - (\mu_e + \mu_v\frac{\partial}{\partial t})\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = \rho\frac{\partial^2\vec{u}}{\partial t^2}$ THESE TWO FORMS i) AND ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).
8) STRESS - INDEPENDENT POTENTIAL RELATIONS: NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.	8) STRESS - INDEPENDENT POTENTIAL RELATIONS: $\tau_{rr} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2(\mu_e + \mu_v\frac{\partial}{\partial t})[\frac{\partial^2}{\partial r^2}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$ $\tau_{\theta\theta} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2(\mu_e + \mu_v\frac{\partial}{\partial t})[(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r})(\varphi + \frac{\partial\psi}{\partial z}) + \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$ $\tau_{zz} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2(\mu_e + \mu_v\frac{\partial}{\partial t})[\frac{\partial^2}{\partial z^2}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial z}(\nabla^2\chi)]$ $\tau_{\theta z} = [\mu_e + \mu_v\frac{\partial}{\partial t}][\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi) + \frac{\partial^2}{\partial z\partial r}(\nabla^2\chi)]$ $\tau_{rz} = [\mu_e + \mu_v\frac{\partial}{\partial t}][2\frac{\partial^2}{\partial r\partial z}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial r}(\nabla^2\psi) - \frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi)]$ $\tau_{r\theta} = [\mu_e + \mu_v\frac{\partial}{\partial t}][2\frac{\partial^2}{\partial r\partial\theta}(\frac{1}{r}(\varphi + \frac{\partial\psi}{\partial z})) + (\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\nabla^2\chi]$ THESE ARE EQS. (5) WITH $\Delta = \nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQS. 6
9) TELEGRAPH TYPE EQS. FOR SCALAR & VECTOR POTENTIALS: i) $[4QR + 2S(P+2Q)]\nabla^2\varphi = 2S(3R+2S)\rho\frac{\partial^2\varphi}{\partial t^2}$	9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS i) $[1 + M\frac{\partial}{\partial t}]\nabla^2\varphi = \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2}$, $M = \frac{\lambda_v + 2\mu_v}{[illegible]}$, $c_1^2 = \frac{\lambda_e + 2\mu_e}{[illegible]}$

2

ND [illegible] COORDINATES FOR GENERAL [illegible] TRACE [illegible] IN VARIOUS [illegible] TER

ODEL (PARALLEL) FOR LINDRICAL COORDINATES	[illegible] MAXWELL [illegible] MODEL (SERIES) FOR GENERAL [illegible] DIMENSIONAL [illegible] COORDINATES	D) KELVIN VOIGT VISCOELASTIC MODEL (PARALLEL) IN C DRICAL COORDS FOR THE PLANE STRAIN CASE.
ES τ_{kk} AND ϵ_{kk} : $\mu_v)\frac{\partial}{\partial t}]\epsilon_{kk}$ S, λ_v, μ_v = VISCOUS CONSTANTS	1) RELATION BETWEEN THE TRACES τ_{kk} AND ϵ_{kk}: $\tau_{kk}=\frac{(3\lambda_e+2\mu_e)\frac{\partial}{\partial t}}{3\alpha+2\beta+\frac{\partial}{\partial t}}\epsilon_{kk}$ τ_{kk} = BULK STRESS & ϵ_{kk} = CUBICAL DILATATION = $\nabla\cdot\vec{u}$	1) DEFINITION OF THE PLANE STRAIN CASE: THIS SITUATION, WHICH IS APPROPRIATE FOR INFINITELY LO THUS, HERE THERE IS NO z-DEPENDENCE AND $u_r=u_r(r,\theta,t)$ HAVING $\varphi=\varphi(r,\theta,t)$, $\psi_r=0$, $\psi_\theta=0$ & $\psi_z=\psi_z(r,\theta,t)$ HAVING $\varphi=\varphi(r,\theta,t)$, $\psi=0$ & $\chi=\chi(r,\theta,t)$ NOTE TH & THE FORMULATION IS INDEPENDENT OF ψ. ANY OF THESE
$\frac{\partial}{\partial t}$, $Q=\mu_e+\mu_v\frac{\partial}{\partial t}$ ∴ $2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{ij}$ 1, 2, r, $\beta=\frac{1}{2}$, $=\mu_e$, $\lambda_1=\lambda_v$, $\mu_1=\mu_v$	2) CONSTITUTIVE RELATIONS. HERE, $R=\alpha$, $S=\beta+\frac{1}{2}\frac{\partial}{\partial t}$, $P=\lambda_e\frac{\partial}{\partial t}$, $Q=\mu_e\frac{\partial}{\partial t}$ ∴ $\tau_{ij}=\frac{\frac{\partial}{\partial t}}{2\beta+\frac{\partial}{\partial t}}\left\{\left[\frac{\lambda_e(2\beta+\frac{\partial}{\partial t})-2\alpha\mu_e}{3\alpha+2\beta+\frac{\partial}{\partial t}}\right]\delta_{ij}\epsilon_{kk}+2\mu_e\epsilon_{ij}\right\}$ CLEARLY, $\mu=0$, $\mu_1=\mu_e$, $\mu_k=0$ ∀k≠1, $\lambda=0$, $\lambda_1=\lambda_e$, $\lambda_k=0$ ∀k≠1 $\alpha\neq0$, $\alpha_k=0$ ∀k=1,2,…r, $\beta\neq0$, $\beta_1=0.5$, $\beta_k=0$ ∀k≠1.	2) CONSTITUTIVE RELATIONS: $\tau_{rr}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{rr}$ $\tau_{\theta\theta}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{\theta\theta}$ $\tau_{zz}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta$ HERE $\Delta=\epsilon$, $\epsilon_{rr}=\frac{\partial u_r}{\partial r}$ $\tau_{r\theta}=[\mu_e+\mu_v\frac{\partial}{\partial t}]2\epsilon_{r\theta}$
RELATIONS: $\epsilon_{zz}=\frac{\partial u_z}{\partial z}$, $2\epsilon_{\theta z}=\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}$, $2\epsilon_{rz}=\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}$, $2\epsilon_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}$		3) STRAIN DISPLACEMENT RELATIONS: $\epsilon_{rr}=\frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}$ [illegible], $2\epsilon_{r\theta}=\frac{1}{r}$
POTENTIAL RELATIONS $(\nabla^2\chi)=\vec\nabla\varphi+\vec\nabla\times[(\vec\nabla\psi)\times\hat e_z+\vec\nabla(\frac{\partial\chi}{\partial z})-\hat e_z\nabla^2\chi]=\vec\nabla\varphi+\vec\nabla\times[\vec\nabla\times(\psi\hat e_z)+\vec\nabla\times\{\vec\nabla\times(\hat e_z\chi)\}]$. $u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}[\varphi+\frac{\partial\chi}{\partial z}]+\frac{\partial}{\partial r}(\nabla^2\chi)$, $u_z=\frac{\partial}{\partial z}[\varphi+\frac{\partial\chi}{\partial z}]-\nabla^2\chi$ $z,t)$, $\chi=\chi(r,\theta,z,t)$, $\vec\nabla=\hat e_r\frac{\partial}{\partial r}+\hat e_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat e_z\frac{\partial}{\partial z}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$. & $\nabla\cdot\vec\psi=0$.		4) DISPLACEMENT-INDEPENDENT POTENTIAL RELATION [illegible] $u_r=\frac{\partial\varphi}{\partial r}-\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)$, $u_\theta=\frac{1}{r}\frac{\partial\varphi}{\partial\theta}+\frac{\partial}{\partial r}(\nabla^2\chi)$, $u_r=\frac{\partial\varphi}{\partial r}+\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}$, $u_\theta=\frac{1}{r}\frac{\partial\varphi}{\partial\theta}-\frac{\partial\psi_z}{\partial r}$, $u_z=$ HERE, $\varphi=\varphi(r,\theta,t)$, $\chi=\chi(r,\theta,t)$. $\vec\nabla=\hat e_r\frac{\partial}{\partial r}+\hat e_\theta$
LATIONS: $\mu_v\frac{\partial}{\partial t})\epsilon_{ij}$ WHERE $=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}$.	5) STRESS DISPLACEMENT RELATIONS: $\tau_{ij}=\frac{\frac{\partial}{\partial t}}{2\beta+\frac{\partial}{\partial t}}\left\{\left[\frac{\lambda_e(2\beta+\frac{\partial}{\partial t})-2\alpha\mu_e}{3\alpha+2\beta+\frac{\partial}{\partial t}}\right]\delta_{ij}\Delta+2\mu_e\epsilon_{ij}\right\}$ WHERE, ϵ_{ij} ARE GIVEN IN 3) AND $\Delta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}=\nabla\cdot\vec u$	5) STRESS-DISPLACEMENT RELATIONS: $\tau_{rr}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\frac{\partial u_r}{\partial r}$ $\tau_{\theta\theta}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta})$, $\tau_{zz}=[\lambda_e+$ $\tau_{r\theta}=[\mu_e+\mu_v\frac{\partial}{\partial t}](\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r})$, $\Delta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}$
TIAL RELATIONS $]=\frac{\partial^2\varphi}{\partial r^2}+\frac{\partial^3\chi}{\partial r^2\partial z}-\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\nabla^2\chi)+\frac{1}{r^2}\frac{\partial}{\partial\theta}(\nabla^2\chi)$. $\frac{\partial}{\partial\theta}(\nabla^2\chi)]+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial z}[\frac{\partial\chi}{\partial r}+\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2}]+\frac{1}{r}\frac{\partial}{\partial\theta}\{\frac{\partial}{\partial r}(\nabla^2\chi)-\frac{1}{r}\nabla^2\chi\}$. $\frac{\partial^2\varphi}{\partial z^2}-\frac{\partial}{\partial z}[\frac{\partial^2\chi}{\partial r^2}+\frac{1}{r}\frac{\partial\chi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}]$. $\psi)+\frac{\partial^2}{\partial z\partial r}(\nabla^2\chi)=\frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z}+\frac{1}{r}\frac{\partial}{\partial\theta}[2\frac{\partial^2\chi}{\partial z^2}-\nabla^2\psi]+\frac{\partial^2}{\partial r\partial z}(\nabla^2\chi)$. $-\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\nabla^2\chi)=2\frac{\partial^2\varphi}{\partial r\partial z}+\frac{\partial}{\partial r}\{2\frac{\partial^2\chi}{\partial z^2}-\nabla^2\psi\}-\frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}(\nabla^2\chi)$ $\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi)=2\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial\varphi}{\partial\theta}]-\frac{2}{r}\frac{\partial}{\partial\theta}[\frac{\partial^2\chi}{\partial r\partial z}-\frac{1}{r}\frac{\partial\chi}{\partial z}]+[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi)$. [THESE ARE EQUATIONS (4) SUBSTITUTED INTO EQS. (3).]		6) STRAIN-INDEPENDENT POTENTIAL RELATIONS: $\epsilon_{rr}=\frac{\partial}{\partial r}[\frac{\partial\varphi}{\partial r}-\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)]$, $\epsilon_{\theta\theta}=\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+$ THE OTHER STRAINS (ϵ_{zz}, ϵ_{zr}, $\epsilon_{z\theta}$) VANISH. IN THIS CASE, THE RESULT IS, $\epsilon_{rr}=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial^2\psi_z}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\psi_z}{\partial\theta}$, $\epsilon_{\theta\theta}=\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}-\frac{1}{r}$ THESE RELATIONS ARE FOUND BY SUBSTITUTING EQS
$+\mu_v)\frac{\partial}{\partial t}]\vec\nabla(\nabla\cdot\vec u)=\rho\frac{\partial^2\vec u}{\partial t^2}$ $+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r}=\rho\frac{\partial^2u_r}{\partial t^2}$ $+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta}=\rho\frac{\partial^2u_\theta}{\partial t^2}$ $\mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial z}=\rho\frac{\partial^2u_z}{\partial t^2}$ OVE IN 4) AND 5). $)-(\mu_e+\mu_v\frac{\partial}{\partial t})\nabla\times(\nabla\times\vec u)=\rho\frac{\partial^2\vec u}{\partial t^2}$ EQUIVALENT. ∇^2 GIVEN IN (4).	7) FIELD EQUATIONS: i) $[1+\frac{1}{2\beta}\frac{\partial}{\partial t}][\nabla^2\vec u+\frac{\lambda_e+\mu_e}{\mu_e}\nabla(\nabla\cdot\vec u)-\frac{1}{c_2^2}\frac{\partial^2\vec u}{\partial t^2}]+\frac{3\alpha}{2\beta}[\nabla^2\vec u+\frac{1}{3}\nabla(\nabla\cdot\vec u)-\frac{1}{c_2^2}\frac{\partial^2\vec u}{\partial t^2}]=\frac{1}{c_2^2}\frac{\partial^3\vec u}{\partial t^3}+\frac{3\alpha+2\beta}{c_2^2}\frac{\partial^2\vec u}{\partial t^2}$. ($c_2^2=\frac{\mu_e}{\rho}$) THIS CAN BE OPENED UP INTO COMPONENTS BY MEANS OF $\nabla^2\vec u=[\nabla^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}]\hat e_r+[\nabla^2u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}]\hat e_\theta+[\nabla^2u_z]\hat e_z$ $\nabla(\nabla\cdot\vec u)=\frac{\partial\Delta}{\partial r}\hat e_r+\frac{1}{r}\frac{\partial\Delta}{\partial\theta}\hat e_\theta+\frac{\partial\Delta}{\partial z}\hat e_z$ & Δ AS GIVEN ABOVE IN 5). ii) $[1+\frac{1}{2\beta}\frac{\partial}{\partial t}][(\frac{\lambda_e+2\mu_e}{\mu_e})\nabla(\nabla\cdot\vec u)-\nabla\times(\nabla\times\vec u)-\frac{1}{c_2^2}\frac{\partial^2\vec u}{\partial t^2}]+\frac{3\alpha}{2\beta}[\frac{4}{3}\nabla(\nabla\cdot\vec u)-\nabla\times(\nabla\times\vec u)-\frac{1}{c_2^2}\frac{\partial^2\vec u}{\partial t^2}]=\frac{1}{c_2^2}[\frac{\partial^3\vec u}{\partial t^3}+(3\alpha+2\beta)\frac{\partial^2\vec u}{\partial t^2}]$ THESE TWO FORMS i) AND ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).	7) FIELD EQUATIONS: i): $[\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r}=$ $[\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta}$ WHERE, $\Delta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}$ THE SECOND FORM (ii) IS NOT AS USEFUL
TIAL RELATIONS: $+\frac{\partial\chi}{\partial z})-\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$ $+\frac{1}{r}\frac{\partial}{\partial r}(\varphi+\frac{\partial\chi}{\partial z})+\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$ $(\varphi+\frac{\partial\chi}{\partial z})-\frac{\partial}{\partial z}(\nabla^2\chi)]$ $(\nabla^2\psi)+\frac{\partial^2}{\partial z\partial r}(\nabla^2\chi)]$ $\psi)-\frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi)]$ $-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\nabla^2\chi]$ AND ϵ_{ij} AS GIVEN IN EQS (6)	8) STRESS-INDEPENDENT POTENTIAL RELATIONS: $\tau_{rr}=M_1\{\lambda_e\nabla^2\varphi+2\mu_e[\frac{\partial^2}{\partial r^2}(\varphi+\frac{\partial\chi}{\partial z})-\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]\}+N_1$ $\tau_{\theta\theta}=M_1\{\lambda_e\nabla^2\varphi+2\mu_e[(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r})(\varphi+\frac{\partial\chi}{\partial z})+\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]\}+N_1$ $\tau_{zz}=M_1\{\lambda_e\nabla^2\varphi+2\mu_e[\frac{\partial^2}{\partial z^2}(\varphi+\frac{\partial\chi}{\partial z})-\frac{\partial}{\partial z}(\nabla^2\chi)]\}+N_1$ $\tau_{\theta z}=M_1\mu_e[\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}(\varphi+\frac{\partial\chi}{\partial z})-\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi)+\frac{\partial^2}{\partial z\partial r}(\nabla^2\chi)]$ $\tau_{rz}=M_1\mu_e[2\frac{\partial^2}{\partial r\partial z}(\varphi+\frac{\partial\chi}{\partial z})-\frac{\partial}{\partial r}(\nabla^2\psi)-\frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi)]$ $\tau_{r\theta}=M_1\mu_e[2\frac{\partial^2}{\partial r\partial\theta}\{\frac{1}{r}(\varphi+\frac{\partial\chi}{\partial z})\}+(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\nabla^2\chi)]$ WHERE, $M_1=\frac{\frac{\partial}{\partial t}(2\beta-\frac{\partial}{\partial t})}{4\beta^2-\frac{\partial^2}{\partial t^2}}$, $N_1=\frac{-\alpha(3\lambda_e+2\mu_e)}{(3\alpha+2\beta)^2-\frac{\partial^2}{\partial t^2}}\left[\frac{-(3\alpha+4\beta)\frac{\partial^2}{\partial t^2}+2\beta[(3\alpha+2\beta)\frac{\partial}{\partial t}+\frac{1}{2\beta}\frac{\partial^3}{\partial t^3}]}{4\beta^2-\frac{\partial^2}{\partial t^2}}\right]\nabla^2\varphi$ THESE ARE EQS. (5) WITH $\Delta=\nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQS. (6).	8) STRESS-INDEPENDENT POTENTIAL RELATIONS: $\tau_{rr}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi+2[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{\partial^2\varphi}{\partial r^2}-\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2$ $\tau_{\theta\theta}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi+2[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial}{\partial r}($ $\tau_{zz}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi$ $\tau_{r\theta}=[\mu_e+\mu_v\frac{\partial}{\partial t}][2\frac{\partial^2}{\partial r\partial\theta}(\frac{\varphi}{r})+(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\nabla^2\chi]$ TO HAVE THESE RELATIONS IN TERMS OF φ AND ψ_z REPLACE $\nabla^2\chi$ BY $-\psi_z$. ∇^2 IS GIVEN ABOVE IN THESE ARE EQS (5) WITH $\Delta=\nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQ
CALAR & VECTOR POTENTIALS	9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS:	9) TELEGRAPH EQS. FOR THE [illegible] POTENTIALS

3

RIOUS (…) CA… …TERE… T.

…T VISCOELASTIC MODEL (PARALLEL) IN CYLIN-…S. FOR THE PLANE-STRAIN SUBCASE.

E) MAXWELL VISCOELASTIC MODEL (SERIES) IN CYLINDRICAL COORDINATES FOR THE PLANE-STRAIN SUBCASE.

F) KELVIN-VOIGT VISCOELASTIC M…

…OF THE PLANE-STRAIN CASE:

…, WHICH IS APPROPRIATE FOR INFINITELY LONG BODIES (ie, CYLINDERS), OCCURS WHEN $u_z = 0$ AND $\frac{\partial}{\partial z}$(ANY VARIABLE) $= 0$. …RE IS NO z-DEPENDENCE AND $u_r = u_r(r,\theta,t)$, $u_\theta = u_\theta(r,\theta,t)$. IN TERMS OF φ, ψ_r, ψ_θ, ψ_z, PLANE-STRAIN AMOUNTS TO …(r,θ,t), $\psi_r = 0$, $\psi_\theta = 0$ & $\psi_z = \psi_z(r,\theta,t)$. OR IN TERMS OF THE INDEPENDENT POTENTIALS φ, ψ, χ, IT AMOUNTS TO …(r,θ,t), $\psi = 0$ & $\chi = \chi(r,\theta,t)$. NOTE THAT BY EQS. B.11-B, $\psi_z = -\nabla^2_{r\theta}\chi$. SO IN PLANE-STRAIN ONLY φ & χ APPEAR …TION IS INDEPENDENT OF ψ. ANY OF THESE EQUIVALENT DEFINITIONS YIELDS. $\epsilon_{zz} = 0$, $\epsilon_{zr} = 0$, & $\epsilon_{z\theta} = 0$.

…VE RELATIONS:

$\frac{\partial}{\partial t}]\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t})\epsilon_{rr}$

$\frac{\partial}{\partial t}]\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t})\epsilon_{\theta\theta}$

$\lambda_v\frac{\partial}{\partial t}]\Delta$ HERE $\Delta = \epsilon_{rr} + \epsilon_{\theta\theta} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$

$\frac{\partial}{\partial t}]\, 2\epsilon_{r\theta}$

…ISPLACEMENT RELATIONS:

$,\quad \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}, \quad 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad \epsilon_{zz} = 0, \quad 2\epsilon_{zr} = 0, \quad 2\epsilon_{z\theta} = 0$

(EVERY STRAIN-COMPONENT WITH SUBINDEX z, VANISHES.)

…ENT-INDEPENDENT POTENTIAL RELATIONS:

$\ldots + \hat{e}_z \cdot [\nabla_{r\theta}(\nabla^2_{r\theta}\chi)] = \nabla\varphi - \nabla\times[\hat{e}_z\nabla^2_{r\theta}\chi] = \vec{\nabla}\varphi + \vec{\nabla}\times[\vec{\nabla}\times\{\vec{\nabla}\times(\hat{e}_z\chi)\}]$ OR IN COMPONENT-FORM,

$\ldots\frac{\partial}{\partial\theta}(\nabla^2\chi), \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \frac{\partial}{\partial r}(\nabla^2\chi), \quad u_z = 0$ IN TERMS OF φ AND χ. OR,

$\ldots\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}, \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi_z}{\partial r}, \quad u_z = 0$ IN TERMS OF φ AND ψ_z. $(\psi_z = -\nabla^2_{r\theta}\chi)$

$\ldots(r,\theta,t), \quad \chi = \chi(r,\theta,t). \quad \vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$

…ISPLACEMENT RELATIONS:

$\ldots]\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t})\frac{\partial u_r}{\partial r}$

$\ldots]\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t})(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}), \quad \tau_{zz} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\Delta$

$\ldots](\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}), \quad \Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}.$

…EPENDENT POTENTIAL RELATIONS: (THESE ARE EQS (4) SUBSTITUTED INTO EQS. (3).)

$\ldots - \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)], \quad \epsilon_{\theta\theta} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)], \quad 2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}(\frac{\varphi}{r}) + (\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\nabla^2\chi$

…STRAINS (ϵ_{zz}, ϵ_{zr}, $\epsilon_{z\theta}$) VANISH. TO HAVE THESE RELATIONS IN TERMS OF φ AND ψ_z, WE REPLACE $\nabla^2\chi$ BY $-\psi_z$…, THE RESULT IS,

$\ldots\frac{\partial^2\psi_z}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi_z}{\partial\theta}, \quad \epsilon_{\theta\theta} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{1}{r}\frac{\partial^2\psi_z}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial\psi_z}{\partial\theta}, \quad 2\epsilon_{r\theta} = \frac{2}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{\partial^2\psi_z}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_z}{\partial\theta^2}.$

…TIONS ARE FOUND BY SUBSTITUTING EQS. 4) INTO EQS. 3).

…TIONS:

$\ldots[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}$

$\ldots[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2}$

$\ldots = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$

…FORM ii) IS NOT AS USEFUL.

…DEPENDENT POTENTIAL RELATIONS:

$\ldots\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e + \mu_v\frac{\partial}{\partial t}][\frac{\partial^2\varphi}{\partial r^2} - \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$

$\ldots\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e + \mu_v\frac{\partial}{\partial t}][\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$

$\ldots\frac{\partial}{\partial t}]\nabla^2\varphi$

$\ldots\frac{\partial}{\partial t}][2\frac{\partial^2}{\partial r\partial\theta}(\frac{\varphi}{r}) + (\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\nabla^2\chi]$

…SE RELATIONS IN TERMS OF φ AND ψ_z WE …χ BY $-\psi_z$. ∇^2 IS GIVEN ABOVE IN (7). …S. (8) WITH $\Delta = \nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQS. (6).

…EQS. FOR THE SCALAR & VECTOR POTENTIALS:

$\ldots\frac{1}{\ldots}\frac{\partial^2\varphi}{\ldots}, \quad ii)\ldots(1 + N\frac{\partial}{\ldots})\nabla^2\vec{\psi} = \frac{1}{\ldots}\frac{\partial^2\psi}{\partial t^2},$

2) CONSTITUTIVE RELATIONS:

$$\begin{cases} \tau_{rr} = M_1\{\lambda_e\Delta + 2\mu_e\epsilon_{rr}\} + N_1 \\ \tau_{\theta\theta} = M_1\{\lambda_e\Delta + 2\mu_e\epsilon_{\theta\theta}\} + N_1 \\ \tau_{zz} = M_1\{\lambda_e\Delta\} + N_1 \\ \tau_{r\theta} = M_1 2\mu_e\epsilon_{r\theta} \end{cases}$$

(HERE M_1 AND N_1 ARE THE SAME OPERATORS GIVEN IN C-8) OR E-8))

WHERE Δ IS GIVEN BELOW (ie, $\epsilon_{rr} + \epsilon_{\theta\theta}$) IN (5).

5) STRESS-DISPLACEMENT RELATIONS:

$$\begin{cases} \tau_{rr} = M_1\{\lambda_e\Delta + 2\mu_e[\frac{\partial u_r}{\partial r}]\} + N_1 \\ \tau_{\theta\theta} = M_1\{\lambda_e\Delta + 2\mu_e[\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}]\} + N_1 \\ \tau_{zz} = M_1\{\lambda_e\Delta\} + N_1 \\ \tau_{r\theta} = M_1\mu_e[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}], \quad \Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \end{cases}$$

(HERE M_1 & N_1 ARE THE SAME OPERATORS GIVEN IN C-8) or E-8…

7) FIELD EQUATIONS:

i)

$$[3\alpha + 2\beta + \frac{\partial}{\partial t}][\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}] + [\alpha + \frac{\lambda_e + \mu_e}{\mu_e}(2\beta + \frac{\partial}{\partial t})]\frac{\partial\Delta}{\partial r} = \frac{1}{c_s^2}[(3\alpha + 4\beta + \frac{\partial}{\partial t})\frac{\partial^2 u_r}{\partial t^2} + 2\beta(2\beta + 3\alpha)\frac{\partial u_r}{\partial t}].$$

$$[3\alpha + 2\beta + \frac{\partial}{\partial t}][\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}] + [\alpha + \frac{\lambda_e + \mu_e}{\mu_e}(2\beta + \frac{\partial}{\partial t})]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \frac{1}{c_s^2}[(3\alpha + 4\beta + \frac{\partial}{\partial t})\frac{\partial^2 u_\theta}{\partial t^2} + 2\beta(2\beta + 3\alpha)\frac{\partial u_\theta}{\partial t}].$$

WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN (4) & (5). ($c_s^2 = \mu_e/\rho$.)
FORM ii) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{cases} \tau_{rr} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[\frac{\partial^2\varphi}{\partial r^2} - \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]\} + N_1 \\ \tau_{\theta\theta} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]\} + N_1 \\ \tau_{zz} = M_1\{\lambda_e\nabla^2\varphi\} + N_1, \quad \tau_{\theta z} = 0, \quad \tau_{zr} = 0, \\ \tau_{r\theta} = M_1\mu_e\{2\frac{\partial^2}{\partial r\partial\theta}(\frac{\varphi}{r}) + [\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi)\} \end{cases}$$

WHERE,

$$M_1 = \frac{\frac{\partial}{\partial t}(2\beta - \frac{\partial}{\partial t})}{4\beta^2 - \frac{\partial^2}{\partial t^2}},$$

(TO HAVE THESE IN TERMS OF φ & ψ_z, WE REPLACE $\nabla^2\chi$ BY $-\psi_z$)

$$N_1 = -\left[\frac{\alpha(3\lambda_e + 2\mu_e)}{(3\alpha + 2\beta)^2 - \frac{\partial^2}{\partial t^2}}\right]\left[\frac{-(3\alpha + 4\beta)\frac{\partial^2}{\partial t^2} + 2\beta[(3\alpha + 2\beta)\frac{\partial}{\partial t} + \frac{1}{2\beta}\frac{\partial^3}{\partial t^3}]}{4\beta^2 - \frac{\partial^2}{\partial t^2}}\right]\nabla^2\varphi$$

9) TELEGRAPH EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i) $[2\beta + 4\alpha\frac{\lambda_e}{\ldots} + \frac{\partial}{\partial t}]\nabla^2\varphi = \frac{1}{c_s^2}\{[4\beta + 3\alpha + \frac{\partial}{\partial t}]\frac{\partial^2\varphi}{\partial t^2} + 2\beta(3\alpha + 2\beta)\frac{\partial\varphi}{\partial t}\}$

1) RELATION BETWEEN THE TRAC…

$\tau_{kk} = [(3\lambda_e + \ldots$

τ_{kk} = BULK OR HYDROSTATIC ST…

2) CONSTITUTIVE RELATIONS:

$\tau_{ij} = (\lambda_e + \lambda_v\frac{\partial}{\partial t})\delta_{ij}\epsilon_{kk} + 2(\mu_e + \mu_v\frac{\partial}{\partial t}\ldots$

$$\begin{cases} \tau_{rr} = (\lambda_e + \lambda_v\frac{\partial}{\partial t})\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t}\ldots \\ \tau_{\phi\phi} = (\lambda_e + \lambda_v\frac{\partial}{\partial t})\Delta + 2(\mu_e + \mu_v\frac{\partial}{\partial t}\ldots \\ \tau_{r\phi} = 2(\mu_e + \mu_v\frac{\partial}{\partial t})\epsilon_{r\phi} \end{cases}$$

3) STRAIN-DISPLACEMENT RELAT…

$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad \epsilon_{\phi\phi} = \ldots$

4) DISPLACEMENT-INDEPENDENT P…

$\vec{u} = \nabla[\varphi + \psi + \vec{r}\cdot\vec{\nabla}\psi] - \vec{r}\nabla^2\ldots$

$$\begin{cases} u_r = \frac{\partial}{\partial r}[\varphi + \frac{\partial}{\partial r}(r\psi)] - r\nabla^2\psi = \ldots \\ u_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}[\varphi + \frac{\partial}{\partial r}(r\psi)] - \frac{1}{\sin\theta}\frac{\partial}{\ldots} \\ u_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[\varphi + \frac{\partial}{\partial r}(r\psi)] + \frac{\partial}{\partial\theta}(\ldots \end{cases}$$

5) STRESS-DISPLACEMENT RELAT…

$\tau_{ij} = (\lambda_e + \lambda_v\frac{\partial}{\partial t})\delta_{ij}\Delta + 2(\mu_e + \mu_v\ldots$

WHERE THE ϵ_{ij} ARE GIVEN ABOVE…

6) STRAIN-INDEPENDENT POTENT…

$\epsilon_{rr} = \frac{\partial^2}{\partial r^2}[\varphi + \frac{\partial}{\partial r}(r\psi)] + \frac{\partial}{\partial r}[r\nabla^2\psi] = \ldots$

$\epsilon_{\theta\theta} = [\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}][\varphi + \frac{\partial}{\partial r}(r\psi)] - \nabla^2\ldots$

$\epsilon_{\phi\phi} = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} + \ldots$

$2\epsilon_{\theta\phi} = \frac{2}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\theta\partial\phi} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\varphi}{\partial\phi} + \frac{2}{r^2\sin\theta}\ldots$

$2\epsilon_{r\phi} = \frac{2}{r\sin\theta}\frac{\partial^2\varphi}{\partial r\partial\phi} - \frac{2}{r^2\sin\theta}\frac{\partial\varphi}{\partial\phi} - \frac{1}{r\sin\theta}\ldots$

$2\epsilon_{r\theta} = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial r} - \frac{2}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^3} - \frac{1}{r^2\sin^2\theta}\ldots$

THESE ARE FOUND BY SUBSTITUTING E…

7) FIELD EQUATIONS.

i) $[\mu_e + \mu_v\frac{\partial}{\partial t}]\nabla^2\vec{u} + [\lambda_e + \mu\ldots$

$$\begin{cases} [\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \ldots \\ [\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta}\ldots \\ [\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\ldots \end{cases}$$

ii) $[\lambda_e + 2\mu_e + (\lambda_v + 2\mu_v)\frac{\partial}{\partial t}]\vec{\nabla}(\ldots$

THE DILATATION Δ IS GIVEN IN …

8) STRESS-INDEPENDENT POTE…

$\tau_{rr} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e + \mu_v\ldots$

$\tau_{\theta\theta} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e + \mu_v\ldots$

$\tau_{\phi\phi} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e + \mu_v\ldots$

$\tau_{\theta\phi} = [\mu_e + \mu_v\frac{\partial}{\partial t}][\frac{2}{r}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi})(\ldots$

$\tau_{r\phi} = [\mu_e + \mu_v\frac{\partial}{\partial t}][\frac{2}{\sin\theta}\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\phi}(\varphi\ldots$

$\tau_{r\theta} = [\mu_e + \mu_v\frac{\partial}{\partial t}][2\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi + \ldots$

9) TELEGRAPH EQUATIONS FOR…

i) $(1 + M\frac{\partial}{\partial t})\nabla^2\varphi = \frac{1}{c_p^2}\frac{\partial^2\varphi}{\partial t^2}$

4

C MO VISCOELASTIC MODEL (PARALLEL) IN GENERAL SPHERICAL COORDINATES

TRAC WEEN THE TRACES τ_{kk} AND ϵ_{kk} OF THE STRESS AND STRAIN TENSORS.

$3\lambda_e +$ $$\tau_{kk} = [(3\lambda_e+2\mu_e)+(3\lambda_v+2\mu_v)\frac{\partial}{\partial t}]\epsilon_{kk}$$

STR HYDROSTATIC STRESS , ϵ_{kk} = CUBICAL DILATATION = Δ

S: E RELATIONS:

$\mu_v\frac{\partial}{\partial t})$ $)\delta_{ij}\epsilon_{kk}+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{ij}$ WHERE $\Delta=\epsilon_{kk}=\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{\phi\phi}=u_{k,k}$ OR,

$\mu_v\frac{\partial}{\partial t})$ $)\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{rr}$, $\tau_{\theta\theta}=(\lambda_e+\lambda_v\frac{\partial}{\partial t})\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{\theta\theta}$

$\mu_v\frac{\partial}{\partial t})$ $)\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{\phi\phi}$, $\tau_{\theta\phi}=2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{\theta\phi}$

$\frac{\partial}{\partial t})\epsilon_{r\phi}$, $\tau_{r\theta}=2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{r\theta}$.

LATIC CEMENT RELATIONS:

$\epsilon_{\phi\phi}=$ $\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$, $\epsilon_{\phi\phi}=\frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$, $2\epsilon_{\theta\phi}=\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{u_\phi}{r}\cot\theta$, $2\epsilon_{r\phi}=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}$, $2\epsilon_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}$

NT PO INDEPENDENT POTENTIAL RELATIONS:

$\nabla^2\psi$ $+\vec{r}\cdot\vec{\nabla}\psi]-\vec{r}\nabla^2\psi+\vec{r}\times[\vec{\nabla}(\nabla^2\chi)]$ OR IN COMPONENT-FORM,

$\psi=$ $(r\psi)]-r\nabla^2\psi=\frac{\partial\varphi}{\partial r}-\frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2}-\frac{\cot\theta}{r}\frac{\partial\psi}{\partial\theta}-\frac{1}{r\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$.

$\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}$ $\frac{\partial}{\partial r}(r\psi)]-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)=\frac{1}{r}\frac{\partial\varphi}{\partial\theta}+\frac{\partial^2\psi}{\partial r\partial\theta}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)$.

$\frac{\partial}{\partial\theta}($ $[\varphi+\frac{\partial}{\partial r}(r\psi)]+\frac{\partial}{\partial\theta}(\nabla^2\chi)=\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}+\frac{\partial}{\partial\theta}(\nabla^2\chi)$.

LATI ACEMENT RELATIONS:

$+\mu_v\frac{\partial}{\partial t}$ $)\delta_{ij}\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{ij}$. $\Delta=\vec{\nabla}\cdot\vec{u}=$ [illegible] $+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\theta}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$.

BOVE ARE GIVEN ABOVE IN 3) IN TERMS OF THE DISPLACEMENT-FIELD COMPONENTS.

ENTI ENDENT POTENTIAL RELATIONS:

$=\frac{\partial^2}{\partial r}$ $+\frac{\partial}{\partial r}[r\nabla^2\psi]=\frac{\partial^2\varphi}{\partial r^2}-\frac{1}{r\sin^2\theta}\frac{\partial^3\psi}{\partial r\partial\phi^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}-\frac{\cot\theta}{r}\frac{\partial^2\psi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial^3\psi}{\partial r\partial\theta^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$.

$\nabla^2\psi$ $[\varphi+\frac{\partial}{\partial r}(r\psi)]-\nabla^2\psi-\frac{1}{r}\frac{\partial}{\partial\theta}[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)]=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial^3\psi}{\partial r\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$
$-\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}(\nabla^2\chi)+\frac{\cot\theta}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)$.

$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{1}{r\sin^2\theta}\frac{\partial^3\psi}{\partial r\partial\phi^2}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\cot\theta}{r}\frac{\partial^2\psi}{\partial r\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi\partial\theta}(\nabla^2\chi)-\frac{\cot\theta}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)$.

[illegible] $+\frac{2}{r\sin\theta}\frac{\partial^2\psi}{\partial\theta\partial\phi}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\phi}+\frac{2}{r\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta\partial\phi}-\frac{2\cot\theta}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi}+[\frac{1}{r}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r}\frac{\partial}{\partial\theta}-\frac{1}{r\sin^2\theta}\frac{\partial^2}{\partial\phi^2}](\nabla^2\chi)$.

[illegible] $-\frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\phi}-\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi}-\frac{\cot\theta}{r\sin\theta}\frac{\partial^2\psi}{\partial\theta\partial\phi}-\frac{2}{r^2\sin\theta}\frac{\partial\psi}{\partial\phi}-[\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{\partial^2}{\partial r\partial\theta}](\nabla^2\chi)$.

$-\frac{1}{r}\frac{\partial^3\psi}{\partial\theta^3}-\frac{1}{r\sin^2\theta}\frac{\partial^3\psi}{\partial\theta\partial\phi^2}+\frac{\partial^3\psi}{\partial r^2\partial\theta}-\frac{\cot\theta}{r}\frac{\partial^2\psi}{\partial\theta^2}+\frac{2\cot\theta}{r\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}+[\frac{1}{r\sin^2\theta}-\frac{2}{r}]\frac{\partial\psi}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}[\frac{\partial}{\partial r}-\frac{1}{r}](\nabla^2\chi)$.

SUBSTITUTING EQS (6) INTO EQS. (3).

ONS.

$]\nabla^2\vec{u}+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\vec{\nabla}(\vec{\nabla}\cdot\vec{u})=\rho\frac{\partial^2\vec{u}}{\partial t^2}$ OR IN COMPONENT-FORM,

$-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r}=\rho\frac{\partial^2u_r}{\partial t^2}$

$+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta}=\rho\frac{\partial^2u_\theta}{\partial t^2}$

$-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\phi}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}=\rho\frac{\partial^2u_\phi}{\partial t^2}$

$](\vec{\nabla}$ $(\lambda_v+2\mu_v)\frac{\partial}{\partial t}]\vec{\nabla}(\vec{\nabla}\cdot\vec{u})-[\mu_e+\mu_v\frac{\partial}{\partial t}]\vec{\nabla}\times(\vec{\nabla}\times\vec{u})=\rho\frac{\partial^2\vec{u}}{\partial t^2}$

IN 8) Δ IS GIVEN IN 8) ABOVE AND, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$.

OTEN ENDENT POTENTIAL RELATIONS:

$\nabla^2\varphi+2[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{\partial^2}{\partial r^2}(\varphi+\frac{\partial}{\partial r}(r\psi))-\frac{\partial}{\partial r}(r\nabla^2\psi)]$

$\nabla^2\varphi+2[\mu_e+\mu_v\frac{\partial}{\partial t}][(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\varphi+\frac{\partial}{\partial r}(r\psi))-\nabla^2\psi-\frac{1}{r}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi))]$

$\nabla^2\varphi+2[\mu_e+\mu_v\frac{\partial}{\partial t}][(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2})(\varphi+\frac{\partial}{\partial r}(r\psi))-\nabla^2\psi+\frac{\partial}{\partial\theta}(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi))]$.

$[\frac{2}{r^2}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi})(\varphi+\frac{\partial}{\partial r}(r\psi))+(\frac{1}{r}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r}\frac{\partial}{\partial\theta}-\frac{1}{r\sin^2\theta}\frac{\partial^2}{\partial\phi^2})(\nabla^2\chi)]$.

$\frac{2}{r\sin\theta}\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\phi}(\varphi+\frac{\partial}{\partial r}(r\psi))\}-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\psi)+\frac{\partial}{\partial\theta}(\frac{\partial}{\partial r}-\frac{1}{r})(\nabla^2\chi)]$. (THESE ARE EQS. 6) SUBSTITUTED INTO EQS 2).)

$(\varphi+$ $[2\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi+\frac{\partial}{\partial r}(r\psi))\}-\frac{\partial}{\partial\theta}(\nabla^2\psi)-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\frac{\partial}{\partial r}-\frac{1}{r})(\nabla^2\chi)]$.

OR ATIONS FOR THE SCALAR AND VECTOR POTENTIALS φ AND $\vec{\psi}$:

$=\frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $(1+N\frac{\partial}{\partial t})\vec{\nabla}\times(\vec{\nabla}\times\vec{\psi})+\frac{1}{c_2^2}\frac{\partial^2\vec{\psi}}{\partial t^2}=0$ WHERE,

6) KELVIN-VOIGT VISCOELASTIC MODEL (PARALLEL) IN SPHERICAL COORDS. FOR THE AXIALLY SYMMETRIC SPHERICAL SUBCASE

1) DEFINITION OF CASE: THIS SUBCASE OCCURS WHEN $u_\phi=0$ AND $\frac{\partial}{\partial\phi}$[ANY VARIABLE] = 0. HENCE, $u_r=u_r(r,\theta,t)$, $u_\theta=u_\theta(r,\theta,t)$. IN TERMS OF φ AND $\psi_r,\psi_\theta,\psi_\phi$ THIS AMOUNTS TO HAVING $\varphi=\varphi(r,\theta,t)$, $\psi_r=0$, $\psi_\theta=0$, $\psi_\phi=\psi_\phi(r,\theta,t)$. IN TERMS OF φ, ψ & χ THIS IS EQUIVALENT TO HAVING $\varphi=\varphi(r,\theta,t)$, $\psi=\psi(r,\theta,t)$, $\chi=0$. [HERE $\psi_\phi=-\frac{\partial\psi}{\partial\theta}$]. THIS YIELDS. $\epsilon_{\theta\phi}=\epsilon_{r\phi}=0$ BUT $\epsilon_{\phi\phi}\neq0$. THIS IS NOT PLANE-STRAIN.

2) CONSTITUTIVE RELATIONS:

$$\begin{cases}\tau_{rr}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{rr}\\ \tau_{\theta\theta}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{\theta\theta}\\ \tau_{\phi\phi}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2(\mu_e+\mu_v\frac{\partial}{\partial t})\epsilon_{\phi\phi}\\ \tau_{r\theta}=[\mu_e+\mu_v\frac{\partial}{\partial t}]2\epsilon_{r\theta}\end{cases}\quad,\quad \tau_{\theta\phi}=\tau_{r\phi}=0 \quad \Delta=\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{\phi\phi}.$$

3) STRAIN-DISPLACEMENT RELATIONS: $(2\epsilon_{\theta\phi}=2\epsilon_{r\phi}=0)$

$$\epsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \epsilon_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad \epsilon_{\phi\phi}=\frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta,\quad 2\epsilon_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}$$

4) DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS:

$$\vec{u}=\vec{\nabla}_{r\theta}(\varphi+\vec{r}\cdot\vec{\nabla}_{r\theta}\psi)-\vec{r}\,\nabla^2_{r\theta}\psi \quad , \quad \text{OR}, \quad (\vec{r}=r\hat{e}_r.)$$

$$\begin{cases}u_r=\frac{\partial}{\partial r}[\varphi+\frac{\partial}{\partial r}(r\psi)]-r\nabla^2_{r\theta}\psi=\frac{\partial\varphi}{\partial r}-(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cot\theta}{r})\frac{\partial\psi}{\partial\theta}\\ u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}[\varphi+\frac{\partial}{\partial r}(r\psi)]=\frac{1}{r}\frac{\partial\varphi}{\partial\theta}+(\frac{\partial}{\partial r}+\frac{1}{r})\frac{\partial\psi}{\partial\theta}\\ u_\phi=0\end{cases}$$

5) STRESS-DISPLACEMENT RELATIONS:

$$\begin{cases}\tau_{rr}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2[\mu_e+\mu_v\frac{\partial}{\partial t}]\frac{\partial u_r}{\partial r}\\ \tau_{\theta\theta}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}]\\ \tau_{\phi\phi}=[\lambda_e+\lambda_v\frac{\partial}{\partial t}]\Delta+2[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta]\\ \tau_{r\theta}=[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}]\end{cases}$$ [Δ AS IN 7)]

6) STRAIN-INDEPENDENT POTENTIAL RELATIONS:

$\epsilon_{rr}=\frac{\partial^2}{\partial r^2}[\varphi+\frac{\partial}{\partial r}(r\psi)]-\frac{\partial}{\partial r}[r\nabla^2_{r\theta}\psi]=\frac{\partial^2\varphi}{\partial r^2}-[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}+\frac{\cot\theta}{r}\frac{\partial}{\partial r}-\frac{\cot\theta}{r^2}]\frac{\partial\psi}{\partial\theta}$.

$\epsilon_{\theta\theta}=[\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}][\varphi+\frac{\partial}{\partial r}(r\psi)]-\nabla^2_{r\theta}\psi=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}-(\frac{\cot\theta}{r^2}-\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta})\frac{\partial\psi}{\partial\theta}$.

$\epsilon_{\phi\phi}=[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}][\varphi+\frac{\partial}{\partial r}(r\psi)]-\nabla^2_{r\theta}\psi=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta}-(\frac{1}{r^2}\frac{\partial}{\partial\theta}-\frac{\cot\theta}{r}\frac{\partial}{\partial r})\frac{\partial\psi}{\partial\theta}$.

$2\epsilon_{\theta\phi}=0$, $2\epsilon_{r\phi}=0$,

$2\epsilon_{r\theta}=2\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi+\frac{\partial}{\partial r}(r\psi))]-\frac{\partial}{\partial\theta}(\nabla^2_{r\theta}\psi)=\frac{2}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\varphi}{\partial\theta}+$
$+(\frac{\partial^2}{\partial r^2}-\frac{2}{r^2}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta})\frac{\partial\psi}{\partial\theta}$.

7) FIELD EQUATIONS:

i) $[\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2_{r\theta}u_r-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2u_\theta\cot\theta}{r^2}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r}=$
$=\rho\frac{\partial^2u_r}{\partial t^2}$

$[\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2_{r\theta}u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}]+[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta}=\rho\frac{\partial^2u_\theta}{\partial t^2}$.

WHERE, $\nabla^2_{r\theta}=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}$

$\Delta=\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\theta}{r}\cot\theta$

FORM ii) WILL NOT BE GIVEN HERE.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr}=(\lambda_e+\lambda_v\frac{\partial}{\partial t})\nabla^2\varphi+2(\mu_e+\mu_v\frac{\partial}{\partial t})[\frac{\partial^2}{\partial r^2}(\varphi+\frac{\partial}{\partial r}(r\psi))-\frac{\partial}{\partial r}(r\nabla^2_{r\theta}\psi)]$

$\tau_{\theta\theta}=(\lambda_e+\lambda_v\frac{\partial}{\partial t})\nabla^2\varphi+2(\mu_e+\mu_v\frac{\partial}{\partial t})[(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\varphi+\frac{\partial}{\partial r}(r\psi))-\nabla^2_{r\theta}\psi]$

$\tau_{\phi\phi}=(\lambda_e+\lambda_v\frac{\partial}{\partial t})\nabla^2\varphi+2(\mu_e+\mu_v\frac{\partial}{\partial t})[(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta})(\varphi+\frac{\partial}{\partial r}(r\psi))-\nabla^2_{r\theta}\psi]$

$\tau_{\theta\phi}=0$, $\tau_{r\phi}=0$,

$\tau_{r\theta}=(\mu_e+\mu_v\frac{\partial}{\partial t})[2\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi+\frac{\partial}{\partial r}(r\psi))\}-\frac{\partial}{\partial\theta}(\nabla^2_{r\theta}\psi)]$.

ALTHOUGH $\psi_\phi=-\frac{\partial\psi}{\partial\theta}$ IT IS NOT CONVENIENT TO EXPRESS NEITHER THESE RELATIONS NOR G-4) IN TERMS OF φ AND ψ_ϕ.
(THESE ARE EQS. 6) SUBSTITUTED INTO EQS. 2)) $(\Delta=\nabla^2_{r\theta}\varphi)$.

9) TELEGRAPH EQS. FOR SCALAR & VECTOR POTENTIALS φ & $\vec{\psi}$:

i) $(1+M\frac{\partial}{\partial t})\nabla^2_{r\theta}\varphi=\frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $(1+N\frac{\partial}{\partial t})\vec{\nabla}\times(\vec{\nabla}\times\vec{\psi})+\frac{1}{c_2^2}\frac{\partial^2\vec{\psi}}{\partial t^2}=0$

NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM	$2\epsilon_{\theta z} = \frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi) + \frac{\partial^2}{\partial z\partial r}(\nabla^2\chi) = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial}{\partial\theta}[2$ $2\epsilon_{rz} = 2\frac{\partial^2}{\partial r\partial z}[\varphi + \frac{\partial\psi}{\partial z}] - \frac{\partial}{\partial r}(\nabla^2\psi) - \frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi) = 2\frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial}{\partial r}\{2\frac{\partial^2\psi}{\partial z^2}$ $2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}\{\frac{1}{r}[\varphi + \frac{\partial\psi}{\partial z}]\} + [\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi) = 2\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial\varphi}{\partial\theta}] +$	
7) <u>THE FIELD EQUATIONS</u> : (FOR THE DISPLACEMENT FIELD) i): $[QR + 2S(P+Q)]\vec\nabla(\vec\nabla\cdot\vec u) + Q(3R+2S)\nabla^2\vec u = 2S(3R+2S)\rho\ddot{\vec u}$ ii): $2[2QR + S(P+2Q)]\vec\nabla(\vec\nabla\cdot\vec u) - Q(3R+2S)\vec\nabla\times(\vec\nabla\times\vec u) = 2S(3R+2S)\rho\ddot{\vec u}$ *NOTE*: $\nabla^2\vec u = \vec\nabla(\vec\nabla\cdot\vec u) - \vec\nabla\times(\vec\nabla\times\vec u)$ Q, R, S, P ARE OPERATORS DEPENDING ON THE MODEL.	7) <u>FIELD EQUATIONS</u> : i): $[\mu_e + \mu_v\frac{\partial}{\partial t}]\nabla^2\vec u + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\vec\nabla(\vec\nabla\cdot\vec u) = \rho\frac{\partial^2\vec u}{\partial t^2}$ OR, IN COMPONENT FORM: $\left\{\begin{array}{l}[\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}\\ [\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2}\\ [\mu_e + \mu_v\frac{\partial}{\partial t}][\nabla^2 u_z] + [\lambda_e + \mu_e + (\lambda_v + \mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2}\end{array}\right.$ WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN 4) AND 5). ii): $[\lambda_e + 2\mu_e + (\lambda_v + 2\mu_v)\frac{\partial}{\partial t}]\vec\nabla(\vec\nabla\cdot\vec u) - (\mu_e + \mu_v\frac{\partial}{\partial t})\vec\nabla\times(\vec\nabla\times\vec u) = \rho\frac{\partial^2\vec u}{\partial t^2}$ THESE TWO FORMS i) AND ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).	7) <u>FIELD</u> i) $[1+$ $+\frac{1}{3}$ THIS CA $\nabla^2\vec u = [\nabla$ $\vec\nabla(\vec\nabla\cdot\vec u)$ ii) $[1+$ $+\frac{3g}{2\beta}[$ THESE T
8) <u>STRESS - INDEPENDENT POTENTIAL RELATIONS</u>: NO GENERAL EXPRESSION CAN BE GIVEN VALID IN ANY COORD. SYSTEM.	8) <u>STRESS - INDEPENDENT POTENTIAL RELATIONS</u>: $\tau_{rr} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2(\mu_e + \mu_v\frac{\partial}{\partial t})[\frac{\partial^2}{\partial r^2}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$ $\tau_{\theta\theta} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2(\mu_e + \mu_v\frac{\partial}{\partial t})[(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r})(\varphi + \frac{\partial\psi}{\partial z}) + \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$ $\tau_{zz} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2(\mu_e + \mu_v\frac{\partial}{\partial t})[\frac{\partial^2}{\partial z^2}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial z}(\nabla^2\chi)]$ $\tau_{\theta z} = [\mu_e + \mu_v\frac{\partial}{\partial t}][\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi) + \frac{\partial^2}{\partial z\partial r}(\nabla^2\chi)]$ $\tau_{rz} = [\mu_e + \mu_v\frac{\partial}{\partial t}][2\frac{\partial^2}{\partial r\partial z}(\varphi + \frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial r}(\nabla^2\psi) - \frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi)]$ $\tau_{r\theta} = [\mu_e + \mu_v\frac{\partial}{\partial t}][2\frac{\partial^2}{\partial r\partial\theta}(\frac{1}{r}[\varphi + \frac{\partial\psi}{\partial z}]) + (\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\nabla^2\chi]$ THESE ARE EQS. (5) WITH $\Delta = \nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQS.(6)	8) <u>STRES</u> $\tau_{rr} = M_1\{$ $\tau_{\theta\theta} = M_1\{$ $\tau_{zz} = M_1\{$ $\tau_{\theta z} = M_1$ $\tau_{rz} = M_1$ $\tau_{r\theta} = M_1$ $M_1 = \frac{\frac{\partial}{\partial t}(2}{4\beta^2-}$ THESE A
9) <u>TELEGRAPH - TYPE EQS. FOR SCALAR & VECTOR POTENTIALS.</u> i) $[4QR + 2S(P+2Q)]\nabla^2\varphi = 2S(3R+2S)\rho\frac{\partial^2\varphi}{\partial t^2}$ ii) $-Q[\vec\nabla\times(\vec\nabla\times\vec\psi)] = 2S\rho\frac{\partial^2\vec\psi}{\partial t^2}$ EQ. ii) CAN ALSO BE WRITTEN AS FOLLOWS: $\nabla^2\vec\psi = \frac{2S\rho}{Q}\frac{\partial^2\vec\psi}{\partial t^2}$ AND $\vec\nabla\cdot\vec\psi = 0$.	9) <u>TELEGRAPH - TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS</u>: i) $[1 + M\frac{\partial}{\partial t}]\nabla^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, $M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $c_d^2 = \frac{\lambda_e + 2\mu_e}{\rho}$ ii) $-[1 + N\frac{\partial}{\partial t}]\vec\nabla\times(\vec\nabla\times\vec\psi) = \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}$, $N = \frac{\mu_v}{\mu_e}$, $c_s^2 = \frac{\mu_e}{\rho}$. EQ. ii) CAN ALSO BE WRITTEN AS FOLLOWS: $[1 + N\frac{\partial}{\partial t}]\nabla^2\vec\psi = \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}$ AND $\vec\nabla\cdot\vec\psi = 0$.	9) <u>TELEG</u> i) $[2\beta$ EQ. ii) ∇^2
10) <u>TELEGRAPH - TYPE EQS. FOR INDEP. SCALAR POTENTIALS</u>: i) $[4QR + 2S(P+2Q)]\nabla^2\varphi = 2S(3R+2S)\rho\frac{\partial^2\varphi}{\partial t^2}$ ii) $Q\nabla^2\psi = 2S\rho\frac{\partial^2\psi}{\partial t^2}$ iii) $Q\nabla^2\chi = 2S\rho\frac{\partial^2\chi}{\partial t^2}$	10) <u>TELEGRAPH EQS. FOR INDEPENDENT SCALAR POTENTIALS</u>: i) $[1 + M\frac{\partial}{\partial t}]\nabla^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $[1 + N\frac{\partial}{\partial t}]\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$, iii) $[1 + N\frac{\partial}{\partial t}]\nabla^2\chi = \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}$ WHERE, $M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$.	10) <u>TELE</u> i): $[2\beta+$ ii) ∇^2 FOR c_s^2 ∇^2
11) <u>SOLENOIDAL SOLUTION OF VECTOR TELEGRAPH - EQN.</u> THE SOLUTION EXISTS ONLY IN FIVE COORDINATE SYSTEMS (CYLINDRICALS & SPHERICALS INCLUDED) BUT IT VARIES WITH THE SYSTEM USED, AND NO EXPRESSION EXISTS VALID FOR ALL SYSTEMS. NO GENERAL EXPRESSIONS FOR RELATIONS (4), (6) OR (8) ABOVE CAN BE GIVEN FOR THIS REASON. (SEE: P. MORSE & H. FESHBACH "METHODS OF THEORETICAL PHYSICS" VOL. 2, CHAPTER 13. McGRAW-HILL, 1953)	11) <u>SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH - EQUATION</u> IT CAN BE SHOWN THAT THE SOLENOIDAL (ie, $\vec\nabla\cdot\vec\psi = 0$) SOLUTION ($= \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}$) IS THE SOLUTION OF $[1 + N\frac{\partial}{\partial t}]\vec\nabla\times(\vec\nabla\times\vec\psi) + \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2} = 0$ AND $\vec\psi = \vec\nabla\times(\psi\hat e_z) + \vec\nabla\times[\vec\nabla\times(\hat e_z\chi)] = (\vec\nabla\psi)\times\hat e_z - \vec\nabla(\frac{\partial\chi}{\partial z})$ WHERE $\psi(r,\theta,z,t)$ AND $\chi(r,\theta,z,t)$ ARE TWO <u>NEW</u> SCALAR FUNCTIONS EQUATIONS ie, $[1 + N\frac{\partial}{\partial t}]\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$ AND $[1 + N\frac{\partial}{\partial t}]\nabla^2\chi = \frac{1}{c_s}$ $\psi_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\chi}{\partial r\partial z}$, $\psi_\theta = -\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\chi}{\partial\theta\partial z}$, $\psi_z = -$ WHICH RELATE THE THREE CYLINDRICAL COMPONENTS $\psi_r, \psi_\theta, \psi_z$ OF T ψ AND χ. SUBSTITUTING $\vec\psi$ FROM THE SOLUTION (A) INTO CYLINDRICAL COMPONENTS OF THE DISPLACEMENT FIELD AND THE SC AS GIVEN ABOVE IN EQS. (4). THE SOLUTION (A) OR (B) INVOLVES ON WAVE-EQ. (OR TELEGRAPH-EQ. <u>WITHOUT</u> DAMPING) PROVIDED THAT ψ A	
12) <u>REMARKS</u>: SEE THE COMPARTMENTS TO THE RIGHT.	<u>REMARKS</u>: a) THE BASIC RESULT OF THIS TABLE, AND THAT IS WHY IT WAS ORGANIZE SATISFIED IN EACH ONE OF THE CASES CONSIDERED AND THEN THE DI BOUNDARY VALUE PROBLEMS, AND WHICH IS COMMON IN OTHER DISCI THE READER FAMILIAR WITH ELECTRODYNAMIC THEORY WILL QUICKLY N b) IT IS EVIDENT THAT IN ALL THE CASES IN THIS CHART WE RECOVER c) IN CYLINDRICAL COORDS. THE PLANE-STRESS RESULTS CAN BE QUICK d) THE CASE OF PLANE-STRAIN (OR PLANE-STRESS IN VIEW OF c) ABOVE) WI e) A CASE OF SOME INTEREST, WHICH REDUCES THE TWO VISCOUS CONSTAN	

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$$\frac{\partial^2\varphi}{\partial\theta\partial z}+\frac{1}{r}\frac{\partial}{\partial\theta}\left[2\frac{\partial^2\psi}{\partial z^2}-\nabla^2\psi\right]+\frac{\partial^2}{\partial r\partial z}(\nabla^2\chi).$$

$$\frac{\partial^2\varphi}{\partial r\partial z}+\frac{\partial}{\partial r}\left\{2\frac{\partial^2\psi}{\partial z^2}-\nabla^2\psi\right\}-\frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}(\nabla^2\chi).$$

$$=2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right]+\frac{2}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial^2\psi}{\partial r\partial z}-\frac{1}{r}\frac{\partial\psi}{\partial z}\right]+\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right](\nabla^2\chi).$$

INTO EQS. (3).

$$\epsilon_{rr}=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial^2\psi_z}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\psi_z}{\partial\theta},\qquad \epsilon_{\theta\theta}=\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}-\frac{1}{r}\frac{\partial^2\psi_z}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial\psi_z}{\partial\theta},$$

THESE RELATIONS ARE FOUND BY SUBSTITUTING EQS. 4) INTO EQS. 3).

7) FIELD EQUATIONS:

i) $$\left[1+\frac{1}{2\beta}\frac{\partial}{\partial t}\right]\left[\nabla^2\vec{u}+\frac{\lambda_e+\mu_e}{\mu_e}\vec\nabla(\vec\nabla\cdot\vec{u})-\frac{1}{c_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\right]+\frac{3\alpha}{2\beta}\left[\nabla^2\vec{u}+\frac{1}{3}\vec\nabla(\vec\nabla\cdot\vec{u})-\frac{1}{c_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\right]=\frac{1}{c_s^2}\frac{\partial^2\vec{u}}{\partial t^2}+\frac{3\alpha+2\beta}{c_s^2}\frac{\partial\vec{u}}{\partial t}.\quad\left(c_s^2=\frac{\mu_e}{\rho}\right)$$

THIS CAN BE OPENED UP INTO COMPONENTS BY MEANS OF

$$\nabla^2\vec{u}=\left[\nabla^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{e}_r+\left[\nabla^2u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right]\hat{e}_\theta+\left[\nabla^2u_z\right]\hat{e}_z$$

$$\vec\nabla(\vec\nabla\cdot\vec{u})=\frac{\partial\Delta}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial\Delta}{\partial\theta}\hat{e}_\theta+\frac{\partial\Delta}{\partial z}\hat{e}_z$$ & Δ AS GIVEN ABOVE IN 5).

ii) $$\left[1+\frac{1}{2\beta}\frac{\partial}{\partial t}\right]\left[\left(\frac{\lambda_e+2\mu_e}{\mu_e}\right)\vec\nabla(\vec\nabla\cdot\vec{u})-\vec\nabla\times(\vec\nabla\times\vec{u})-\frac{1}{c_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\right]+\frac{3\alpha}{2\beta}\left[\frac{4}{3}\vec\nabla(\vec\nabla\cdot\vec{u})-\vec\nabla\times(\vec\nabla\times\vec{u})-\frac{1}{c_s^2}\frac{\partial^2\vec{u}}{\partial t^2}\right]=\frac{1}{c_s^2}\left[\frac{\partial^2\vec{u}}{\partial t^2}+(3\alpha+2\beta)\frac{\partial\vec{u}}{\partial t}\right]$$

THESE TWO FORMS i) AND ii) ARE EQUIVALENT. ∇^2 GIVEN IN (4).

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\tau_{rr}=M_1\left\{\lambda_e\nabla^2\varphi+2\mu_e\left[\frac{\partial^2}{\partial r^2}\left(\varphi+\frac{\partial\psi}{\partial z}\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)\right)\right]\right\}+N_1$$

$$\tau_{\theta\theta}=M_1\left\{\lambda_e\nabla^2\varphi+2\mu_e\left[\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\varphi+\frac{\partial\psi}{\partial z}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)\right)\right]\right\}+N_1$$

$$\tau_{zz}=M_1\left\{\lambda_e\nabla^2\varphi+2\mu_e\left[\frac{\partial^2}{\partial z^2}\left(\varphi+\frac{\partial\psi}{\partial z}\right)-\frac{\partial}{\partial z}(\nabla^2\psi)\right]\right\}+N_1$$

$$\tau_{\theta z}=M_1\mu_e\left[\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}\left(\varphi+\frac{\partial\psi}{\partial z}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\psi)+\frac{\partial^2}{\partial z\partial r}(\nabla^2\chi)\right]$$

$$\tau_{rz}=M_1\mu_e\left[2\frac{\partial^2}{\partial r\partial z}\left(\varphi+\frac{\partial\psi}{\partial z}\right)-\frac{\partial}{\partial r}(\nabla^2\psi)-\frac{1}{r}\frac{\partial^2}{\partial z\partial\theta}(\nabla^2\chi)\right]$$

$$\tau_{r\theta}=M_1\mu_e\left[2\frac{\partial^2}{\partial r\partial\theta}\left\{\frac{1}{r}\left(\varphi+\frac{\partial\psi}{\partial z}\right)\right\}+\left\{\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}(\nabla^2\chi)\right]$$ WHERE,

$$M_1=\frac{\frac{\partial}{\partial t}\left(2\beta-\frac{\partial}{\partial t}\right)}{4\beta^2-\frac{\partial^2}{\partial t^2}},\quad N_1=\frac{-\alpha(3\lambda_e+2\mu_e)}{(3\alpha+2\beta)^2-\frac{\partial^2}{\partial t^2}}\left[\frac{-(3\alpha+4\beta)\frac{\partial^2}{\partial t^2}+2\beta\left[(3\alpha+2\beta)\frac{\partial}{\partial t}+\frac{1}{2\beta}\frac{\partial^3}{\partial t^3}\right]}{4\beta^2-\frac{\partial^2}{\partial t^2}}\right]\nabla^2\varphi$$

THESE ARE EQS. (5) WITH $\Delta=\nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQS. (6).

9) TELEGRAPH-TYPE EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i) $$\left[2\beta+4\alpha\frac{c_s^2}{c_d^2}+\frac{\partial}{\partial t}\right]\nabla^2\varphi=\frac{1}{c_d^2}\left[\left(3\alpha+4\beta+\frac{\partial}{\partial t}\right)\frac{\partial^2\varphi}{\partial t^2}+2\beta(3\alpha+2\beta)\frac{\partial\varphi}{\partial t}\right]$$

ii) $$-\vec\nabla\times(\vec\nabla\times\vec\psi)=\frac{1}{c_s^2}\left(2\beta+\frac{\partial}{\partial t}\right)\frac{\partial\vec\psi}{\partial t}$$

EQ. ii) CAN ALSO BE WRITTEN AS FOLLOWS,

$$\nabla^2\vec\psi=\frac{1}{c_s^2}\left(2\beta+\frac{\partial}{\partial t}\right)\frac{\partial\vec\psi}{\partial t}\quad\text{AND}\quad\vec\nabla\cdot\vec\psi=0$$

10) TELEGRAPH EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:

i): $$\left[2\beta+4\alpha\frac{c_s^2}{c_d^2}+\frac{\partial}{\partial t}\right]\nabla^2\varphi=\frac{1}{c_d^2}\left\{\left[3\alpha+4\beta+\frac{\partial}{\partial t}\right]\frac{\partial^2\varphi}{\partial t^2}+2\beta(3\alpha+2\beta)\frac{\partial\varphi}{\partial t}\right\}$$

ii) $$\nabla^2\psi=\frac{1}{c_s^2}\left(2\beta+\frac{\partial}{\partial t}\right)\frac{\partial\psi}{\partial t},\qquad \text{iii) }\nabla^2\chi=\frac{1}{c_s^2}\left(2\beta+\frac{\partial}{\partial t}\right)\frac{\partial\chi}{\partial t}.$$

FOR $c_s^2 \ll c_d^2$ (ie, RUBBER) EQ. (i) REDUCES TO:

$$\nabla^2\varphi=\frac{1}{c_d^2}\left[\frac{\partial^2\varphi}{\partial t^2}+(3\alpha+2\beta)\frac{\partial\varphi}{\partial t}\right]\qquad c_d^2=\frac{\lambda_e+2\mu_e}{\rho}\ \ \&\ \ c_s^2=\frac{\mu_e}{\rho}.$$

…PH-EQUATION IN CYLINDRICAL COORDINATES:

…O) SOLUTION OF THE VECTOR TELEGRAPH EQUATION (VIZ, $\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\vec\psi-\frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}=0$ AND THAT IT IS GIVEN BY EITHER ONE OF THE FOLLOWING,

$\vec\psi=\hat{e}_z-\vec\nabla\left(\frac{\partial\chi}{\partial z}\right)-\hat{e}_z\nabla^2\chi$ (A) [CYLINDRICAL COORDS. ONLY],

…LAR FUNCTIONS (ie, ψ IS NOT $|\vec\psi|$) SATISFYING SCALAR TELEGRAPH … $N\frac{\partial}{\partial t}]\nabla^2\chi=\frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}$. THE SOLUTION (A) IN COMPONENT FORM IS,

$$\psi_z=-\frac{\partial^2\chi}{\partial r^2}-\frac{1}{r}\frac{\partial\chi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}.\qquad (B)$$

…$\psi_r,\psi_\theta,\psi_z$ OF THE VECTOR POTENTIAL $\vec\psi$ TO THE TWO SCALAR FUNCTIONS …) INTO $\vec{u}=\vec\nabla\varphi+\vec\nabla\times\vec\psi$ YIELDS THE RELATIONS BETWEEN THE …LD AND THE SCALAR FUNCTIONS (ie, INDEPENDENT POTENTIALS) φ,ψ,χ …(B) INVOLVES ONLY SPACE OPERATORS, SO IT ALSO HOLDS FOR THE VECTOR …VIDED THAT ψ AND χ THEN SATISFY SCALAR WAVE-EQS. (ie $\left[\nabla^2-\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right]{\psi \atop \chi}=0$

7) FIELD EQUATIONS:

i):

$$\begin{cases}\left[\mu_e+\mu_v\frac{\partial}{\partial t}\right]\left[\nabla^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]+\left[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}\right]\frac{\partial\Delta}{\partial r}=\rho\frac{\partial^2u_r}{\partial t^2}\\ \left[\mu_e+\mu_v\frac{\partial}{\partial t}\right]\left[\nabla^2u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right]+\left[\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}\right]\frac{1}{r}\frac{\partial\Delta}{\partial\theta}=\rho\frac{\partial^2u_\theta}{\partial t^2}\end{cases}$$

WHERE, $\Delta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$.

THE SECOND FORM ii) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{cases}\tau_{rr}=\left[\lambda_e+\lambda_v\frac{\partial}{\partial t}\right]\nabla^2\varphi+2\left[\mu_e+\mu_v\frac{\partial}{\partial t}\right]\left[\frac{\partial^2\varphi}{\partial r^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)\right)\right]\\ \tau_{\theta\theta}=\left[\lambda_e+\lambda_v\frac{\partial}{\partial t}\right]\nabla^2\varphi+2\left[\mu_e+\mu_v\frac{\partial}{\partial t}\right]\left[\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi)\right)\right]\\ \tau_{zz}=\left[\lambda_e+\lambda_v\frac{\partial}{\partial t}\right]\nabla^2\varphi\\ \tau_{r\theta}=\left[\mu_e+\mu_v\frac{\partial}{\partial t}\right]\left[2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right)+\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\nabla^2\chi\right]\end{cases}$$

TO HAVE THESE RELATIONS IN TERMS OF φ AND ψ_z WE REPLACE $\nabla^2\chi$ BY $-\psi_z$. ∇^2 IS GIVEN ABOVE IN (7). THESE ARE EQS. (5) WITH $\Delta=\nabla^2\varphi$ AND ϵ_{ij} AS GIVEN IN EQS. (6).

9) TELEGRAPH EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i): $\left(1+M\frac{\partial}{\partial t}\right)\nabla^2\varphi=\frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\vec\psi=\frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}$,

WHERE, $M=\frac{\lambda_v+2\mu_v}{\lambda_e+2\mu_e}$, $N=\frac{\mu_v}{\mu_e}$, $c_d^2=\frac{\lambda_e+2\mu_e}{\rho}$, $c_s^2=\frac{\mu_e}{\rho}$, $\vec\nabla\cdot\vec\psi=0$

HERE, $\vec\psi=\hat{e}_z\psi_z(r,\theta,t)=-\hat{e}_z\nabla^2_{r\theta}\chi(r,\theta,t)$. HENCE, $\vec\nabla\cdot\vec\psi=0$, AND EQ. ii) CAN BE WRITTEN IN EITHER ALTERNATIVE FORM GIVEN IN 10).

10) TELEGRAPH-EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:

i) $$\left(1+M\frac{\partial}{\partial t}\right)\nabla^2\varphi=\frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}.$$

ii) $$\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\psi_z=\frac{1}{c_s^2}\frac{\partial^2\psi_z}{\partial t^2}\quad\text{OR,}\quad-\nabla^2\left\{\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\chi-\frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}\right\}=0$$

WHERE M, N, c_s^2, c_d^2 ARE GIVEN ABOVE IN 9).

11) SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH EQUATION IN PLANE…

THE SOLUTION FOLLOWS THAT GIVEN TO THE LEFT FOR GENERAL CYLIND… …DENCE. THEREFORE, ONE CAN ANALOGOUSLY SHOW THAT THE SOLEN…

$$\vec\psi=\vec\nabla\times\left[\vec\nabla\times(\hat{e}_z\chi(r,\theta,t))\right]=-\hat{e}_z\nabla^2\chi$$

PROVIDED THAT $\chi(r,\theta,t)$ SATISFIES THE SCALAR TELEGRAPH EQUAT… SINCE THE LAPLACIAN AND THE CURL OPERATORS ACT ON A z-INDEPENDEN… CASE.

SOLUTION (A) IN COMPONENT-FORM IS: $\psi_r=0$, $\psi_\theta=0$, $\psi_z=-\nabla^2$… A z-COMPONENT WHICH IS z-INDEPENDENT. SUBSTITUTING SLN. (A) IN… POTENTIAL RELATIONS GIVEN IN EQS. (4) ABOVE. WE REPEAT THAT… THE TELEGRAPH-EQUATION REDUCES TO THE WAVE-EQUATION, AS IT OCCU… SCALAR WAVE-EQUATION $\left[\nabla^2-\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right]\chi=0$.

7) FIELD EQ…

i)

$[3\alpha+2\beta+$…

$[3\alpha+2\beta+$…

WHERE Δ A…
FORM ii)…

8) STRESS…

$\tau_{rr}=M$…

$\tau_{\theta\theta}=M$…

$\tau_{zz}=M$…

$\tau_{r\theta}=$…

$M_1=\frac{\frac{\partial}{\partial t}(}{4\beta}$…

$N_1=-\left[\frac{\alpha(3}{(3\alpha+}\right.$…

9) TELEGR…

i) $[2\beta+4\alpha$…

ii) ∇…

FOR $c_s^2\ll$…

i)

10) TELEG…

THE EQ…

$-\nabla^2\{$…

…WAS ORGANIZED AS SHOWN ABOVE, IS THAT IF WE INTRODUCE INDEPENDENT SCALAR POTENTIALS WHICH SATISFY THE TELEGRAPH-TYPE EQS. 10)… …ND THEN THE DISPLACEMENT AND STRESS FIELD COMPONENTS ARE GIVEN IN TERMS OF THE POTENTIALS BY EQS. 4) AND 8) RESPECTIVELY. THIS… …N OTHER DISCIPLINES (ie, ELECTRODYNAMICS), IS APPARENTLY NOVEL IN VISCOELASTICITY, AND THIS IS WHY THIS CHART WAS DEVELOPPED FOR A… …WILL QUICKLY NOTICE THE ANALOGY OF THE PRESENTATION IN THIS CHART WITH THAT OF ELECTROMAGNETIC THEORY IN A COULOMB (ie, $\vec\nabla\cdot\vec\psi=0$)… …T WE RECOVER THE SIMPLER RESULTS OF DYNAMIC ELASTICITY IN THE ABSENCE OF VISCOSITY. (ie, $\lambda_v=0$, $\mu_v=0$ FOR THE KELVIN-VOIGT MODEL, A… …S CAN BE QUICKLY OBTAINED FROM THE PLANE-STRAIN RESULTS IN CASES D) & E) BY THE STANDARD TRICK OF REPLACING λ_e IN THE PLANE-STRAIN CASE… …OF C) ABOVE) WITH AXIAL SYMMETRY ABOUT THE z-AXIS, CAN BE OBTAINED FROM D) & E) BY SETTING $u_\theta=0$ & $\frac{\partial}{\partial\theta}$[ANY VARIABLE]$=0$. THEN, THERE IS ONLY… …SCOUS CONSTANTS TO ONE, OCCURS WHEN THE BULK VISCOSITY IS ZERO OR NEARLY SO. THIS AMOUNTS TO HAVING $3\lambda_e+2\mu_e=0$ (KELVIN-VOIGT) O…

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$2\epsilon \quad \frac{\partial u}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial\theta}$, $2\epsilon_{r\theta} = \frac{2}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$.

3). QS. 4) INTO EQS. 3).

D EQ

$2\beta + \quad \frac{\partial\Delta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}$

7) FIELD EQUATIONS:

i)

$$[3\alpha+2\beta+\tfrac{\partial}{\partial t}][\nabla^2 u_r - \tfrac{u_r}{r^2} - \tfrac{2}{r^2}\tfrac{\partial u_\theta}{\partial\theta}] + [\alpha + \tfrac{\lambda_e+\mu_e}{\mu_e}(2\beta+\tfrac{\partial}{\partial t})]\tfrac{\partial\Delta}{\partial r} =$$
$$= \frac{1}{c_s^2}\left[(3\alpha+4\beta+\tfrac{\partial}{\partial t})\frac{\partial^2 u_r}{\partial t^2} + 2\beta(2\beta+3\alpha)\frac{\partial u_r}{\partial t}\right] .$$
$$[3\alpha+2\beta+\tfrac{\partial}{\partial t}][\nabla^2 u_\theta - \tfrac{u_\theta}{r^2} + \tfrac{2}{r^2}\tfrac{\partial u_r}{\partial\theta}] + [\alpha + \tfrac{\lambda_e+\mu_e}{\mu_e}(2\beta+\tfrac{\partial}{\partial t})]\tfrac{1}{r}\tfrac{\partial\Delta}{\partial\theta} =$$
$$= \frac{1}{c_s^2}\left[(3\alpha+4\beta+\tfrac{\partial}{\partial t})\frac{\partial^2 u_\theta}{\partial t^2} + 2\beta(2\beta+3\alpha)\frac{\partial u_\theta}{\partial t}\right] .$$

WHERE Δ AND ∇^2 ARE GIVEN ABOVE IN (4) & (5). $(c_s^2 = \mu_e/\rho.)$
FORM ii) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{cases} \tau_{rr} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[\frac{\partial^2\varphi}{\partial r^2} - \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]\} + N_1 \\ \tau_{\theta\theta} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]\} + N_1 \\ \tau_{zz} = M_1\{\lambda_e\nabla^2\varphi\} + N_1 \quad . \quad \tau_{\theta z} = 0 , \quad \tau_{zr} = 0 , \\ \tau_{r\theta} = M_1\mu_e\{2\frac{\partial^2}{\partial r\partial\theta}(\frac{\varphi}{r}) + [\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi)\} \end{cases} \quad \text{WHERE,}$$

$M_1 = \dfrac{\frac{\partial}{\partial t}(2\beta - \frac{\partial}{\partial t})}{4\beta^2 - \frac{\partial^2}{\partial t^2}}$, (TO HAVE THESE IN TERMS OF φ & ψ_z, WE REPLACE $\nabla^2\chi$ BY $-\psi_z$)

$$N_1 = -\left[\frac{\alpha(3\lambda_e+2\mu_e)}{(3\alpha+2\beta)^2 - \frac{\partial^2}{\partial t^2}}\right]\left[\frac{-(3\alpha+4\beta)\frac{\partial^2}{\partial t^2} + 2\beta[(3\alpha+2\beta)\frac{\partial}{\partial t} + \frac{1}{2\beta}\frac{\partial^3}{\partial t^3}]}{4\beta^2 - \frac{\partial^2}{\partial t^2}}\right]\nabla^2\varphi$$

$(\nabla^2\chi))]$

$(\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\chi))]$

ψ_z WE IN (7). EQS. (6).

9) TELEGRAPH EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i) $[2\beta + 4\alpha\frac{c_s^2}{c_d^2} + \frac{\partial}{\partial t}]\nabla^2\varphi = \frac{1}{c_d^2}\{[4\beta+3\alpha+\frac{\partial}{\partial t}]\frac{\partial^2\varphi}{\partial t^2} + 2\beta(3\alpha+2\beta)\frac{\partial\varphi}{\partial t}\}$

ii) $\nabla^2\vec{\psi} = \frac{1}{c_s^2}(2\beta+\frac{\partial}{\partial t})\frac{\partial\vec{\psi}}{\partial t}$ AND $\nabla\cdot\vec{\psi} = 0$.

• FOR $c_s^2 \ll c_d^2$ (ie, $\mu_e \ll \lambda_e$ OR $\nu \approx 1/2$, ie, RUBBER) EQ.(i) IS:

i) $\nabla^2\varphi = \frac{1}{c_d^2}[\frac{\partial^2\varphi}{\partial t^2} + (3\alpha+2\beta)\frac{\partial\varphi}{\partial t}]$

$\nabla\cdot\vec{\psi} = 0$, $\nabla\cdot\vec{\psi} = 0$, FORM

10) TELEGRAPH-EQS FOR THE INDEPENDENT SCALAR POTENTIALS:

THE EQ. FOR φ IS THE SAME AS ABOVE IN 9). THE EQ. FOR χ IS:

$-\nabla^2\{\nabla^2\chi - \frac{1}{c_s^2}(2\beta+\frac{\partial}{\partial t})\frac{\partial\chi}{\partial t}\} = 0$ OR IN TERMS OF ψ_z,

$\nabla^2\psi_z = \frac{1}{c_s^2}(2\beta+\frac{\partial}{\partial t})\frac{\partial\psi_z}{\partial t}$

$\nabla^2\{ \quad \chi - \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}\} = 0$

PH EQUATION ... COORDINATES

FOR GENERAL CYLINDRICAL COORDINATES BUT NOW WITHOUT THE z DEPEN-
OW THAT THE SOLENOIDAL SOLUTION OF $(1+N\frac{\partial}{\partial t})\nabla^2\vec{\psi} = \frac{1}{c_s^2}\frac{\partial^2\vec{\psi}}{\partial t^2}$ IS,
$t))] = -\hat{e}_z\nabla^2\chi$ (A),
R TELEGRAPH EQUATION, VIZ. $(1+N\frac{\partial}{\partial t})\nabla^2\chi = \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}$. NOTE THAT
CT ON A z-INDEPENDENT FUNCTION, THEY ARE ALSO z-INDEPENDENT IN THIS
, $\psi_\theta = 0$, $\psi_z = -\nabla^2\chi(r,\theta,t)$ (B). SO THE VECTOR POTENTIAL HAS ONLY
STITUTING SLN. (A) INTO $\vec{u} = \nabla\varphi + \nabla\times\vec{\psi}$ YIELDS THE DISPLACEMENT-
VE. WE REPEAT THAT ALL THIS HOLDS IN THE ABSENCE OF DAMPING WHEN
EQUATION, AS IT OCCURS IN ELASTICITY, PROVIDED THAT χ THEN SATISFIES A

GRAPH-TYPE EQS. 10) THEN THE FIELD-EQS 7) ARE AUTOMATICALLY
RESPECTIVELY. THIS TECHNIQUE, WHICH IS BASIC TO SET-UP AND SOLVE
AS DEVELOPPED FOR ALL THESE CASES AND VISCOELASTIC MODELS.
LOMB (ie, $\nabla\cdot\vec{\psi} = 0$) RATHER THAN A LORENTZ (ie, $\nabla\cdot\vec{\psi} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0$) GAUGE.
ELVIN-VOIGT MODEL, AND/OR $\alpha = 0$, $\beta = 0$ FOR THE MAXWELL MODEL).
THE PLANE-STRAIN CASES BY $\bar{\lambda}_e$ DEFINED TO BE: $\bar{\lambda}_e = 2\lambda_e\mu_e/(\lambda_e+2\mu_e)$.
O. THEN, THERE IS ONLY r-DEPENDENCE (ie, $u_r = u_r(r,t)$ & $\varphi = \varphi(r,t)$) ∴ $u_r = \frac{\partial\varphi}{\partial r}$.
$=0$ (KELVIN-VOIGT) OR $3\alpha+2\beta = 0$ (MAXWELL) SO, FORMULAS SIMPLIFY A BIT.

$2\epsilon_{\theta\phi} = \frac{2}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\theta\partial\phi} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\varphi}{\partial\phi} + \frac{2}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\theta\partial\phi} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta\partial\phi} - \frac{2\cot\theta}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi} + [\frac{1}{r}$

$2\epsilon_{r\phi} = \frac{2}{r\sin\theta}\frac{\partial^2\varphi}{\partial r\partial\phi} - \frac{2}{r^2\sin\theta}\frac{\partial\varphi}{\partial\phi} - \frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial\phi\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\phi} - \frac{1}{r^2\sin^3\theta}\frac{\partial^3\psi}{\partial\phi^3} - \frac{\cot\theta}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi\partial\theta} - \frac{2}{r^2\sin\theta}$

$2\epsilon_{r\theta} = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial r} - \frac{2}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^3} - \frac{1}{r^2\sin^2\theta}\frac{\partial^3\psi}{\partial\theta\partial\phi^2} + \frac{\partial^3\psi}{\partial r^2\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{2\cot\theta}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + [\frac{1}{r^2\sin^2\theta}$

THESE ARE FOUND BY SUBSTITUTING EQS (4) INTO EQS. (3).

7) FIELD EQUATIONS:

i) $[\mu_e + \mu_v\frac{\partial}{\partial t}]\nabla^2\vec{u} + [\lambda_e + \mu_e + (\lambda_v+\mu_v)\frac{\partial}{\partial t}]\nabla(\nabla\cdot\vec{u}) = \rho\frac{\partial^2\vec{u}}{\partial t^2}$ OR I

$$\begin{cases} [\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}] + [\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}] \\ [\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}] + [\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial}{\partial\theta} \\ [\mu_e+\mu_v\frac{\partial}{\partial t}][\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\phi}] + [\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r\sin\theta} \end{cases}$$

ii) $[\lambda_e + 2\mu_e + (\lambda_v+2\mu_v)\frac{\partial}{\partial t}]\nabla(\nabla\cdot\vec{u}) - [\mu_e+\mu_v\frac{\partial}{\partial t}]\nabla\times(\nabla\times\vec{u}) = \rho\frac{\partial^2\vec{u}}{\partial t^2}$

THE DILATATION Δ IS GIVEN IN 5) ABOVE AND, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{\partial^2}{\partial r^2}(\varphi + \frac{\partial}{\partial r}(r\psi)) - \frac{\partial}{\partial r}(r\nabla^2\psi)]$

$\tau_{\theta\theta} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e+\mu_v\frac{\partial}{\partial t}][(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\varphi + \frac{\partial}{\partial r}(r\psi)) - \nabla^2\psi - \frac{1}{r}$

$\tau_{\phi\phi} = [\lambda_e + \lambda_v\frac{\partial}{\partial t}]\nabla^2\varphi + 2[\mu_e+\mu_v\frac{\partial}{\partial t}][(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2})(\varphi + \frac{\partial}{\partial r}($

$\tau_{\theta\phi} = [\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{2}{r^2}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi})(\varphi + \frac{\partial}{\partial r}(r\psi)) + (\frac{1}{r}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2}{\partial\phi^2})$

$\tau_{r\phi} = [\mu_e+\mu_v\frac{\partial}{\partial t}][\frac{2}{\sin\theta}\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\phi}(\varphi + \frac{\partial}{\partial r}(r\psi))\} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\psi) + \frac{\partial}{\partial\theta}(\frac{\partial}{\partial r} - \frac{1}{r}$

$\tau_{r\theta} = [\mu_e+\mu_v\frac{\partial}{\partial t}][2\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi + \frac{\partial}{\partial r}(r\psi))\} - \frac{\partial}{\partial\theta}(\nabla^2\psi) - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\frac{\partial}{\partial r} - \frac{1}{r})($

9) TELEGRAPH EQUATIONS FOR THE SCALAR AND VECTOR POTENTIAL

i) $(1 + M\frac{\partial}{\partial t})\nabla^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $(1 + N\frac{\partial}{\partial t})\nabla\times(\nabla\times\vec{\psi}) +$

$M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c_s^2 =$

THESE EQS. ARE OBTAINED BY SUBSTITUTING $\vec{u} = \nabla\varphi + \nabla\times\vec{\psi}$ INT
OF THE FIELD EQS. 7). EQ. ii) CAN ALSO BE WRITTEN AS
ii)' $(1 + N\frac{\partial}{\partial t})\nabla^2\vec{\psi} = \frac{1}{c_s^2}\frac{\partial^2\vec{\psi}}{\partial t^2}$ AND $\nabla\cdot\vec{\psi} = 0$

10) TELEGRAPH EQUATIONS FOR THE INDEPENDENT SCALAR POTENT

i) $(1 + M\frac{\partial}{\partial t})\nabla^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $(1 + N\frac{\partial}{\partial t})\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$.

WHERE, $M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$

11) SOLENOIDAL SOLUTION OF THE VECTOR TELEGRAPH EQ. N S

IT CAN BE SHOWN THAT THE SOLUTION OF EQS. 9) ii)' ABOVE IS GI
$\vec{\psi} = \nabla\times(\hat{r}\psi) + \nabla\times[\nabla\times(\vec{r}\chi)] = (\nabla\psi)\times\vec{r} + \nabla[\frac{\partial}{\partial r}(r\chi$
PROVIDED THAT THE TWO NEW SCALAR FUNCTIONS $\psi(r,\theta,\phi,t)$
SCALAR TELEGRAPH EQS (ie, $(1+N\frac{\partial}{\partial t})\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$ AND $(1+N\frac{\partial}{\partial t}$
(NOTE THAT ψ IS NOT $|\vec{\psi}|$). OPENING UP THE SOLUTION (A

$\psi_r = -\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial\chi}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}$, $\psi_\theta = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{1}{r}\frac{\partial\chi}{\partial\theta} + \frac{\partial^2\chi}{\partial r\partial\theta}$

WHICH RELATE THE 3 SPHERICAL COMPONENTS OF THE VECTOR POTE
SOLENOIDAL BY THE GAUGE CONDITION $\nabla\cdot\vec{\psi} = 0$) TO THE 2 SCAL
SOLUTION (A) IS VALID IN SPHERICALS ONLY, AND WHEN IT IS SUBST
THE RESULT IS THE DISPLACEMENT - INDEPENDENT POTENTIAL REL
AUXILIARY RELATIONS NEEDED TO PROVE THIS ARE: $\nabla\times[(\nabla\psi)\times\vec{r}] =$
$\nabla\times[\vec{r}\nabla^2\chi] = [\nabla(\nabla^2\chi)]\times\vec{r}$. IN THE ABSENCE OF VISCOSITY (
EQ. (A) ABOVE IS THE SOLENOIDAL SOLUTION OF THE VECTOR WA
PROVIDED THAT THEN ψ AND χ SATISFY 2 SCALAR WAVE-EQ

12) REMARKS:

i) REMARKS a), b), e) OF THE CYLINDRICAL CASES TO THE LEFT ALS
ii) NOTE AGAIN THAT IN SPITE OF THE APARENT GENERALITY OF THE
ABOVE WITH ITS VECTOR OPERATORS, IT IS ONLY VALID IN SPHER
FURTHER THAT THE TWO PARTS (ie, $\nabla\times(\vec{r}\psi)$ AND $\nabla\times[\nabla\times(\vec{r}$
THEY ARE USUALLY PERPENDICULAR) EVEN WHEN $\psi = \chi$.

iii) THERE IS NO WAY TO HAVE A PLANE-STRAIN OR PLANE-STRESS
THE CLOSEST WE CAN GET TO THEM IS THE AXIALLY SYMMETRIC C

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$\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi} - \frac{}{r^2}\frac{\partial\psi}{\partial\phi} + \frac{\cot\theta}{r}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi\partial\theta}(\nabla^2\chi) - \frac{\cot\theta}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi).$

$\frac{\partial^2\psi}{\partial\theta\partial\phi} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{2}{r\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta\partial\phi} - \frac{2\cot\theta}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi} + \left[\frac{1}{r}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\nabla^2\chi).$

$\frac{\partial^3\psi}{\partial\phi\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\phi} - \frac{1}{r^2\sin^3\theta}\frac{\partial^3\psi}{\partial\phi^3} - \frac{\cot\theta}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial\psi}{\partial\phi} - \left[\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial r\partial\theta}\right](\nabla^2\chi).$

$\sin^2\theta\,\frac{\partial^3\psi}{\partial\theta\partial\phi^2} + \frac{\partial^3\psi}{\partial r^2\partial\theta} - \frac{\cot\theta}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{2\cot\theta}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \left[\frac{1}{r^2\sin^2\theta} - \frac{2}{r^2}\right]\frac{\partial\psi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial}{\partial r} - \frac{1}{r}\right](\nabla^2\chi).$

NG EQS (4) INTO EQS (3).

$\lambda_e + \mu_e + (\lambda_v+\mu_v)\frac{\partial}{\partial t}]\,\nabla(\vec\nabla\cdot\vec u) = \rho\frac{\partial^2\vec u}{\partial t^2}$ OR IN COMPONENT- FORM,

$- \frac{2u_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}] + [\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{\partial\Delta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}$

$\frac{u_\theta}{\sin^2\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}] + [\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2}$

$\frac{2}{\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\phi}] + [\lambda_e+\mu_e+(\lambda_v+\mu_v)\frac{\partial}{\partial t}]\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi} = \rho\frac{\partial^2 u_\phi}{\partial t^2}$

$]\,\vec\nabla(\vec\nabla\cdot\vec u) - [\mu_e+\mu_v\frac{\partial}{\partial t}]\,\vec\nabla\times(\nabla\times\vec u) = \rho\frac{\partial^2\vec u}{\partial t^2}$

IN 5) ABOVE AND, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$.

OTENTIAL RELATIONS:

$+\mu_v\frac{\partial}{\partial t}]\left[\frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - \frac{\partial}{\partial r}(r\nabla^2\psi)\right]$

$+\mu_v\frac{\partial}{\partial t}]\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - \nabla^2\psi - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)\right)\right]$.

$+\mu_v\frac{\partial}{\partial t}]\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - \nabla^2\psi + \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)\right)\right]$.

$\frac{\partial}{\partial\phi})\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) + \left(\frac{1}{r}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)(\nabla^2\chi)]$.

$\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right)\} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\psi) + \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)(\nabla^2\chi)]$.

$\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right)\} - \frac{\partial}{\partial\theta}(\nabla^2\psi) - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)(\nabla^2\chi)]$.

[THESE ARE EQS. 6) SUBSTITUTED INTO EQS. 2).]

OR THE SCALAR AND VECTOR POTENTIALS φ AND $\vec\psi$:

, ii) $\left(1+N\frac{\partial}{\partial t}\right)\vec\nabla\times(\vec\nabla\times\vec\psi) + \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2} = 0$ WHERE,

$N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e+2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$.

BY SUBSTITUTING $\vec u = \vec\nabla\varphi + \vec\nabla\times\vec\psi$ INTO THE ALTERNATIVE FORMS

EQ. ii) CAN ALSO BE WRITTEN AS FOLLOWS,

$\nabla^2\vec\psi = \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}$ AND $\vec\nabla\cdot\vec\psi = 0$

FOR THE INDEPENDENT SCALAR POTENTIALS φ, ψ, χ:

, ii) $\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$. iii) $\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\chi = \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}$.

$N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e+2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$ & ∇^2 AS IN EQ. 7).

OF THE VECTOR TELEGRAPH EQ. N GENERAL SPHERICAL COORDS;

SOLUTION OF EQS. 9) ii)' ABOVE IS GIVEN BY,

$+ \vec\nabla\times[\vec\nabla\times(\vec r\chi)] = (\vec\nabla\psi)\times\vec r + \vec\nabla\left[\frac{\partial}{\partial r}(r\chi)\right] - \vec r\,\nabla^2\chi$ (A) $\vec r = r\hat e_r$

NEW SCALAR FUNCTIONS $\psi(r,\theta,\phi,t)$ & $\chi(r,\theta,\phi,t)$ SATISFY

(ie, $(1+N\frac{\partial}{\partial t})\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$ AND $(1+N\frac{\partial}{\partial t})\nabla^2\chi = \frac{1}{c_s^2}\frac{\partial^2\chi}{\partial t^2}$.)

1). OPENING UP THE SOLUTION (A) ABOVE WE FIND, (B)

$\frac{1}{r\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}$, $\psi_\theta = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{1}{r}\frac{\partial\chi}{\partial\theta} + \frac{\partial^2\chi}{\partial r\partial\theta}$, $\psi_\phi = -\frac{\partial\psi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial^2\chi}{\partial r\partial\phi}$.

ICAL COMPONENTS OF THE VECTOR POTENTIAL $\vec\psi$ (WHICH IS ALSO

E CONDITION $\vec\nabla\cdot\vec\psi = 0$) TO THE 2 SCALARS ψ AND χ.

PHERICALS ONLY, AND WHEN IT IS SUBSTITUTED INTO $\vec u = \vec\nabla\varphi + \vec\nabla\times\vec\psi$

CEMENT - INDEPENDENT POTENTIAL RELATIONS IN EQS. 4) ABOVE.

DED TO PROVE THIS ARE: $\vec\nabla\times[(\vec\nabla\psi)\times\vec r] = \vec\nabla\psi - \vec r\,\nabla^2\psi - \vec\nabla(\vec\nabla\psi\cdot\vec r)$ AND

$\times\vec r$. IN THE ABSENCE OF VISCOSITY (ie, M = N = 0, ie, ELASTICITY)

ENOIDAL SOLUTION OF THE VECTOR WAVE-EQ. $\left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\vec\psi = 0$

AND χ SATISFY 2 SCALAR WAVE-EQS. (ie, $\left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\chi = 0$ ...).

THE CYLINDRICAL CASES TO THE LEFT ALSO HOLD HERE.

ITE OF THE APARENT GENERALITY OF THE SOLENOIDAL SOLUTION 11)A)

OPERATORS, IT IS ONLY VALID IN SPHERICAL COORDINATES. NOTE

O PARTS (ie, $\vec\nabla\times(\vec r\psi)$ AND $\vec\nabla\times[\vec\nabla\times(\vec r\chi)]$) ARE DIFFERENT (ie,

PENDICULAR) EVEN WHEN $\psi = \chi$.

E A PLANE-STRAIN OR PLANE-STRESS SITUATION IN SPHERICAL COORDS

ST TO THEM IS THE AXIALLY SYMMETRIC CASE 8) TO THE RIGHT.

$2\epsilon_{\theta\phi} = 0$, $2\epsilon_{r\phi} = 0$,

$$2\epsilon_{r\theta} = 2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right)\right] - \frac{\partial}{\partial\theta}(\nabla_{r\theta}^2\psi) = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial r} - \frac{2}{r^2}\frac{\partial\varphi}{\partial\theta} + \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\right)\frac{\partial\psi}{\partial\theta} .$$

7) FIELD EQUATIONS:

i) $\left[\mu_e + \mu_v\frac{\partial}{\partial t}\right]\left[\nabla_{r\theta}^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{r^2}\right] + \left[\lambda_e + \mu_e + (\lambda_v+\mu_v)\frac{\partial}{\partial t}\right]\frac{\partial\Delta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}$

$\left[\mu_e + \mu_v\frac{\partial}{\partial t}\right]\left[\nabla_{r\theta}^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta}\right] + \left[\lambda_e + \mu_e + (\lambda_v+\mu_v)\frac{\partial}{\partial t}\right]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2}$.

WHERE, $\nabla_{r\theta}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}$

$\Delta = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\theta}{r}\cot\theta$

FORM ii) WILL NOT BE GIVEN HERE.

8) STRESS- INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\nabla^2\varphi + 2\left(\mu_e + \mu_v\frac{\partial}{\partial t}\right)\left[\frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - \frac{\partial}{\partial r}(r\,\nabla_{r\theta}^2\psi)\right]$

$\tau_{\theta\theta} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\nabla^2\varphi + 2\left(\mu_e + \mu_v\frac{\partial}{\partial t}\right)\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - \nabla_{r\theta}^2\psi\right]$

$\tau_{\phi\phi} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\nabla^2\varphi + 2\left(\mu_e + \mu_v\frac{\partial}{\partial t}\right)\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - \nabla_{r\theta}^2\psi\right]$

$\tau_{\theta\phi} = 0$, $\tau_{r\phi} = 0$,

$\tau_{r\theta} = \left(\mu_e + \mu_v\frac{\partial}{\partial t}\right)\left[2\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial\theta}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right)\right\} - \frac{\partial}{\partial\theta}(\nabla_{r\theta}^2\psi)\right]$.

ALTHOUGH $\psi_\phi = -\frac{\partial\psi}{\partial\theta}$ IT IS NOT CONVENIENT TO EXPRESS NEITHER THESE RELATIONS NOR 6-4) IN TERMS OF φ AND ψ_ϕ.

(THESE ARE EQS. 6) SUBSTITUTED INTO EQS. 2)) ($\Delta = \nabla_{r\theta}^2\varphi$).

9) TELEGRAPH EQS. FOR SCALAR & VECTOR POTENTIALS φ & $\vec\psi$:

i) $\left(1 + M\frac{\partial}{\partial t}\right)\nabla_{r\theta}^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $\left(1 + N\frac{\partial}{\partial t}\right)\vec\nabla\times(\vec\nabla\times\vec\psi) + \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2} = 0$

WHERE M, N, c_d^2, c_s^2 ARE GIVEN TO THE LEFT IN EQS. F-9.).

EQ. ii) CAN ALSO BE WRITTEN IN THE FORM:

ii)' $\left(1 + N\frac{\partial}{\partial t}\right)\nabla_{r\theta}^2\vec\psi = \frac{1}{c_s^2}\frac{\partial^2\vec\psi}{\partial t^2}$ & $\vec\nabla_{r\theta}\cdot\vec\psi = 0$

$\nabla_{r\theta}^2$ IS GIVEN IN 7).

10) TELEGRAPH EQS. FOR THE INDEPENDENT SCALAR POTENTIALS:

i) $\left(1 + M\frac{\partial}{\partial t}\right)\nabla_{r\theta}^2\varphi = \frac{1}{c_d^2}\frac{\partial^2\varphi}{\partial t^2}$, ii) $\left(1 + N\frac{\partial}{\partial t}\right)\nabla_{r\theta}^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$.

WHERE $M, N, c_d^2, c_s^2, \nabla^2$ ARE AS BEFORE. SINCE $\psi_\phi = -\frac{\partial\psi}{\partial\theta}$. THE TELEGRAPH-TYPE EQ. FOR ψ_ϕ IS:

iii) $\left(1 + N\frac{\partial}{\partial t}\right)\left[\nabla_{r\theta}^2\psi_\phi - \frac{\psi_\phi}{r^2\sin^2\theta}\right] = \frac{1}{c_s^2}\frac{\partial^2\psi_\phi}{\partial t^2}$.

WHICH IS NOT AS CONVENIENT AS ii).

11) SOLENOIDAL SOLUTION OF VECTOR TELEGRAPH EQUATION:

THE SOLUTION OF EQS. 9 ii)' IN SPHERICAL COORDS. WITH AXIAL SYMMETRY IS NOW GIVEN BY: $\vec\psi = \vec\nabla_{r\theta}\times(\vec r\psi) = (\vec\nabla_{r\theta}\psi)\times\vec r$. (A) WHICH IN COMPONENT FORM IS: $\psi_r = 0$, $\psi_\theta = 0$, $\psi_\phi = -\frac{\partial\psi}{\partial\theta}$. (B) (NOTE AGAIN THAT ψ IS NOT $|\vec\psi|$.) PROVIDED THAT $\psi(r,\theta,t)$ SATISFIES A SCALAR TELEGRAPH-EQ. $\left(1+N\frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}$. SUBSTITUTING SOLUTION (A) INTO $\vec u = \vec\nabla_{r\theta}\varphi + \vec\nabla_{r\theta}\times\vec\psi$ YIELDS, $\vec u = \vec\nabla_{r\theta}\left[\varphi + \frac{\partial}{\partial r}(r\psi)\right] - \vec r\,\nabla_{r\theta}^2\psi$ AS GIVEN ABOVE IN EQS. 4).

AGAIN, IN THE ABSENCE OF VISCOSITY (ie, M=N=0, ie, ELASTICITY) EQ. (A) IS THE SOLENOIDAL SOLUTION OF THE VECTOR WAVE-EQ. $\left(\nabla_{r\theta}^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\vec\psi = 0$ IN SPHERICAL COORDS. WITH AXIAL SYMMETRY, PROVIDED THAT THE SCALAR FUNCTION $\psi(r,\theta,t)$ SATISFIES A SCALAR WAVE-EQUATION. (ie, $\left[\nabla_{r\theta}^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right]\psi = 0$).

NOTE THAT EQ 10 iii) FOR ψ_ϕ IS MORE COMPLICATED THAN 10 ii), HENCE, EQS. 4) & 8) ARE BEST LEFT IN TERMS OF φ & ψ.

12) REMARKS:

i) REMARKS a), b), c) OF THE CYLINDRICAL CASES TO THE LEFT ALSO HOLD HERE.

ii) THE MAXWELL MODEL IN SPHERICAL COORDS. WILL BE LEFT AS AN EXERCISE FOR THE READER.

iii) THIS ENTIRE CHART, VALID FOR ARBITRARY TIME-DEPENDENCE, WILL BE PARTICULARIZED FOR HARMONIC TIME-DEPENDENCE OF THE FORM $e^{-i\omega t}$ IN A COMPANION CHART. (G. GAUNAURD, '75)

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TABLE 2: EQUATIONS OF LINEAR DYNAMIC VISCOELASTICITY IN CYLINDRICAL AND SPHERICAL COORD

(I) KELVIN VOIGT VISCOELASTIC MODEL [illegible]

[illegible]

1) RELATION BETWEEN THE TRACES τ_{kk} & ϵ_{kk}:

$$\tau_{kk} = [3\lambda_e + 2\mu_e - i\omega(3\lambda_v + 2\mu_v)]\epsilon_{kk}$$

τ_{kk} = BULK STRESS , $\epsilon_{kk} = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = \Delta$ = DILATATION.

1) RELATION BETWEEN THE TRACES τ_{kk} AND ϵ_{kk}:

$$\tau_{kk} = \frac{3\lambda_e + 2\mu_e}{\omega^2 + (3\alpha + 2\beta)^2}[\omega^2 - i\omega(3\alpha + 2\beta)]\epsilon_{kk}$$

τ_{kk} = BULK STRESS , ϵ_{kk} = DILATATION Δ, α, β = VISCOUS CONST

2) CONSTITUTIVE RELATIONS:

HERE $R = 0$, $S = 1/2$, $P = \lambda_e - i\omega\lambda_v$, $Q = \mu_e - i\omega\mu_v$, HENCE,

$$\tau_{ij} = (\lambda_e - i\omega\lambda_v)\delta_{ij}\epsilon_{kk} + 2(\mu_e - i\omega\mu_v)\epsilon_{ij}$$

HERE, λ_e, μ_e = ELASTIC (LAME) CONSTANTS,
λ_v, μ_v = VISCOUS CONSTANTS.

2) CONSTITUTIVE RELATIONS:

HERE $R = \alpha$, $S = \beta -$ [illegible] ω, $P = i\omega\lambda_e$, $Q = -i\omega\mu_e$, ∴

$$\tau_{ij} = \left(\frac{\omega^2 - i\omega 2\beta}{\omega^2 + 4\beta^2}\right)[\lambda_e\delta_{ij}\epsilon_{kk} + 2\mu_e\epsilon_{ij}] + N_1 \qquad \text{WHERE,}$$

$$N_1 = -\frac{\alpha(3\lambda_e + 2\mu_e)}{\omega^2 + (3\alpha + 2\beta)^2}\left[\frac{\omega^2(3\alpha + 4\beta) - i\omega 2\beta(3\alpha + 2\beta - \omega^2/2\beta)}{\omega^2 + 4\beta^2}\right]\delta_{ij}\Delta$$

3) STRAIN - DISPLACEMENT RELATIONS:

$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$, $\epsilon_{zz} = \frac{\partial u_z}{\partial z}$, $2\epsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}$, $2\epsilon_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}$, $2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$

THESE ARE THE SAME AS IN TABLE 1.

4) DISPLACEMENT - INDEPENDENT POTENTIAL RELATIONS:

$$\vec{u} = \vec{\nabla}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2[\psi\hat{e}_z + (\nabla\chi)\times\hat{e}_z] \qquad \text{OR IN COMPONENT FORM,}$$

$$u_r = \frac{\partial}{\partial r}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + \frac{K_2^2}{r}\frac{\partial\chi}{\partial\theta}, \quad u_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\varphi + \frac{\partial\psi}{\partial z}\right) - K_2^2\frac{\partial\chi}{\partial r}, \quad u_z = \frac{\partial}{\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\psi.$$

HERE, $\varphi = \varphi(r,\theta,z)e^{-i\omega t}$, $\psi = \psi(r,\theta,z)e^{-i\omega t}$, $\chi = \chi(r,\theta,z)e^{-i\omega t}$, $\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}$.

5) STRESS - DISPLACEMENT RELATIONS:

$$\tau_{ij} = (\lambda_e - i\omega\lambda_v)\delta_{ij}\Delta + 2(\mu_e - i\omega\mu_v)\epsilon_{ij} \qquad \text{WHERE,}$$

$2\epsilon_{ij} = u_{i,j} + u_{j,i}$ ARE GIVEN IN 3) AND, $\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = \nabla\cdot\vec{u}$

<u>NOTE</u>: COMMAS IN SUBINDICES DENOTE COVARIANT DERIVATIVES.

5) STRESS - DISPLACEMENT RELATIONS:

$$\tau_{ij} = \left(\frac{\omega^2 - i\omega 2\beta}{\omega^2 + 4\beta^2}\right)[\lambda_e\delta_{ij}\Delta + 2\mu_e\epsilon_{ij}] + N_1$$

WHERE N_1 IS GIVEN IN 2) , $2\epsilon_{ij} = u_{i,j} + u_{j,i}$ ARE GIVEN IN 3) AND
$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = u_{k,k}$

6) STRAIN - INDEPENDENT POTENTIAL RELATIONS:

$$\epsilon_{rr} = \frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right) = \frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^3\psi}{\partial z\partial r^2} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\nabla^2\chi) + \frac{1}{r^2}\frac{\partial}{\partial\theta}(\nabla^2\chi)$$

$$\epsilon_{\theta\theta} = \left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\varphi + \frac{\partial\psi}{\partial z}\right) - K_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right) = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial z}\left[\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2}\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial r}(\nabla^2\chi) - \frac{1}{r}\nabla^2\chi\right]$$

$$\epsilon_{zz} = \frac{\partial^2}{\partial z^2}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial\psi}{\partial z} = \frac{\partial^2\varphi}{\partial z^2} - \frac{\partial}{\partial z}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]$$

$$2\epsilon_{\theta z} = \frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + \frac{K_2^2}{r}\frac{\partial\psi}{\partial\theta} - K_2^2\frac{\partial^2\chi}{\partial r\partial z} = \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial}{\partial\theta}\left[2\frac{\partial^2\psi}{\partial z^2} - \nabla^2\psi\right] + \frac{\partial^2}{\partial r\partial z}(\nabla^2\chi).$$

$$2\epsilon_{rz} = 2\frac{\partial^2}{\partial r\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial\psi}{\partial r} + \frac{K_2^2}{r}\frac{\partial^2\chi}{\partial\theta\partial z} = 2\frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial}{\partial r}\left[2\frac{\partial^2\psi}{\partial z^2} - \nabla^2\psi\right] - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}(\nabla^2\chi).$$

$$2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}\left[\frac{1}{r}\left(\varphi + \frac{\partial\psi}{\partial z}\right)\right] - K_2^2\left(\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right) = 2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right] + \frac{2}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial^2\psi}{\partial r\partial z} - \frac{1}{r}\frac{\partial\psi}{\partial z}\right] + \left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right](\nabla^2\chi).$$

7) FIELD EQUATIONS:

i): $$\nabla^2\vec{u} + \left[1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}\right]\nabla(\nabla\cdot\vec{u}) + K_2^2\vec{u} = 0 \qquad \text{WHERE,}$$

$K_2^2 = \rho\omega^2/(\mu_e - i\omega\mu_v)$. THIS EQ. CAN BE OPENED UP BY MEANS OF

$$\nabla^2\vec{u} = \left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right]\hat{e}_r + \left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right]\hat{e}_\theta + [\nabla^2 u_z]\hat{e}_z ,$$

AND $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$.

THE GRADIENT OPERATOR & THE DILATATION Δ ARE GIVEN ABOVE.

ii). $$-\nabla\times(\nabla\times\vec{u}) + \left[2 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}\right]\nabla(\nabla\cdot\vec{u}) + K_2^2\vec{u} = 0$$

<u>NOTE</u> : $\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u})$.

7) FIELD EQUATIONS:

i) $(1 - \frac{i\omega}{2\beta})(\nabla^2\vec{u} + \frac{\lambda_e + \mu_e}{\mu_e}\nabla(\nabla\cdot\vec{u}) +$ [illegible] $) -$ [illegible] $[\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla\cdot\vec{u}) -$ [illegible] $] -$ [illegible]
OR IN COMPONENT FORM, $+$ [illegible] $\vec{u} = 0$

$$\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right] + V\frac{\partial\Delta}{\partial r} - Wu_r = 0,$$

$$\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right] + \frac{V}{r}\frac{\partial\Delta}{\partial\theta} - Wu_\theta = 0, \qquad [\nabla^2 u_z] + V\frac{\partial\Delta}{\partial z} - Wu_z = 0,$$

WHERE, $V = \frac{\alpha\mu_e + (\lambda_e + \mu_e)(2\beta - i\omega)}{\mu_e(3\alpha + 2\beta - i\omega)}$, $W =$ [illegible]

ii). $(1 - \frac{i\omega}{2\beta})[$ [illegible] $\nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u}) +$ [illegible] $] +$ [illegible] $[\frac{4}{3}\nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u})$
$+$ [illegible] $] -$ [illegible] $[1 + i($ [illegible] $)]\vec{u} = 0$

$\Delta = \nabla\cdot\vec{u}$ IS GIVEN IN 5).

8) STRESS - INDEPENDENT POTENTIAL RELATIONS:

$$\tau_{rr} = (\lambda_e - i\omega\lambda_v)\nabla^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right].$$

$$\tau_{\theta\theta} = (\lambda_e - i\omega\lambda_v)\nabla^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\varphi + \frac{\partial\psi}{\partial z}\right) - K_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]$$

$$\tau_{zz} = (\lambda_e - i\omega\lambda_v)\nabla^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\frac{\partial^2}{\partial z^2}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial\psi}{\partial z}\right].$$

$$\tau_{\theta z} = (\mu_e - i\omega\mu_v)\left[\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + \frac{K_2^2}{r}\frac{\partial\psi}{\partial\theta} - K_2^2\frac{\partial^2\chi}{\partial r\partial z}\right].$$

$$\tau_{rz} = (\mu_e - i\omega\mu_v)\left[2\frac{\partial^2}{\partial r\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial\psi}{\partial r} + \frac{K_2^2}{r}\frac{\partial^2\chi}{\partial\theta\partial z}\right]$$

$$\tau_{r\theta} = (\mu_e - i\omega\mu_v)\left[2\frac{\partial^2}{\partial r\partial\theta}\left\{\frac{1}{r}\left(\varphi + \frac{\partial\psi}{\partial z}\right)\right\} - K_2^2\left\{\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right\}\right].$$

THESE ARE EQS 5) WITH $\Delta = \nabla^2\varphi = -K_1^2\varphi$ AND ϵ_{ij} AS GIVEN IN 6).

8) STRESS - INDEPENDENT POTENTIAL RELATIONS:

$$\tau_{rr} = M_1\left\{\lambda_e\nabla^2\varphi + 2\mu_e\left[\frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]\right\} + N_1$$

$$\tau_{\theta\theta} = M_1\left\{\lambda_e\nabla^2\varphi + 2\mu_e\left[\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\varphi + \frac{\partial\psi}{\partial z}\right) - K_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]\right\} + N_1$$

$$\tau_{zz} = M_1\left\{\lambda_e\nabla^2\varphi + 2\mu_e\left[\frac{\partial^2}{\partial z^2}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial\psi}{\partial z}\right]\right\} + N_1$$

$$\tau_{\theta z} = M_1\mu_e\left[\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + \frac{K_2^2}{r}\frac{\partial\psi}{\partial\theta} - K_2^2\frac{\partial^2\chi}{\partial r\partial z}\right]$$

$$\tau_{r\theta} = M_1\mu_e\left[2\frac{\partial^2}{\partial r\partial\theta}\left\{\frac{1}{r}\left(\varphi + \frac{\partial\psi}{\partial z}\right)\right\} - K_2^2\left\{\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right\}\right]$$

$$\tau_{rz} = M_1\mu_e\left[2\frac{\partial^2}{\partial r\partial z}\left(\varphi + \frac{\partial\psi}{\partial z}\right) + K_2^2\frac{\partial\psi}{\partial r} + \frac{K_2^2}{r}\frac{\partial^2\chi}{\partial\theta\partial z}\right], \qquad M_1 = \frac{\omega^2 - i\omega 2\beta}{\omega^2 + 4\beta^2}.$$

$$N_1 = -\frac{\alpha K_1^2\varphi(3\lambda_e + 2\mu_e)}{\omega^2 + (3\alpha + 2\beta)^2}\left\{\frac{\omega^2(3\alpha + 4\beta) - i\omega 2\beta(3\alpha + 2\beta - \omega^2/2\beta)}{\omega^2 + 4\beta^2}\right\}$$

9) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i) $(\nabla^2 + K_1^2)\varphi = 0$, ii) $(\nabla^2 + K_2^2)\vec{\psi} = 0$ & $\nabla\cdot\vec{\psi} = 0$

9) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:

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2

…L AND SPHERICAL COORDINATES

TIME-DEPENDENCE OF THE FORM $e^{-i\omega t}$ (i.e., HARMONIC) IN SIX CASES

$+2\beta)]\epsilon_{kk}$

$N_1\Delta$, α,β = VISCOUS CONST.

$Q = -i\omega\mu_e$, N_1 WHERE,

$(3\alpha+2\beta-\omega^2/2\beta)]\delta_{ij}\Delta$

$+ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$ {THESE ARE THE SAME AS IN TABLE 1

$+N_1$

$+u_{j,i}$ ARE GIVEN IN 3) AND $u_{i,j}$

$\frac{1}{r}\nabla^2\chi]$

$](\nabla^2\chi)$.

$[\nabla^2 u_z] + V\frac{\partial\Delta}{\partial z} - W u_z = 0$,

$\vec{u} = 0$

---

1) [illegible]:

THIS IS THE CASE WHEN $u_z = 0$ AND $\frac{\partial}{\partial z}$[ANY VARIABLE] = 0. IT IS APPROPRIATE FOR INFINITELY LONG CYLINDERS. HERE NO z-DEPENDENCE, HENCE; $u_r = u_r(r,\theta)e^{-i\omega t}$ AND $u_\theta = u_\theta(r,\theta)e^{-i\omega t}$. IN TERMS OF φ AND $\psi_r, \psi_\theta, \psi_z$ IT AMOUNTS TO HAVIN $\varphi = \varphi(r,\theta)e^{-i\omega t}$, $\psi_r = 0$, $\psi_\theta = 0$, & $\psi_z = \psi_z(r,\theta)e^{-i\omega t}$. IN TERMS OF THE INDEPENDENT POTENTIALS, PLANE-STRAIN AMOU HAVING $\varphi = \varphi(r,\theta)e^{-i\omega t}$, $\psi = 0$, $\chi = \chi(r,\theta)e^{-i\omega t}$. NOTE THAT BY EQS. B 11) B) $\psi_z = -e^{-i\omega t}\nabla^2_{r\theta}\chi$. THUS, PLANE-STRAIN ARE INDEPENDENT OF z. ANY OF THESE EQUIVALENT DEFINITIONS YIELDS $\epsilon_{zz} = \epsilon_{zr} = \epsilon_{z\theta} = 0$.

2) CONSTITUTIVE RELATIONS:

$$\begin{cases} \tau_{rr} = [\lambda_e - i\omega\lambda_v]\Delta + 2(\mu_e - i\omega\mu_v)\epsilon_{rr} \\ \tau_{\theta\theta} = [\lambda_e - i\omega\lambda_v]\Delta + 2(\mu_e - i\omega\mu_v)\epsilon_{\theta\theta} \\ \tau_{zz} = [\lambda_e - i\omega\lambda_v]\Delta \qquad (\tau_{rz} = \tau_{\theta z} = 0) \\ \tau_{r\theta} = [\mu_e - i\omega\mu_v]2\epsilon_{r\theta} \qquad (\Delta = \epsilon_{rr} + \epsilon_{\theta\theta}) \end{cases}$$

2) CONSTITUTIVE RELATIONS:

$\tau_{rr} = M_1\{\lambda_e\Delta + 2\mu_e\epsilon_{rr}\} + N_1$

$\tau_{\theta\theta} = M_1\{\lambda_e\Delta + 2\mu_e\epsilon_{\theta\theta}\} + N_1$

$\tau_{zz} = M_1\{\lambda_e\Delta\} + N_1$, $\tau_{z\theta} = 0$, $\tau_{zr} = 0$,

$\tau_{r\theta} = \mu_e M_1 2\epsilon_{r\theta}$. $(\Delta = \epsilon_{rr} + \epsilon_{\theta\theta})$

M_1 & N_1 ARE GIVEN BELOW IN 8).

3) STRAIN-DISPLACEMENT RELATIONS:

$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$, $2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$, $\epsilon_{zz} = 0$, $\epsilon_{zr} = 0$, $\epsilon_{z\theta} = 0$.

{THESE ARE THE SAME AS IN TABLE 1}

4) DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS:

$\vec{u} = \nabla\varphi + \kappa_2^2(\nabla\chi)\times\hat{e}_z$ OR IN COMPONENT FORM,

$u_r = \frac{\partial\varphi}{\partial r} + \frac{\kappa_2^2}{r}\frac{\partial\chi}{\partial\theta}$, $u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \kappa_2^2\frac{\partial\chi}{\partial r}$, $u_z = 0$

OR IN TERMS OF φ & ψ_z (SINCE $\kappa_2^2\chi = -\nabla^2\chi = \psi_z$), $u_r = \frac{\partial\varphi}{\partial r} + \frac{1}{r}\frac{\partial\psi_z}{\partial\theta}$, $u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi_z}{\partial r}$, $u_z = 0$.

5) STRESS-DISPLACEMENT RELATIONS:

$\tau_{rr} = [\lambda_e - i\omega\lambda_v]\Delta + 2(\mu_e - i\omega\mu_v)\frac{\partial u_r}{\partial r}$

$\tau_{\theta\theta} = [\lambda_e - i\omega\lambda_v]\Delta + 2(\mu_e - i\omega\mu_v)\left[\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right]$, $\tau_{zz} = [\lambda_e - i\omega\lambda_v]\Delta$,

$\tau_{r\theta} = [\mu_e - i\omega\mu_v]\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right]$ $\left(\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)$.

5) STRESS-DISPLACEMENT RELATIONS:

$\tau_{rr} = M_1\{\lambda_e\Delta + 2\mu_e\frac{\partial u_r}{\partial r}\} + N_1$, $\tau_{zz} = M_1\lambda_e\Delta$

$\tau_{\theta\theta} = M_1\{\lambda_e\Delta + 2\mu_e[\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}]\} + N_1$, $\tau_{z\theta} = \tau_{zr} =$

$\tau_{r\theta} = \mu_e M_1[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}]$, $\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$

6) STRAIN-INDEPENDENT POTENTIAL RELATIONS:

$$\begin{cases} \epsilon_{rr} = \frac{\partial^2\varphi}{\partial r^2} + \kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right) \\ \epsilon_{\theta\theta} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right) \\ 2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right) - \kappa_2^2\left(\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right) \\ \epsilon_{zz} = \epsilon_{\theta z} = \epsilon_{zr} = 0 \end{cases}$$

{TO HAVE THESE IN TERMS OF φ & ψ_z WE REPLACE $\kappa_2^2\chi$ BY ψ_z. i.e., →}

$$\begin{cases} \epsilon_{rr} = \frac{\partial^2\varphi}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}\right) \\ \epsilon_{\theta\theta} = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}\right) \\ 2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right) - \left(\frac{\partial^2\psi_z}{\partial r^2} - \frac{1}{r}\frac{\partial\psi_z}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi_z}{\partial\theta^2}\right) \\ \epsilon_{zz} = \epsilon_{zr} = \epsilon_{z\theta} = 0 \end{cases}$$

7) FIELD EQUATIONS:

$$\text{i)}\begin{cases} [\mu_e - i\omega\mu_v]\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right] + [\lambda_e + \mu_e - i\omega(\lambda_v + \mu_v)]\frac{\partial\Delta}{\partial r} + \rho\omega^2 u_r = 0 \\ [\mu_e - i\omega\mu_v]\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right] + [\lambda_e + \mu_e - i\omega(\lambda_v + \mu_v)]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + \rho\omega^2 u_\theta = 0 \end{cases}$$

$\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

NOTE: $\kappa_2^2 = \rho\omega^2/(\mu_e - i\omega\mu_v)$

FORM ii) IS NOT AS USEFUL.

7) FIELD EQUATIONS:

$$\text{i)}\begin{cases} \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} + \left[\frac{[illegible]}{\mu_e(3\alpha + 2\beta - i\omega)}\right]\frac{\partial\Delta}{\partial r} + \kappa_2^2 u_r \\ \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} + \left[\frac{[illegible]}{\mu_e(3\alpha + 2\beta - i\omega)}\right]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + \kappa \end{cases}$$

$\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, $\kappa_2^2 =$

FORM ii) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = -[\lambda_e - i\omega\lambda_v]\kappa_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\frac{\partial^2\varphi}{\partial r^2} + \kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]$

$\tau_{\theta\theta} = -[\lambda_e - i\omega\lambda_v]\kappa_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]$

$\tau_{zz} = -[\lambda_e - i\omega\lambda_v]\kappa_1^2\varphi$

$\tau_{rz} = \tau_{\theta z} = 0$

$\tau_{r\theta} = [\mu_e - i\omega\mu_v]\left[2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right) - \kappa_2^2\left(\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right)\right]$

TO HAVE THESE IN TERMS OF φ & ψ_z WE REPLACE $\kappa_2^2\chi$ BY ψ_z.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = M_1\{-\lambda_e\kappa_1^2\varphi + 2\mu_e[\frac{\partial^2\varphi}{\partial r^2} + \kappa_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]\} + N_1$

$\tau_{\theta\theta} = M_1\{-\lambda_e\kappa_1^2\varphi + 2\mu_e[\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} - \kappa_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]\} + N_1$

$\tau_{zz} = M_1\{-\lambda_e\kappa_1^2\varphi\} + N_1$, $\tau_{z\theta} = 0$, $\tau_{zr} = 0$

$\tau_{r\theta} = \mu_e M_1[2\frac{\partial^2}{\partial r\partial\theta}(\frac{\varphi}{r}) - \kappa_2^2(\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2})]$ WHE

$N_1 =$ [illegible] $[\frac{\omega^2(3\alpha + 4\beta)}{\omega^2 + 4\beta^2}$ [illegible] $(3\alpha + 2\beta - \omega^2/2\beta)$

$M_1 = \frac{\omega^2 [illegible]}{\omega^2 + 4\beta^2}$ {TO HAVE THESE IN TERMS OF φ AND ψ_z WE REPLACE $\kappa_2^2\chi$ BY ψ_z}

9) HELMHOLTZ'S EQUATIONS FOR SCALAR AND VECTOR POTENTIALS:

i) $(\nabla^2 + \kappa_1^2)\varphi = 0$, ii) $(\nabla^2 + \kappa_2^2)\vec{\psi} = 0$ & $\nabla\cdot\vec{\psi} = 0$

9) HELMHOLTZ'S EQUATIONS FOR SCALAR & VECTOR POTENTIALS

3

) IN SIX CASES OF INTEREST. (SEE REMARK 1) BELOW).

[illegible] STRAIN	1) [illegible] ORDINATES	[illegible]
LONG CYLINDERS, HERE THERE IS IT AMOUNTS TO HAVING PLANE-STRAIN AMOUNTS TO THUS, PLANE-STRAIN PROBLEM	2) [illegible] τ_{kk} [illegible] $\tau_{kk}=[(3\lambda_e+2\mu_e)-i\omega(3\lambda_v+2\mu_v)]\epsilon_{kk}$ τ_{kk} = BULK STRESS, ϵ_{kk} = DILATATION	OF φ AN ψ_θ TO HAVING FIELDS ϵ
$=0$, $\tau_{zr}=0$, $\epsilon_{rr}+\epsilon_{\theta\theta})$	[illegible] RELATIONS: $\tau_{ij}=(\lambda_e-i\omega\lambda_v)\delta_{ij}\epsilon_{kk}+2(\mu_e-i\omega\mu_v)\epsilon_{ij}$ $\Delta=\epsilon_{kk}=\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{\phi\phi}=\vec\nabla\cdot\vec u=u_{k,k}$ OR, $\tau_{rr}=(\lambda_e-i\omega\lambda_v)\Delta+2(\mu_e-i\omega\mu_v)\epsilon_{rr}$ $\tau_{\theta\phi}=2(\mu_e-i\omega\mu_v)\epsilon_{\theta\phi}$ $\tau_{\theta\theta}=(\lambda_e-i\omega\lambda_v)\Delta+2(\mu_e-i\omega\mu_v)\epsilon_{\theta\theta}$ $\tau_{r\phi}=2(\mu_e-i\omega\mu_v)\epsilon_{r\phi}$ $\tau_{\phi\phi}=(\lambda_e-i\omega\lambda_v)\Delta+2(\mu_e-i\omega\mu_v)\epsilon_{\phi\phi}$ $\tau_{r\theta}=2(\mu_e-i\omega\mu_v)\epsilon_{r\theta}$.	$\tau_{rr}=[\lambda$ $\tau_{\theta\theta}=[\lambda$ $\tau_{\phi\phi}=[\lambda_e-i\omega\lambda$ $\tau_{r\theta}=[\mu_e-i\omega\mu$
THESE ARE THE SAME AS IN TABLE 1).	3) STRAIN-DISPLACEMENT RELATIONS: $\epsilon_{rr}=\frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$, $\epsilon_{\phi\phi}=\frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$, $2\epsilon_{\theta\phi}=\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{u_\phi}{r}\cot\theta$, $2\epsilon_{r\phi}=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}$, $2\epsilon_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}$ THE ϵ ARE THE SAME AS IN TABLE 1	3) STRAIN $\epsilon_{rr}=\frac{\partial u_r}{\partial r}$, $\epsilon_{\theta\theta}=$
$u_z=0$.	4) [illegible] $\vec u=\nabla[\varphi+\frac{\partial}{\partial r}(r\psi)]+k_2^2[\vec r\psi+\nabla\times(\vec r\chi)]=\nabla[\varphi+\frac{\partial}{\partial r}(r\psi)]-\vec r\nabla^2\psi+\vec r\times[\nabla(\nabla^2\chi)]$. $u_r=\frac{\partial}{\partial r}[\varphi+\frac{\partial}{\partial r}(r\psi)]+k_2^2r\psi=\frac{\partial}{\partial r}[\varphi+\frac{\partial}{\partial r}(r\psi)]-r\nabla^2\psi$. $u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}[\varphi+\frac{\partial}{\partial r}(r\psi)]-\frac{k_2^2}{\sin\theta}\frac{\partial\chi}{\partial\phi}=\frac{1}{r}\frac{\partial}{\partial\theta}[\varphi+\frac{\partial}{\partial r}(r\psi)]-\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\nabla^2\chi)$. $u_\phi=\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[\varphi+\frac{\partial}{\partial r}(r\psi)]+k_2^2\frac{\partial\chi}{\partial\theta}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[\varphi+\frac{\partial}{\partial r}(r\psi)]+\frac{\partial}{\partial\theta}(\nabla^2\chi)$.	$\vec u=\nabla[\varphi+\frac{\partial}{\partial r}(r\psi)]$ $u_r=\frac{\partial}{\partial r}[\varphi+\frac{\partial}{\partial r}(r\psi)]$
RELATIONS: $\tau_{zz}=M_1\Delta+N_1$ $+N_1$, $\tau_{z\theta}=\tau_{zr}=0$, $\Delta=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$ {M_1 & N_1 AS [illegible]	5) [illegible] RELATIONS: $\tau_{ij}=[\lambda_e-i\omega\lambda_v]\delta_{ij}\Delta+2(\mu_e-i\omega\mu_v)\epsilon_{ij}$, $\Delta=\nabla\cdot\vec u=\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\theta}{r}\cot\theta+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$. WHERE THE ϵ_{ij} ARE GIVEN ABOVE IN EQS. 3) IN TERMS OF THE DISPLACEMENT COMPONENTS.	5) $\tau_{rr}=[\lambda_e-i\omega\lambda_v]\Delta$ $\tau_{\theta\theta}=[\lambda_e-i\omega\lambda_v]\Delta$ $\tau_{\phi\phi}=[\lambda_e-i\omega\lambda_v]\Delta$ $\tau_{r\theta}=[\mu_e-i\omega\mu_v]$
$\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{1}{r^2}\frac{\partial u_r}{\partial\theta})$	6) [illegible] $\epsilon_{rr}=\frac{\partial^2}{\partial r^2}[\varphi+\frac{\partial}{\partial r}(r\psi)]+k_2^2\frac{\partial}{\partial r}(r\psi)=\frac{\partial^2\varphi}{\partial r^2}+r\frac{\partial^3\psi}{\partial r^3}+3\frac{\partial^2\psi}{\partial r^2}-k_2^2[r\frac{\partial\psi}{\partial r}+\psi]$. $\epsilon_{\theta\theta}=$ [illegible] $-k_2^2[\psi-\frac{1}{r\sin\theta}\frac{\partial^2\chi}{\partial\theta\partial\phi}-\frac{\cot\theta}{r}\frac{\partial\chi}{\partial\phi}]$. $\epsilon_{\phi\phi}=$ [illegible] $2\epsilon_{\theta\phi}=$ [illegible] $2\epsilon_{r\phi}=$ [illegible] $+k_2^2[\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}-\frac{\partial}{\partial\theta}(\frac{\partial\chi}{\partial r}-\frac{\chi}{r})]$. $2\epsilon_{r\theta}=$ [illegible]	6) $\epsilon_{rr}=\frac{\partial^2}{\partial r^2}[\varphi+\frac{\partial}{\partial r}(r\psi)]$ $\epsilon_{\theta\theta}=[\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}$ $\epsilon_{\phi\phi}=[\frac{1}{r}\frac{\partial}{\partial r}-$ $2\epsilon_{r\theta}=2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta}$ WE WILL SEE BELOW TERMS OF [illegible]
$\frac{\partial\Delta}{\partial r}+k_2^2u_r=0$ $\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+k_2^2u_\theta=0$ $k_1^2=\frac{\rho\omega^2}{\mu_e-i\omega\mu_v}$	7) FIELD EQUATIONS: i) $\nabla^2\vec u+[1+\frac{\lambda_e-i\omega\lambda_v}{\mu_e-i\omega\mu_v}]\nabla(\nabla\cdot\vec u)+k_2^2\vec u=0$ WHERE, $k_2^2=\frac{\omega^2}{c_2^2(1-i\omega N)}$, $N=\frac{\mu_v}{\mu_e}$, $c_2^2=\frac{\mu_e}{\rho}$. THIS CAN BE OPENED UP BY MEANS OF, $\nabla\Delta=\frac{\partial\Delta}{\partial r}\hat e_r+\frac{1}{r}\frac{\partial\Delta}{\partial\theta}\hat e_\theta+\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}\hat e_\phi$, Δ AS GIVEN IN EQ. 5), $\nabla^2\vec u=[\nabla^2u_r-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2\cot\theta}{r^2}u_\theta-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}]\hat e_r+[\nabla^2u_\theta-\frac{u_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}]\hat e_\theta+$ $+[\nabla^2u_\phi-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}]\hat e_\phi$ AND, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$. ii) $-\nabla\times(\nabla\times\vec u)+[2+\frac{\lambda_e-i\omega\lambda_v}{\mu_e-i\omega\mu_v}]\nabla(\nabla\cdot\vec u)+k_2^2\vec u=0$ WHERE k_2^2 IS AS ABOVE.	7) FIELD EQUATIONS i) $[\nabla_1^2u_r-\frac{2u_r}{r^2}-$ $[\nabla_1^2u_\theta-\frac{u_\theta}{r^2\sin^2\theta}$ WHERE, FORM (ii) IS NO
POTENTIAL RELATIONS $(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]\}+N_1$ $-k_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]\}+N_1$ $=0$, $\tau_{z\theta}=0$ $-\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2})$; WHERE, [illegible] IN TERMS OF φ AND ψ_z k_2^2 BY [illegible]	8) STRESS-STRAIN-POTENTIAL RELATIONS: $\tau_{rr}=-(\lambda_e-i\omega\lambda_v)k_1^2\varphi+2(\mu_e-i\omega\mu_v)[\frac{\partial^2}{\partial r^2}(\varphi+\frac{\partial}{\partial r}(r\psi))+k_2^2\frac{\partial}{\partial r}(r\psi)]$ $\tau_{\theta\theta}=-(\lambda_e-i\omega\lambda_v)k_1^2\varphi+2(\mu_e-i\omega\mu_v)[(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\varphi+\frac{\partial}{\partial r}(r\psi))+k_2^2(\psi+\frac{1}{r}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial\chi}{\partial\phi}))]$ $\tau_{\phi\phi}=-(\lambda_e-i\omega\lambda_v)k_1^2\varphi+2(\mu_e-i\omega\mu_v)[(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2})(\varphi+\frac{\partial}{\partial r}(r\psi))+k_2^2(\psi-\frac{\partial}{\partial\theta}(\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}))]$ $\tau_{\theta\phi}=(\mu_e-i\omega\mu_v)[\frac{2}{r^2}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi})(\varphi+\frac{\partial}{\partial r}(r\psi))-k_2^2(\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2}-\frac{\cot\theta}{r}\frac{\partial\chi}{\partial\theta}-\frac{1}{r\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2})]$ $\tau_{r\phi}=(\mu_e-i\omega\mu_v)[\frac{2}{\sin\theta}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\phi})(\varphi+\frac{\partial}{\partial r}(r\psi))+k_2^2(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}-\frac{\partial}{\partial\theta}(\frac{\partial\chi}{\partial r}-\frac{\chi}{r}))]$ $\tau_{r\theta}=(\mu_e-i\omega\mu_v)[2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial\theta})(\varphi+\frac{\partial}{\partial r}(r\psi))+k_2^2\{\frac{\partial\psi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^2\chi}{\partial r\partial\phi}\}]$ THERE ARE [illegible] $\vec u=\nabla\varphi+\nabla\times\vec\psi$ THEN, $\Delta=\nabla\cdot\vec u=$ [illegible]	8) STRESS-STRAIN $\tau_{rr}=-[\lambda_e-i\omega\lambda_v]$ $\tau_{\theta\theta}=-[\lambda_e-i\omega\lambda_v]$ $\tau_{\phi\phi}=-[\lambda_e-i\omega\lambda_v]$ $\tau_{\theta\phi}=0$, $\tau_{r\phi}=$ $\tau_{r\theta}=[\mu_e-i\omega\mu_v]$ WE WILL SEE BELOW TERMS OF [illegible]
[illegible] EQUATIONS $(\nabla^2+k_2^2)\vec\psi=0$ & $\nabla\cdot\vec\psi=0$	9) [illegible] EQS. FOR THE SCALAR AND VECTOR POTENTIALS φ AND $\vec\psi$: i) $(\nabla^2+k_1^2)\varphi=0$, ii) $(\nabla^2+k_2^2)\vec\psi=0$ & $\nabla\cdot\vec\psi=0$	9) [illegible] i) $(\nabla_1^2+k_1^2)\varphi=0$

4

OR.

SEE ARE SAME ... TABLE

EQ. 5).

[illegible] OF INTEREST [illegible] THIS TABLE.

[illegible] TO HAVING $\psi(r,\theta)$, [illegible] IN TERMS OF [illegible] THIS AMOUNT [illegible] TO HAVING [illegible] AND [illegible] HERE $e_{\phi\phi} = \dots$. THIS [illegible]

$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(E-2\mu)}{3\mu-E} = k - \frac{2}{3}\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{3k\nu}{1+\nu} = \frac{3k(3k-E)}{9k-E}$.

THE [illegible] IN TERMS OF λ,μ OR E,μ OR k,μ OR E,ν OR k,ν OR k,E.

$\mu = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(k-\lambda) = \frac{E}{2(1+\nu)} = \frac{3k(1-2\nu)}{2(1+\nu)} = \frac{3kE}{9k-E}$.

$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{3k-\lambda} = \frac{E}{2\mu} - 1 = \frac{3k-2\mu}{2(3k+\mu)} = \frac{3k-E}{6k}$.

$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9k(k-\lambda)}{3k-\lambda} = 2\mu(1+\nu) = \frac{9k\mu}{3k+\mu} = 3k(1-2\nu)$

$k = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\nu)}{3\nu} = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{E\mu}{3(3\mu-E)} = \frac{E}{3(1-2\nu)}$.

[illegible] ARE NOT SO USEFUL.

[illegible] RELATIONS BETWEEN VARIOUS ELASTIC CONSTANTS:

$\frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}$, $\frac{\lambda+2\mu}{2\mu} = \frac{1-\nu}{1-2\nu}$, $\frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu}$, $\frac{\lambda}{\lambda+2\mu} = \frac{\nu}{1-\nu}$,

$\lambda+2\mu = \frac{2\mu(1-\nu)}{1-2\nu}$, $\lambda+\mu = \frac{E}{2(1+\nu)(1-2\nu)}$, $\frac{\lambda}{\lambda+\mu} = 2\nu$,

$\frac{\lambda+\mu}{\lambda+2\mu} = \frac{1}{2(1-\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$.

IF WE DEFINE: $\bar{\lambda} = \frac{2\lambda\mu}{\lambda+2\mu} = \frac{E\nu}{1-\nu^2}$, THEN,

$\bar{\lambda}+\mu = \frac{E}{2(1-\nu)} = \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}$, $\frac{\bar{\lambda}+\mu}{\mu} = \frac{1+\nu}{1-\nu}$, $\frac{\bar{\lambda}}{\bar{\lambda}+2\mu} = \nu = \frac{\lambda}{2(\lambda+\mu)}$

$\bar{\lambda}+2\mu = \frac{E}{1-\nu^2} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}$, $\frac{\bar{\lambda}+2\mu}{2\mu} = \frac{1}{1-\nu} = \frac{2(\lambda+\mu)}{\lambda+2\mu}$.

$\bar{\lambda}$ IS USEFUL BECAUSE REPLACING λ BY $\bar{\lambda}$ IN THE PLANE-STRAIN EQUATIONS, WE GET THE PLANE STRESS EQUATIONS. CONVERSELY, IF WE REPLACE THE E & ν OF PLANE STRESS BY $E/(1-\nu^2)$ AND $\nu/(1-\nu)$ RESPECTIVELY, WE GET THE PLANE-STRAIN EQUATIONS.

[illegible]

$\tau_{rr} = [(\lambda_e - i\omega\lambda_v)\Delta + 2(\mu_e - i\omega\mu_v)e_{rr}]$

$\tau_{\theta\theta} = [(\lambda_e - i\omega\lambda_v)\Delta + 2(\mu_e - i\omega\mu_v)e_{\theta\theta}]$

$\tau_{\phi\phi} = [(\lambda_e - i\omega\lambda_v)\Delta + 2(\mu_e - i\omega\mu_v)e_{\phi\phi}]$, $\tau_{\theta\phi} = 0$, $\tau_{r\phi} = 0$.

$\tau_{r\theta} = [\mu_e - i\omega\mu_v]\,2e_{r\theta}$ $\Delta = e_{rr} + e_{\theta\theta} + e_{\phi\phi}$

3) [illegible]

$e_{rr} = \frac{\partial u_r}{\partial r}$, [illegible] $\quad e_{\phi\phi} = \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta$ [illegible]

[illegible] FORM.

$u_r = \frac{\partial}{\partial r}[\psi + \frac{\partial}{\partial r}(r\Phi)] + k_2^2 r\Phi$, $u_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}[\psi + \frac{\partial}{\partial r}(r\Phi)]$, [illegible]

[illegible] (Δ AS DEFINED [illegible])

$\tau_{rr} = [\lambda_e - i\omega\lambda_v]\Delta + [illegible]$

$\tau_{\theta\theta} = [illegible]$

$\tau_{\phi\phi} = [illegible]$

$\tau_{r\theta} = [\mu_e - i\omega\mu_v][illegible]$

6) [illegible]

$e_{rr} = \frac{\partial^2}{\partial r^2}[\psi + \frac{\partial}{\partial r}(r\Phi)] + k_2^2\frac{\partial}{\partial r}(r\Phi)$

$e_{\theta\theta} = [\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}][\psi + \frac{\partial}{\partial r}(r\Phi)] + k_2^2\Phi$

$e_{\phi\phi} = [\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}][\psi + \frac{\partial}{\partial r}(r\Phi)] + k_2^2\Phi$ $\quad (e_{\theta\phi} = e_{r\phi} = 0)$

$2e_{r\theta} = 2\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\theta}(\psi + \frac{\partial}{\partial r}(r\Phi))] + k_2^2\frac{\partial\Phi}{\partial\theta}$.

WE WILL SEE BELOW THAT IT IS NOT CONVENIENT TO HAVE THESE IN TERMS OF [illegible] A HARDER EQ. TO SOLVE.

7) EQUATIONS OF MOTION:

i) $[\nabla_{r\theta}^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{r^2}] + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}]\frac{\partial\Delta}{\partial r} + k_2^2 u_r = 0$

$[\nabla_{r\theta}^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta}] + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}]\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + k_2^2 u_\theta = 0$

WHERE, $\nabla_{r\theta}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}$.

$\Delta = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\theta}{r}\cot\theta$

$k_2^2 = \rho\omega^2/(\mu_e - i\omega\mu_v)$

FORM (i) IS NOT AS USEFUL.

8) STRESS - DISPLACEMENT - POTENTIAL RELATIONS:

$\tau_{rr} = -[\lambda_e - i\omega\lambda_v]k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\{\frac{\partial^2}{\partial r^2}(\varphi + \frac{\partial}{\partial r}(r\Phi)) + k_2^2\frac{\partial}{\partial r}(r\Phi)\}$

$\tau_{\theta\theta} = -[\lambda_e - i\omega\lambda_v]k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\{(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\varphi + \frac{\partial}{\partial r}(r\Phi)) + k_2^2\Phi\}$

$\tau_{\phi\phi} = -[\lambda_e - i\omega\lambda_v]k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\{(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta})(\varphi + \frac{\partial}{\partial r}(r\Phi)) + k_2^2\Phi\}$

$\tau_{\theta\phi} = 0$, $\tau_{r\phi} = 0$.

$\tau_{r\theta} = [\mu_e - i\omega\mu_v][2\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi + \frac{\partial}{\partial r}(r\Phi))\} + k_2^2\frac{\partial\Phi}{\partial\theta}]$

WE WILL SEE BELOW THAT IT IS NOT CONVENIENT TO HAVE THESE IN TERMS OF [illegible] AND [illegible] HARDER [illegible]

9) [illegible]

i) $(\nabla^2 + k_1^2)\varphi = 0$, ii) $(\nabla^2 + k_2^2)\vec{\Phi} = 0$ & $\nabla\cdot\vec{\Phi} = 0$

[illegible]

GRADIENT: $\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{e}_\phi$

DIVERGENCE: $\nabla\cdot\vec{u} = \frac{\partial u_r}{\partial r} + \frac{2}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\cot\theta}{r}u_\theta + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$.

LAPLACIAN: $\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$.

CURL: $\nabla\times\vec{u} = [\frac{\cot\theta}{r}u_\phi + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}]\hat{e}_r + [\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}]\hat{e}_\theta + [\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}]\hat{e}_\phi$

GRADIENT OF DIVERGENCE: $\nabla(\nabla\cdot\vec{u}) = \{[illegible]\}\hat{e}_r + \{[illegible]\}\hat{e}_\theta + \{[illegible]\}\hat{e}_\phi$

VECTOR LAPLACIAN: $\nabla^2\vec{u} = [\nabla^2 u_r - [illegible]]\hat{e}_r + [\nabla^2 u_\theta - [illegible]]\hat{e}_\theta + [\nabla^2 u_\phi - [illegible]]\hat{e}_\phi$

CURL OF CURL: $\nabla\times(\nabla\times\vec{u}) = [illegible]\hat{e}_r + \{[illegible]\}\hat{e}_\theta + \{[illegible]\}\hat{e}_\phi$.

SOLUTION OF SCALAR HELMHOLTZ EQ. $(\nabla^2 + k^2)\psi(r,\theta,\phi) = 0$.

$\psi(r,\theta,\phi) = \left\{ \begin{matrix} j_n(kr) \\ y_n(kr) \end{matrix} \right\} \left\{ \begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix} \right\} \left\{ \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \right\}$

m, n = INTEGERS, P-FUNCTIONS ARE LEGENDRE FUNCTIONS AND THE j-FUNCTIONS ARE PROPORTIONAL TO THE SPHERICAL BESSEL FUNCTIONS, i.e., $j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z)$, $y_n(z) = \sqrt{\frac{\pi}{2z}}Y_{n+\frac{1}{2}}(z)$.

9) [illegible] CYLINDRICAL [illegible]

GRADIENT: $\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z$.

DIVERGENCE: $\nabla\cdot\vec{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$.

LAPLACIAN: $\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$.

CURL: $\nabla\times\vec{u} = [\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}]\hat{e}_r + [\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}]\hat{e}_\theta + [\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}]\hat{e}_z$.

GRADIENT OF DIVERGENCE: $\nabla(\nabla\cdot\vec{u}) = [illegible]\hat{e}_r + [illegible]\hat{e}_\theta + [illegible]\hat{e}_z$.

VECTOR LAPLACIAN: $\nabla^2\vec{u} = [\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}]\hat{e}_r + [illegible]$

$2\epsilon_{\theta z} = \frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}(\varphi+\frac{\partial\psi}{\partial z}) + \frac{K_2^2}{r}\frac{\partial\psi}{\partial\theta} - K_2^2\frac{\partial^2\chi}{\partial r\partial z}$. $= \frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial}{\partial\theta}[2\frac{\partial^2\psi}{\partial z^2} - \nabla^2\psi] + \frac{\partial^2}{\partial r\partial z}(\nabla^2\chi)$.

$2\epsilon_{rz} = 2\frac{\partial^2}{\partial r\partial z}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial\psi}{\partial r} + \frac{K_2^2}{r}\frac{\partial^2\chi}{\partial\theta\partial z}$. $= 2\frac{\partial^2\varphi}{\partial r\partial z} + \frac{\partial}{\partial r}[2\frac{\partial^2\psi}{\partial z^2} - \nabla^2\psi] - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}(\nabla^2\chi)$.

$2\epsilon_{r\theta} = 2\frac{\partial^2}{\partial r\partial\theta}[\frac{1}{r}(\varphi+\frac{\partial\psi}{\partial z})] - K_2^2(\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2})$. $= 2\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial\varphi}{\partial\theta}] + \frac{2}{r}\frac{\partial}{\partial\theta}[\frac{\partial^2\psi}{\partial r\partial z} - \frac{1}{r}\frac{\partial\psi}{\partial z}] + [\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}](\nabla^2\chi)$.

$2\epsilon_{r\theta}$

$\epsilon_{zz} =$

7) FI[illegible] EQUATIONS:

i): $\nabla^2\vec{u} + [1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}]\nabla(\vec\nabla\cdot\vec u) + K_2^2\vec u = 0$ WHERE,

$K_2^2 = \rho\omega^2/(\mu_e - i\omega\mu_v)$. THIS EQ. CAN BE OPENED UP BY MEANS OF

$\nabla^2\vec u = [\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}]\hat e_r + [\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}]\hat e_\theta + [\nabla^2 u_z]\hat e_z$,

AND $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$.

THE GRADIENT OPERATOR & THE DILATATION Δ ARE GIVEN ABOVE.

ii): $-\vec\nabla\times(\vec\nabla\times\vec u) + [2 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}]\vec\nabla(\vec\nabla\cdot\vec u) + K_2^2\vec u = 0$

NOTE : $\nabla^2\vec u = \vec\nabla(\vec\nabla\cdot\vec u) - \vec\nabla\times(\vec\nabla\times\vec u)$.

7) FIELD EQUATIONS:

i): $(1-\frac{i\omega}{2\beta})(\nabla^2\vec u + \frac{\lambda_e+\mu_e}{\mu_e}\vec\nabla(\vec\nabla\cdot\vec u) + \frac{\omega^2}{c_s^2}\vec u) + \frac{3\alpha}{2\beta}[\nabla^2\vec u + \frac{1}{3}\vec\nabla(\vec\nabla\cdot\vec u) + \frac{\omega^2}{c_s^2}\vec u] + \frac{\omega^2}{c_s^2}\vec u + \frac{i\omega(3\alpha+2\beta)}{c_s^2}\vec u = 0$

OR IN COMPONENT FORM,

$[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}] + V\frac{\partial\Delta}{\partial r} + Wu_r = 0$,

$[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}] + \frac{V}{r}\frac{\partial\Delta}{\partial\theta} + Wu_\theta = 0$, $[\nabla^2 u_z] + V\frac{\partial\Delta}{\partial z} + Wu_z = 0$,

WHERE, $V = \frac{\alpha\mu_e + (\lambda_e+\mu_e)(2\beta - i\omega)}{\mu_e(3\alpha+2\beta-i\omega)}$, $W = \frac{\omega^2(3\alpha+4\beta-i\omega)+2\beta i\omega(3\alpha+2\beta)}{c_s^2(3\alpha+2\beta-i\omega)}$

ii). $(1-\frac{i\omega}{2\beta})[\frac{\lambda_e+2\mu_e}{\mu_e}\vec\nabla(\vec\nabla\cdot\vec u) - \vec\nabla\times(\vec\nabla\times\vec u) + \frac{\omega^2}{c_s^2}\vec u] + \frac{3\alpha}{2\beta}[\frac{4}{3}\vec\nabla(\vec\nabla\cdot\vec u) - \vec\nabla\times(\vec\nabla\times\vec u) + \frac{\omega^2}{c_s^2}\vec u] + \frac{\omega^2}{c_s^2}[1 + i(\frac{3\alpha+2\beta}{\omega})]\vec u = 0$

$\Delta = \vec\nabla\cdot\vec u$ IS GIVEN IN 5).

7) FIELD

i) $\{[\mu_e$

$\{[\mu_e$

$\Delta = \frac{\partial}{\partial}$

NOTE :

FORM

8) STRESS - INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = (\lambda_e - i\omega\lambda_v)\nabla^2\varphi + 2(\mu_e - i\omega\mu_v)[\frac{\partial^2}{\partial r^2}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]$.

$\tau_{\theta\theta} = (\lambda_e - i\omega\lambda_v)\nabla^2\varphi + 2(\mu_e - i\omega\mu_v)[(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r})(\varphi+\frac{\partial\psi}{\partial z}) - K_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]$

$\tau_{zz} = (\lambda_e - i\omega\lambda_v)\nabla^2\varphi + 2(\mu_e - i\omega\mu_v)[\frac{\partial^2}{\partial z^2}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial\psi}{\partial z}]$.

$\tau_{\theta z} = (\mu_e - i\omega\mu_v)[\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}(\varphi+\frac{\partial\psi}{\partial z}) + \frac{K_2^2}{r}\frac{\partial\psi}{\partial\theta} - K_2^2\frac{\partial^2\chi}{\partial r\partial z}]$.

$\tau_{rz} = (\mu_e - i\omega\mu_v)[2\frac{\partial^2}{\partial r\partial z}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial\psi}{\partial r} + \frac{K_2^2}{r}\frac{\partial^2\chi}{\partial\theta\partial z}]$

$\tau_{r\theta} = (\mu_e - i\omega\mu_v)[2\frac{\partial^2}{\partial r\partial\theta}\{\frac{1}{r}(\varphi+\frac{\partial\psi}{\partial z})\} - K_2^2\{\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\}]$.

THESE ARE EQS 5) WITH $\Delta = \nabla^2\varphi = -K_1^2\varphi$ AND ϵ_{ij} AS GIVEN IN 6).

8) STRESS - INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[\frac{\partial^2}{\partial r^2}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]\} + N_1$

$\tau_{\theta\theta} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r})(\varphi+\frac{\partial\psi}{\partial z}) - K_2^2\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})]\} + N_1$

$\tau_{zz} = M_1\{\lambda_e\nabla^2\varphi + 2\mu_e[\frac{\partial^2}{\partial z^2}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial\psi}{\partial z}]\} + N_1$

$\tau_{\theta z} = M_1\mu_e[\frac{2}{r}\frac{\partial^2}{\partial\theta\partial z}(\varphi+\frac{\partial\psi}{\partial z}) + \frac{K_2^2}{r}\frac{\partial\psi}{\partial\theta} - K_2^2\frac{\partial^2\chi}{\partial r\partial z}]$

$\tau_{r\theta} = M_1\mu_e[2\frac{\partial^2}{\partial r\partial\theta}\{\frac{1}{r}(\varphi+\frac{\partial\psi}{\partial z})\} - K_2^2\{\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\frac{\partial\chi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\}]$

$\tau_{rz} = M_1\mu_e[2\frac{\partial^2}{\partial r\partial z}(\varphi+\frac{\partial\psi}{\partial z}) + K_2^2\frac{\partial\psi}{\partial r} + \frac{K_2^2}{r}\frac{\partial^2\chi}{\partial\theta\partial z}]$, $M_1 = \frac{\omega^2 - i\omega 2\beta}{\omega^2+4\beta^2}$.

$N_1 = \frac{\alpha\nabla^2\varphi(3\lambda_e+2\mu_e)}{\omega^2+(3\alpha+2\beta)^2}\{\frac{\omega^2(3\alpha+4\beta) - i\omega 2\beta(3\alpha+2\beta-\omega^2/2\beta)}{\omega^2+4\beta^2}\}$

8) STRES

$\tau_{rr} = -[$

$\tau_{\theta\theta} = -[$

$\tau_{zz} = -[$

$\tau_{rz} =$

$\tau_{r\theta} = [$

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9) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i) $(\nabla^2 + K_1^2)\varphi = 0$, ii) $(\nabla^2 + K_2^2)\vec\psi = 0$ & $\vec\nabla\cdot\vec\psi = 0$

WHERE, $K_1^2 = \omega^2/c_d^2(1 - i\omega M)$, $K_2^2 = \omega^2/c_s^2(1 - i\omega N)$,

$M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e+2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$.

NOTE THESE ARE HELMHOLTZ'S EQS. WITH COMPLEX K'S.

9) HELMHOLTZ'S EQS. FOR THE SCALAR & VECTOR POTENTIALS:

i) $(\nabla^2 + K_1^2)\varphi = 0$, ii) $(\nabla^2 + K_2^2)\vec\psi = 0$ & $\vec\nabla\cdot\vec\psi = 0$

WHERE, $K_1^2 = \frac{\omega^2}{c_d^2}\{\frac{\omega^2 + 4\beta^2 + 4\alpha(3\alpha+4\beta)c_s^2/c_d^2}{\omega^2 + (2\beta + 4\alpha c_s^2/c_d^2)^2}\} +$

$+ i\omega\{\frac{(\omega^2+4\beta^2)(3\alpha+2\beta)/c_d^2 + 4\alpha[2\beta(3\alpha+2\beta) - \omega^2]c_s^2/c_d^4}{\omega^2 + (2\beta + 4\alpha c_s^2/c_d^2)^2}\}$ [illegible] $\frac{\omega^2}{c_s^2}[1 + i(\frac{3\alpha+2\beta}{\omega})]$

AND, $K_2^2 = \frac{\omega^2}{c_s^2}[1 + i\frac{2\beta}{\omega}]$ $c_s^2 = \mu_e/\rho$, $c_d^2 = (\lambda_e + 2\mu_e)/\rho$.

9) HELM

i) (∇

WHERE

$M = \frac{\lambda_e}{\lambda_e}$

HERE

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10) HELMHOLTZ'S EQS. FOR INDEPENDENT SCALAR POTENTIALS:

i) $(\nabla^2 + K_1^2)\varphi = 0$, ii) $(\nabla^2 + K_2^2)\psi = 0$, iii) $(\nabla^2 + K_2^2)\chi = 0$,

WHERE $K_1^2 = \omega^2/c_d^2(1 - i\omega M)$, $K_2^2 = \omega^2/c_s^2(1 - i\omega N)$

AND M, N, c_d^2, c_s^2 ARE AS GIVEN ABOVE IN 9).

10) [illegible]

i) $(\nabla^2 + K_1^2)\varphi = 0$, ii) $(\nabla^2 + K_2^2)\psi = 0$, iii) $(\nabla^2 + K_2^2)\chi = 0$

WHERE K_1^2 AND K_2^2 ARE AS GIVEN ABOVE IN 9).

THE LIMIT $c_s^2 \ll c_d^2$ (OR $\mu_e \ll \lambda_e$) IMPLIES THAT THE MATERIAL IS RUBBER. ONLY THEN WILL K_1^2 DEPEND ON c_d^2 ALONE.

i) $(\nabla^2$

WHERE

11) [illegible]

IT CAN BE SHOWN THAT THE SOLENOIDAL (ie, $\vec\nabla\cdot\vec\psi = 0$) SOLUTION OF $(\nabla^2 + K_2^2)\vec\psi = 0$, WHICH IS THE SOLUTION OF $\vec\nabla\times(\vec\nabla\times\vec\psi) = K_2^2\vec\psi$, IS GIVEN BY, $\vec\psi = \vec\nabla\times(\psi\hat e_z) + \vec\nabla\times[\vec\nabla\times(\hat e_z\chi)] = (\vec\nabla\psi)\times\hat e_z + \vec\nabla(\frac{\partial\chi}{\partial z}) - \hat e_z K_2^2\chi$ (A) IN CYLINDRICAL COORDS ONLY.

HERE $\psi(r,\theta,z)$ AND $\chi(r,\theta,z)$ ARE TWO SCALAR FUNCTIONS WHICH SATISFY SCALAR HELMHOLTZ'S EQS. 10) ii) & iii).

EQ. (A) IN COMPONENT-FORM, IS:

$\psi_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\chi}{\partial r\partial z}$, $\psi_\theta = -\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\chi}{\partial\theta\partial z}$, $\psi_z = \frac{\partial^2\chi}{\partial z^2} + K_2^2\chi$ (B),

WHICH RELATE THE THREE CYLINDRICAL COMPONENTS OF THE VECTOR POTENTIAL $\vec\psi$ TO THE TWO SCALAR FUNCTIONS ψ AND χ. SUBSTITUTING $\vec\psi$ FROM EQ.(A) INTO $\vec u = \vec\nabla\varphi + \vec\nabla\times\vec\psi$ YIELDS THE DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS GIVEN ABOVE IN 4). NOTE THAT ψ IS NOT $|\vec\psi|$. IN THE ABSENCE OF VISCOSITY (ie, M = N = 0, ie, ELASTICITY) EQ (A) ABOVE IS THE SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQ. NOW WITH A REAL, RATHER THAN COMPLEX PROPAGATION CONSTANT. HENCE, FOR VIBRATIONS HARMONIC IN TIME, DAMPING IS ACCOUNTED FOR BY MEANS OF COMPLEX PROPAGATION CONSTANTS K_1^2 & K_2^2 RELATED TO THE ELASTIC AND VISCOUS MATERIAL CONSTANTS BY THE FORMULAS ABOVE IN 9).

K_1 IS ASSOCIATED WITH LONGITUDINAL OR DILATATIONAL WAVES AND K_2 WITH TRANSVERSE OR SHEAR WAVES.

IN CYLINDRICAL COORDS. ONE CAN CHECK THAT, $\nabla^2(\vec\nabla\varphi) = \vec\nabla(\nabla^2\varphi)$ AND THAT, $\vec\nabla\times(\nabla^2\vec\psi) = \nabla^2(\vec\nabla\times\vec\psi)$.

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$= K_2^2\vec\psi$

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12) REMARKS:

a) THE BASIC RESULT OF THIS TABLE, AND THAT IS WHY IT WAS ORGANIZED AS SHOWN ABOVE, IS THAT IF WE INTRODUCE INDEPENDENT SCALAR POTEN SATISFIED IN EACH ONE OF THE CASES CONSIDERED, AND THEN THE DISPLACEMENT AND STRESS FIELD COMPONENTS ARE GIVEN IN TERMS OF THE PROBLEMS, AND WHICH IS COMMON IN OTHER DISCIPLINES (ie, ELECTRODYNAMICS), IS APPARENTLY NOVEL IN VISCOELASTICITY, AND WITH ELECTRODYNAMIC THEORY WILL QUICKLY NOTICE THE ANALOGY OF THE PRESENTATION IN THIS TABLE WITH THAT OF ELECTROMAGNET

b) IT IS EVIDENT THAT IN ALL THE CASES IN THIS CHART, WE RECOVER THE SIMPLER RESULTS OF DYNAMICAL ELASTICITY, IN THE ABSENCE (

c) IN CYLINDRICAL COORDS. THE PLANE-STRESS RESULTS CAN BE QUICKLY OBTAINED FROM THE PLANE STRAIN RESULTS IN CASES D&E) ABOVE B

d) IN THE CASE OF PLANE-STRAIN (OR PLANE-STRESS IN VIEW OF C) ABOVE) WITH AXIAL SYMMETRY ABOUT THE z-AXIS, CAN BE OBTAINED FROM D&E) BY

e) A CASE OF SOME INTEREST WHICH REDUCES THE TWO VISCOUS CONSTANTS TO ONE, OCCURS WHEN THE BULK-VISCOSITY IS ZERO OR NEARLY SO. THIS AM

$$2\epsilon_{r\theta}=2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right)-\kappa_2^2\left(\frac{\partial^2\chi}{\partial r^2}-\frac{1}{r}\frac{\partial\chi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right)$$

[BY ψ_z, ie, —]

$$\epsilon_{zz}=\epsilon_{\theta z}=\epsilon_{zr}=0$$

$$2\epsilon_{r\theta}=2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right)-\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)$$

$$\epsilon_{zz}=\epsilon_{zr}=\epsilon_{z\theta}=0$$

[illegible] EQUATIONS:

i) [illegible]

[illegible] $+\rho\omega^2 u_\theta=0$

$\Delta=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

NOTE: $\kappa_2^2=\rho\omega^2/(\mu_e-i\omega\mu_v)$

FORM ii) IS NOT AS USEFUL

7) FIELD EQUATIONS:

i) $\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\left[\frac{\alpha\mu_e+(\lambda_e+\mu_e)(2\beta-i\omega)}{\mu_e(3\alpha+2\beta-i\omega)}\right]\frac{\partial\Delta}{\partial r}+\kappa_2^2u_r=0$

$\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\left[\frac{\alpha\mu_e+(\lambda_e+\mu_e)(2\beta-i\omega)}{\mu_e(3\alpha+2\beta-i\omega)}\right]\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+\kappa_2^2u_\theta=0$

$\Delta=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, $\kappa_2^2=\frac{\rho\omega^2}{\mu_e-i\omega\mu_v}$

FORM ii) IS NOT AS USEFUL.

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8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr}=-[\lambda_e-i\omega\lambda_v]\kappa_1^2\varphi+2(\mu_e-i\omega\mu_v)\left[\frac{\partial^2\varphi}{\partial r^2}+\kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]$

$\tau_{\theta\theta}=-[\lambda_e-i\omega\lambda_v]\kappa_1^2\varphi+2(\mu_e-i\omega\mu_v)\left[\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}-\kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]$

$\tau_{zz}=-[\lambda_e-i\omega\lambda_v]\kappa_1^2\varphi$

$\tau_{rz}=\tau_{\theta z}=0$

$\tau_{r\theta}=[\mu_e-i\omega\mu_v]\left[2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right)-\kappa_2^2\left(\frac{\partial^2\chi}{\partial r^2}-\frac{1}{r}\frac{\partial\chi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right)\right]$

TO HAVE THESE IN TERMS OF φ & ψ_z WE REPLACE $\kappa_2^2\chi$ BY ψ_z.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr}=M_1\left\{-\lambda_e\kappa_1^2\varphi+2\mu_e\left[\frac{\partial^2\varphi}{\partial r^2}+\kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]\right\}+N_1$

$\tau_{\theta\theta}=M_1\left\{-\lambda_e\kappa_1^2\varphi+2\mu_e\left[\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}-\kappa_2^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)\right]\right\}+N_1$

$\tau_{zz}=M_1\{-\lambda_e\kappa_1^2\varphi\}+N_1$, $\tau_{z\theta}=0$, $\tau_{zr}=0$

$\tau_{r\theta}=\mu_e M_1\left[2\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi}{r}\right)-\kappa_2^2\left(\frac{\partial^2\chi}{\partial r^2}-\frac{1}{r}\frac{\partial\chi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right)\right]$ WHERE,

$N_1=\frac{\alpha\kappa_1^2\varphi(3\lambda_e+2\mu_e)}{\omega^2+(3\alpha+2\beta)^2}\left[\frac{\omega^2(3\alpha+4\beta)-i\omega2\beta(3\alpha+2\beta-\omega^2/2\beta)}{\omega^2+4\beta^2}\right]$

$M_1=\frac{\omega^2-i\omega2\beta}{\omega^2+4\beta^2}$

TO HAVE THESE IN TERMS OF φ AND ψ_z WE REPLACE $\kappa_2^2\chi$ BY ψ_z

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9) HELMHOLTZ'S EQUATIONS FOR SCALAR AND VECTOR POTENTIALS:

i) $(\nabla^2+\kappa_1^2)\varphi=0$, ii) $(\nabla^2+\kappa_2^2)\vec{\psi}=0$ & $\vec{\nabla}\cdot\vec{\psi}=0$

WHERE, $\kappa_1^2=\omega^2/c_d^2(1-i\omega M)$, $\kappa_2^2=\omega^2/c_s^2(1-i\omega N)$

$M=\frac{\lambda_v+2\mu_v}{\lambda_e+2\mu_e}$, $N=\frac{\mu_v}{\mu_e}$, $c_d^2=\frac{\lambda_e+2\mu_e}{\rho}$, $c_s^2=\frac{\mu_e}{\rho}$.

HERE $\vec{\psi}=\hat{e}_z\psi_z(r,\theta)=-\hat{e}_z\nabla^2\chi(r,\theta)=\hat{e}_z\kappa_2^2\chi$, HENCE ii) ABOVE IS EQUIVALENT TO EITHER ALTERNATIVE FORM GIVEN IN 10).

9) HELMHOLTZ'S EQS. FOR SCALAR & VECTOR POTENTIALS φ & $\vec{\psi}$:

i) $(\nabla^2+\kappa_1^2)\varphi=0$, ii) $(\nabla^2+\kappa_2^2)\vec{\psi}=0$ & $\vec{\nabla}\cdot\vec{\psi}=0$

$$\kappa_1^2=\frac{\omega^2}{c_d^2}\left\{\frac{\omega^2-[illegible]+4\alpha(3\alpha+4\beta)c_s^2/c_d^2}{\omega^2+(2\beta+4\alpha c_s^2/c_d^2)^2}\right\}+i\omega\left\{\frac{(\omega^2+4\beta^2)(3\alpha+2\beta)c_d^2+4\alpha[2\beta(3\alpha+2\beta)-\omega^2]c_s^2/c_d^2}{\omega^2+(2\beta+4\alpha c_s^2/c_d^2)^2}\right\}\xrightarrow[c_s^2\ll c_d^2]{}\frac{\omega^2}{c_d^2}\left[1+i\left(\frac{3\alpha+2\beta}{\omega}\right.\right.$$

$$\kappa_2^2=\frac{\omega^2}{c_s^2}\left[1+i\frac{2\beta}{\omega}\right]$$

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10) HELMHOLTZ'S EQUATION FOR INDEPENDENT SCALAR POTENTIALS:

i) $(\nabla^2+\kappa_1^2)\varphi=0$ ii) $(\nabla^2+\kappa_2^2)\psi_z=0$

WHERE κ_1^2, κ_2^2, M, N, c_d^2, c_s^2, ARE AS GIVEN ABOVE IN 9).

ii) IS EQUIVALENT TO : $-\nabla^2[(\nabla^2+\kappa_2^2)\chi]=0$

10) HELMHOLTZ'S EQS. FOR INDEPENDENT SCALAR POTENTIALS:

i) $(\nabla^2+\kappa_1^2)\varphi=0$ ii) $(\nabla^2+\kappa_2^2)\psi_z=0$

WHERE κ_1^2 & κ_2^2 ARE AS GIVEN ABOVE IN 9). SINCE $\psi_z=\kappa_2^2\chi$ EQ. ii) IS EQUIVALENT TO $-\nabla^2[(\nabla^2+\kappa_2^2)\chi]=0$.
THE CONDITION $c_s^2\ll c_d^2$ MEANS THE MATERIAL IS RUBBER.
ONLY FOR $c_s^2\ll c_d^2$ WILL κ_1^2 DEPEND ON c_d^2 ALONE.

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11) SOLENOIDAL SOLUTION OF VECTOR HELMHOLTZ'S EQUATION IN PLANE POLAR COORDINATES:

THE SOLUTION FOLLOWS THAT GIVEN TO THE LEFT FOR GENERAL CYLINDRICAL COORDINATES BUT NOW WITHOUT THE z-DEPENDENCE. HENCE, ONE CAN ALSO SHOW THAT THE SOLENOIDAL SOLUTION OF $(\nabla^2+\kappa_2^2)\vec{\psi}=0$, WHICH IS ALSO THE SOLUTION OF $\vec{\nabla}\times(\vec{\nabla}\times\vec{\psi})=\kappa_2^2\vec{\psi}$, IS GIVEN BY,

$$\vec{\psi}=\vec{\nabla}_{r\theta}\times(\vec{\nabla}_{r\theta}\times(\hat{e}_z\chi))=\hat{e}_z\kappa_2^2\chi \qquad (A)$$

WHERE $\chi(r,\theta)$ IS A SCALAR FUNCTION SATISFYING THE SCALAR HELMHOLTZ EQUATION $(\nabla^2+\kappa_2^2)\chi=0$. EQUATION (A) IN COMPONENT FORM IS , $\psi_r=0$, $\psi_\theta=0$, $\psi_z=\kappa_2^2\chi$ (B), WHICH RELATE THE CYLINDRICAL COMPONENTS OF $\vec{\psi}$ TO THE SCALAR FUNCTION χ. SUBSTITUTING SOLUTION (A) INTO $\vec{u}=\vec{\nabla}\varphi+\vec{\nabla}\times\vec{\psi}$ YIELDS $\vec{u}=\vec{\nabla}\varphi-\kappa_2^2(\vec{\nabla}\chi)\times\hat{e}_z$ WHICH ARE THE DISPLACEMENT-INDEPENDENT POTENTIAL RELATIONS FOUND IN EQS. 4) ABOVE.

WE REPEAT THAT IN THE ABSENCE OF VISCOSITY (ie, M = N = 0 , viz, ELASTICITY), SOLUTION (A) IS THEN THE SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQUATION WITH A REAL RATHER THAN COMPLEX PROPAGATION CONSTANT, PROVIDED THAT χ SATISFIES A SCALAR HELMHOLTZ EQ. WITH THE SAME REAL PROPAGATION CONSTANT $k_2=\omega/c_s$. HENCE, DAMPING DUE TO VISCOSITY IS ACCOUNTED FOR IN THESE MODELS BY MEANS OF COMPLEX WAVENUMBERS κ_1^2 AND κ_2^2 RELATED TO THE MATERIAL PROPERTIES AS SHOWN ABOVE IN 9). κ_1 & κ_2 ARE ASSOCIATED WITH LONGITUDINAL AND TRANSVERSE WAVES RESPECTIVELY. NOTE THAT IN THE MAXWELL MODEL THE PREVIOUS STATEMENT IS TRUE ONLY WHEN $c_s^2\ll c_d^2$. OTHERWISE κ_1 "CONTAINS" BOTH TYPES OF WAVES.

AR POTENTIALS WHICH SATISFY THE HELMHOLTZ'S EQS. WITH COMPLEX κ's IN 10), THEN THE FIELD EQS. 7) ARE AUTOMATICALLY
IMS OF THE POTENTIALS BY EQS. 4) & 8) RESPECTIVELY. THIS TECHNIQUE, WHICH IS BASIC TO SET-UP AND SOLVE BOUNDARY VALUE
CITY, AND THAT IS WHY THIS CHART WAS DEVELOPPED FOR ALL THESE CASES AND VISCOELASTIC MODELS. THE READER FAMILIAR
ROMAGNETIC THEORY IN A COULOMB ($\vec{\nabla}\cdot\vec{\psi}=0$) RATHER THAN A LORENTZ (ie, $\vec{\nabla}\cdot\vec{\psi}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0$) GAUGE.
ABSENCE OF VISCOSITY. (ie, $\lambda_v=\mu_v=0$ FOR THE KELVIN-VOIGT MODEL AND/OR $\alpha=\beta=0$ FOR THE MAXWELL MODEL.)
E) ABOVE BY THE STANDARD TRICK OF REPLACING λ_e IN THE PLANE-STRAIN CASES BY $\bar{\lambda}_e$ DEFINED TO BE : $\bar{\lambda}_e=2\lambda_e\mu_e/(\lambda_e+2\mu_e)$.
M D) & E) BY SETTING $u_\theta=0$ AND $\frac{\partial}{\partial\theta}$[ANY VARIABLE] = 0 , THEN THERE IS ONLY r-DEPENDENCE. (ie, $u_r=u_r(r)e^{i\omega t}$ & $\varphi=\varphi(r)$ & $u_r=\partial\varphi/\partial r$.
30. THIS AMOUNTS TO HAVING $3\lambda_v+2\mu_v=0$ (KELVIN-VOIGT) OR $3\alpha+2\beta=0$ (MAXWELL), SO THE FORMULAS THEN SIMPLIFY A LITTLE.

$2\epsilon_{\theta\phi} = \frac{2}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\theta\partial\phi} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\varphi}{\partial\phi} + \frac{2}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\theta\partial\phi} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{2}{r\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta\partial\phi} - \frac{2\cot\theta}{r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi} - k_2^2\left[\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial\chi}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}\right]$

[illegible] $+ k_2^2\left[\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{\partial}{\partial\theta}\left(\frac{\partial\chi}{\partial r} - \frac{\chi}{r}\right)\right]$.

[illegible] $+ k_2^2\left[\frac{\partial\psi}{\partial\theta}\right.$ [illegible] $\left. + \frac{1}{\sin\theta}\frac{\partial^2\chi}{\partial r\partial\phi}\right]$.

[illegible]

i) $\nabla^2\vec{u} + \left[1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}\right]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + k_2^2\vec{u} = 0$ WHERE, $k_2^2 = \frac{\omega^2}{c_s^2(1-i\omega N)}$, $N = \frac{\mu_v}{\mu_e}$, $c_s^2 = \frac{\mu_e}{\rho}$.

THIS CAN BE OPENED UP BY MEANS OF, $\nabla\Delta = \frac{\partial\Delta}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\Delta}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}\hat{e}_\phi$, Δ AS GIVEN IN EQ. 5),

$\nabla^2\vec{u} = \left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}u_\theta - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right]\hat{e}_r + \left[\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right]\hat{e}_\theta +$

$+\left[\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right]\hat{e}_\phi$ AND, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$.

ii) $-\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) + \left[2 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}\right]\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + k_2^2\vec{u} = 0$ WHERE k_2^2 IS AS ABOVE.

8) STRESS-INDEPENDENT POTENTIAL RELATIONS:

$\tau_{rr} = -(\lambda_e - i\omega\lambda_v)k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) + k_2^2\frac{\partial}{\partial r}(r\psi)\right]$

$\tau_{\theta\theta} = -(\lambda_e - i\omega\lambda_v)k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) + k_2^2\left(\psi + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\chi}{\partial\phi}\right)\right)\right]$

$\tau_{\phi\phi} = -(\lambda_e - i\omega\lambda_v)k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) + k_2^2\left(\psi - \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}\right)\right)\right]$

$\tau_{\theta\phi} = (\mu_e - i\omega\mu_v)\left[\frac{2}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) - k_2^2\left(\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial\chi}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}\right)\right]$

$\tau_{r\phi} = (\mu_e - i\omega\mu_v)\left[\frac{2}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\phi}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) + k_2^2\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{\partial}{\partial\theta}\left(\frac{\partial\chi}{\partial r} - \frac{\chi}{r}\right)\right)\right]$

$\tau_{r\theta} = (\mu_e - i\omega\mu_v)\left[2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right) + k_2^2\left\{\frac{\partial\psi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial^2\chi}{\partial r\partial\phi}\right\}\right]$

THESE ARE EQS. 6) SUBSTITUTED INTO 2) IF, $\vec{u} = \nabla\varphi + \vec{\nabla}\times\vec{\psi}$ THEN, $\Delta = \nabla\cdot\vec{u} = \nabla^2\varphi = -k_1^2\varphi$.

9) HELMHOLTZ'S EQS. FOR THE SCALAR AND VECTOR POTENTIALS φ AND $\vec{\psi}$:

i) $(\nabla^2 + k_1^2)\varphi = 0$, ii) $(\nabla^2 + k_2^2)\vec{\psi} = 0$ & $\vec{\nabla}\cdot\vec{\psi} = 0$

WHERE, $k_1^2 = \frac{\omega^2}{c_d^2(1-i\omega M)}$, $k_2^2 = \frac{\omega^2}{c_s^2(1-i\omega N)}$, $M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e}$, $N = \frac{\mu_v}{\mu_e}$, $c_d^2 = \frac{\lambda_e + 2\mu_e}{\rho}$, $c_s^2 = \frac{\mu_e}{\rho}$.

THESE ARE HELMHOLTZ EQS. WITH COMPLEX PROPAGATION CONSTANTS. THE VECTOR EQ. IS FURTHER SUBJECTED TO THE SOLENOIDAL (GAUGE) CONDITION SHOWN.

10) HELMHOLTZ'S EQUATIONS FOR THE INDEPENDENT SCALAR POTENTIALS:

i) $(\nabla^2 + k_1^2)\varphi = 0$, ii) $(\nabla^2 + k_2^2)\psi = 0$, iii) $(\nabla^2 + k_2^2)\chi(r,\theta,\phi) = 0$.

WHERE THE COMPLEX k_1^2 & k_2^2 ARE AS GIVEN IN 9) AND ∇^2 AS GIVEN IN 7).

11) SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQ. IN GENERAL SPHERICAL COORDINATES:

THE SOLENOIDAL SOLUTION OF $(\nabla^2 + k_2^2)\vec{\psi} = 0$ IS THE SOLUTION OF $\vec{\nabla}\times(\vec{\nabla}\times\vec{\psi}) = k_2^2\vec{\psi}$ AND IT IS,

$$\vec{\psi} = \mathrm{curl}\,\vec{r}\psi + \mathrm{curl\,curl}\,\vec{r}\chi = (\vec{\nabla}\psi)\times\vec{r} + \vec{\nabla}\chi + k_2^2\vec{r}\chi + \vec{\nabla}(\vec{\nabla}\chi\cdot\vec{r}) = (\nabla\psi)\times\vec{r} + \vec{\nabla}\left[\frac{\partial}{\partial r}(r\chi)\right] - \vec{r}\,\nabla^2\chi \qquad (A)$$

PROVIDED $\psi(r,\theta,\phi)$ & $\chi(r,\theta,\phi)$ SATISFY SCALAR HELMHOLTZ EQS. $(\nabla^2 + k_2^2)\psi$ (or χ) $= 0$.

NOTE THAT ψ IS NOT $|\vec{\psi}|$. OPENING SOLUTION (A) INTO ITS COMPONENTS WE FIND,

$$\psi_r = \frac{\partial^2}{\partial r^2}(r\chi) + rk_2^2\chi = -\frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial\chi}{\partial\theta} - \frac{1}{r\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}, \quad \psi_\theta = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{1}{r}\frac{\partial\chi}{\partial\theta} + \frac{\partial^2\chi}{\partial r\partial\theta}, \quad \psi_\phi = -\frac{\partial\psi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial^2\chi}{\partial r\partial\phi}. \qquad (B)$$

WHICH RELATE ψ_r, ψ_θ, ψ_ϕ TO ψ AND χ.

USING THE AUXILIARY RELATIONS $\nabla\times[(\vec{\nabla}\psi)\times\vec{r}] = \vec{\nabla}\psi + k_2^2\vec{r}\psi + \nabla(\vec{\nabla}\psi\cdot\vec{r})$ AND $\nabla\times(\vec{r}\chi) = (\vec{\nabla}\chi)\times\vec{r}$ WE CAN NOW SUBSTITUTE SOLUTION (A) INTO $\vec{u} = \nabla\varphi + \vec{\nabla}\times\vec{\psi}$ TO OBTAIN $\vec{u} = \vec{\nabla}\left[\varphi + \frac{\partial}{\partial r}(r\psi)\right] + k_2^2\left[\vec{r}\psi + (\vec{\nabla}\psi)\times\vec{r}\right]$ AS WE HAD IN EQS. 4. IN THE ABSENCE OF VISCOSITY (ie, M = N = 0, viz, ELASTICITY) EQ. (A) ABOVE IS THE SOLENOIDAL SOLUTION OF THE VECTOR HELMHOLTZ EQ. NOW WITH A REAL RATHER THAN COMPLEX PROPAGATION CONSTANT $k_2^2 = \omega^2/c_s^2$ PROVIDED ψ & χ SATISFY SCALAR HELMHOLTZ EQS. WITH THE SAME REAL PROPAGATION CONSTANT k_2^2.

12) REMARKS:

i) REMARKS a), b), e) OF THE CYLINDRICAL CASES TO THE LEFT, ALSO HOLD HERE.

ii) NOTE AGAIN THAT IN SPITE OF THE APPARENT GENERALITY OF THE SOLENOIDAL SOLUTION 11) A) ABOVE WITH ITS VECTOR OPERATORS, IT IS ONLY VALID IN SPHERICAL COORDINATES. NOTE FURTHER THAT THE TWO PARTS (ie, curl $\vec{r}\psi$ AND curl curl $\vec{r}\chi$) ARE DIFFERENT EVEN WHEN $\psi = \chi$. ACTUALLY, THEY ARE PERPENDICULAR WHEN $\psi = \chi$.

iii) THERE IS NO WAY TO HAVE A PLANE-STRAIN OR PLANE-STRESS SITUATION IN SPHERICAL COORDS. THE CLOSEST WE CAN GET TO THEM IS THE AXIALLY-SYMMETRIC CASE G) TO THE RIGHT.

$2\epsilon_{r\theta} = 2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right)\right] + k_2^2\frac{\partial\psi}{\partial\theta}$

WE WILL SEE BELOW THAT IT IS NOT CO
TERMS OF φ & $\psi_\phi = -\frac{\partial\psi}{\partial\theta}$ SINCE ψ_ϕ SAT

7) FIELD EQUATIONS:

i) $\left[\nabla_{r\theta}^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{r^2}\right] + \left[1 + \right.$

$\left[\nabla_{r\theta}^2 u_\theta - \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta}\right] + \left[1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu}\right.$

WHERE, $\nabla_{r\theta}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} +$

$\Delta = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} +$

$k_2^2 = \rho\omega^2/(\mu_e - i\omega\mu_v)$

FORM ii) IS NOT AS USEFUL.

8) STRESS-INDEPENDENT POTENTIAL R

$\tau_{rr} = -[\lambda_e - i\omega\lambda_v]k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\frac{\partial^2}{\partial r^2}\right.($

$\tau_{\theta\theta} = -[\lambda_e - i\omega\lambda_v]k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\left(\frac{1}{r}\right.\right.$

$\tau_{\phi\phi} = -[\lambda_e - i\omega\lambda_v]k_1^2\varphi + 2(\mu_e - i\omega\mu_v)\left[\left(\frac{1}{r}\right.\right.$

$\tau_{\theta\phi} = 0$, $\tau_{r\phi} = 0$.

$\tau_{r\theta} = [\mu_e - i\omega\mu_v]\left[2\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial\theta}\left(\varphi + \frac{\partial}{\partial r}(r\psi)\right)\right.\right.$

WE WILL SEE BELOW THAT IT IS NOT CONV
TERMS OF φ & $\psi_\phi = -\frac{\partial\psi}{\partial\theta}$ SINCE ψ_ϕ SATI

9) HELMHOLTZ'S EQS. FOR THE SCALAR

i) $(\nabla_{r\theta}^2 + k_1^2)\varphi = 0$, ii) (

WHERE, $k_1^2 = \frac{\omega^2}{c_d^2(1-i\omega M)}$, $k_2^2 = \frac{\omega^2}{c_s^2(1-i\omega N)}$

AND, $c_d^2 = (\lambda_e + 2\mu_e)/\rho$, $c_s^2 = \mu_e/\rho$.

THESE ARE HELMHOLTZ EQS. WITH COM
SINCE, $\vec{\psi} = \psi_\phi(r,\theta)\hat{e}_\phi$ AND, $\nabla_{r\theta}^2(\psi_\phi\hat{e}_\phi)$
TURNS OUT THAT THE EQ FOR ψ_ϕ IS: ∇

10) HELMHOLTZ'S EQS. FOR INDEPENDENT

i) $(\nabla_{r\theta}^2 + k_1^2)\varphi = 0$, ii) $(\nabla_{r\theta}^2 + k_2^2)\psi =$

WHERE k_1^2 & k_2^2 ARE THE COMPLEX QUAN
IF WE SET $\psi_\phi = -\frac{\partial\psi}{\partial\theta}$ IN THE LAST EQ. GI
THIS IS WHY IT IS MORE CONVENIENT TO
TERMS OF ψ RATHER THAN ψ_ϕ.

11) SOLENOIDAL SOLUTION OF VECTOR HEL

THE SOLENOIDAL SOLUTION OF $(\nabla_{r\theta}^2 + k_2^2$

$\vec{\psi} = (\vec{\nabla}_{r\theta}\psi)\times\vec{r} = \vec{\nabla}\times(\vec{r}\psi(r,\theta))$

OR IN COMPONENT FORM, $\psi_r = 0$, ψ_θ
(NOTE THAT ψ IS NOT $|\vec{\psi}|$.) PROVIDED THA
SUBSTITUTING SOLUTION (A) INTO $\vec{u} =$
$\vec{u} = \vec{\nabla}_{r\theta}\left[\varphi + \frac{\partial}{\partial r}(r\psi)\right] + k_2^2\vec{r}\psi$ AS WE HAD A
SOLUTION (A) HOLDS IN THE ABSENCE OF V
ELASTICITY). IT IS THEN THE SOLUTION OF
EQ. WITH A REAL PROPAGATION CONSTAN
SATISFIES THE SCALAR HELMHOLTZ EG
GATION CONSTANT. NOTE THAT THE LA
HELMHOLTZ EQ. WITH $k^2 = k_2^2 - 1/r^2\mathrm{si}$
TO SOLVE THAN 10) ii) BUT IT IS STILL
IS POSSIBLE FOR k^2 AS GENERAL AS, k
$+ \frac{1}{r^2\sin^2\theta}h(\phi)$ WHERE f, g, h, ARE CONSTA
FUNCTIONS OF THE INDICATED

12) REMARKS:

i) REMARKS a), b) & e) OF THE CYLINDRI
ALSO HOLD HERE. REMARK a) IS THE
ii) THE MAXWELL MODEL IN SPHERICAL CO
AS AN EXERCISE TO THE READER.
iii) THIS ENTIRE CHART, WHICH IS VALID FO
DENCE OF THE FORM $e^{-i\omega t}$, IS A PARTIC
COMPANION "TABLE 1" VALID FOR ARBIT
(G. GAUNAURD, 1975).

EQ. (A)

ABOVE
THAT
UALLY,
OORDS.

$\tau_{\theta\theta} = [\frac{1}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}][\varphi + \frac{\partial}{\partial r}(r\psi)]$ [illegible]

[illegible]

FORM [illegible]

[illegible] RELATIONS:

$\tau_{rr} =$ [illegible]

$\tau_{\theta\theta} =$ [illegible]

$\tau_{\phi\phi} =$ [illegible]

$\tau_{\theta\phi} = 0$, $\tau_{r\phi} = 0$.

$\tau_{r\theta} =$ [illegible]

WE [illegible] CONVENIENT TO HAVE THESE IN TERMS OF [illegible]

[illegible] $(\nabla_0^2 + k_s^2)\vec\psi = 0$ & $\nabla\cdot\vec\psi = 0$

WHERE, [illegible]

AND [illegible]

THESE ARE [illegible] COMPLEX PROPAGATION CONSTS. SINCE, [illegible] TURNS OUT [illegible] $\nabla_0^2\psi_\phi + [k_s^2 - \frac{1}{r^2\sin^2\theta}]\psi_\phi = 0$

[illegible] SCALAR POTENTIALS:

[illegible]

WHERE [illegible] ARE THE COMPLEX QUANTITIES GIVEN ABOVE IN 9). IF WE SET $\psi_\phi = -\frac{\partial\chi}{\partial\theta}$ IN THE LAST EQ. GIVEN IN 9) WE GET 10)(c). THIS [illegible] CONVENIENT TO HAVE EQS 4), 6) & 8) IN TERMS OF [illegible], RATHER THAN [illegible].

11) [illegible] VECTOR HELMHOLTZ EQUATION:

THE [illegible] IS NOW GIVEN BY, [illegible]

OR IN COMPONENT FORM, [illegible]

NOTE THAT [illegible] THAT [illegible] SATISFIES [illegible] ELDS.

THE [illegible] HAD ABOVE IN EQS. 4),

[illegible] IN THE ABSENCE OF VISCOSITY (i.e., DYNAMICAL ELASTICITY) [illegible] OF THE VECTOR HELMHOLTZ'S EQ. [illegible] CONSTANT PROVIDED THAT [illegible] SATISFIES THE SCALAR HELMHOLTZ EQ. WITH THE SAME PROPAGATION CONSTANT. NOTE THAT THE LAST EQ. GIVEN IN 10) IS HELMHOLTZ EQ. WITH $K^2 = K_s^2 - 1/r^2\sin^2\theta$. THIS EQ. IS HARDER TO SOLVE THAN [illegible] BUT [illegible] STILL SEPARABLE. SEPARATION IS POSSIBLE FOR K^2 AS GENERAL AS, $K^2 = K_0^2 - f(r) - \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\phi)$ WHERE [illegible] ARE CONSTANTS OR ARBITRARY FUNCTIONS OF THE INDICATED VARIABLES.

12) REMARKS:

i) REMARKS a), b) & e) OF THE CYLINDRICAL CASES TO THE LEFT ALSO HOLD HERE. REMARK d) IS THE MOST IMPORTANT ONE

ii) THE MAXWELL MODEL IN SPHERICAL COORDINATES WILL BE LEFT AS AN EXERCISE TO THE READER.

iii) THIS ENTIRE CHART, WHICH IS VALID FOR HARMONIC TIME-DEPENDENCE OF THE FORM $e^{-i\omega t}$, IS A PARTICULAR CASE OF THE COMPANION TABLE 1 VALID FOR ARBITRARY TIME-DEPENDENCE. (G. GAUNAURD, 1975).

[illegible] $]\hat{e}_\theta + [\frac{\partial\psi_\theta}{\partial r} + \frac{\psi_\theta}{r} - \frac{1}{r}\frac{\partial\psi_r}{\partial\theta}]\hat{e}_\phi$

GRADIENT OF DIVERGENCE: $\nabla(\nabla\cdot\vec\psi) = \{\frac{\partial^2\psi_r}{\partial r^2} + \frac{2}{r}\frac{\partial\psi_r}{\partial r} - \frac{2\psi_r}{r^2} + \frac{1}{r}\frac{\partial^2\psi_\theta}{\partial r\partial\theta}$ [illegible] $- \frac{\cot\theta}{r^2}\psi_\theta +$ [illegible] $- \frac{1}{r^2\sin\theta}\frac{\partial\psi_\phi}{\partial\phi}\}\hat{e}_r +$ [illegible] $\}\hat{e}_\phi$

VECTOR LAPLACIAN: $\nabla^2\vec\psi = [\nabla^2\psi_r - \frac{2\psi_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\psi_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial\psi_\phi}{\partial\phi}]\hat{e}_r + [\nabla^2\psi_\theta - \frac{\psi_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial\psi_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial\psi_\phi}{\partial\phi}]\hat{e}_\theta + [\nabla^2\psi_\phi - \frac{\psi_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial\psi_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial\psi_\theta}{\partial\phi}]\hat{e}_\phi$

CURL OF CURL: $\nabla\times(\nabla\times\vec\psi) = [-\frac{1}{r^2}\frac{\partial^2\psi_r}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi_r}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi_r}{\partial\phi^2} +$ [illegible] $+ \frac{1}{r^2}\frac{\partial\psi_\theta}{\partial\theta} - \frac{\cot\theta}{r^2}\psi_\theta + \frac{1}{r}\frac{\partial^2\psi_\theta}{\partial\theta\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial\psi_\phi}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial^2\psi_\phi}{\partial\phi\partial r}]\hat{e}_r +$ [illegible] $- \frac{2}{r}\frac{\partial\psi_\theta}{\partial r} - \frac{\partial^2\psi_\theta}{\partial r^2} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi_\theta}{\partial\phi^2} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial\psi_\phi}{\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi_\phi}{\partial\phi\partial\theta}\}\hat{e}_\theta + \{\frac{1}{r\sin\theta}\frac{\partial^2\psi_r}{\partial r\partial\phi} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial\psi_\theta}{\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi_\theta}{\partial\theta\partial\phi} - \frac{\partial^2\psi_\phi}{\partial r^2} - \frac{2}{r}\frac{\partial\psi_\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi_\phi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi_\phi}{\partial\theta} + \frac{\psi_\phi}{r^2\sin^2\theta}\}\hat{e}_\phi$.

SOLUTION OF SCALAR HELMHOLTZ EQ. $(\nabla^2 + k^2)\varphi(r,\theta,\phi) = 0$.

$$\varphi(r,\theta,\phi) = \begin{Bmatrix} r^{-1/2}J_{n+1/2}(kr) \\ r^{-1/2}Y_{n+1/2}(kr) \end{Bmatrix}\begin{Bmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{Bmatrix}\begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$$

[illegible] INTEGERS, θ-FUNCTIONS ARE LEGENDRE FUNCTIONS AND THE r-FUNCTIONS ARE PROPORTIONAL TO THE SPHERICAL BESSEL FUNCTIONS, i.e., $j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z)$, $y_n(z) = \sqrt{\frac{\pi}{2z}}Y_{n+\frac{1}{2}}(z)$.

4) CYLINDRICAL COORDINATES OPERATORS:

GRADIENT: $\nabla\varphi = \frac{\partial\varphi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{e}_\theta + \frac{\partial\varphi}{\partial z}\hat{e}_z$.

DIVERGENCE: $\nabla\cdot\vec\psi = \frac{\partial\psi_r}{\partial r} + \frac{\psi_r}{r} + \frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta} + \frac{\partial\psi_z}{\partial z}$.

LAPLACIAN: $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}$.

CURL: $\nabla\times\vec\psi = [\frac{1}{r}\frac{\partial\psi_z}{\partial\theta} - \frac{\partial\psi_\theta}{\partial z}]\hat{e}_r + [\frac{\partial\psi_r}{\partial z} - \frac{\partial\psi_z}{\partial r}]\hat{e}_\theta + [\frac{\partial\psi_\theta}{\partial r} + \frac{\psi_\theta}{r} - \frac{1}{r}\frac{\partial\psi_r}{\partial\theta}]\hat{e}_z$

GRADIENT OF DIVERGENCE: $\nabla(\nabla\cdot\vec\psi) = [\frac{\partial^2\psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_r}{\partial r} - \frac{\psi_r}{r^2} + \frac{1}{r}\frac{\partial^2\psi_\theta}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi_\theta}{\partial\theta} + \frac{\partial^2\psi_z}{\partial r\partial z}]\hat{e}_r + [\frac{1}{r}\frac{\partial^2\psi_r}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial\psi_r}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi_\theta}{\partial\theta^2} + \frac{1}{r}\frac{\partial^2\psi_z}{\partial\theta\partial z}]\hat{e}_\theta + [\frac{\partial^2\psi_r}{\partial z\partial r} + \frac{1}{r}\frac{\partial\psi_r}{\partial z} + \frac{1}{r}\frac{\partial^2\psi_\theta}{\partial z\partial\theta} + \frac{\partial^2\psi_z}{\partial z^2}]\hat{e}_z$.

VECTOR LAPLACIAN: $\nabla^2\vec\psi = [\nabla^2\psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial\psi_\theta}{\partial\theta}]\hat{e}_r + [\nabla^2\psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial\psi_r}{\partial\theta}]\hat{e}_\theta + [\nabla^2\psi_z]\hat{e}_z$

CURL OF CURL: $\nabla\times(\nabla\times\vec\psi) = [\frac{1}{r}\frac{\partial^2\psi_\theta}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial\psi_\theta}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2\psi_r}{\partial\theta^2} - \frac{\partial^2\psi_r}{\partial z^2} + \frac{\partial^2\psi_z}{\partial z\partial r}]\hat{e}_r + [\frac{1}{r}\frac{\partial^2\psi_z}{\partial z\partial\theta} - \frac{\partial^2\psi_\theta}{\partial z^2} - \frac{\partial^2\psi_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial\psi_\theta}{\partial r} + \frac{\psi_\theta}{r^2} + \frac{1}{r}\frac{\partial^2\psi_r}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi_r}{\partial\theta}]\hat{e}_\theta + [\frac{\partial^2\psi_r}{\partial r\partial z} - \frac{\partial^2\psi_z}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_r}{\partial z} - \frac{1}{r}\frac{\partial\psi_z}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi_z}{\partial\theta^2} + \frac{1}{r}\frac{\partial^2\psi_\theta}{\partial\theta\partial z}]\hat{e}_z$.

SOLUTION OF SCALAR HELMHOLTZ' EQ. $(\nabla^2 + k^2)\varphi(r,\theta,z) = 0$.

$$\varphi(r,\theta,z) = \begin{Bmatrix} J_n(\lambda r) \\ Y_n(\lambda r) \end{Bmatrix}\begin{Bmatrix} e^{-\gamma z} \\ e^{+\gamma z} \end{Bmatrix}\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

WHERE n = INTEGER AND $\lambda^2 = \frac{\omega^2}{c^2} + \gamma^2 = k^2 + \gamma^2$.
IF $\delta^2 = k^2 - \gamma^2$, THEN THERE ARE TWO ALTERNATIVES,

$$\varphi = \begin{Bmatrix} J_n(\delta r) \\ Y_n(\delta r) \end{Bmatrix}\begin{Bmatrix} \cos\gamma z \\ \sin\gamma z \end{Bmatrix}\begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} \quad \text{OR,} \quad \varphi = \begin{Bmatrix} I_n(\delta r) \\ K_n(\delta r) \end{Bmatrix}\begin{Bmatrix} \cos\gamma z \\ \sin\gamma z \end{Bmatrix}\begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}$$

FOR $\delta > 0$ OR $\delta < 0$ RESPECTIVELY. (n = INTEGER).

REFERENCES: ALTHOUGH THE TECHNIQUE OF INTRODUCING INDEPENDENT SCALAR POTENTIALS WHICH SATISFY TELEGRAPH EQS. (OR HELMHOLTZ'S EQS WITH COMPLEX PROPAGATION CONSTANTS) SO THAT ALL THE STRESSES & DISPLACEMENTS ARE DETERMINED FROM THOSE POTENTIALS, IS APPARENTLY NOVEL IN VISCOELASTICITY, THERE IS MUCH OTHER LITERATURE ON VISCOELASTIC MODELS. FOR EXAMPLE,

1) WITOLD NOWACKI, "DYNAMICS OF ELASTIC SYSTEMS", JOHN WILEY & SONS, INC, 1963 (CHAP. 2)
2) A. CEMAL ERINGEN, "MECHANICS OF CONTINUA" JOHN WILEY & SONS, INC. 1967, (CHAPTER 9).

NOT MUCH APPLIED MATERIAL IS GIVEN IN THESE REFERENCES BEYOND THE FIELD EQUATIONS & CONSTITUTIVE RELATIONS, USUALLY IN GENERAL TENSOR FORM. OTHER MORE GENERAL VISCOELASTIC MODELS (SUCH AS THE "STANDARD" VISCOELASTIC MODEL, WHICH IS A PARALLEL COMBINATION OF A MAXWELL AND A KELVIN-VOIGT MODELS AT EACH MATERIAL POINT OF THE BODY, AND WHICH IS VERY COMPLICATED TO HANDLE) ARE DISCUSSED IN GENERAL IN THE ABOVE REFERENCES.

SOME OF THE BASIC EQS. OF STATIC ELASTICITY (i.e, NO TIME-DEPENDENCE & NO VISCOSITY) IN SIX ORTHOGONAL COORDINATE SYSTEMS HAVE BEEN DERIVED BY STANLEY HALPERSON (M.S. THESIS, CATHOLIC UNIVERSITY, WASH. D.C., APRIL 1967.). THIS IS ONE OF THE NICEST COMPILATIONS OF THIS SORT WE HAVE SEEN.

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in a more or less complicated fashion depending on the model being used. For example, the relation is more complicated for the Maxwell than for the Kelvin model, in any coordinate system. These tables provide us with the methodology and information needed to solve boundary-value-problems of viscoelasto-dynamics for spherical or cylindrical geometries, in the Kelvin-Voigt or Maxwell models. The approach and material covered in these tables seems to be novel as applied to viscoelasticity. The reader familiar with electro-magnetic theory will quickly notice the analogy of the presentation in this table to that of electrodynamic theory in the less familiar Coulomb ($\vec{\nabla} \cdot \vec{\psi} = 0$), rather than Lorentz (ie, $\vec{\nabla} \cdot \vec{\psi} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$) gauge.

To verify that all the entries in the tables check with each other, we will illustrate the use of the formulas by analyzing one example. We arbitrarily select column D, which deals with the Kelvin-Voigt model in cylindrical coordinates for the subcase of plane-strain.

1. Definition of Plane-Strain

In plane-strain, the strain tensor is two-dimensional and all strain components with a subindex z (axial coordinate) vanish. The stress-tensor is not, and there is a nonvanishing normal stress τ_{zz}. Another equivalent way to define this situation is by stating that the axial displacement u_z vanishes, and that the other two displacement components u_r and u_θ are z-independent. Plane-strain is a particularly useful approximation for bodies which are very long in the z-direction.

2. Constitutive Relations

These are the relations between the stress and strain field components. There are three normal and one shear-stress components related to two normal and one shear-strain components. Stress and strain are independent concepts. Only when these two notions are linked through constitutive relations, is that a viscoelastic theory is formed. The first of these four relations is,

$$\tau_{rr} = (\lambda_e - i\omega\lambda_v) \ \Delta + 2(\mu_e - i\omega\mu_v) \ \varepsilon_{rr}$$

where Δ is the trace of the strain tensor (ie, $\varepsilon_{rr} + \varepsilon_{\theta\theta}$) and it is called the dilatation. The other three relations are shown in Table 2.

3. Strain-Displacement Relations

Since the elastic or viscous constants do not appear in these relations, they are the same as in elasticity, namely,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2\varepsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

all other strain-components vanishing by definition of plane-strain.

4. Displacement - Independent Potential Relation

These relations are intimately linked to the solenoidal solution of the vector Helmholtz equation in plane polar coordinates which is discussed below in item (11). The relations can be written as a single vector equation as follows,

$$\vec{u} = \vec{\nabla}\phi + (\vec{\nabla}\psi_z) \times \hat{e}_z$$

where ϕ and ψ_z are the two independent scalar potentials needed in this plane-strain formulation. It is shown in item (11) that $\psi_z = \kappa_2^2 \chi$, where χ is a solution of the scalar Helmholtz's equation $(\nabla^2 + \kappa_2^2)\chi = 0$.

5. Stress - Displacement Relations

If the strain-displacement relations (3) are substituted into the constitutive relations (2), the result is the stress-displacement relations. The first such relation is,

$$\tau_{rr} = \left[\lambda_e - i\omega\lambda_v\right]\Delta + 2\left[\mu_e - i\omega\mu_v\right]\frac{\partial u_r}{\partial r}$$

where the dilatation Δ is now expressed in terms of the displacement components as follows,

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} .$$

6. Strain - Independent Potential Relations

If the displacement-independent potential relations (4) are substituted into the strain-displacement relations (3), the result is the strain-potential relation. The first of these three is,

$$\epsilon_{rr} = \frac{\partial^2\phi}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}\right)$$

and analogously for the other two. Here ϕ and ψ_z are the two independent scalar potentials. Note that ψ_z is the axial component of the vector potential $\vec{\psi}$. Further, $\psi_z = \kappa_2^2\chi = -\nabla^2\chi$ where χ is another independent scalar potential which can also be used here. (See item (11) below)

7. Field Equations

These are the Navier equations given initially in (1). For harmonic time-dependence they can be written as follows,

$$\nabla^2 \vec{u} + \left[1 + \frac{\lambda_e - i\omega\lambda_v}{\mu_e - i\omega\mu_v}\right] \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \kappa_2^2 \vec{u} = 0$$

where $\kappa_2^2 = \rho\omega^2 [\mu_e - i\omega\mu_v]^{-1}$, and Δ = dilatation $= \vec{\nabla} \cdot \vec{u}$. For plane-strain in cylindricals, the quantities $\nabla^2 \vec{u}$ and $\vec{\nabla}\Delta$ take on the following simplified forms,

$$\nabla^2 \vec{u} = \left\{\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right\} \hat{\varepsilon}_r + \left\{\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right\} \hat{\varepsilon}_\theta$$

$$\text{grad } \Delta = \frac{\partial \Delta}{\partial r} \hat{\varepsilon}_r + \frac{1}{r}\frac{\partial \Delta}{\partial \theta} \hat{\varepsilon}_\theta$$

and also we have,

$$\vec{u} = u_r \hat{\varepsilon}_r + u_\theta \hat{\varepsilon}_\theta$$

and, $$\nabla^2 u_r = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right) u_r (r,\theta).$$

8. Stress - Independent Potential Relations

If the strain-potential relations (6) are substituted into the constitutive relations (2), we obtain the stress-potential relations. One form of the first of these relations is,

$$\tau_{rr} = \left[\lambda_e - i\omega\lambda_v\right] \Delta + 2\left[\mu_e - i\omega\mu_v\right] \left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial \psi_z}{\partial \theta}\right]\right\}$$

where, $\Delta = \vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot [\vec{\nabla}\phi + \vec{\nabla} \times \vec{\psi}] = \nabla^2\phi = -\kappa_1^2 \phi.$

The second form can be found by setting, $\psi_z = \kappa_2^2 \chi$. The formulas for the other three stresses present in this case can be found in Table 2.

9. Helmholtz's Equations For The Scalar And Vector Potentials

Substituting $u = \text{grad } \phi + \text{curl } \vec{\psi}$ into the Navier equations written in the alternative form,

$$(\lambda^* + 2\mu^*) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) - \mu^* \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) + \mu^* \kappa_2^2 \vec{u} = 0$$

where $\lambda^* = \lambda_e - i\omega\lambda_v$, $\mu^* = \mu_e - i\omega\mu_v$, and $\mu^* \kappa_2^2 = \rho\omega^2$ we eventually find,

$(\lambda^* + 2\mu^*)\, \vec{\nabla} \{\nabla^2\phi + \kappa_1{}^2\phi\} + \mu^*\, \vec{\nabla} \times \{\nabla^2\vec{\psi} - \vec{\nabla}(\vec{\nabla}\cdot\vec{\psi}) + \kappa_2{}^2\,\vec{\psi}\} = 0$ where $\kappa_1{}^2 = \rho\omega^2\,[\lambda_e + 2\mu_e - i\omega(\lambda_v + 2\mu_v)]^{-1}$ and $\kappa_2{}^2$ is as we defined it above in item (7). It is obvious that the above expression is satisfied if,

$\nabla^2\phi + \kappa_1{}^2\phi = 0$ and $\nabla^2\psi + \kappa_2{}^2\,\vec{\psi} = 0$ provided that $\vec{\nabla}\cdot\vec{\psi} = 0$ (ie, "Coulomb" gauge).

10. Helmholtz's Equations For The Independent Scalar Potentials

Since in this case, $\vec{\psi} = \hat{\varepsilon}_z\,\psi_z$ then, $\nabla^2\,\vec{\psi} = \hat{\varepsilon}_z\,\nabla^2\,\psi_z$, and the result is,

$$(\nabla^2 + \kappa_1{}^2)\,\phi = 0, \qquad (\nabla^2 + \kappa_2{}^2)\,\psi_z = 0$$

11. Solenoidal Solution of The Vector Helmholtz Equation

It is not hard to show that the solenoidal solution of $(\nabla^2 + \kappa_2{}^2)\,\vec{\psi} = 0$

is given by, $\vec{\psi} = \vec{\nabla} \times [\vec{\nabla} \times (\hat{\varepsilon}_z\,\chi)] = \hat{\varepsilon}_z\,\kappa_2{}^2\,\chi$ where $\chi\,(r,\theta)$ is a scalar function satisfying the scalar Helmholtz's equation $(\nabla^2 + \kappa_2{}^2)\,\chi = 0$. In component form, the above relation is $\psi_r = 0$, $\psi_\theta = 0$, and $\psi_z = \kappa_2{}^2\,\chi$. These are the relations between the three cylindrical components of the vector potential $\vec{\psi}$ and the scalar function $\chi\,(r,\theta)$. Note that, indeed, $\vec{\nabla}\cdot\vec{\psi} = 0$. Note that in the absence of viscosity (ie, $\lambda_v = 0$, $\mu_v = 0$) the complex propagation constants κ_1 and κ_2 become real. Thus viscous damping is accounted for in these models by complex propagation constants, related to the material "constants" as shown above in items (7) and (9).

12. Remarks

a) In cylindrical coordinates the plane-stress results are derivable from the plane-strain results by replacing the elastic constant λ_e in the plane-strain results, by the fictitious "constant" λ_e defined to be, $\lambda_e = 2\lambda_e\mu_e \,|(\lambda_e + 2\mu_e)$. In spherical coordinates there is no way to define the plane-stress subcase. In plane-stress, the stress tensor is two-dimensional, which means that all stress-components with a subindex z, vanish. The strain-tensor is not two-dimensional and there is an ε_{zz} nonvanishing strain. This case is ideally suited for bodies which are very thin in one direction, the z-direction.

b) Plane-strain (or plane-stress) with axial-symmetry about the z-direction can be obtained from the results in the eleven items above by merely setting $u_\theta = 0$ and $\frac{\partial}{\partial\theta}$ [any variable] = 0. There is only radial dependence in this situation.

c) All the above applies to the Kelvin-Voigt model in cylindrical (actually, plane polar) coordinates, for the plane-strain subcase. This is Column D of Table 2. There are twelve cases covered in Tables 1 and 2 and this one was intended as an example to show how the various entries are derived from the others, how they check with each other, and how one could proceed to drive any other case. We have used the information in these tables to rigorously set up and solve acoustic scattering problems where the scattering bodies are elastic cylinders and spheres with their outer surface coated with layers of viscoelastically absorbing materials.[6] One reason to have written this report is to gather these fundamental viscoelasticity relations, which are not available elsewhere to this degree of detail and approach, in one document that we could refer to in future work as the place where the theoretical background is derived and presented, in the form in which we will use it.

d) Some authors have stated that in order to solve problems involving absorptive bodies one must deal with Helmholtz's equation with a complex propagation constant κ. This is indeed correct but there is much more to it than just that. Just with a complex κ we would not know how the real and imaginary parts of κ are related to the elastic and viscous material constants of the solid in the various viscoelastic models that one could use. In fact, we would not know this relationship in any model. Furthermore, we would not know how to relate the stresses and displacements in the body to the solution of that Helmholtz equation with a complex κ. Hence, although the idea is correct, in practice one really needs to derive all the detailed information contained in these tables, and that is why we developed them. Careful examination of these tables shows that the problem is really harder than anticipated, since we must solve not one but several (ie, three) Helmholtz's equations with various (ie, two) complex propagation constants which are different, and then go through various other sets of equations (ie, 4 and 8) to obtain the displacements and stresses from the solutions of those Helmholtz's equations. We finally point out that this procedure yields different results in each one of the various models and cases presented there.

e) Column A, Table 1 shows some general formulas valid for all coordinate systems. Note, however, that since the solution of the vector telegraph equation varies with the coordinate system, not many general entries can be filled in that column.

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[6] G. C. Gaunaurd, "Sound Scattering from an Elastic Cylinder Covered With a Viscoelastic Coating", JASA 58, S101, 1975. Also, Proc. of III, U.S.-Fed. Republic of Germany, Hydroacoustics Symposium, Munich, Germany, Vol 1, Part II, pp 4-31, May 1975 (U)

or also, $$\phi(x,t) = A\ e^{\pm \kappa x + i\omega t} \quad \text{where,} \quad \kappa^2 = -\frac{\omega^2}{C_p^2\ (1 + i\omega P)} \tag{3}$$

Here C_p is real and the $\pm$ signs in front of κx describe waves travelling to the right or left of some origin. (If one were interested in time-dependence $\exp(-i\omega t)$ we would replace i by $-i$ in all these results.)

Now let us consider the standard one-dimensional wave equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{C^2}\ \frac{\partial^2\phi}{\partial t^2} \tag{4}$$

The solution of eq. (4) for harmonic time-dependence $\exp(i\omega t)$ can analogously be written as,

$$\phi(x,t) = A\ e^{i[\pm \kappa x + \omega t]} \quad \text{where} \quad \kappa^2 = \frac{\omega^2}{C^2} \tag{5}$$

or also, $$\phi(x,t) = A\ e^{\pm \kappa x + i\omega t} \quad \text{where} \quad \kappa^2 = \frac{\omega^2}{C^2}\ .$$

It is obvious that solutions (2) and (5) and also (3) and (6) can be made equal by setting $C^2 = C_p^2\ (1 + i\omega P)$. It is clear that we can solve the telegraph equation with a real propagation speed C_p by ignoring the damping term, ie, by solving the wave-equation with a suitably chosen complex propagation speed. Complex speeds are artificially produced by fictitious complex elastic constants that one can introduce for this purpose.

Let us now look at the case of shear waves in a solid. Define a complex shear modulus $\mu^* = \mu' + i\mu'' = \mu'\ (1 + i\delta)$, where $\delta = \mu''/\mu'$.

The equations of linear viscoelasticity are the same of those of elasticity (ie, no viscosity) if λ_e and μ_e in the elasticity equations are replaced by the

quantities, $$\lambda^* = \lambda_e + i\omega\lambda_v \qquad \mu^* = \mu_e + i\omega\mu_v \tag{7}$$

NOTE: This is true only in the Kelvin-Voigt viscoelastic model with assumed time-dependence of the form $\exp(i\omega t)$.

Clearly $\mu_e = \mu'$ and $\mu_v = \mu''/\omega$. Thus, for shear waves,

$$P = N = \frac{\mu_v}{\mu_e} = \frac{\mu''}{\omega\mu'} = \frac{\delta}{\omega} \quad \text{and} \quad C \rightarrow C_s\sqrt{1 + i\omega N} = \sqrt{\frac{\mu' + i\mu''}{\rho}} = \sqrt{\frac{\mu^*}{\rho}} \tag{8}$$

Hence, for harmonic shear waves in the Kelvin model, by solving the wave-equation, with a complex shear modulus, we have the solution of the telegraph equation. This can also be done in the Kelvin model for longitudinal (or dilatational) waves.

It turns out that the Kelvin model with harmonic time-dependence assumed, is <u>the only case</u> where the viscosity coefficients λ_v, μ_v of the viscoelastic solid are linearly proportional to the imaginary parts λ'', μ'' of the "complex elastic constants" λ^*, μ^*, the proportionality factors being $1/\omega$. It is also the only case where the elastic constants λ_e, μ_e of the solid are just the real parts λ', μ' of the "complex elastic constants."

To show this, let us now consider the Maxwell model, where the springs and dashpots at each material point are now connected in series. It can be quickly shown that the field equations and constitutive relations of the Maxwell solid are the same as those of elasticity, provided that the elastic constants λ_e, μ_e of the elasticity equations are replaced by the operators,

$$\lambda_e \rightarrow \frac{\frac{\partial}{\partial t}}{2\beta + \frac{\partial}{\partial t}} \left\{ \frac{\lambda_e \left(2\beta + \frac{\partial}{\partial t}\right) - 2\alpha\mu_e}{3\alpha + 2\beta + \frac{\partial}{\partial t}} \right\}, \qquad \mu_e \rightarrow \frac{\mu_e \frac{\partial}{\partial t}}{2\beta + \frac{\partial}{\partial t}} \tag{9}$$

where α, β are the viscosity coefficients of the Maxwell model (which are analogous to λ_v, μ_v of the Kelvin model). For time-dependence of the form $\exp(-i\omega t)$ we can call these operators by the name "complex elastic constants" λ^*, μ^* ie,

$$\lambda^* = \frac{i\omega}{i\omega - 2\beta} \left\{ \frac{\lambda_e (2\beta - i\omega) - 2\alpha\mu_e}{3\alpha + 2\beta - i\omega} \right\}, \qquad \mu^* = \frac{\mu_e i\omega}{i\omega - 2\beta} \tag{10}$$

It is obvious that if there is no viscosity (ie, $\alpha = 0, \beta = 0$) these quantities λ^*, μ^* would reduce to λ_e, μ_e respectively. It is also clear that α and β are now <u>not</u> proportional to the imaginary parts of λ^* and μ^* respectively. Furthermore, λ_e and μ_e are not the real parts of λ^* and μ^* anymore. Thus the "trick" of the complex elastic constants does not work here at all. The "trick" also fails for the standard viscoelastic model, or any more complicated model which contains at least one Maxwell element. In fact it fails even when there is no Maxwell element in the model provided there is more than one Kelvin element. Thus, only for one single Kelvin element will it "work."

For the reasons given above we believe that the most clear, systematic and natural way to handle viscoelastic problems of this sort is to measure the (real) elastic constants of the material independently from the (real) viscous coefficients and then use those numerical values in the field-equations when analytically solving them. Of course, we must keep the damping terms in the field equations. The final result will be complex, but this fact is due to the presence of the damping terms rather than because of any "complex moduli" we want to fictitiously introduce because of our insistance on ignoring the damping terms instead. We have seen that the complex moduli "trick" becomes exceedingly difficult, in fact impossible, in any case other than the Kelvin-Voigt model with harmonic time-dependence.

1. The Correspondence Theorem

A very useful theorem that permits us to relate solutions of elastic problems to those of viscoelasticity (in any model) is the Correspondence Theorem.[7] This theorem has a conceptually very simple statement, but it is very difficult to apply it to actual cases. The theorem basically says that if we want the solution to a viscoelasticity problem (static or dynamic, and in any model) we should first solve the "corresponding" elastic problem without viscosity. Then we Laplace-transform the solution. Then we replace the elastic constants appearing in that Laplace-transformed solution, by certain "memory functions" (which vary with the viscoelastic model being used). The result of this replacement is the Laplace transform of the solution of the "corresponding" viscoelastic problem. Inverting it, we have the solution of the viscoelastic problem we wanted to solve. We here recall the Laplace Transform pair,

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt \quad , \qquad f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) e^{st} ds \quad .$$

The prescription given by the Correspondence Theorem is very straight forward. It turns out in practice that when the elastic constants are replaced by those "memory functions", the resulting Laplace-transformed solution that must now be inverted is quite formidable in most cases of interest (ie, dynamic cases). We should point out that the memory functions depend on the Laplace transform variable s in a more or less complicated manner depending on whatever viscoelastic model one uses. See equations (7) or (10) with $i\omega$ replaced by s, for the Kelvin or the Maxwell model respectively.

I don't really want to discuss the Correspondence Theorem or its applications. My point is that those "memory functions" mentioned in it, are precisely the equivalent or the analogue of those "complex-elastic constants" that some authors try to introduce fictitiously in real-space rather than in the Laplace-space, before inversion to the real time-domain. Hence, the way to use those "complex elastic constants" in a way that "works," is really in the light of the correspondence theorem. Unfortunately, that is quite a difficult task.

We have already pointed out that another method which "works" is to measure the (real) elastic and viscous constants of the material, each set independently of the other, and then solve the field-equations with the damping terms included, with those numerical values measured for the constants. This method is also general and works for any viscoelastic model and it is, incidentally, the way viscous flow problems are attacked in Fluid Mechanics.

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[7] See A. Cemal Eringen, "Continuum Mechanics," John Wiley and Sons, Inc., 1967, Article 9.12, p. 368.

Observation of Table 1 and 2 shows why the telegraph-type equations we have discussed here are so important in viscoelasto-dynamical problems. Note that the telegraph-type equations for the potentials ϕ, ψ, χ are of the type given here in eq. (1), for the Kelvin-Voigt model only. Note, for exmaple, that for the Maxwell model they are substantially different. (Table 1, eqs. C-10). The equation for ϕ is,

$$\left[2\beta + 4\alpha\frac{C_s^2}{C_d^2} + \frac{\partial}{\partial t}\right] \nabla^2\phi = \frac{1}{C_d^2} \left\{\left[3\alpha + 4\beta + \frac{\partial}{\partial t}\right] \frac{\partial^2\phi}{\partial t^2} + 2\beta\,(3\alpha + 2\beta)\,\frac{\partial\phi}{\partial t}\right\} \qquad (11)$$

which is of the same "type" but not quite of the form given here in eq. (1).

We can conceptually always introduce complex moduli (as in eqs. (7) or (10) or their analogues for the "standard" model) provided we work in Laplace space as prescribed by the Correspondence Theorem. If we introduce them in the real time-domain and then try to physically identify the real (or imaginary) parts of these complex constants with the real elastic constants (or the viscous coefficients), then the process only "works" in the Kelvin-Voigt model. In this sense, "works" means that the equivalence of both approaches can be established.

One way to experimentally measure the absorption losses of material samples is the impedance tube or pulse tube. It is customary for the literature on this technique to report measurements of quantities such as E' and E" or μ' and μ" etc... This technique is a good source for the popularity of "complex moduli". We should note that the lossy samples tested in this fashion are always characterized by the Kelvin-Voigt viscoelastic model with harmonic time-dependence, an assumption that may not always be justified.

2. Conclusions

We summarize our points as follows,

(a) The continuum (infinite number of degrees-of-freedom) approach to viscoelasticity as a field-theory is the only way to go today. Discrete approaches leave much to be desired and are simplistically unrealistic.

(b) It looks like for some time to come, we will be dealing with the Kelvin-Voigt and Maxwell models, since any other model presents too many analytical difficulties.

(c) A systematic way to set up viscoelastic boundary-value-problems and solve the field equations in these two models, is presented here for cylinder and sphere problems. This approach is shown in Tables 1 and 2 which are self-explanatory.

(d) The most general method available to solve static or dynamic problems of viscoelasticity in any model is by means of the Correspondence Theorem. Here we must work in Laplace-space and the inversions are hard. This approach is totally equivalent (at least for the Kelvin and Maxwell models) to our approach described above in (3) and in Tables 1 and 2.

(e) Another technique, which holds for <u>any</u> model, is also given here. It consists of solving those resulting "telegraph-type" equations for the scalar potentials keeping in them the damping terms, but using real values for the elastic and viscous constants which are to be experimentally measured as it is done in Fluid Mechanics.

(f) We claim this technique is more systematic and less confusing than to ignore the damping terms introducing instead "complex elastic moduli". The complex moduli "trick" only works for the Kelvin model with harmonic time-dependence anyway.

(g) It seems clearer to us when talking about elastic constants, say, Young's modulus E, to think of the slope of the stress-strain curve as found in a tension test, than of that "complex Young's modulus" which contains the "loss" in its imaginary part, all because some want to solve wave equations rather than telegraph equations, particularly in the only situation when one is no harder to solve than the other.

(h) The determination of the solid's elastic constants should be kept separate and independent from the determination of the viscosity coefficients. I know this can be done in some instances, but I am not aware of how plausible this recommendation can be in all instances.

(i) Finally it is worth stating that the pulse tube measurements implicitly describe the viscoelastic "losses" in the sample by the Kelvin-Voigt model, an observation that escaped me (and perhaps others) until recently.

(j) Equation (10) can be used to express the complex shear and dilational moduli for the Maxwell model in terms of the viscosity coefficients and the elastic constants of the model as follows,

$$\mu^* = \frac{\mu_e}{1 + \left(\frac{2\beta}{\omega}\right)^2} \left[1 - i\left(\frac{2\beta}{\omega}\right)\right] \qquad \text{and}$$

$$\begin{aligned}\lambda^* + 2\mu^* = {} & \frac{1}{1 + \left(\frac{2\beta}{\omega}\right)^2} \cdot \frac{1}{1 + \left(\frac{3\alpha + 2\beta}{\omega}\right)^2} \Bigg\{\Bigg[\left[\lambda_e + 2\mu_e\right] \left[1 + \left(\frac{2\beta}{\omega}\right)^2\right] + \\ & + \frac{4\alpha\mu_e}{\omega} \left(1 + \frac{2\beta}{\omega}\right)\Bigg] - i \left(\frac{3\alpha + 2\beta}{\omega}\right) \Bigg[\lambda_e \left\{1 + \left(\frac{2\beta}{\omega}\right)^2\right\} + \\ & + 2\mu_e \left\{\frac{4\beta}{\omega} \cdot \frac{\alpha + \beta}{\omega} \quad \frac{\alpha + 2\beta}{\omega}\right\}\Bigg]\Bigg\}.\end{aligned}$$

These relations may be useful when trying to interpret "losses" in the light of the Maxwell model if one day one wishes, or the need arises to do so. It is obvious that these relations are considerably more complicated than the analogous ones for the Kelvin model, which are,

$$\mu^* = \mu_e \left[1 + i\omega N\right] \quad , \quad \lambda^* + 2\mu^* = (\lambda_e + 2\mu_e)\left[1 + i\omega N\right] \qquad \text{where,}$$

$$N = \frac{\mu_v}{\mu_e} \quad \text{and} \quad M = \frac{\lambda_v + 2\mu_v}{\lambda_e + 2\mu_e} \quad .$$

(See Table 2, column A, item (9).)

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