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Inverse Source Problem: Eigenfunction Analysis of Bojarski's Integral Equation

by

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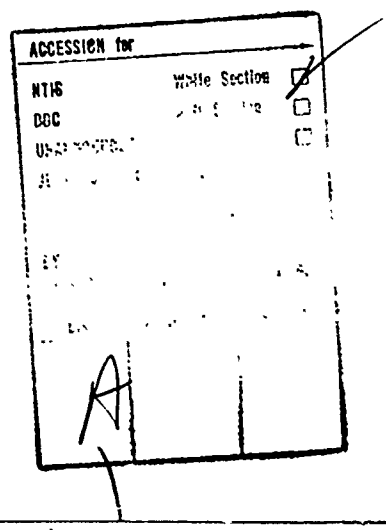
Abstract.

An integral equation elsewhere employed to solve inverse source problems is discussed from the viewpoint of Hilbert Space theory. The eigenfunctions and eigenvalues are determined and the null space is explicitly shown to be infinite dimensional. An existence criterion is established and an application is made to the problem of determining sources which radiate maximum power for given input power.



1. Introduction.

Recently, N. N. Bojarski (1974) conjectured that inverse source problems for the wave equation could be investigated by means of a Fredholm integral equation of first kind which took the form,

$$\int_D K(\underline{r} - \underline{r}')\ \rho(\underline{r}')\ d^3r' = \Theta(\underline{r}). \tag{1.1}$$

Here, $\rho(\underline{r}')$ denoted the unknown source whose support, however, was assumed *a priori* to be located within a domain D. Furthermore, $\Theta(\underline{r})$ denoted a function determined by observations of the field on the boundary of D.

Later that year, N. Bleistein and N. N. Bojarski (1974) presented an invited talk at the Summer Institute on Inverse Problems at the University of California at Irvine, in which they gave a rigorous derivation of (1.1), identifying the kernel K as the difference of the free space outgoing and incoming Green's functions. The equation as thus formulated has a unique solution, but it was soon realized that the source function ρ, which arose from a time transform of the wave equation, would in general depend on the transform variable, ω. Thus, the treatment and numerical results presented by these authors only applied to the case in which the time-dependent source had the form, $\rho(\underline{r})\ \delta(t)$, where $\delta(t)$ represents the Dirac delta function.

Accordingly, the present authors (1975) gave a derivation of the integral equation valid for general time-dependent sources and showed that in this general case, the source *cannot* be determined uniquely. We gave several characterizations of the

non-uniqueness and also gave illustrations of some type of additional *a priori* information about the source, which would allow its unique determination. Furthermore, we extended the theory to the Maxwell Equation system.

In a sequel, now in preparation, we will present additional examples of both the source determination and source synthesis. Many of these applications proceed most easily by treating the spatial Fourier transform of the integral equation. However, we decided that, since the integral equation is of quite classical type, a study of its eigenfunction structure from the Hilbert Space viewpoint would be illuminating and it is that study we present here.

2. Notation and Derivation of the Integral Equation.

We deal with functions of space and time denoted by $F(\underline{r},t)$, where

$$\underline{r} = (x,y,z), \quad r = |\underline{r}|, \quad \hat{r} = \underline{r}/r. \tag{2.1}$$

The time transforms of such functions are denoted by the corresponding lower case letters,

$$f(\underline{r},\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} F(\underline{r},t)dt \tag{2.2}$$

and we shall occasionally also have reference to the time-space transform, denoted by a tilde as,

$$\tilde{f}(\underline{k},\omega) = \iiint f(\underline{r},\omega)e^{-i\underline{k}\cdot\underline{r}}d^3r. \tag{2.3}$$

We consider the wave equation,

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\, U(\underline{r},t) = -F(\underline{r},t) \tag{2.4}$$

and assume that we know *a priori* that the source F has spatial support within a sphere of radius a and also that the source has temporal support in the interval*, $-t_0 < t < t_0$. We assume that

$$U(\underline{r},t) \equiv 0 \text{ for } t < -t_0. \tag{2.5}$$

Upon time transformation, we obtain

$$\left[\nabla^2 + (\frac{\omega}{c})^2\right] u(\underline{r},\omega) = -f(\underline{r},\omega) \tag{2.6}$$

with u satisfying the outgoing radiation condition,

$$u(\underline{r},\omega) \sim \frac{1}{4\pi r} e^{-i\frac{\omega}{c}r} u_0(\hat{r},\omega) \text{ as } r \to \infty . \tag{2.7}$$

* This can be weakened somewhat.

Here, u_o is a quantity analogous to the scattering cross section called the phase and range normalized scattering amplitude.

We now present a simplified derivation of the generalized Bojarski integral equation which proceeds immediately from the Green's Identity,

$$(2.8)\quad \iiint_D \left\{u(\underline{r}',\omega)\, Lv(\underline{r}',\underline{r},\omega) - v(\underline{r}',\underline{r},\omega)\, Lu(\underline{r}',\omega)\right\} d^3r'$$

$$= \iint_{\partial D} \hat{n}' \cdot \left[u\nabla' v - v\nabla' u\right] ds'.$$

Here

$$(2.9)\qquad L = \nabla'^2 + \left(\frac{\omega}{c}\right)^2,$$

∂D denotes an (observation) boundary which lies outside the sphere $r \leq a$ (the support of the source). D is the interior of ∂D and $\hat{n}'$ is the unit outward normal to D. We choose v as a solution of the homogeneous Helmholtz equation.

$$(2.10)\qquad \left[\nabla^2 + \left(\frac{\omega}{c}\right)^2\right] v(\underline{r},\underline{r}',\omega) = 0$$

which is regular at its source point $\underline{r} = \underline{r}'$. The simplest choice of v is

$$(2.11)\qquad v = j_0\left(\frac{\omega}{c} R\right),\ R = |\underline{r} - \underline{r}'|;$$

$$j_0(x) = \frac{\sin x}{x}.$$

Here, j_0 is the spherical Bessel Function of first kind and order zero.

On using (2.10) and (2.6) in the Green's Identity (2.8), we have at once the integral equation

$$Kf = \Theta(\underline{r},\omega) \tag{2.12}$$

where

$$Kf = \iiint_D j_0\left(\frac{\omega r}{c}\right) f(\underline{r}',\omega)\, d^3r' \tag{2.13}$$

and

$$\Theta = \iint_{\partial D} \hat{n}' \cdot \left[u(\underline{r}',\omega)\nabla' j_0\left(\frac{\omega r}{c}\right) - j_0\left(\frac{\omega r}{c}\right) \nabla' u(\underline{r}',\omega)\right] ds'. \tag{2.14}$$

The integral equation (2.12) is valid for all $\underline{r}$, and since f has support in a sphere of radius a, the integration in (2.13) can be regarded as extending either over all space or just over the a-sphere as we find conceptually convenient.

The integral equation (2.12) differs only slightly from Bojarski's original conception. The kernel and Θ differ only by a common multiplicative factor from his, and most importantly Θ is still an observable quantity computed from observations of u and $\partial u/\partial n$ on the single surface ∂D.

As mentioned in the introduction, most of the practical applications of the integral equation follow most easily from its spatial Fourier Transform, which we remark in passing, is given by

$$\hat{f}(\underline{k},ck) = \hat{\Theta}(\underline{k}) \tag{2.15}$$

where

$$\hat{\Theta}(\underline{k}) = -\iint_{\partial D} e^{-i\underline{k}\cdot\underline{r}}\, \hat{n}' \cdot \left[ik\underline{u}(\underline{r}',ck) + \nabla' u(\underline{r}',ck)\right] ds' \sim u_0(\hat{k},ck) \text{ as } r \to \infty. \tag{2.16}$$

3. Eigenfunction Analysis of the Integral Equation.

We study the problem

$$K\psi(\underline{r}',\omega) = \lambda\psi(\underline{r},\omega), \quad |\underline{r}| \leq a, \quad |\underline{r}'| \leq a, \tag{3.1}$$

where K is defined by (2.13) and ψ is a *regular* function. Since the kernel, $j_0(\frac{\omega}{c}|\underline{r} - \underline{r}'|)$, of K is a symmetric, continuous function on a compact product space, we are dealing with the most classic case of symmetric Hilbert-Schmidt theory. Below, in statements 1 - 4, we shall summarize the principal conclusions of this theory and, when appropriate, give the specializations of these results to our kernel. Note that our eigenfunctions exist only in the a-sphere and that the precise value of a is somewhat arbitrary. Also, we assume that

$$\psi(\underline{r},\omega) \in L_2[a], \tag{3.2}$$

where the notation indicates that ψ is square-integrable on the a-sphere.

We begin our review of the theory with the following definitions.

1. E = linear manifold generated by the eigenfunctions for $\lambda \neq 0$.
2. N = null space of K (i.e., the solutions of $Kf = 0$).
3. E and N are subspaces; in fact, $L_2[a] = E \oplus N$. This result means that for any $f \in L_2[a]$,
4. $f = \Sigma f_n \psi_n + h$, where the f_n are the unique Fourier Coefficients of the eigenfunctions which generate E, while h is an arbitrary element of N.

For our kernel, the eigenfunctions generating E are explicitly given by

$$\psi_{\ell m}(\underline{r},\omega) = \frac{1}{N_\ell} j_\ell(\tfrac{\omega r}{c}) Y_{\ell m}(\theta,\phi), \tag{3.3}$$

$$\ell = 0,1,2 \ldots ; \quad |m| \leq \ell .$$

The corresponding eigenvalues are given by

$$\lambda_{\ell m} = \lambda_\ell = 4\pi N_\ell^{\,2} \tag{3.4}$$

and the normalization factor, N_ℓ, is given by

$$N_\ell^{\,2}(\omega,a) = \int_0^a j_\ell^{\,2}(\tfrac{\omega r}{c}) r^2 dr \tag{3.5}$$

$$= \frac{a^3}{2}\left\{ j_\ell^{\,2}(\tfrac{\omega a}{c}) - j_{\ell-1}(\tfrac{\omega a}{c}) j_{\ell+1}(\tfrac{\omega a}{c}) \right\}.$$

To establish (3.3), recall that the kernel, j_0, was chosen as a regular solution to the homogeneous Helmholtz Equation. Thus, upon applying the operation, $\nabla^2 + (\frac{\omega}{c})^2$, to both sides of (3.1), we have at once

$$\lambda\left[\nabla^2 + (\tfrac{\omega}{c})^2\right]\psi = 0 . \tag{3.6}$$

In determining E, we assume $\lambda \neq 0$, so it is clear that any eigenfunction in E is a regular solution to the homogeneous Helmholtz equation. If we expand ψ in the complete set, $Y_{\ell m}(\theta,\phi)$, of regular spherical harmonics as

$$\psi(\underline{r},\omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}(r,\omega) Y_{\ell m}(\theta,\phi), \tag{3.7}$$

we find that $f_{\ell m}(r,\omega)$ satisfies the second order spherical Bessel *ordinary* differential equation. The only $L_2[a]$

solution is $j_\ell(\frac{\omega}{c}r)$, so that $\psi_{\ell m}(\underline{r},\omega) \propto j_\ell(\frac{\omega r}{c})Y_{\ell m}(\theta,\phi)$, which concludes the proof of (3.3).

If we expand our kernel in spherical harmonics as

$$(3.8)\quad j_0(\frac{\omega}{c}|\underline{r}-\underline{r}'|) = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} j_\ell(\frac{\omega r}{c})j_\ell(\frac{\omega r'}{c}) \cdot \psi_{\ell m}(\theta,\phi)\psi_{\ell m}^*(\theta,\phi),$$

(see Jackson, 1962), it is easy to compute $K\psi_{\ell m}$, and hence obtain the eigenvalues $\lambda_{\ell m}$ given in (3.4).

To any function f in $L_2[a]$, we can associate both an expansion in spherical harmonics and an eigen-expansion. To distinguish these, we henceforth use $f_{\ell m}$ to denote the spherical harmonic coefficients and $\overline{f}_{\ell m}$ to denote the eigen-coefficients. Thus,

$$(3.9)\quad f(\underline{r},\omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(r,\omega)Y_{\ell m}(\theta,\phi)$$

and

$$(3.10)\quad f(\underline{r},\omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \overline{f}_{\ell m}(\omega)\psi_{\ell m}(\underline{r},\omega).$$

The relation between these coefficients is

$$(3.11)\quad \overline{f}_{\ell m}(\omega) = \int_0^a f_{\ell m}(r,\omega)j_\ell(\frac{\omega r}{c})r^2dr,$$

also

$$(3.12)\quad f_{\ell m}(r,\omega) = \iint f(\underline{r},\omega)Y_{\ell m}^*(\theta,\phi)d\Omega$$

and

$$(3.13)\quad \overline{f}_{\ell m}(\omega) = \iiint f(\underline{r},\omega)\psi_{\ell m}^*(\underline{r},\omega)r^2drd\Omega.$$

Here, $d\Omega$ denotes an integration over the unit sphere.

The general expansion result (4) can now be explicitly expressed for our kernel as

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \overline{f}_{\ell m}\psi_{\ell m} + h, \quad f\epsilon L_2(a) \tag{3.14}$$

where the $\overline{f}_{\ell m}$ are the unique eigen-coefficients (3.13), and h is an arbitrary element of N.

We now turn to the characterization of the range of K. We write

$$\mathfrak{R}(K) = \text{range of } K, \tag{3.15}$$

and have

5. $\mathfrak{R}(K)$ is a linear manifold in E, which is *not* closed.

Furthermore, we have

6. $g\epsilon\mathfrak{R}(K) \rightarrow$ the eigen-series associated with g is uniformly convergent (instead of merely L_2 convergent).

This latter result requires that the iterated kernel be bounded and continuous, which is certainly true here since we deal with a continuous kernel on a compact set. The second part of (5) requires only that the kernel be Hilbert-Schmidt and there exist infinitely many eigenfunctions (i.e., that K be non-degenerate).

The fact that $\mathfrak{R}(K)$ is not closed means that the alternative theorem does not apply, but since $N = \mathfrak{R}^{\perp}$ always holds, we do have

7. E = closure ($\mathfrak{R}$).

Using our eigenfunctions, we see at once that the solutions of

(3.16) $$Kf = g(\underline{r},\omega), \quad g\epsilon\mathcal{R}(K), \quad r \leq a$$

are

(3.17) $$f(\underline{r},\omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\overline{g}_{\ell m}}{\lambda_\ell} \psi_{\ell m} + h$$

where h is an arbitrary element of N. Thus, the range may be characterized by

(3.18) $$g\epsilon\mathcal{R}(K) \leftrightarrow \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\overline{g}_{\ell m}}{\lambda_\ell} \psi_{\ell m} \epsilon L_2[a]$$

or by

(3.19) $$g\epsilon\mathcal{R}(K) \leftrightarrow \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left|\frac{\overline{g}_{\ell m}}{\lambda_\ell}\right|^2 < +\infty ,$$

where the $\overline{g}_{\ell m}$ are the eigen-coefficients of g. The existence criterion (3.19) is quite stringent for, while it is also true for non-degenerate symmetric compact operations that the sequence of eigenvalues, λ_n, satisfy $\lambda_n \to 0$ as $n \to \infty$, we have for our operation extremely rapid convergence to zero. In fact, the explicit expressions (3.4) and (3.5) for the eigenvalues yield

(3.20) $$\lambda_\ell = \frac{\pi a^3}{4\ell^2}\left(\frac{\omega ae}{2c\ell}\right)^{2\ell}\left[1 + O\left(\frac{1}{\ell}\right)\right] = O\left(\frac{1}{\ell^{2\ell+3}}\right), \quad \ell \to \infty.$$

Thus, in a sense, $\mathcal{R}(K)$ is much smaller than E.

In the following table, we give some typical values for $\lambda_\ell/2\pi a^3$ so that the reader can judge the noise amplification inherent in an eigenfunction solution such as (3.17).

ℓ \ $\frac{\omega a}{c}$	.1	1	10	100
0	.6653	.5454	$.9544\times10^{-2}$	---
1	$.4438\times10^{-3}$	$.3850\times10^{-1}$	$.1040\times10^{-1}$	---
2	$.4122\times10^{-7}$	$.1136\times10^{-2}$	$.9174\times10^{-2}$	---
3	$.2014\times10^{-10}$	$.1840\times10^{-4}$	$.9452\times10^{-2}$	---
5	$.1423\times10^{-18}$	$.1332\times10^{-9}$	$.7783\times10^{-2}$	$.9950\times10^{-4}$
10	---	---	$.6130\times10^{-3}$	$.9943\times10^{-4}$
15	---	---	$.8437\times10^{-7}$	$.9929\times10^{-4}$

Table of values for $\frac{\lambda_\ell}{2\pi a^3} = j_\ell{}^2(\frac{\omega a}{c}) - j_{\ell-1}(\frac{\omega a}{c}) j_{\ell+1}(\frac{\omega a}{c})$

We conclude this section by providing several characterizations of the null space N. Aside from the obvious characterizations, $Kf = 0$ and $f \epsilon E^1$, we have

$$(3.21) \qquad f \epsilon N \leftrightarrow \bar{f}_{\ell m} = 0; \quad \ell = 0,1,2, \ldots ; \quad |m| \leq \ell,$$

$$(3.22) \qquad f \epsilon N \leftrightarrow \tilde{f}(\underline{k}, ck) = 0,$$

and

$$(3.23) \qquad f \epsilon N \leftrightarrow f \text{ is a non-radiating source}$$

By the phrase "non-radiating source", we mean one whose field, outside the source region $r \leq a$, vanishes identically. This last characterization identifies the mathematical concept of a null space with those sources which are physically undetectable outside the source region. The proofs of statements (3.21)-(3.23) appear in Bleistein and Cohen (1975). Here, we show in a constructive way that (3.21) implies that N is infinite dimensional. Consider the functions

$$(3.24) \quad h(\underline{r},\omega) = \left\{ f(r,\omega) - \frac{j_{\ell'}(\frac{\omega r}{c}) \int_0^a f(r',\omega) j_{\ell'}(\frac{\omega r'}{c}) r'^2 dr'}{\int_0^a j_{\ell'}^2(\frac{\omega r'}{c}) r'^2 dr'} \right\} Y_{\ell' m'}(\theta,\phi).$$

For these functions, we have

$$(3.25) \qquad \bar{h}_{\ell m} = \int_0^a \int h(\underline{r},\omega) \psi_{\ell m}(\underline{r},\omega) r^2 d\Omega dr$$

$$= \frac{1}{N_\ell} \int_0^a \left\{ f(r,\omega) - \frac{j_\ell(\frac{\omega r}{c}) \int_0^a f(r',\omega) j_{\ell'}(\frac{\omega r'}{c}) r'^2 dr'}{\int_0^a j_{\ell'}^2(\frac{\omega r}{c}) r'^2 dr'} \right\}$$

$$\cdot j_\ell(\frac{\omega r}{c}) r^2 dr \; \delta_{\ell,\ell'} \delta_{m,m'}$$

$$= \frac{\delta_{\ell\ell'} \delta_{mm'}}{N_\ell} \left\{ \int_0^a f j_\ell r^2 dr - \frac{\int_0^a j_\ell^2 r^2 dr \int_0^a f j_\ell r^2 dr}{\int_0^a j_\ell^2 r^2 dr} \right\}$$

$$= 0$$

Hence, for each $f(r,\omega)$ and each harmonic ℓ,m, there is a solution in the null space. For example, if we choose the (0,0) harmonic and $f = r^k$, $k = 0,1,2, \ldots$, we obtain an infinite family for which the quadratures can be explicitly performed. We also note that if $f(r,\omega)$ is an entire function of order one in ω, so is $h(r,\omega)$. Thus, the inverse transform $H(\underline{r},t)$ would have compact support in time and would, thus, be a legitimate source if only $f(r,\omega)$ vanishes for $r > a$.

4. Existence and Uniqueness for the Inverse Source Integral Equation.

We can now state the fundamental existence theorem for the integral equation (2.12)

Theorem 1: The equation $Kf = \Theta$ has solutions if and only if

$$(4.1)\quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left| \iint_{\partial D} \hat{n}' \cdot \left[u\nabla'\psi_{\ell m}^* - \psi_{\ell m}^* \nabla' u \right] ds' \right|^2 < \infty.$$

Moreover, if the equation has solutions, then it has the infinite family of solutions,

$$(4.2)\quad f = h + \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \iint_{\partial D} \hat{n}' \cdot \left[u\nabla'\psi_{\ell m}^* - \psi_{\ell m}^* \nabla' u \right] ds' \psi_{\ell m}, \quad h \epsilon N.$$

Proof: $Kf = \Theta$ has solutions if and only if $\Theta \epsilon R(K)$, hence, by (3.18), if and only if

$$(4.3)\quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left| \frac{\overline{\Theta}_{\ell m}}{\lambda_\ell} \right|^2 < \infty .$$

If it does have solutions, then by (3.17) they are given by

$$(4.4)\quad f = h + \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\overline{\Theta}_{\ell m}}{\lambda_\ell} \psi_{\ell m}, \quad h \epsilon N.$$

Thus, to establish the theorem, we need only show that

$$(4.5)\quad \overline{\Theta}_{\ell m} = \lambda_\ell \iint_{\partial D} \hat{n}' \cdot \left[u\nabla'\psi_{\ell m}^* - \psi_{\ell m}^* \nabla' u \right] ds'.$$

This can be done by recalling the definition of Θ in (2.14) and realizing that the expansion (3.8) for $j_0(\frac{\omega}{c} R)$ can be written in terms of the eigenfunction as,

$$(4.6)\quad j_0\left(\frac{\omega r}{c}\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \lambda_\ell \psi_{\ell m} \psi_{\ell m}^* .$$

Using this, we obtain

$$(4.7)\qquad \Theta(\underline{r},\omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \lambda_\ell \iint_{\partial D} \hat{n}' \cdot \left[u \cdot \nabla' \psi_{\ell m}^*(\underline{r}',\omega) - \psi_{\ell m}^*(\underline{r}',\omega)\nabla' u\right] ds' \cdot \psi_{\ell m}(\underline{r},\omega)$$

which immediately gives (4.5) and establishes the theorem.

We remark that if Θ is computed from data from an actual source $f \epsilon L_2[a]$, then the existence criterion in Theorem 1 is always satisfied because the integral equation itself implies that

$$(4.8)\qquad \overline{\Theta}_{\ell m} = \lambda_\ell \overline{f}_{\ell m}$$

and hence

$$(4.9)\qquad \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left|\frac{\Theta_{\ell m}}{\lambda_\ell}\right|^2 = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left|\overline{f}_{\ell m}\right|^2 < \infty ,$$

since $f \epsilon L_2[a]$.

We further point out that because of the uniform convergence of the eigen-expansion of functions in the range and the final remark made at the end of the last section, infinitely many solutions of the form (4.2) can be found which will give rise to physical space-time sources $F(\underline{r},t)$ if the hypothesis of the theorem is met.

5. An Application.

As mentioned in the introduction, we plan a sequel in which various applications of equation (2.12) will be presented. Naturally, the key to these applications is the restriction of f to subsets M of $L_2[a]$, such that the problem

$$Kf = \Theta \qquad f \in M \tag{5.1}$$

has a unique solution. We have already given some examples of suitable choices of M in Bleistein and Cohen, 1975. Here, we confine ourselves to mentioning just one such application. Namely, we point out that if Θ satisfies the existence criterion and, furthermore, if we impose *a priori* that $f \in E$, then clearly the unique solution is

$$f(\underline{r},\omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\Theta_{\ell m}}{\lambda_\ell} \psi_{\ell m} \, . \tag{5.2}$$

The restriction $f \in E$ is not devoid of interest, for by the characterization (3.23) of the null space, any component of $f \in E^{\perp}$ does not radiate. Thus, any energy used in producing such a component is "wasted". In this sense, the solution (5.2) is the most economical of the family of solutions (4.2) and, thus, is of particular interest in the synthesis problem.

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