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A PROGRAM FOR THE STABILITY ANALYSIS OF PIPE POISEUILLE FLOW

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THESIS

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OF
PIPE POISEUILLE FLOW

by

Richard Howard Johnston III

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Recent research by Harrison on the stability of parallel flows resulted in a successful solution of plane Poiseuille flow but produced unexplained anomalies for pipe flow. The purpose of the research in this paper was to find and correct errors in Harrison's initial analysis of the pipe flow problem.

A minor error in Harrison's numerical method was corrected. Moreover, it proved necessary to reanalyze the conditions on the axis of symmetry of the pipe. These changes finally made it possible to obtain reasonable results.

Owing to time limitations, the number of solutions obtained using the corrected program was sufficient only to confirm its general validity. However, the results obtained are significant in that they disclose instabilities which are known to exist but which have not been accounted for in previous theoretical investigations.

A Program for the Stability Analysis

of

Pipe Poiseuille Flow

by

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Lieutenant, United States Navy
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ABSTRACT

Recent research by Harrison on the stability of parallel flows resulted in a successful solution of plane Poiseuille flow but produced unexplained anomalies for pipe flow. The purpose of the research in this paper was to find and correct errors in Harrison's initial analysis of the pipe flow problem.

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TABLE OF SYMBOLS

All quantities below are in dimensionless form.

$D, D^2, \ldots$	The partial derivatives with respect to r in cylindrical coordinates.
e	Base of natural logarithms. Exponentiation is also denoted by exp( ).
$\bar{e}_x, \bar{e}_r, \bar{e}_\theta$	Unit vectors along the x,r, and θ axes in cylindrical coordinates.
F,G,H	Components of the velocity vector potential defined in equation (2-1).
i	$+\sqrt{-1}$, the imaginary unit. Also used as an index in the finite difference mesh in Section IV.
n	The number of interior points in the finite difference mesh of Section IV. Also used as the imaginary part of B, i.e. B=in.
Re	Reynolds number based on the mean velocity and pipe radius.
t	Time

T, t, P, Q Shorthand notation for commonly occurring groups of symbols defined in equations (2-18), (2-19), (3-42), and (3-43).

$\bar{v}$ Velocity vector of the perturbation flow defined by equation (2-3).

$\bar{W}$ Complex vector potential of perturbation velocity defined by equations (2-1) and (2-2).

x, r, θ Cylindrical coordinates.

A $A_R + iA_I$ Complex wave number of the perturbation in the x direction.

B iB_I Complex wave number of the perturbation in the θ direction

γ $\gamma_R + \gamma_I$ Complex frequency of the perturbation.

$\bar{\Gamma}$ The vorticity transport equation (2-5) expressed in abbreviated notation as defined by equation (2-6).

$\Gamma_x, \Gamma_r, \Gamma_\theta$ The components of $\bar{\Gamma}$ in cylindrical coordinates defined by equation (2-6).

$\bar{\omega}$ Vorticity vector of the perturbation flow defined in equation (2-4).

∇	Linear vector operator (nabla).
x	Vector cross-product operator.
[ ]	Brackets enclosing a matrix.
{ }	Brackets enclosing a column vector.

I. INTRODUCTION

The research reported in this thesis was undertaken in an attempt to isolate sources of error causing obviously incorrect computer solutions for the three-dimensional linearized vorticity transport equation for pipe Poiseuille flow. Solutions obtained using the theory and program developed in Ref. 1 had indicated decreasing flow stability with decreasing Reynolds number, a result which is clearly inconsistent with theory and experimental data.

Initial analysis of that program confirmed that it was basically correct as presented in Ref. 1, but the nature of the solutions obtained indicated one or more errors in the formulation of the problem to be solved. Analysis of this problem formulation was conducted in three phases. The first phase consisted of ensuring accurate expression of the vorticity transport equation in terms of a complex velocity vector potential. The second and most significant phase of the analysis required extensive examination of the conditions required to satisfy the vorticity transport equation at the singular point on the axis of symmetry of the pipe. The third phase entailed establishing a finite difference approximation of this equation and its boundary conditions for adaptation of the problem to a standard form for computer solution.

II. THE VORTICITY TRANSPORT EQUATION

The governing equations for laminar flow of an incompressible fluid with constant viscosity are the continuity equation and the Navier-Stokes equation. Taking the curl ($\nabla \times$) of the latter equation and introducing a perturbation velocity ($\bar{v}$) and vorticity ($\bar{\omega}$) leads to the linearized perturbation vorticity transport equation as derived in Appendix A of Ref. 1.

Expressed in terms of the complex velocity vector potential, $\bar{W}$, which has the form

$$\bar{W}(x,r,\theta,t) = (\bar{e}_x F(r) + \bar{e}_r G(r) + \bar{e}_\theta H(r))e^X \tag{2-1}$$

where

$$X = Ax + B\theta + \gamma t \tag{2-2}$$

and

$$\bar{v} = \nabla \times \bar{W} \tag{2-3}$$

$$\bar{\omega} = \nabla \times \bar{v} \ , \tag{2-4}$$

the vorticity transport equation becomes three simultaneous fourth-order differential equations of the form

$$[M_4]\begin{Bmatrix} D^4F \\ D^4G \\ D^4H \end{Bmatrix} + [M_3]\begin{Bmatrix} D^3F \\ D^3G \\ D^3H \end{Bmatrix} + ([M_2] + \gamma[N_2])\begin{Bmatrix} D^2F \\ D^2G \\ D^2H \end{Bmatrix} +$$

$$([M_1] + \gamma[N_1])\begin{Bmatrix} DF \\ DG \\ DH \end{Bmatrix} + ([M_0] + \gamma[N_0])\begin{Bmatrix} F \\ G \\ H \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2-5}$$

as derived in Appendix E of Ref. 1.

Equations (2-5) may be expressed in the abbreviated form

$$\bar{\Gamma} = \begin{Bmatrix} \Gamma_x \\ \Gamma_r \\ \Gamma_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2-6}$$

where $\bar{\Gamma}$ appears to be a set of three coupled equations in the components of $\bar{W}$. As shown in Appendix B of Ref. 1, equations (2-6) actually express only two independent conditions. Therefore, an appropriate linear combination of Γ_x and Γ_θ allows expression of equations (2-5) as a set of two equations in three unknowns. The linear combination

$$\frac{-B}{r}\Gamma_x + A\Gamma_\theta = 0 \tag{2-7}$$

does not, in general, uncouple $\bar{\Gamma}$ but does reduce the highest order derivative of the component G(r) in equations (2-5) to second order.

In a manner similar to that just described for $\bar{\Gamma}$, Appendix C of Ref. 1 illustrates the redundancy of the three components of $\bar{W}$. This allows arbitrary selection of one of these components as being uniformly zero for all r. As will be seen later, the maximum benefits from the consequences of the linear combination expressed by equation (2-7) are obtained if

$$F(r) = 0 \;. \tag{2-8}$$

Incorporation of equations (2-7) and (2-8) in equation (2-5) results in expression of the vorticity transport equation in the form

$$[M_4'] \begin{Bmatrix} D^4G \\ D^4H \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3G \\ D^3H \end{Bmatrix} + ([M_2'] - \gamma[N_2']) \begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} +$$

$$([M_1'] - \gamma[N_1']) \begin{Bmatrix} DG \\ DH \end{Bmatrix} + ([M_0'] - \gamma[N_0']) \begin{Bmatrix} G \\ H \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2-9}$$

where

$$[M_4'] = \begin{bmatrix} 0 & \frac{-A}{Re} \\ 0 & 0 \end{bmatrix} \qquad (2\text{-}10)$$

$$[M_3'] = \begin{bmatrix} 0 & \frac{-2A}{rRe} \\ 0 & \frac{B}{rRe} \end{bmatrix} \qquad (2\text{-}11)$$

$$[M_2'] = \begin{bmatrix} \frac{-4AB}{r^2 Re} & A\left(T + \frac{1}{Re}\left\{\frac{3}{r^2} - t\right\}\right) \\ \frac{-t}{Re} & \frac{2B}{r^2 Re} \end{bmatrix} \qquad (2\text{-}12)$$

$$[M_1'] = \begin{bmatrix} \frac{4AB}{r^3 Re} & \frac{A}{r}\left(\frac{1}{Re}\left\{\frac{3}{r^2}\left[B^2 - 1\right] - A^2\right\} + T\right) \\ \frac{1}{rRe}\left(\frac{B^2}{r^2} - A^2\right) & \frac{-B}{r}\left(T + \frac{1}{r^2 Re}\right) \end{bmatrix} \qquad (2\text{-}13)$$

$$[M_0'] = \qquad (2\text{-}14)$$

$$\begin{bmatrix} \frac{2AB}{r^2}\left(T - \frac{1}{Re}\left\{t + \frac{2}{r^2}\right\}\right) & A\left(Tt + \frac{1}{r^2}\left\{\frac{1}{Re}\left[t + \frac{3}{r^2}\right] - T\right\}\right) \\ Tt + \frac{1}{r^2 Re}\left(A^2 - \frac{B^2}{r^2}\right) & B\left(\frac{1}{r^2}\left\{\frac{1}{Re}\left[2A^2 + \frac{1}{r^2}\right] - T\right\} - 4A\right) \end{bmatrix}$$

$$[N_2'] = \begin{bmatrix} 0 & -A \\ 0 & 0 \end{bmatrix} \qquad (2\text{-}15)$$

$$[N_1'] = \begin{bmatrix} 0 & \frac{-A}{r} \\ 0 & \frac{B}{r} \end{bmatrix} \qquad (2\text{-}16)$$

$$[N_0'] = \begin{bmatrix} \frac{-2AB}{r^2} & A\left(\frac{1}{r^2} - t\right) \\ -t & \frac{B}{r^2} \end{bmatrix} \qquad (2\text{-}17)$$

and

$$t \equiv A^2 + \frac{B^2}{r^2} \qquad (2\text{-}18)$$

$$T \equiv 2A(1 - r^2) - \frac{t}{Re} \qquad (2\text{-}19)$$

This statement of the vorticity transport equation is identical to that published in Ref. 1. Note that equation (2-8) makes it unnecessary to include F(r) and its derivatives in equations (2-9). The coefficients of this function and its derivatives are included in the arrays represented by equations (2-15) through (2-22) of Ref. 1 and have been verified. The reader's attention is called to a misprint ocurring in Ref. 1 concerning the development to this point. Equation (2-22) of Ref. 1 should be corrected to read as follows.

$$[N_0] = \begin{bmatrix} \frac{-Bt}{r} & \frac{2AB}{r^2} & A\left(t - \frac{1}{r^2}\right) \\ 0 & t & \frac{-B}{r^2} \end{bmatrix} \qquad (2\text{-}20)$$

III. BOUNDARY AND SPECIAL CONDITIONS

A. BOUNDARY CONDITIONS AT THE WALL

The boundary conditions at the wall, r=1, are derived from the fact that the axial, radial, and angular components of the flow velocity at the wall are identically zero. These components are obtained respectively from the three components of the velocity vector resulting from equation (2-3).

As shown in part two of Appendix F in Ref 1 for the case F(r)=0, these components imply the following boundary conditions at the wall.

$$G(1) = 0 \tag{3-1a}$$

$$H(1) = 0 \tag{3-1b}$$

$$DH(1) = 0 \tag{3-1c}$$

B. CONDITIONS ON THE AXIS OF SYMMETRY

It is important to note that the line of points constituting the axis of symmetry is not a boundary but rather a line consisting of an infinite number of singular points in the flow.

It should be noted in passing that all functions are periodic with respect to θ. Consequently, as shown in part one of Appendix G in Ref. 1, the wave number of the perturbation, B, must be a pure imaginary number such that

$$B = ni \ , \quad n = 0,1,2,\ldots \tag{3-2}$$

Hence references to B in this paper shall be understood to indicate a pure imaginary number.

The primary condition to be imposed is that the vorticity transport equation be satisfied at the point r=0. Inspection of equations (2-9) through (2-19) shows that, in general, these are not satisfied for direct substitution of r=0. To resolve this situation, consider the following development.

The coefficients of the functions G(r) and H(r) and their derivatives with respect to r up to fourth order in equations (2-9) through (2-19) may be expressed in the following form.

$$[M_4'] = [M_{4-0}] \tag{3-3}$$

$$[M_3'] = [M_{3-1}]1/r \tag{3-4}$$

$$[M_2'] = [M_{2-2}]1/r^2 + [M_{2-0}] + [M_{2+2}]r^2 \tag{3-5}$$

$$[M_1'] = [M_{1-3}]1/r^3 + [M_{1-1}]1/r + [M_{1+1}]r \tag{3-6}$$

$$[M_0'] = [M_{0-4}]1/r^4 + [M_{0-2}]1/r^2 + [M_{0-0}] + [M_{0+2}]r^2 \tag{3-7}$$

$$[N_2'] = [N_{2-0}] \tag{3-8}$$

$$[N_1'] = [N_{1-1}]1/r \tag{3-9}$$

$$[N_0'] = [N_{0-2}]1/r^2 + [N_{0-0}] \tag{3-10}$$

where

$$[M_{4-0}] = \begin{bmatrix} 0 & \frac{-A}{Re} \\ 0 & 0 \end{bmatrix} \tag{3-11}$$

$$[M_{3-1}] = \begin{bmatrix} 0 & \frac{-2A}{Re} \\ 0 & \frac{B}{Re} \end{bmatrix} \tag{3-12}$$

$$[M_{2-2}] = \begin{bmatrix} \frac{-4AB}{Re} & \frac{A}{Re}(3 - 2B^2) \\ \frac{-B^2}{Re} & \frac{2B}{Re} \end{bmatrix} \tag{3-13}$$

$$[M_{2-0}] = \begin{bmatrix} 0 & 2A^2\left(1 - \frac{A}{Re}\right) \\ \frac{-A^2}{Re} & 0 \end{bmatrix} \qquad (3\text{-}14)$$

$$[M_{2+2}] = \begin{bmatrix} 0 & -2A^2 \\ 0 & 0 \end{bmatrix} \qquad (3\text{-}15)$$

$$[M_{1-3}] = \begin{bmatrix} \frac{4AB}{Re} & \frac{A}{Re}(2B^2 - 3) \\ \frac{B^2}{Re} & \frac{B}{Re}(B^2 - 1) \end{bmatrix} \qquad (3\text{-}16)$$

$$[M_{1-1}] = \begin{bmatrix} 0 & 2A^2\left(1 - \frac{A}{Re}\right) \\ \frac{-A^2}{Re} & -AB\left(2 - \frac{A}{Re}\right) \end{bmatrix} \qquad (3\text{-}17)$$

$$[M_{1+1}] = \begin{bmatrix} 0 & -2A^2 \\ 0 & 2AB \end{bmatrix} \qquad (3\text{-}18)$$

$$[M_{0-4}] = \begin{bmatrix} \frac{-4AB}{Re}(B^2 + 1) & \frac{A}{Re}\left(3 + B^2(2 - B^2)\right) \\ \frac{-B^2}{Re}(B^2 + 1) & \frac{B}{Re}(B^2 + 1) \end{bmatrix} \qquad (3\text{-}19)$$

$$[M_{0-2}] = \begin{bmatrix} 4A^2B\left(1 - \frac{A}{Re}\right) & 2A^2\left(1 - \frac{A}{Re}\right)(B^2 - 1) \\ 2AB^2\left(1 - \frac{A}{Re}\right) + \frac{A^2}{Re} & AB\left(\frac{3A}{Re} - 2\right) \end{bmatrix} \qquad (3\text{-}20)$$

$$[M_{0-0}] = \begin{bmatrix} -4A^2B & A^2\left(2(1 - B^2) + A^2\left\{2 - \frac{A}{Re}\right\}\right) \\ A\left(A^2\left\{2 - \frac{A}{Re}\right\} - 2B^2\right) & -2AB \end{bmatrix} \qquad (3\text{-}21)$$

$$[M_{0+2}] = \begin{bmatrix} 0 & -2A^4 \\ -2A^3 & 0 \end{bmatrix} \qquad (3\text{-}22)$$

$$[N_{2-0}] = \begin{bmatrix} 0 & -A \\ 0 & 0 \end{bmatrix} \tag{3-23}$$

$$[N_{1-1}] = \begin{bmatrix} 0 & -A \\ 0 & B \end{bmatrix} \tag{3-24}$$

$$[N_{0-2}] = \begin{bmatrix} -2AB & A(1 - B^2) \\ -B^2 & B \end{bmatrix} \tag{3-25}$$

$$[N_{0-0}] = \begin{bmatrix} 0 & -A^3 \\ -A^2 & 0 \end{bmatrix} \tag{3-26}$$

The Maclaurin series representation of the functions G(r) and H(r) and their derivatives may be expressed according to the following scheme.

$$\begin{Bmatrix} G \\ H \end{Bmatrix} = \begin{bmatrix} G(r) + DG(r) + D^2G(r) + D^3G(r) + \ldots. \\ H(r) + DH(r) + D^2H(r) + D^3H(r) + \ldots. \end{bmatrix} \{V\} \tag{3-27}$$

$$\begin{Bmatrix} DG \\ DH \end{Bmatrix} = \begin{bmatrix} DG(r) + D^2G(r) + D^3G(r) + D^4G(r) + \ldots. \\ DH(r) + D^2H(r) + D^3H(r) + D^4H(r) + \ldots. \end{bmatrix} \{V\} \tag{3-28}$$

$$\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} = \begin{bmatrix} D^2G(r) + D^3G(r) + D^4G(r) + D^5G(r) + \ldots. \\ D^2H(r) + D^3H(r) + D^4H(r) + D^5H(r) + \ldots. \end{bmatrix} \{V\} \tag{3-29}$$

where

$$\{V\} = \begin{Bmatrix} 1 \\ r \\ r^2/2! \\ r^3/3! \\ r^4/4! \\ \vdots \end{Bmatrix} \tag{3-30}$$

Substitution of equations (3-3) through (3-30) into equation (2-9) converts the vorticity transport equation to the following form.

$$\left[M_{4-0}\right]\begin{bmatrix} D^4G(r) + \cdots \\ D^4H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} + \left[M_{3-1}\right]\begin{bmatrix} D^3G(r) + D^4G(r) + \cdots \\ D^3H(r) + D^4H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r \\ 1 \\ \vdots \end{Bmatrix}$$

$$+ \left[M_{2-2}\right]\begin{bmatrix} D^2G(r) + D^3G(r) + D^4G(r) + \cdots \\ D^2H(r) + D^3H(r) + D^4H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r^2 \\ 1/r \\ 1/2 \\ \vdots \end{Bmatrix}$$

$$+ \left[M_{2-0}\right]\begin{bmatrix} D^2G(r) + \cdots \\ D^2H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} + \left[M_{2+2}\right]\begin{bmatrix} D^2G(r) + \cdots \\ D^2H(r) + \cdots \end{bmatrix}\begin{Bmatrix} r^2 \\ \vdots \end{Bmatrix}$$

$$+ \left[M_{1-3}\right]\begin{bmatrix} DG(r) + D^2G(r) + D^3G(r) + D^4G(r) + \cdots \\ DH(r) + D^2H(r) + D^3H(r) + D^4H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r^3 \\ 1/r^2 \\ 1/2r \\ 1/6 \\ \vdots \end{Bmatrix}$$

$$+ \left[M_{1-1}\right]\begin{bmatrix} DG(r) + D^2G(r) + \cdots \\ DH(r) + D^2H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r \\ 1 \\ \vdots \end{Bmatrix} + \left[M_{1+1}\right]\begin{bmatrix} DG(r) + \cdots \\ DH(r) + \cdots \end{bmatrix}\begin{Bmatrix} r \\ \vdots \end{Bmatrix}$$

$$+\left[M_{0-4}\right]\begin{bmatrix} G(r) + DG(r) + D^2G(r) + D^3G(r) + D^4G(r) + \cdots \\ H(r) + DH(r) + D^2H(r) + D^3H(r) + D^4H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r^4 \\ 1/r^3 \\ 1/2r^2 \\ 1/6r \\ 1/24 \\ \vdots \end{Bmatrix}$$

$$+\left[M_{0-2}\right]\begin{bmatrix} G(r) + DG(r) + D^2G(r) + \cdots \\ H(r) + DH(r) + D^2H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r^2 \\ 1/r \\ 1/2 \\ \vdots \end{Bmatrix}$$

$$+\left[M_{0-0}\right]\begin{bmatrix} G(r) + \cdots \\ H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} + \left[M_{0+2}\right]\begin{bmatrix} G(r) + \cdots \\ H(r) + \cdots \end{bmatrix}\begin{Bmatrix} r^2 \\ \vdots \end{Bmatrix}$$

$$-\gamma\left[N_{2-0}\right]\begin{bmatrix} D^2G(r) + \cdots \\ D^2H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} - \gamma\left[N_{1-1}\right]\begin{bmatrix} DG(r) + \cdots \\ DH(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r \\ 1 \\ \vdots \end{Bmatrix}$$

$$-\gamma\left[N_{0-2}\right]\begin{bmatrix} G(r) + DG(r) + D^2G(r) + \cdots \\ H(r) + DH(r) + D^2H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1/r^2 \\ 1/r \\ 1/2 \\ \vdots \end{Bmatrix}$$

$$-\left[N_{0-0}\right]\begin{bmatrix} G(r) + \cdots \\ H(r) + \cdots \end{bmatrix}\begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3\text{-}31)$$

Since equations (3-31) must be satisfied at all points in the flow, they must be satisfied in the limit as r approaches zero. A method of determining the conditions to be satisfied at the singular point, r=0, is described below.

Noting that as r approaches zero those terms in equations (3-31) containing r to the first power or greater may be neglected, these equations may be regrouped as a power series in r and expressed in the following abbreviated form.

$$\left[C_1\right]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} r^{-4} + \left[C_2\right]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} r^{-3}$$
$$+ \left(\left[C_3\right]\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + \left[C_4\right]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix}\right) r^{-2}$$
$$+ \left(\left[C_5\right]\begin{Bmatrix} D^3G(0) \\ D^3H(0) \end{Bmatrix} + \left[C_6\right]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix}\right) r^{-1}$$
$$+ \left(\left[C_7\right]\begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + \left[C_8\right]\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + \left[C_9\right]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix}\right) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3\text{-}32)$$

where

$$\left[C_1\right] = \left[M_{0-4}\right] \quad (3\text{-}33)$$
$$= \begin{bmatrix} \frac{-4ABP}{Re} & \frac{-AP(P-4)}{Re} \\ \frac{-B^2P}{Re} & \frac{BP}{Re} \end{bmatrix}$$

$$\left[C_2\right] = \left[M_{1-3}\right] + \left[M_{0-4}\right] \quad (3\text{-}34)$$
$$= \begin{bmatrix} \frac{-4AB^3}{Re} & \frac{-AB^2(P-5)}{Re} \\ \frac{-B^4}{Re} & \frac{2B^3}{Re} \end{bmatrix}$$

$$[C_3] = [M_{2-2}] + [M_{1-3}] + (1/2)[M_{0-4}] \tag{3-35}$$

$$= \begin{bmatrix} \frac{-2ABP}{Re} & \frac{-AP}{2Re}(P - 4) \\ \frac{-B^2P}{2Re} & \frac{3BP}{2Re} \end{bmatrix}$$

$$[C_4] = [M_{0-2}] - \gamma[N_{0-2}] \tag{3-36}$$

$$= \begin{bmatrix} 2ABQ & AQ(P - 2) \\ B^2Q + \frac{A^2}{Re} & -B\left(Q - \frac{A^2}{Re}\right) \end{bmatrix}$$

$$[C_5] = [M_{3-1}] + [M_{2-2}] + (1/2)[M_{1-3}] + (1/6)[M_{0-4}] \tag{3-37}$$

$$= \begin{bmatrix} \frac{-2AB}{3Re}(P + 3) & \frac{-AB^2}{6Re}(B^4 + 4) \\ \frac{-B^2}{6Re}(P + 3) & \frac{2B}{3Re}(P + 3) \end{bmatrix}$$

$$[C_6] = [M_{1-1}] + [M_{0-2}] - \gamma\left([N_{1-1}] + [N_{0-2}]\right) \tag{3-38}$$

$$= \begin{bmatrix} 2ABQ & AB^2Q \\ B^2Q & -2BQ \end{bmatrix}$$

$$[C_7] = [M_{4-0}] + [M_{3-1}] + (1/2)[M_{2-2}] + (1/6)[M_{1-3}] + (1/24)[M_{0-4}]$$

$$= \begin{bmatrix} \frac{AB}{6Re}(P + 8) & \frac{-A(P + 8)(P + 4)}{24Re} \\ \frac{-B^2}{24Re}(P + 8) & \frac{5B(P + 8)}{24Re} \end{bmatrix} \tag{3-39}$$

$$[C_8] = [M_{2-0}] + [M_{1-1}] + (1/2)[M_{0-2}] -$$

$$\gamma\left([N_{2-0}] + [N_{1-1}] + (1/2)[N_{0-2}]\right) \tag{3-40}$$

$$= \begin{bmatrix} ABQ & \frac{AQ(P + 2)}{2} \\ \frac{1}{2}\left(B^2Q - \frac{3A^2}{Re}\right) & \frac{-B}{2}\left(3Q + \frac{A^2}{Re}\right) \end{bmatrix}$$

$$[C_9] = [M_{0-0}] - \gamma [N_{0-0}] \tag{3-41}$$

$$= \begin{bmatrix} -4AB^2 & A^2\left(A\left\{Q + \frac{A^2}{Re}\right\} - 2(P-2)\right) \\ A\left(A\left\{Q + \frac{A^2}{Re}\right\} - 2B^2\right) & -2AB \end{bmatrix}$$

and

$$P = B^2 + 1 \tag{3-42}$$

$$Q = 2A\left(1 - \frac{A}{Re}\right) + \gamma \tag{3-43}$$

As a consequence of equations (3-32), the following conditions must be met in the limit as r approaches zero.

$$[C_1]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-44}$$

$$[C_2]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-45}$$

$$[C_3]\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + [C_4]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-46}$$

$$[C_5]\begin{Bmatrix} D^3G(0) \\ D^3H(0) \end{Bmatrix} + [C_6]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-47}$$

$$[C_7]\begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + [C_8]\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + [C_9]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-48}$$

As discussed in Section II, the vorticity transport equation as expressed by equations (2-9) through (2-19) is, in general, a coupled set of differential equations. However, if B=0 these equations do uncouple. For this special case, the first of equations (2-9) becomes a homogeneous fourth-order differential equation in H(r) and the second becomes a homogeneous second-order differential equation in G(r). Thus, the conditions expressed by equations (3-44) through (3-48) may now be examined for the special case B=0 and the general case $B > 0$.

1. B=0

Inspection of equations (3-45) and (3-47) shows that they are identically satisfied for all values of the applicable odd-order derivatives of G(0) and H(0) for this case. Since the remaining equations uncouple for this case, the conditions on G(0) and H(0) may be studied independently.

a. H(0)

Sequential inspection of the first of equations (3-44), (3-46), and (3-48) verifies that there are three conditions to be enforced. These three conditions are as follows.

$$H(0) = 0 \tag{3-49}$$

$$D^2H(0) = 0 \tag{3-50}$$

$$D^4H(0) = 0 \tag{3-51}$$

b. G(0)

Although the second of equations (3-44) is identically satisfied for arbitrary G(0), sequential inspection of the second of equations (3-46) and (3-48) yields the following two conditions which must be enforced.

$$G(0) = 0 \tag{3-52}$$

$$D^2G(0) = 0 \tag{3-53}$$

2. $B > 0$

With the exception of improbable special cases for which the determinants of any or all of the arrays $[C_1]$ through $[C_9]$ may be zero, the conditions to be met for this case are as follows.

$$G(0) = H(0) = 0 \tag{3-54}$$

$$DG(0) = DH(0) = 0 \tag{3-55}$$

$$D^2G(0) = D^2H(0) = 0 \tag{3-56}$$

$$D^3G(0) = D^3H(0) = 0 \tag{3-57}$$

$$D^4G(0) = D^4H(0) = 0 \tag{3-58}$$

The conditions expressed above represent a marked departure from the conditions previously thought to exist at $r=0$. Before the situation was properly understood, it was thought that the required conditions could be deduced from considerations of single-valuedness at this point. This approach is discussed in part two of Appendix G in Ref. 1 but must now be discarded as insufficient. In the interest of accuracy, the reader's attention is called to an error in equation (G-29) of that development. That equation should be corrected to read as follows.

$$v_1(0) = -iw_1(0) \tag{3-59}$$

IV. NUMERICAL METHODS

Regrouping equations (2-9) allows expression of the equations to be solved in the following format.

$$[M_4']\begin{Bmatrix} D^4G \\ D^4H \end{Bmatrix} + [M_3']\begin{Bmatrix} D^3G \\ D^3H \end{Bmatrix} + [M_2']\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} + [M_1']\begin{Bmatrix} DG \\ DH \end{Bmatrix} + [M_0']\begin{Bmatrix} G \\ H \end{Bmatrix}$$

$$- \gamma\left([N_2']\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} + [N_1']\begin{Bmatrix} DG \\ DH \end{Bmatrix} + [N_0']\begin{Bmatrix} G \\ H \end{Bmatrix}\right) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4\text{-}1)$$

The coefficient arrays are defined by equations (2-10) through (2-19).

As discussed in Section III, the nature of this system of equations depends on the value of the complex perturbation wave number, B. For the special case B=0, the first of equations (4-1) becomes a homogeneous fourth-order differential equation in H(r) and the second becomes a homogeneous second-order differential equation in G(r). For the general case of $B > 0$, equations (4-1) do not uncouple and must be solved simultaneously.

An additional influence of B is reflected in the character of the conditions at the singular point r=0. These conditions, as developed in Section III, are summarized below.

For the special case B=0, the first of equations (4-1) must simultaneously satisfy the conditions

$$H(0) = 0 \quad (4\text{-}2)$$

$$D^2H(0) = 0 \quad (4\text{-}3)$$

$$D^4H(0) = 0 \quad (4\text{-}4)$$

and the second of equations (4-1) must similarly satisfy the following conditions.

$$G(0) = 0 \tag{4-5}$$

$$D^2G(0) = 0 \tag{4-6}$$

For the general case where $B > 0$, equations (4-1) must simultaneously satisfy the conditions below.

$$G(0) = H(0) = 0 \tag{4-7}$$

$$DG(0) = DH(0) = 0 \tag{4-8}$$

$$D^2G(0) = D^2H(0) = 0 \tag{4-9}$$

$$D^3G(0) = D^3H(0) = 0 \tag{4-10}$$

$$D^4G(0) = D^4H(0) = 0 \tag{4-11}$$

The boundary conditions at the wall, r=1, are a consequence only of zero-velocity viscous effects at that point and thus do not vary with B. These conditions are as follows.

$$G(1) = 0 \tag{4-12}$$

$$H(1) = 0 \tag{4-13}$$

$$DH(1) = 0 \tag{4-14}$$

Using the method of finite differences, the functions G(r) and H(r) may be approximated by a finite number of discrete, evenly spaced unknowns. As shown in Figure 4-1 below, the non-dimensionalized radius of the pipe may be divided into a one-dimensional computational mesh of uniform spacings consisting of n interior points, n+1 intervals, and n+2 total points, including the boundary point at r=1 and the singular point at r=0. The uniform spacing between each of the mesh points is δ as defined below.

$$\delta = 1/(n+1) \tag{4-15}$$

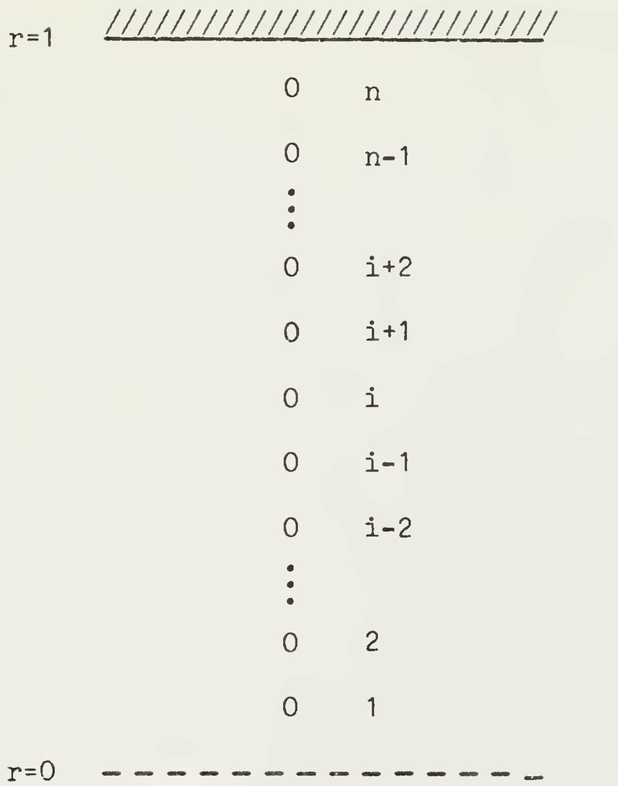


Figure 4-1 Finite Difference Mesh

By taking the Taylor series expansion of the function H(r) about the i^{th} mesh point in terms of the values of this function at the mesh points i+2, i+1, i-1, and i-2 the second-order central difference approximations of the derivatives of H(r) at the i^{th} point are found to have the following form.

$$DH_i = (H_{i+1} - H_{i-1})/2\delta \qquad (4\text{-}16)$$

$$D^2H_i = (H_{i+1} - 2H_i + H_{i-1})/\delta^2 \qquad (4\text{-}17)$$

$$D^3H_i = (H_{i+2} - 2(H_{i+1} - H_{i-1}) - H_{i-2})/2\delta^3 \qquad (4\text{-}18)$$

$$D^4H_i = (H_{i+2} - 4H_{i+1} + 6H_i - 4H_{i-1} + H_{i-2})/\delta^4 \qquad (4\text{-}19)$$

where

$$i = n, n-1, \ldots, 3, 2, 1 \qquad (4\text{-}20)$$

The approximations for the derivatives of G(r) are determined in an identical manner. The error of equations (4-16) through (4-19) is of the order of magnitude δ^2. The order of magnitude of this error may be reduced by expanding the method of derivation of these equations to include more adjacent points on either side of the i^{th}, central, point.

It is important to note a discrepancy appearing in Ref. 1 concerning the development to this point. Careful comparison of Figure 3-1 in Ref. 1 with Figure 4-1 in this paper reveals that the direction of the labeling schemes used to depict the finite difference mesh has been reversed. Note also that equations (3-2) in Ref. 1 compare exactly with equations (4-16) through (4-19) above. When these equations are used in conjunction with the labeling scheme used in Figure 3-1 in Ref. 1, the signs of the odd-order derivative approximations are reversed. This was a major factor in producing the erroneous solutions obtained from the program presented in that paper.

Substitution of the central difference approximations for the derivatives of the functions G and H in each of equations (4-1) results in a set of n linear, algebraic difference equations in terms of the unknown values for each of these functions at each of the n interior points of the finite difference mesh depicted in Figure 4-1. Since each of these equations is of the form of a linear combination of the i^{th}, central, mesh point and the two adjacent points on either side of this central point, this system of equations consists of banded coefficient arrays multiplying vectors containing the unknown values of the functions at each of the n interior points. By using this technique the problem is converted to an eigenvalue problem of the general form

$$[X]\{V\} - \gamma[Y]\{V\} = \{0\} \qquad (4\text{-}21)$$

with the basic composition of the arrays $[X]$ and $[Y]$ and the vector $\{V\}$ as illustrated in Figure 4-2 below.

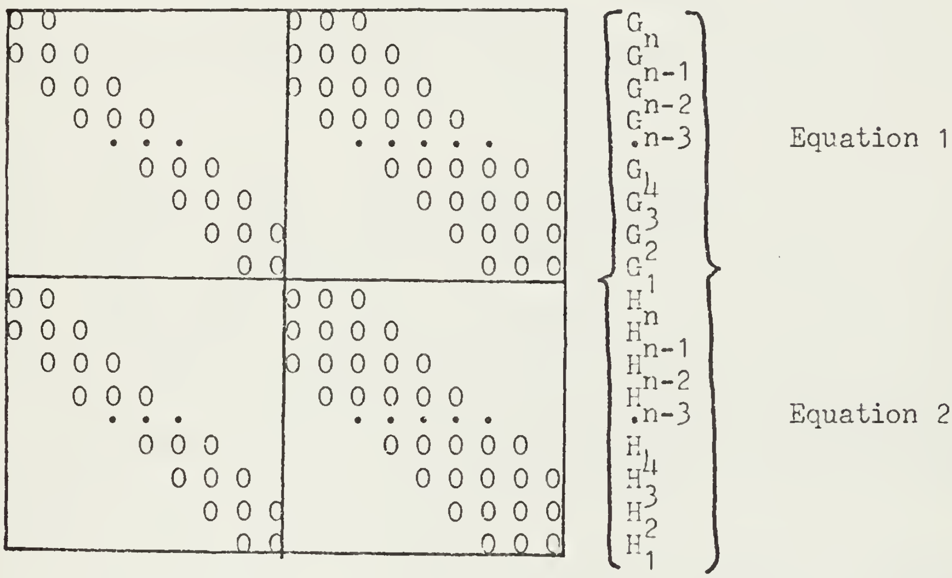


Figure 4-2 Basic Composition of the Coefficient Arrays and the Vector of Unknowns.

The exact composition of $[X]$ and $[Y]$ depends upon the value of B. These arrays are established by the subroutine MSET2 in conjunction with the function subprograms CHM1E1 and CHM1E2, which compute the numerical value for each element in these arrays. The function subprogram CHM1E1 provides those coefficients required from the first of equations (4-1) and CHM1E2 provides those coefficients required from the second of these equations. For the special case B=0, all coefficients contained in the upper right and lower left quadrants of Figure 4-2 would be zero.

Of particular interest are the difference equations whose central points are adjacent to the boundary at r=1, point n, and the singular point at r=0, point 1. To amplify this matter, consider the homogeneous differential equation in H(r) resulting from setting B=0 in equations (4-1). Use of equation (4-19) to evaluate the fourth derivative of this function at point n of the finite difference mesh reveals that

$$D^4H_n = (H_{n+2} - 4H_{n+1} + 6H_n - 4H_{n-1} + H_{n-2})/\delta^4. \qquad (4\text{-}22)$$

Similarly, for i=1 equation (4-19) becomes

$$D^4H_1 = (H_3 - 4H_2 + 6H_1 - 4H_0 + H_{-1})/\delta^4. \qquad (4\text{-}23)$$

Since $0 \leq r \leq 1$, equations (4-22) and (4-23) present apparent problems in that H_{n+2} is located at $r=1+\delta$ and H_{-1} is located at $r=-\delta$, i.e. beyond the allowable range of values for r.

These inconsistencies are resolved using the boundary conditions expressed by equations (4-12) through (4-14) and the conditions at r=0 expressed by equations (4-2) through (4-4). Using the labeling convention of Figure 4-1 and equation (4-13) it is easily confirmed that

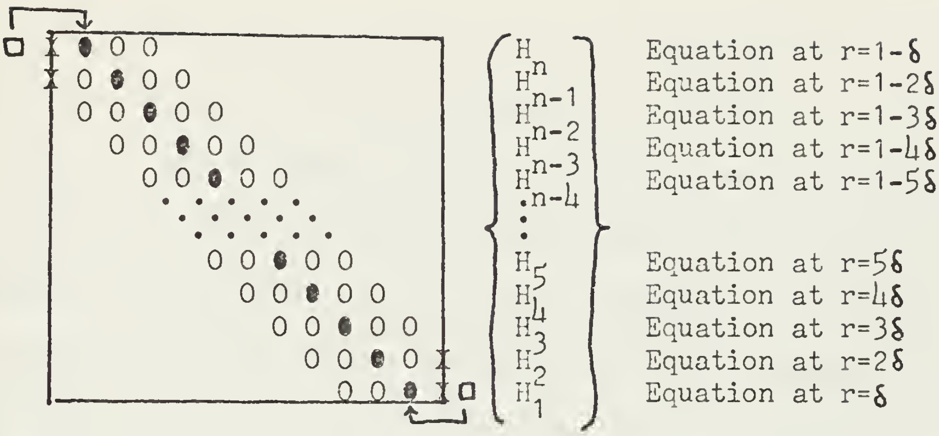
$$H(1) = H_{n+1} = 0 \,. \qquad (4\text{-}24)$$

Comparison of equations (4-14) and (4-16) yields the following relationships.

$$DH(1) = DH_{n+1} = (H_{n+2} - H_n)/2 = 0 \qquad (4\text{-}25)$$

$$H_{n+2} = H_n \qquad (4\text{-}26)$$

Equation (4-26) expresses the virtual point at $r=1+\delta$ in terms of the interior point n. Equation (4-22) may now be expressed exclusively in terms of boundary and interior points as follows (see depiction of equation at $r=1-\delta$ in Figure 4-3 below).



● Central point of the difference equation.
O Other points in the difference equation.
X Boundary or singular point where the value of the variable is zero.
□ Virtual point whose coefficient is combined with the coefficient of the point indicated by an arrow.

Figure 4-3 Illustration of Basic Method Used to Include Virtual Points in Difference Equations.

The virtual point at $r=-\delta$ can be expressed in terms of the interior point at $r=\delta$ in a similar manner. Expressed in terms of the labeling convention of Figure 4-1, equation (4-2) becomes

$$H_0 = 0 \; . \tag{4-27}$$

Comparison of equations (4-3) and (4-17) shows that

$$D^2H_0 = (H_1 - 2H_0 + H_{-1})/\delta^2 = 0 \; . \tag{4-28}$$

Direct substitution of equation (4-27) in equation (4-28) implies

$$H_{-1} = -H_1 \; . \tag{4-29}$$

This leads to a more appropriate expression of equation (4-23) in terms of interior points as follows.

$$D^4H_1 = (H_3 - 4H_2 + 6H_1 - 4H_0 - H_1)/\delta^4 \tag{4-30}$$

Consider now the homogeneous second-order differential equation in G(r) for the special case B=0. Since the highest order derivative of this function contained in equations (4-1) is second-order, inspection of equations (4-16) and (4-17) verifies the fact that no virtual points are required in the vicinities of the wall or the axis of symmetry.

As previously mentioned, the general case of $B > 0$ requires solution of equations (4-1) as a system of coupled equations. Implementation of the conditions at $r=0$ for this problem as given by equations (4-7) through (4-11) requires special attention.

The central difference approximation of the fourth derivative of H at $r=\delta$ requires a virtual point at $r=-\delta$. Expression of this virtual point in terms of the interior point at $r=\delta$ is complicated by the requirement that the five conditions for the function H given by equations (4-7) through (4-11) be satisfied simultaneously. Using equation (4-16) to satisfy equation (4-8) seems to imply that

$$H_{-1} = H_1 \ . \tag{4-31}$$

On the other hand, as previously shown in the development of equation (4-29)

$$H_{-1} = -H_1 \ . \tag{4-32}$$

This contradiction between equations (4-31) and (4-32) poses a problem. However, the Maclaurin series approximation for the function H(r) and its derivatives in the vicinity of $r=0$ is helpful in resolving this contradiction. These approximations are as shown below.

$$H(r) = A_0 + A_1 r + A_2\frac{r^2}{2!} + A_3\frac{r^3}{3!} + A_4\frac{r^4}{4!} + A_5\frac{r^5}{5!} + O(r^6) \tag{4-33}$$

$$DH(r) = A_1 + A_2 r + A_3\frac{r^2}{2!} + A_4\frac{r^3}{3!} + A_5\frac{r^4}{4!} + O(r^5) \tag{4-34}$$

$$D^2H(r) = A_2 + A_3 r + A_4\frac{r^2}{2!} + A_5\frac{r^3}{3!} + O(r^4) \tag{4-35}$$

$$D^3H(r) = A_3 + A_4 r + A_5\frac{r^2}{2!} + O(r^3) \tag{4-36}$$

$$D^4H(r) = A_4 + A_5 r + O(r^2) \tag{4-37}$$

A comparison of equations (4-33) through (4-37) with equations (4-7) through (4-11) indicates that at r=0

$$A_0 = A_1 = A_2 = A_3 = A_4 = 0 \;. \tag{4-38}$$

Substitution of equations (4-38) into equation (4-33) yields the following sixth-order approximation for H(r) in the limit as r approaches zero.

$$H(r) = A_5\frac{r^5}{5!} + O(r^6) \tag{4-39}$$

Evaluation of equation (4-39) for $r=\pm\delta$ yields

$$H(\delta) = A_5\frac{\delta^5}{5!} \tag{4-40}$$

and

$$H(-\delta) = A_5\frac{(-\delta)^5}{5!} \tag{4-41}$$

Inspection of equations (4-40) and (4-41) quickly confirms that enforcement of equations (4-7) through (4-11) requires that

$$H_{-1} = -H_1 \;. \tag{4-42}$$

Note that the use of equation (4-39) as an approximation for the function H(r) in equation (4-11) does not alter the second-order accuracy of the latter approximation. This may be conceptualized by a symbolic substitution of equation (4-39) in equation (4-11) as shown below.

$$D^4H_1 = ((H_3 - 4H_2 + 6H_1 - 4H_0 + H_{-1}) + O(\delta^6))/\delta^4 \tag{4-43}$$

$$= (H_3 - 4H_2 + 6H_1 - 4H_0 + H_{-1})/\delta^4 + O(\delta^2) \tag{4-44}$$

With the formulation of the problem as discussed to this point accomplished, i.e. in the form of equation (4-21), the method used for the remainder of the solution is summarized in the following steps:

1.) Subroutine CDMTIN inverts the second coefficient array in equation (4-21), array [Y]. CDMTIN was obtained from the IBM Library routine CMTRIN by modifying it to make it applicable to double precision arrays.

2.) Both coefficient arrays, [X] and [Y], are then premultiplied by $[Y]^{-1}$. Since multiplication of an array by its inverse invariably results in the identity matrix, [I], only the product $[Y]^{-1}[X]$ is computed using subroutine MULM. This converts the eigenvalue problem of equation (4-21) to the more conventional form

$$([Z] - \gamma[I])\{V\} = \{0\} . \qquad (4\text{-}45)$$

where

$$[Z] = [Y]^{-1}[X] \qquad (4\text{-}46)$$

3.) Since all the programs available for solving equations (4-45) require that the real and imaginary parts of the elements of [Z] be presented in separate arrays, subroutine DSPLIT is called to accomplish this.

4.) The eigenvalues of equations (4-45) are computed by subroutines EHESSC and ELRH1C which are available through the International Mathematical and Statistical Library. These subroutines reduce the matrix [Z] into the complex upper Hessenberg form and then solve for the eigenvalues. The results are then passed back to the main program, R2.

5.) The eigenvalues thus returned are listed on the computer output and plotted on the complex plane using subroutine PLOTP from the IBM Library.

V. RESULTS

Owing to time constraints, the number of computer solutions obtained was sufficient only to confirm the general validity of the program presented in this paper. For this reason, a detailed discussion of flow stability will not be undertaken here. However, the following brief discussion should provide sufficient information for interpreting the data presented in this section.

The characteristics of the flow which govern the stability are defined by the input variables A, B, and Reynolds number. A complete set of eigenvalues, γ, is obtained for each chosen combination of these variables. Recalling the form of the perturbation velocity vector potential, $\bar{W}$, as presented in Section II,

$$\bar{W}(x,r,\theta,t) = (\bar{e}_x F(r) + \bar{e}_r G(r) + \bar{e}_\theta H(r))\exp(Ax + B\theta + \gamma t) \qquad (5\text{-}1)$$

where

$$\gamma = \gamma_R + \gamma_I , \qquad (5\text{-}2)$$

it is readily seen that positive values for the real part, γ_R, of the complex perturbation frequency represent an exponential growth rate in time and negative values of γ_R represent an exponential decay rate. The algebraically greatest value of γ_R contained in the solution set is therefore used as the stability criterion. The corresponding eigenvalue is said to be the least stable root. The flow is considered to be stable, neutrally stable, or unstable with respect to stationary coordinates according to whether the least stable root is less than, equal to, or greater than zero, respectively. For a discussion of stability with respect to moving coordinates see part D of Section II in Ref. 1.

All computations were made using a half channel mesh containing 30 interior points and with A_I held constant at a value of one. Tables I through V present the real part of the least stable root for each of the solutions obtained using the program developed in this paper. In general, these solutions are encouraging in that those errors which had previously caused solutions to exhibit decreasing stability with decreasing Reynolds number seem to have been resolved. Results appear to be reasonable in that they exhibit stability characteristics which are qualitatively similar to those obtained for the plane flow problem as detailed in Section IV of Ref. 1. Specifically, for the range of Reynolds numbers investigated, all flows were stable for $A_R=0$ while instabilities were found for $A_R=-0.05$. These results are significant in that previous investigations of fully developed pipe flow have never accounted for the instabilities that are known to exist.

Re	$A_R = 0$	$A_R = -0.05$
100	$-3.44\ 10^{-1}$	$-2.62\ 10^{-1}$
200	$-2.88\ 10^{-1}$	$-1.95\ 10^{-1}$
500	$-1.80\ 10^{-1}$	$-8.45\ 10^{-2}$
1000	$-1.27\ 10^{-1}$	$-2.99\ 10^{-2}$
2000	$-8.93\ 10^{-2}$	$8.57\ 10^{-3}$
5000	$-5.61\ 10^{-2}$	$4.26\ 10^{-2}$
10000	$-3.95\ 10^{-2}$	$5.96\ 10^{-2}$

Table I γ_R for $B_I = 0$.

Re	$A_R = 0$	$A_R = -0.05$
100	$-3.05\ 10^{-1}$	$-2.31\ 10^{-1}$
200	$-2.73\ 10^{-1}$	$-2.02\ 10^{-1}$
500	$-1.84\ 10^{-1}$	$-9.31\ 10^{-2}$
1000	$-1.33\ 10^{-1}$	$-3.98\ 10^{-2}$
2000	$-9.58\ 10^{-2}$	$-3.94\ 10^{-4}$
5000	$-5.98\ 10^{-2}$	$3.74\ 10^{-2}$
10000	$-4.13\ 10^{-2}$	$5.68\ 10^{-2}$

Table II γ_R for $B_I = 1$

Re	$A_R = 0$	$A_R = -0.05$
100	$-4.91\ 10^{-1}$	$-3.38\ 10^{-1}$
200	$-3.36\ 10^{-1}$	$-2.49\ 10^{-1}$
500	$-2.09\ 10^{-1}$	$-1.19\ 10^{-1}$
1000	$-1.48\ 10^{-1}$	$-5.48\ 10^{-2}$
2000	$-1.05\ 10^{-1}$	$-9.54\ 10^{-3}$
5000	$-6.64\ 10^{-2}$	$3.06\ 10^{-2}$
10000	$-4.71\ 10^{-2}$	$5.08\ 10^{-2}$

Table III γ_R for $B_I = 2$

Re	$A_R = 0$	$A_R = -0.05$
100	$-4.94\ 10^{-1}$	$-4.44\ 10^{-1}$
200	$-4.27\ 10^{-1}$	$-3.82\ 10^{-1}$
500	$-2.89\ 10^{-1}$	$-2.01\ 10^{-1}$
1000	$-2.05\ 10^{-1}$	$-1.13\ 10^{-1}$
2000	$-1.45\ 10^{-1}$	$-5.05\ 10^{-2}$
5000	$-9.16\ 10^{-2}$	$4.74\ 10^{-3}$
10000	$-6.49\ 10^{-2}$	$3.25\ 10^{-2}$

Table IV γ_R for $B_I = 3$

Re	$A_R = 0$	$A_R = -0.05$
100	$-6.16\ 10^{-1}$	$-5.70\ 10^{-1}$
200	$-4.69\ 10^{-1}$	$-4.39\ 10^{-1}$
500	$-3.44\ 10^{-1}$	$-2.87\ 10^{-1}$
1000	$-2.75\ 10^{-1}$	$-1.74\ 10^{-1}$
2000	$-1.87\ 10^{-1}$	$-9.36\ 10^{-2}$
5000	$-1.18\ 10^{-1}$	$-2.25\ 10^{-2}$
10000	$-8.36\ 10^{-2}$	$1.33\ 10^{-2}$

Table V γ_R for $B_I = 4$

```
C PROGRAM #R2
C.................................................................
C
C
C                PROGRAM TO PRINT EIGENVALUES
C            FOR THE 3-D CYLINDRICAL FLOW PROBLEM
C
C.......................................................................
C
C         INPUT VALUES ARE N,REY,AR,AI,BR, AND BI.
C                (USING NAMELIST 'LIST')
C
C.......................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*16 A,B,DETERM,G
      COMPLEX*16 XMAT(60,60),YMAT(60,60),WV(60)
      COMPLEX*16 CFM1E1,CFM2E1,CFM1E2,CFM2E2,CGM1E1,CGM2E1,
     *     CGM1E2,CGM2E2,CHM1E1,CHM2E1,CHM1E2,CHM2E2
      REAL*8 GR(60),GI(60)
      REAL*4 GR4(60),GI4(60)
      DIMENSION IVEC(100)
      COMMON / COEFNT / A,B,G,REY,DELR
      NAMELIST / LIST / N,REY,AR,AI,BR,BI
      EXTERNAL   CFM1E1,CFM2E1,CFM1E2,CFM2E2,CGM1E1,CGM2E1,
     *     CGM1E2,CGM2E2,CHM1E1,CHM2E1,CHM1E2,CHM2E2
C
C
C         READ AND VERIFY THE INPUT VARIABLES.
C
C
    1 READ(5,LIST,END=100)
      A = DCMPLX(AR,AI)
      B = DCMPLX(BR,BI)
      MDIM = 60
      N = 30
      WRITE(6,9000)
 9000 FORMAT('1')
      WRITE(6,9001)N,REY,A,B
 9001 FORMAT(//,1X,'N=',I4,/,1X,'REY=',F10.2,/1X,'ALPHA=',2F12.7,8X,
     *      'BETA=',2F12.7,/,1X,'FOR THE CASE F(R)=0.',/)
C
C
C         SET UP THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
C         THE MESH FOR THOSE TERMS IN THE VORTICITY TRANSPORT EQUATION
C         WHICH CONTAIN GAMMA AS A FACTOR.
C
C
```

```
      CALL MSET2(YMAT,N,MDIM,CGM2E1,0,0)
      CALL MSET2(YMAT,N,MDIM,CHM2E1,0,N)
      CALL MSET2(YMAT,N,MDIM,CGM2E2,N,0)
      CALL MSET2(YMAT,N,MDIM,CHM2E2,N,N)
C
C
C        INVERT THE RESULTING ARRAY.
C
C
      CALL COMTIN(MDIM,YMAT,MDIM,DETERM)
C
C
C        SET UP THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
C        THE MESH FOR THOSE TERMS IN THE VORTICITY TRANSPORT EQUATION
C        WHICH DO NOT CONTAIN GAMMA AS A FACTOR.
C
C
      CALL MSET2(XMAT,N,MDIM,CGM1E1,0,0)
      CALL MSET2(XMAT,N,MDIM,CHM1E1,0,N)
      CALL MSET2(XMAT,N,MDIM,CGM1E2,N,0)
      CALL MSET2(XMAT,N,MDIM,CHM1E2,N,N)
C
C
C        PREMULTIPLY XMAT BY THE INVERSE OF YMAT TO CONVERT THE PROBLEM
C        TO THE STANDARD FORM:
C                              A*X - GAMMA*I = 0
C
C
      CALL MULM(YMAT,XMAT,MDIM,MDIM,WV)
C
C
C        SPLIT THE RESULTING ARRAY INTO REAL (XMAT) AND IMAGINARY (YMAT)
C        PARTS.
C
C
      CALL DSPLIT(MDIM,MDIM,YMAT,XMAT,YMAT)
C
C
C        CALCULATE THE EIGENVALUES (GAMMA) FOR THE EQUATION.
C
C
      CALL EHESSC(XMAT,YMAT,1,MDIM,MDIM,MDIM,IVEC)
      CALL ELRH1C(XMAT,YMAT,1,MDIM,MDIM,MDIM,GR,GI,INERR,IER)
      IF(INERR.NE.0) WRITE(6,9010) INERR,IER
 9010 FORMAT('0* * * ERROR NUMBER',I7,' ON EIGENVALUE',
     *     I7,' * * *',///)
C
C
```

```
C          WRITE THE RESULTS.
C
C
      WRITE(6,9002)
 9002 FORMAT(///,16X,'GAMMA REAL',10X,'GAMMA IMAG')
      DO 10 I = 1,MDIM
      GR4(I) = SNGL(GR(I))
      GI4(I) = SNGL(GI(I))
   10 WRITE(6,9003) I,GR(I),GI(I)
 9003 FORMAT('0',I10,1P2D20.10)
C
C
C          PLOT THE RESULTS.
C
C
      WRITE(6,9000)
      CALL PLOTP(GR4,GI4,MDIM,0)
      WRITE(6,9001) N,REY,A,B
C
      GO TO 1
  100 WRITE(6,9000)
      STOP
      END
```

```
C.................SUBROUTINE MSET2(X,N,MDIM,CFMAT,MV,MH).................
C
C     PURPOSE
C
C       MSET2 GENERATES THE ARRAYS REPRESENTING THE CENTRAL DIFFERENCE
C       APPROXIMATION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF
C       THE VELOCITY VECTOR POTENTIAL.
C
C     USAGE
C
C       CALL MSET2(X,N,MDIM,CFMAT,MV,MH)
C
C     DESCRIPTION OF PARAMETERS
C
C       X - THE NAME OF THE ARRAY BEING GENERATED. MUST BE DIMENSIONED
C           IN THE CALLING PROGRAM
C
C       N- THE ROW DIMENSION OF THE MATRIX X.  MUST BE .GE. N.
C
C       MDIM - THE COLUMN DIMENSION OF THE MATRIX X. MUST BE .GE. N.
C
C       CFMAT - THE NAME OF A FUNCTION SUBPROGRAM WITH TWO PARAMETERS, K
C               AND R, INDICATING WHICH TERM OF THE CENTRAL DIFFERENCING
C               SCHEME IS DESIRED, AND THE POSITION OF THE CENTRAL POINT
C               RELATIVE TO THE AXIS OF THE PIPE. CFMAT MUST BE DECLARED
C               EXTERNAL IN THE CALLING PROGRAM.
C
C       MV - ROW INDEX FOR INITIALIZING PLACEMENT OF ELEMENTS INTO
C            MATRIX X.
C
C       MH - COLUMN INDEX FOR INITIALIZING PLACEMENT OF ELEMENTS INTO
C            MATRIX X.
C
C     THE FOLLOWING IS OUTPUT BY MSET2
C
C       X - THE N BY N MATRIX INTO WHICH THE COEFFICIENTS OF THE CENTRAL
C           DIFFERENCING ARE PUT.
C
C     OTHER ROUTINES NEEDED
C
C       FUNCTION SUBPROGRAM NAME PASSED IN THE CALLING PARAMETER 'CFMAT'
C.......................................................................
C
      SUBROUTINE MSET2(X,N,MDIM,CFMAT,MV,MH)
      REAL*8 REY,R,DEL,DFLOAT
      COMPLEX*16 A,B,G
      COMMON / COEFNT / A,B,G,REY,DEL
      COMPLEX*16 CFMAT
```

```
      COMPLEX*16 X(MDIM,MDIM)
C
C
C        DEFINE THE SPACING OF THE INTERIOR MESH POINTS.
C
C
      DEL = 1D0/DFLOAT(N+1)
C
C
C        CHECK IF MATRIX DIMENSIONED LARGE ENOUGH.
C
C
      IF(N+MV.LE.MDIM.AND.N+MH.LE.MDIM) GO TO 1
      WRITE(6,9000)
 9000 FORMAT('0* * * ERROR - ARRAYS NOT DIMENSIONED LARGE',
     *      ' ENOUGH * * *')
      STOP
C
C
C        INITIALIZE ALL ELEMENTS IN THE ARRAY TO ZERO.
C
C
    1 DO 10 I=1,N
      DO 10 J=1,N
   10 X(I+MV,J+MH) = (0D0,0D0)
C
C
C        ESTABLISH THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
C        THE MESH.
C
C
      R = 1D0-DEL
      X(1+MV,1+MH) = CFMAT(3,R)+CFMAT(1,R)
      X(1+MV,2+MH) = CFMAT(4,R)
      X(1+MV,3+MH) = CFMAT(5,R)
C
      R = 1D0-2D0*DEL
      X(2+MV,1+MH) = CFMAT(2,R)
      X(2+MV,2+MH) = CFMAT(3,R)
      X(2+MV,3+MH) = CFMAT(4,R)
      X(2+MV,4+MH) = CFMAT(5,R)
C
      IL = N-2
      DO 20 I=3,IL
      K = I-3
      R = 1D0-DEL*DFLOAT(I)
      DO 20 J=1,5
   20 X(I+MV,K+J+MH) = CFMAT(J,R)
```

```
C
      R = 2*DEL
      X(N-1+MV,N-3+MH) = CFMAT(1,R)
      X(N-1+MV,N-2+MH) = CFMAT(2,R)
      X(N-1+MV,N-1+MH) = CFMAT(3,R)
      X(N-1+MV,N+MH) = CFMAT(4,R)
C
      R = DEL
      X(N+MV,N-2+MH) = CFMAT(1,R)
      X(N+MV,N-1+MH) = CFMAT(2,R)
      X(N+MV,N+MH) = CFMAT(3,R)+CFMAT(5,R)
   30 RETURN
      END
```

```
C.........................FUNCTION CHM1E1(K,R)...........................
C                         (POLAR COORDINATES)
C
C     PURPOSE
C
C
C        SECTICN I
C
C          RETURNS THE VALUES FOR EACH OF THE COEFFICIENTS IN THE LINEAR
C          COMBINATION OF THE FIRST AND THIRD EQUATIONS RESULTING FROM
C          EXPRESSION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF THE
C          VELOCITY VECTOR POTENTIAL.
C
C            DESCRIPTION OF PARAMETERS
C
C              EXAMPLE OF THE CALLING ARGUMENT:
C
C                     C(F,G,H) (4,3,2,1,0) M(1,2)
C
C              C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR
C                         WHICH THE COEFFICIENT IS BEING COMPUTED.
C
C              (4,3,2,1,0) - ORDER OF THE DERIVATIVE OF THE ABOVE
C                            COMPONENT.
C
C              M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A
C                         FACTOR.
C                       2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.
C
C
C        SECTICN II
C
C          RETURNS THE VALUES FOR THE COEFFICIENTS IN THE ARRAYS
C          REPRESENTING THE CENTRAL DIFFERENCE APPROXIMATION OF THE
C          VORTICITY TRANSPORT EQUATION USING THE COEFFICIENTS COMPUTED
C          IN SECTION I.
C
C            DESCRIPTION OF PARAMETERS
C
C              K - INDICATES THE POSITION OF THE PCINT IN THE FINITE
C                  DIFFERENCE MESH RELATIVE TO THE CENTRAL POINT IN THE
C                  CENTRAL DIFFERENCE SCHEME. IF THE CENTRAL DIFFERENCE
C                  IS BEING GENERATED ABOUT THE N-TH POINT IN THE FINITE
C                  DIFFERENCE MESH, THEN K=1 REFERS TO THE POINT N+2, AS
C                  MEASURED TOWARD THE WALL, K=2 REFERS TO THE POINT N+1,
C                  K=3 REFERS TO THE POINT N, K=4 REFERS TO THE POINT
C                  N-1, AND K=5 REFERS TO THE POINT N-2.
C
```

```
C                 EXAMPLE OF THE CALLING ARGUMENT:
C
C                          C(F,G,H) M(1,2) E1
C
C                  C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR
C                             WHICH THE COEFFICIENT IS BEING COMPUTED.
C
C                  M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A
C                             FACTOR.
C                           2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.
C
C                  E1 - REFERS TO THE LINEAR COMBINATION OF THE FIRST AND
C                       THIRD EQUATIONS RESULTING FROM EXPRESSION OF THE
C                       VORTICITY TRANSPORT EQUATION IN TERMS OF THE VELOCITY
C                       VECTOR POTENTIAL.
C
C      USAGE
C
C        CHM1E1 MUST BE DECLARED COMPLEX*16 IN THE CALLING PROGRAM.
C
C      OTHER ROUTINES REQUIRED
C
C        NONE
C.......................................................................
C
      FUNCTION CHM1E1(K,R)
      IMPLICIT COMPLEX*16 (A-H,O-Z)
      COMMON / COEFNT / A,B,G,REY,DEL
      REAL*8 REY,R,DEL
C
C
C         DEFINE THE RECURRING PARAMETERS T1(R) AND T2(R).
C
C
      T1(R)  = A**2+(B/R)**2
      T2(R)  = A*2D0*(1D0-R**2)-T1(R)/REY
C
C
C   SECTION I
C
C
C         COEFFICIENTS OF THE COMPONENT F.
C
C
      CF4M1(R) = B/(R*REY)
      CF3M1(R) = 2D0*B/(R**2*REY)
      CF2M1(R) = (B/R)*((T1(R)-1D0/R**2)/REY-T2(R))
      CF1M1(R) = (B/R**2)*(((1D0-3D0*B**2)/R**2+A**2)/REY-T2(R))
```

```
      CFOM1(R) = (B/R)*(4DO*B**2/(REY*R**4)-T1(R)*T2(R))
C
      CF2M2(R) = B/R
      CF1M2(R) = B/R**2
      CFOM2(R) = B*T1(R)/R
C
C
C        COEFFICIENTS OF THE COMPONENT G.
C
C
      CG2M1(R) = -4DO*A*B/(REY*R**2)
      CG1M1(R) = 4DO*A*B/(REY*R**3)
      CGOM1(R) = (2DO*A*B/R**2)*(T2(R)-(T1(R)+2DO/R**2)/REY)
C
      CGOM2(R) = -2DO*A*B/R**2
C
C
C        COEFFICIENTS OF THE COMPONENT H.
C
C
      CH4M1(R) = -A/REY
      CH3M1(R) = -2DO*A/(R*REY)
      CH2M1(R) = A*(T2(R)+(3DO/R**2-T1(R))/REY)
      CH1M1(R) = (A/R)*((3DO*(B**2-1DO)/R**2-A**2)/REY+T2(R))
      CHOM1(R) = A*(T1(R)*T2(R)+((T1(R)+3DO/R**2)/REY-T2(R))/R**2)
C
      CH2M2(R) = -A
      CH1M2(R) = -A/R
      CHOM2(R) = A*(1DO/R**2-T1(R))
C
C
C SECTION II
C
C
C        CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT H.
C
C
      GO TO (11,12,13,14,15),K
   11 CHM1E1 = CH4M1(R)/DEL**4+CH3M1(R)/(2DO*DEL**3)
      GO TO 100
   12 CHM1E1 = -4DO*CH4M1(R)/DEL**4-2DO*CH3M1(R)/(2DO*DEL**3
     *    )+CH2M1(R)/DEL**2+CH1M1(R)/(2DO*DEL)
      GO TO 100
   13 CHM1E1 = 6DO*CH4M1(R)/DEL**4-2DO*CH2M1(R)/DEL**2
     *     +CHOM1(R)
      GO TO 100
   14 CHM1E1 = -4DO*CH4M1(R)/DEL**4+2DO*CH3M1(R)/(2DO*DEL**3
     *    )+CH2M1(R)/DEL**2-CH1M1(R)/(2DO*DEL)
```

```
      GO TO 100
   15 CHM1E1 = CH4M1(R)/DEL**4-CH3M1(R)/(2D0*DEL**3)
  100 RETURN
C
      ENTRY CHM2E1(K,R)
      GO TO (21,22,23,24,21),K
   21 CHM2E1 = (0D0,0D0)
      GO TO 200
   22 CHM2E1 = CH2M2(R)/DEL**2+CH1M2(R)/(2D0*DEL)
      GO TO 200
   23 CHM2E1 = -2D0*CH2M2(R)/DEL**2+CH0M2(R)
      GO TO 200
   24 CHM2E1 = CH2M2(R)/DEL**2-CH1M2(R)/(2D0*DEL)
  200 RETURN
C
C
C         CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT G.
C
C
      ENTRY CGM1E1(K,R)
      GO TO (31,32,33,34,31),K
   31 CGM1E1 = (0D0,0D0)
      GO TO 300
   32 CGM1E1 = CG2M1(R)/DEL**2+CG1M1(R)/(2D0*DEL)
      GO TO 300
   33 CGM1E1 = -2D0*CG2M1(R)/DEL**2+CG0M1(R)
      GO TO 300
   34 CGM1E1 = CG2M1(R)/DEL**2-CG1M1(R)/(2D0*DEL)
  300 RETURN
C
      ENTRY CGM2E1(K,R)
      GO TO (41,41,42,41,41),K
   41 CGM2E1 = (0D0,0D0)
      GO TO 400
   42 CGM2E1 = CG0M2(R)
  400 RETURN
C
C
C
C         CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT F.
C
C
      ENTRY CFM1E1(K,R)
      GO TO (51,52,53,54,55),K
   51 CFM1E1 = CF4M1(R)/DEL**4+CF3M1(R)/(2D0*DEL**3)
      GO TO 500
   52 CFM1E1 = -4D0*CF4M1(R)/DEL**4-2D0*CF3M1(R)/(2D0*DEL**3
     *   )+CF2M1(R)/DEL**2+CF1M1(R)/(2D0*DEL)
```

```
      GC TO 500
   53 CFM1E1 = 6D0*CF4M1(R)/DEL**4-2D0*CF2M1(R)/DEL**2
     *       +CFOM1(R)
      GC TO 500
   54 CFM1E1 = -4D0*CF4M1(R)/DEL**4+2D0*CF3M1(R)/(2D0*DEL**3
     *      )+CF2M1(R)/DEL**2-CF1M1(R)/(2D0*DEL)
      GC TO 500
   55 CFM1E1 = CF4M1(R)/DEL**4-CF3M1(R)/(2D0*DEL**3)
  50C RETURN
C
      ENTRY CFM2E1(K,R)
      GO TO (61,62,63,64,61),K
   61 CFM2E1 = (0D0,0D0)
      GC TO 600
   62 CFM2E1 = CF2M2(R)/DEL**2+CF1M2(R)/(2D0*DEL)
      GC TO 600
   63 CFM2E1 = -2D0*CF2M2(R)/DEL**2+CFOM2(R)
      GC TO 600
   64 CFM2E1 = CF2M2(R)/DEL**2-CF1M2(R)/(2D0*DEL)
  60C RETURN
      END
```

```
C.........................FUNCTION CHM1E2(K,R)............................
C                         (POLAR COORDINATES)
C
C     PURPOSE
C
C
C        SECTION I
C
C          RETURNS THE VALUES FOR EACH OF THE COEFFICIENTS IN THE SECOND
C          EQUATION RESULTING FROM EXPRESSION OF THE VORTICITY TRANSPORT
C          EQUATION IN TERMS OF THE VELOCITY VECTOR POTENTIAL.
C
C            DESCRIPTION OF PARAMETERS
C
C              EXAMPLE OF THE CALLING ARGUMENT:
C
C                     C(F,G,H) (4,3,2,1,0) M(1,2)
C
C              C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR
C                         WHICH THE COEFFICIENT IS BEING COMPUTED.
C
C              (4,3,2,1,0) - ORDER OF THE DERIVATIVE OF THE ABOVE
C                            COMPONENT.
C
C              M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A
C                         FACTOR.
C                       2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.
C
C
C
C        SECTION II
C
C          RETURNS THE VALUES FOR THE COEFFICIENTS IN THE ARRAYS
C          REPRESENTING THE CENTRAL DIFFERENCE APPROXIMATION OF THE
C          VORTICITY TRANSPORT EQUATION USING THE COEFFICIENTS COMPUTED
C          IN SECTION I.
C
C            DESCRIPTION OF PARAMETERS
C
C              K - INDICATES THE POSITION OF THE POINT IN THE FINITE
C                  DIFFERENCE MESH RELATIVE TO THE CENTRAL POINT IN THE
C                  CENTRAL DIFFERENCE SCHEME. IF THE CENTRAL DIFFERENCE
C                  IS BEING GENERATED ABOUT THE N-TH POINT IN THE FINITE
C                  DIFFERENCE MESH, THEN K=1 REFERS TO THE POINT N+2, AS
C                  MEASURED TOWARD THE WALL, K=2 REFERS TO THE POINT N+1,
C                  K=3 REFERS TO THE POINT N, K=4 REFERS TO THE POINT
C                  N-1, AND K=5 REFERS TO THE POINT N-2.
C
```

```
C               EXAMPLE OF THE CALLING ARGUMENT:
C
C                         C(F,G,H) M(1,2) E2
C
C                C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR
C                           WHICH THE COEFFICIENT IS BEING COMPUTED.
C
C                M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A
C                           FACTOR.
C                         2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.
C
C                E2 - REFERS TO THE SECOND EQUATION RESULTING FROM
C                     EXPRESSION OF THE VORTICITY TRANSPORT EQUATION IN
C                     TERMS OF THE VELOCITY VECTOR POTENTIAL.
C
C     USAGE
C
C       CHM1E2 MUST BE DECLARED COMPLEX*16 IN THE CALLING PROGRAM.
C
C     OTHER ROUTINES REQUIRED
C
C       NONE
C.......................................................................
C
      FUNCTION CHM1E2(K,R)
      IMPLICIT COMPLEX*16(A-H,O-Z)
      COMMON / COEFNT / A,B,C,REY,DEL
      REAL*8 REY,R,DEL
C
C
C        DEFINE THE RECURRING PARAMETERS T1(R) AND T2(R).
C
C
      T1(R) = A**2+(B/R)**2
      T2(R) = A*2D0*(1D0-R**2)-T1(R)/REY
C
C
C SECTION I
C
C
C
C        COEFFICIENTS OF THE COMPONENT F.
C
C
      CF3M1(R) = A/REY
      CF2M1(R) = A/(R*REY)
      CF1M1(R) = -A*(T2(R)+1D0/(REY*R**2))
      CFOM1(R) =(2D0*B**2/R)*(2D0-A/(REY*R**2))
C
```

```
      CF1M2(R) = A
C
C
C        COEFFICIENTS OF THE COMPONENT G.
C
C
      CG2M1(R) = -T1(R)/REY
      CG1M1(R) = (B**2/R**2-A**2)/(R*REY)
      CG0M1(R) = T1(R)*T2(R)+(A**2-B**2/R**2)/(REY*R**2)
C
      CG0M2(R) = -T1(R)
C
C
C        COEFFICIENTS OF THE COMPONENT H.
C
C
      CH3M1(R) = B/(R*REY)
      CH2M1(R) = 2D0*B/(R**2*REY)
      CH1M1(R) = (-B/R)*(T2(R)+1D0/(REY*R**2))
      CH0M1(R) = B*(((2D0*A**2+1D0/R**2)/REY-T2(R))/R**2-4D0*A)
C
      CH1M2(R) = B/R
      CH0M2(R) = B/R**2
C
C
C  SECTION II
C
C
C        CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT H.
C
C
      GO TO (11,12,13,14,15),K
   11 CHM1E2 = CH3M1(R)/(2D0*DEL**3)
      GO TO 100
   12 CHM1E2 = -2D0*CH3M1(R)/(2D0*DEL**3)+CH2M1(R)/DEL**2
     *      +CH1M1(R)/(2D0*DEL)
      GO TO 100
   13 CHM1E2 = -2D0*CH2M1(R)/DEL**2+CH0M1(R)
      GO TO 100
   14 CHM1E2 = 2D0*CH3M1(R)/(2D0*DEL**3)+CH2M1(R)/DEL**2
     *      -CH1M1(R)/(2D0*DEL)
      GO TO 100
   15 CHM1E2 = -CH3M1(R)/(2D0*DEL**3)
  100 RETURN
C
      ENTRY CHM2E2(K,R)
      GO TO (21,22,23,24,21),K
   21 CHM2E2 = (0D0,0D0)
```

```
      GO TO 200
   22 CHM2E2 = CH1M2(R)/(2D0*DEL)
      GO TO 200
   23 CHM2E2 = CHOM2(R)
      GO TO 200
   24 CHM2E2 = -CH1M2(R)/(2D0*DEL)
  200 RETURN
C
C
C          CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT G.
C
C
      ENTRY CGM1E2(K,R)
      GO TO (31,32,33,34,31),K
   31 CGM1E2 = (0D0,0D0)
      GO TO 300
   32 CGM1E2 = CG2M1(R)/DEL**2+CG1M1(R)/(2D0*DEL)
      GO TO 300
   33 CGM1E2 = -2D0*CG2M1(R)/DEL**2+CG0M1(R)
      GO TO 300
   34 CGM1E2 = CG2M1(R)/DEL**2-CG1M1(R)/(2D0*DEL)
  300 RETURN
C
      ENTRY CGM2E2(K,R)
      GO TO (41,41,42,41,41),K
   41 CGM2E2 = (0D0,0D0)
      GO TO 400
   42 CGM2E2 = CG0M2(R)
  400 RETURN
C
C
C          CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT F.
C
C
      ENTRY CFM1E2(K,R)
      GO TO (51,52,53,54,55),K
   51 CFM1E2 = CF3M1(R)/(2D0*DEL**3)
      GO TO 500
   52 CFM1E2 = -2D0*CF3M1(R)/(2D0*DEL**3)+CF2M1(R)/DEL**2
     *       +CF1M1(R)/(2D0*DEL)
      GO TO 500
   53 CFM1E2 = -2D0*CF2M1(R)/DEL**2+CF0M1(R)
      GO TO 500
   54 CFM1E2 = 2D0*CF3M1(R)/(2D0*DEL**3)+CF2M1(R)/DEL**2
     *       -CF1M1(R)/(2D0*DEL)
      GO TO 500
   55 CFM1E2 = -CF3M1(R)/(2D0*DEL**3)
  500 RETURN
```

```
C
      ENTRY CFM2E2(K,R)
      GO TO (61,62,61,64,61),K
   61 CFM2E2 = (0D0,0D0)
      GO TO 600
   62 CFM2E2 = CF1M2(R)/(2D0*DEL)
      GO TO 600
   64 CFM2E2 = -CF1M2(R)/(2D0*DEL)
  600 RETURN
      END
```

```
C.......................................................................
C
C     SUBROUTINE CDMTIN (CATEGORY F-1)
C
C     PURPOSE
C
C        INVERT A COMPLEX*16 MATRIX
C
C     USAGE
C
C        CALL CDMTIN(N,A,NDIM,DETERM)
C
C     DESCRIPTION OF PARAMETERS
C
C        N      - ORDER OF COMPLEX*16 MATRIX TO BE INVERTED
C                 (INTEGER) MAXIMUM 'N' IS 100
C
C        A      - COMPLEX*16 INPUT MATRIX (DESTROYED).  THE
C                 INVERSE OF 'A' IS RETURNED IN ITS PLACE
C
C        NDIM   - THE SIZE TO WHICH 'A' IS DIMENSIONED
C                 (ROW DIMENSION OF 'A' ACTUALLY APPEARING
C                 IN THE DIMENSION STATMENT OF USER'S
C                 CALLING PROGRAM)
C
C        DETERM - COMPLEX*16 VALUE OF DETERMINANT OF 'A'
C                 RETURNED BY CDMTIN.
C
C     REMARKS
C
C        MATRIX 'A' MUST BE A COMPLEX*8 GENERAL MATRIX
C        IF MATRIX 'A' IS SINGULAR THAT MESSAGE IS PRINTED
C        'N' MUST BE .LE. NDIM
C
C     SUBROUTINES AND FUNCTIONS REQUIRED
C
C        ONLY BUILT-IN FORTRAN FUNCTIONS
C
C     METHOD
C
C        GAUSSIAN ELIMINATION WITH COLUMN PIVOTING IS USED.
C        THE DETERMINANT IS ALSO CALCULATED. A DETERMINANT
C                 OF ZERO INDICATES THAT MATRIX 'A' IS
C                 SINGULAR.
C.......................................................................
C
      SUBROUTINE CDMTIN (N,A,NDIM,DETERM)
      IMPLICIT REAL*8 (A-H,O-Z)
```

```
      CCMPLEX*16 A(NDIM,NDIM),PIVOT(100),AMAX,T,SWAP,
     *     DETERM,U
      INTEGER*4 IPIVOT(100),INDEX(100,2)
      REAL*8 TEMP,ALPHA(100)
C
C      INITIALIZATION
C
      DETERM = (1D0,0D0)
      DC 20 J=1,N
      ALPHA(J) = 0D0
      DC 10 I=1,N
   10 ALPHA(J)=ALPHA(J)+A(J,I)*DCONJG(A(J,I))
      ALPHA(J)=DSQRT(ALPHA(J))
   20 IPIVOT(J)=0
      DC 600 I=1,N
C
C      SEARCH FOR PIVOT ELEMENT
C
      AMAX = (0D0,0D0)
      DC 105 J=1,N
      IF (IPIVOT(J)-1) 60,105,60
   60 DC 100 K=1,N
      IF (IPIVOT(K)-1) 80,100,740
   80 TEMP=AMAX*DCONJG(AMAX)-A(J,K)*DCCNJG(A(J,K))
      IF(TEMP) 85, 85, 100
   85 IRCw=J
      ICCLUM=K
      AMAX=A(J,K)
  100 CCNTINUE
  105 CCNTINUE
      IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C
C      INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
      IF(IROW-ICOLUM) 140, 260, 140
  140 DETERM=DETERM
      DO 200 L=1,N
      SWAP=A(IROW,L)
      A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
      SWAP=ALPHA(IROW)
      ALPHA(IROW)=ALPHA(ICOLUM)
      ALPHA(ICOLUM)=SWAP
  260 INDEX(I,1)=IROW
      INDEX(I,2)=ICOLUM
      PIVOT(I)=A(ICOLUM,ICOLUM)
      U=PIVOT(I)
      TEMP=PIVOT(I)*DCONJG(PIVOT(I))
```

```
      IF(TEMP) 330, 720, 330                                            CTIN0990
C                                                                       CTIN1000
C      . CIVIDE PIVOT ROW BY PIVOT ELEMENT                              CTIN1010
C                                                                       CTIN1020
  330 A(ICOLUM,ICOLUM) = (1D0,0D0)
      DC 350 L=1,N                                                      CTIN1040
      L=PIVOT(I)                                                        CTIN1050
  350 A(ICOLUM,L)=A(ICOLUM,L)/U                                         CTIN1060
C                                                                       CTIN1070
C       REDUCE NON-PIVOT ROWS                                           CTIN1080
C                                                                       CTIN1090
  380 DC 550 L1=1,N                                                     CTIN1100
      IF(L1-ICOLUM) 400, 550, 400                                       CTIN1110
  400 T=A(L1,ICOLUM)                                                    CTIN1120
      A(L1,ICOLUM) = (0D0,0D0)
      DC 450 L=1,N                                                      CTIN1140
      U=A(ICOLUM,L)                                                     CTIN1150
  450 A(L1,L)=A(L1,L)-U*T                                               CTIN1160
  550 CCNTINUE                                                          CTIN1170
  600 CCNTINUE                                                          CTIN1180
C                                                                       CTIN1190
C       INTERCHANGE COLUMNS                                             CTIN1200
C                                                                       CTIN1210
  620 DC 710 I=1,N                                                      CTIN1220
      L=N+1-I                                                           CTIN1230
      IF(INDEX(L,1)-INDEX(L,2)) 630, 710, 630                           CTIN1240
  630 JRCW=INDEX(L,1)                                                   CTIN1250
      JCCLUM=INDEX(L,2)                                                 CTIN1260
      DC 705 K=1,N                                                      CTIN1270
      SWAP=A(K,JROW)                                                    CTIN1280
      A(K,JROW)=A(K,JCOLUM)                                             CTIN1290
      A(K,JCOLUM)=SWAP                                                  CTIN1300
  705 CCNTINUE                                                          CTIN1310
  710 CCNTINUE                                                          CTIN1320
      RETURN                                                            CTIN1330
  720 WRITE(6,730)                                                      CTIN1340
  730 FCRMAT(20H  MATRIX IS SINGULAR)                                   CTIN1350
  740 RETURN                                                            CTIN1360
      END
```

```
C.................SUBROUTINE MULM(X1,X2,N,MDIM,TEMPV).....................
C
C     PURPOSE
C
C       PERFORMS THE MATRIX MULTIPLICATION OF A BANDED MATRIX BY A
C       SQUARE MATRIX.  THE RESULT IS RETURNED IN MATRIX X1.
C
C     USAGE
C
C       CALL MULM(X1,X2,N,MDIM,TEMPV)
C
C     DESCRIPTION OF PARAMETERS
C
C       X1 - THE MULTIPLYING MATRIX ON INPUT AND THE RESULTANT PRODUCT
C            ON OUTPUT.
C
C       X2 - THE MULTIPLIED MATRIX.
C
C       N - THE ORDER OF X1 AND X2.
C
C       MDIM - THE DIMENSION OF X1 AND X2 FROM THE CALLING PROGRAM.
C
C       TEMPV - A WORKING VECTOR. MUST BE DIMENSIONED MDIM.
C
C     OTHER ROUTINES REQUIRED
C
C       NONE
C.........................................................................
C
      SUBROUTINE MULM(X1,X2,N,MDIM,TEMPV)
      COMPLEX*16 X1(MDIM,MDIM), X2(MDIM,MDIM), TEMPV(MDIM), TEMP
C
C
C        STORE ROW I OF X1 IN TEMPV.
C
C
      DO 100 I=1,N
      DO 10 J=1,N
   10 TEMPV(J) = X1(I,J)
C
C
C        MULTIPLY COLUMN J OF X2 BY ROW I OF X1 AND STORE IN X1(I,J).
C
C
      DO 30 J=1,N
      TEMP = (0D0,0D0)
C
      DO 40 K=1,N
```

```
   40 TEMP = TEMP + TEMPV(K)*X2(K,J)
C
   3C X1(I,J) = TEMP
  100 CONTINUE
      RETURN
      END
```

```
C.....................SUBROUTINE DSPLIT........................
C
C
C     PURPOSE
C
C       DSPLIT TAKES A MATRIX OF COMPLEX*16 NUMBERS AND
C       SPLITS IT INTO TWO MATRICES, ONE CONTAINING THE REAL
C       PART OF THE ORIGINAL MATRIX, AND ONE CONTAINING THE
C       IMAGINARY PART.
C
C     USAGE
C
C       CALL DSPLIT(N,MDIM,A,AREAL,AIMAG)
C
C     DESCRIPTICN OF PARAMETERS
C
C       N - THE SIZE OF THE MATRIX A, AN N BY N SQUARE
C         MATRIX.
C
C       MDIM - THE COLUMN DIMENSION OF MATRIX A
C
C       A - THE INPUT MATRIX.  MUST BE DIMENSIONED MDIM BY
C         AT LEAST N IN THE CALLING PROGRAM  (COMPLEX*16)
C
C       AREAL,AIMAG - THE OUTPUT MATRICES CONTAINING THE
C         REAL AND IMAGINARY PARTS, RESPECTIVELY, OF
C         MATRIX A.  MUST BE DIMENSIONED (MDIM,MDIM) IN THE
C         CALLING PROGRAM.
C
C     NOTES...
C
C       MATRIX A AND MATRIX AREAL MAY OVERLAP IF THEY ARE
C         DIMENSIONED IN THE CALLING PRCGRAM AS FCLLOWS...
C
C             COMPLEX*16 A(MDIM,MDIM)
C             REAL*8 AREAL(MDIM,MDIM),AIMAG(MDIM,MDIM)
C             EQUIVALENCE(A(1,1),AREAL(1,1))
C
C     OTHER ROUTINES NEEDED
C
C       NONE
C
C
C..............................................................
C
      SUBROUTINE DSPLIT(N,MDIM,A,AR,AI)
      REAL*8 A(2,MDIM,MDIM),AR(MDIM,MDIM),AI(MDIM,MDIM)
C
```

```
      DO 1  J=1,N
      DO 1  I=1,N
      AR(I,J)  =  A(1,I,J)
    1 AI(I,J)  =  A(2,I,J)
C
      RETURN
      END
```

```
C.EHESSC........D...............................................EHEC0010
C                                                               EHEC0020
C   FUNCTION              - REDUCTION OF A COMPLEX MATRIX TO COMPLEX  EHEC0030
C                              UPPER HESSENBERG FORM.           EHEC0040
C   USAGE                 - CALL EHESSC(AR,AI,K,L,N,IA,ID)      EHEC0050
C   PARAMETERS   AR       - INPUT/OUTPUT MATRIX OF DIMENSION N BY N.  EHEC0060
C                              ON INPUT CONTAINS THE REAL COMPONENTS OF  EHEC0070
C                              THE MATRIX TO BE REDUCED.        EHEC0080
C                              ON OUTPUT CONTAINS THE REAL COMPONENTS  EHEC0090
C                              OF THE REDUCED HESSENBERG FORM IN THE  EHEC0100
C                              UPPER TRIANGULAR PORTION (INCLUDING MAIN  EHEC0110
C                              AND SUB-DIAGONAL) AND THE DETAILS OF  EHEC0120
C                              THE REDUCTION IN THE LOWER TRIANGULAR  EHEC0130
C                              PORTION.                         EHEC0140
C                AI       - INPUT/OUTPUT MATRIX OF DIMENSION N BY N  EHEC0150
C                              CONTAINING THE IMAGINARY COUNTERPARTS  EHEC0160
C                              TO AR, ABOVE.                    EHEC0170
C                K        - INPUT SCALAR CONTAINING THE ROW AND COLUMN  EHEC0180
C                              INDEX OF THE STARTING ELEMENT TO BE  EHEC0190
C                              REDUCED BY ROW SCALING.  FOR UNBALANCED  EHEC0200
C                              MATRICES SET K = 1.              EHEC0210
C                L        - INPUT SCALAR CONTAINING THE ROW AND  EHEC0220
C                              COLUMN INDEX OF THE LAST ELEMENT TO BE  EHEC0230
C                              REDUCED BY ROW SCALING. FOR UNBALANCED  EHEC0240
C                              MATRICES SET L = N.              EHEC0250
C                N        - INPUT SCALAR CONTAINING THE ORDER OF  EHEC0260
C                              THE MATRIX TO BE REDUCED.        EHEC0270
C                IA       - INPUT SCALAR CONTAINING ROW DIMENSION  EHEC0280
C                              OF AR AND AI IN THE CALLING PROGRAM.  EHEC0290
C                ID       - OUTPUT VECTOR OF LENGTH L CONTAINING  EHEC0300
C                              DETAILS OF THE TRANSFORMATIONS.  EHEC0310
C   PRECISION             - SINGLE/DOUBLE                       EHEC0320
C   CODE RESPONSIBILITY   - T.J. AIRD/E.W. CHOU                 EHEC0330
C   LANGUAGE              - FORTRAN                             EHEC0340
C...............................................................EHEC0350
C   LATEST REVISION       - FEBRUARY 7, 1973                    EHEC0360
C                                                               EHEC0370
      SUBROUTINE EHESSC (AR,AI,K,L,N,IA,ID)                     EHEC0380
C                                                               EHEC0390
      DIMENSION          AR(IA,1),AI(IA,1),ID(1),T1(2),T2(2)    EHEC0400
      DOUBLE PRECISION   AR,AI,XR,XI,YR,YI,T1,T2,ZERO           EHEC0410
      COMPLEX*16         X,Y                                    EHEC0420
      EQUIVALENCE        (X,T1(1),XR),(T1(2),XI),(Y,T2(1),YR),  EHEC0440
     1                   (T2(2),YI)                             EHEC0450
      DATA               ZERO/0.0D0/                            EHEC0460
      LA=L-1                                                    EHEC0480
      KP1=K+1                                                   EHEC0490
      IF (LA .LT. KP1) GO TO 45                                 EHEC0500
```

```
      DO 40 M=KP1,LA                                                    EHEC0510
         I=M                                                            EHEC0520
         XR=ZERO                                                        EHEC0530
         XI=ZERO                                                        EHEC0540
         DO 5 J=M,L                                                     EHEC0550
            IF (DABS(AR(J,M-1))+DABS(AI(J,M-1)) .LE. DABS(XR)+DABS(XI)) EHEC0560
     1      GO TO 5                                                     EHEC0570
            XR=AR(J,M-1)                                                EHEC0600
            XI=AI(J,M-1)                                                EHEC0610
            I=J                                                         EHEC0620
    5    CCNTINUE                                                       EHEC0630
         ID(M)=I                                                        EHEC0640
         IF (I .EQ. M) GO TO 20                                         EHEC0650
C                                        INTERCHANGE ROWS AND COLUMNS OF EHEC0660
C                                          ARRAYS AR AND AI             EHEC0670
         MM1=M-1                                                        EHEC0680
         DO 10 J=MM1,N                                                  EHEC0690
            YR=AR(I,J)                                                  EHEC0700
            AR(I,J)=AR(M,J)                                             EHEC0710
            AR(M,J)=YR                                                  EHEC0720
            YI=AI(I,J)                                                  EHEC0730
            AI(I,J)=AI(M,J)                                             EHEC0740
            AI(M,J)=YI                                                  EHEC0750
   10    CONTINUE                                                       EHEC0760
         DO 15 J=1,L                                                    EHEC0770
            YR=AR(J,I)                                                  EHEC0780
            AR(J,I)=AR(J,M)                                             EHEC0790
            AR(J,M)=YR                                                  EHEC0800
            YI=AI(J,I)                                                  EHEC0810
            AI(J,I)=AI(J,M)                                             EHEC0820
            AI(J,M)=YI                                                  EHEC0830
   15    CONTINUE                                                       EHEC0840
C                                        END INTERCHANGE                EHEC0850
   20    IF (XR .EQ. ZERO .AND. XI .EQ. ZERO) GO TO 40                  EHEC0860
         MP1=M+1                                                        EHEC0870
         DO 35 I=MP1,L                                                  EHEC0880
            YR=AR(I,M-1)                                                EHEC0890
            YI=AI(I,M-1)                                                EHEC0900
            IF (YR .EQ. ZERO .AND. YI .EQ. ZERO) GO TO 35               EHEC0910
            Y=Y/X                                                       EHEC0920
            AR(I,M-1)=YR                                                EHEC0930
            AI(I,M-1)=YI                                                EHEC0940
            DO 25 J=M,N                                                 EHEC0950
               AR(I,J)=AR(I,J)-YR*AR(M,J)+YI*AI(M,J)                    EHEC0960
               AI(I,J)=AI(I,J)-YR*AI(M,J)-YI*AR(M,J)                    EHEC0970
   25       CONTINUE                                                    EHEC0980
            DO 30 J=1,L                                                 EHEC0990
               AR(J,M)=AR(J,M)+YR*AR(J,I)-YI*AI(J,I)                    EHEC1000
```

```
            AI(J,M)=AI(J,M)+YR*AI(J,I)+YI*AR(J,I)
30        CONTINUE
35      CONTINUE
40  CONTINUE
45  RETURN
    END
```

```
C                                                                       ELR10480
      DIMENSION           HR(IH,1),HI(IH,1),WR(1),WI(1)                 ELR10490
      DIMENSION           T1(2),T2(2),T3(2)                             ELR10500
      COMPLEX*16          X,Y,Z                                         ELR10510
      DOUBLE PRECISION    HR,HI,WR,WI,EPS,ZR,ZI,T1,T2,T3,RDELP          ELR10530
      DOUBLE PRECISION    ZERO,ONE,TWO,SR,SI,XR,XI,YR,YI,TR,TI          ELR10540
      EQUIVALENCE         (X,T1(1),XR),(T1(2),XI),                      ELR10550
     1                    (Y,T2(1),YR),(T2(2),YI),                      ELR10560
     2                    (Z,T3(1),ZR),(T3(2),ZI)                       ELR10570
      DATA                TWO/2.0D0/                                    ELR10580
      DATA                ZERO,ONE,RDELP/0.D0,1.D0,Z3410000000000000/   ELR10590
      INFER=0                                                           ELR10610
      IER=0                                                             ELR10620
      DO 5 I=1,N                                                        ELR10630
         IF (I .GE. K .AND. I .LE. L) GO TO 5                           ELR10640
         WR(I)=HR(I,I)                                                  ELR10650
         WI(I)=HI(I,I)                                                  ELR10660
    5 CONTINUE                                                          ELR10670
      NN=L                                                              ELR10680
      TR=ZERO                                                           ELR10690
      TI=ZERO                                                           ELR10700
C                                SEARCH FOR NEXT EIGENVALUE             ELR10710
   10 IF (NN .LT. K) GO TO 9005                                         ELR10720
      ITS=0                                                             ELR10730
      NM1=NN-1                                                          ELR10740
      IF (NN .EQ. K) GO TO 25                                           ELR10750
C                                LOOK FOR SINGLE SMALL SUB-DIAGONAL     ELR10760
C                                  ELEMENTS                             ELR10770
   15 NPL=NN+K                                                          ELR10780
      DO 20 LL=K,NM1                                                    ELR10790
         M=NPL-LL                                                       ELR10800
         MM1=M-1                                                        ELR10810
         IF (DABS(HR(M,MM1)) + DABS(HI(M,MM1)) .LE.                     ELR10820
     1    RDELP*(DABS(HR(MM1,MM1)) + DABS(HI(MM1,MM1)) +                ELR10830
     2      DABS(HR(M,M)) + DABS(HI(M,M)))) GO TO 30                    ELR10840
   20 CONTINUE                                                          ELR10880
   25 M=K                                                               ELR10890
   30 IF (M .EQ. NN) GO TO 110                                          ELR10900
      IF (ITS .EQ. 30) GO TO 115                                        ELR10910
C                                FORM SHIFT                             ELR10920
      IF (ITS .EQ. 10 .OR. ITS .EQ. 20) GO TO 35                        ELR10930
      SR=HR(NN,NN)                                                      ELR10940
      SI=HI(NN,NN)                                                      ELR10950
      XR=HR(NM1,NN)*HR(NN,NM1)-HI(NM1,NN)*HI(NN,NM1)                    ELR10960
      XI=HR(NM1,NN)*HI(NN,NM1)+HI(NM1,NN)*HR(NN,NM1)                    ELR10970
      IF (XR .EQ. ZERO .AND. XI .EQ. ZERO) GO TO 40                     ELR10980
      YR=(HR(NM1,NM1)-SR)/TWO                                           ELR10990
      YI=(HI(NM1,NM1)-SI)/TWO                                           ELR11000
```

```
      Z=CDSQRT(DCMPLX(YR**2-YI**2+XR,TWO*YR*YI+XI))                     ELR11010
      IF (YR*ZR+YI*ZI  .LT.  ZERO) Z=-Z                                 ELR11030
      X=X/(Y+Z)                                                         ELR11040
      SR=SR-XR                                                          ELR11050
      SI=SI-XI                                                          ELR11060
      GO TO 40                                                          ELR11070
   35 SR=DABS(HR(NN,NM1))+DABS(HR(NM1,NN-2))                            ELR11080
      SI=DABS(HI(NN,NM1))+DABS(HI(NM1,NN-2))                            ELR11090
   40 DC 45 I=K,NN                                                      ELR11120
         HR(I,I)=HR(I,I)-SR                                             ELR11130
         HI(I,I)=HI(I,I)-SI                                             ELR11140
   45 CCNTINUE                                                          ELR11150
      TR=TR+SR                                                          ELR11160
      TI=TI+SI                                                          ELR11170
      ITS=ITS+1                                                         ELR11180
C                                  LOOK FOR TWO CCNSECUTIVE SMALL       ELR11190
C                                    SUB-DIAGONAL ELEMENTS              ELR11200
      XR=DABS(HR(NM1,NM1))+DABS(HI(NM1,NM1))                            ELR11210
      YR=DABS(HR(NN,NM1))+DABS(HI(NN,NM1))                              ELR11220
      ZR=DABS(HR(NN,NN))+DABS(HI(NN,NN))                                ELR11230
      NMJ=NM1-M                                                         ELR11270
      IF (NMJ .EQ. 0) GO TO 55                                          ELR11280
C                                  FOR MM=NN-1 STEP -1 UNTIL M+1 DO     ELR11290
      DC 50 J=1,NMJ                                                     ELR11300
         MM=NN-J                                                        ELR11310
         M1=MM-1                                                        ELR11320
         YI=YR                                                          ELR11330
         YR=DABS(HR(MM,M1))+DABS(HI(MM,M1))                             ELR11340
         XI=ZR                                                          ELR11360
         ZR=XR                                                          ELR11370
         XR=DABS(HR(M1,M1))+DABS(HI(M1,M1))                             ELR11380
         IF (YR.LE.RDELP*ZR/YI*(ZR+XR+XI)) GO TO 60                     ELR11400
   50 CCNTINUE                                                          ELR11410
   55 MM=M                                                              ELR11420
C                                  TRIANGULAR DECCMPOSITION             ELR11430
   60 MP1=MM+1                                                          ELR11440
      DC 85 I=MP1,NN                                                    ELR11450
         IM1=I-1                                                        ELR11460
         XR=HR(IM1,IM1)                                                 ELR11470
         XI=HI(IM1,IM1)                                                 ELR11480
         YR=HR(I,IM1)                                                   ELR11490
         YI=HI(I,IM1)                                                   ELR11500
         IF (DABS(XR)+DABS(XI) .GE. DABS(YR)+DABS(YI)) GO TO 70         ELR11510
C                                  INTERCHANGE ROWS OF HR AND HI        ELR11530
         DO 65 J=IM1,NN                                                 ELR11540
            ZR=HR(IM1,J)                                                ELR11550
            HR(IM1,J)=HR(I,J)                                           ELR11560
            HR(I,J)=ZR                                                  ELR11570
```

```
         ZI=HI(IM1,J)                                                   ELR11580
         HI(IM1,J)=HI(I,J)                                              ELR11590
         HI(I,J)=ZI                                                     ELR11600
   65    CONTINUE                                                       ELR11610
         Z=X/Y                                                          ELR11620
         WR(I)=ONE                                                      ELR11630
         GO TO 75                                                       ELR11640
   7C    Z=Y/X                                                          ELR11650
         WR(I)=-ONE                                                     ELR11660
   75    HR(I,IM1)=ZR                                                   ELR11670
         HI(I,IM1)=ZI                                                   ELR11680
         DO 80 J=I,NN                                                   ELR11690
            HR(I,J)=HR(I,J)-ZR*HR(IM1,J)+ZI*HI(IM1,J)                   ELR11700
            HI(I,J)=HI(I,J)-ZR*HI(IM1,J)-ZI*HR(IM1,J)                   ELR11710
   80    CONTINUE                                                       ELR11720
   85 CCNTINUE                                                          ELR11730
C                                COMPOSITION                            ELR11740
      DC 105 J=MP1,NN                                                   ELR11750
         JM1=J-1                                                        ELR11760
         XR=HR(J,JM1)                                                   ELR11770
         XI=HI(J,JM1)                                                   ELR11780
         HR(J,JM1)=ZERO                                                 ELR11790
         HI(J,JM1)=ZERO                                                 ELR11800
C                                INTERCHANGE COLUMNS CF HR AND HI IF    ELR11810
C                                  NECESSARY                            ELR11820
         IF (WR(J) .LE. ZERO) GO TO 95                                  ELR11830
         DO 90 I=M,J                                                    ELR11840
            ZR=HR(I,JM1)                                                ELR11850
            HR(I,JM1)=HR(I,J)                                           ELR11860
            HR(I,J)=ZR                                                  ELR11870
            ZI=HI(I,JM1)                                                ELR11880
            HI(I,JM1)=HI(I,J)                                           ELR11890
            HI(I,J)=ZI                                                  ELR11900
   90    CONTINUE                                                       ELR11910
   95    DO 100 I=M,J                                                   ELR11920
            HR(I,JM1)=HR(I,JM1)+XR*HR(I,J)-XI*HI(I,J)                   ELR11930
            HI(I,JM1)=HI(I,JM1)+XR*HI(I,J)+XI*HR(I,J)                   ELR11940
  10C    CONTINUE                                                       ELR11950
  105 CONTINUE                                                          ELR11960
      GO TO 15                                                          FLR11970
C                                A ROCT FOUND                           ELR11980
  110 WR(NN)=HR(NN,NN)+TR                                               ELR11990
      WI(NN)=HI(NN,NN)+TI                                               ELR12000
      NN=NM1                                                            ELR12010
      GC TO 10                                                          ELR12020
C                                SET ERROR-NO CONVERGENCE TO AN         ELR12030
C                                  EIGENVALUE AFTER 30 ITERATIONS       FLR12040
  115 INFER=NN                                                          ELR12050
```

```
      IER=129                                                           ELR12060
 9000 CCNTINUE                                                          ELR12070
      CALL UERTST (IER,6HELRH1C)                                        ELR12080
 9005 RETURN                                                            ELR12090
      END                                                               ELR12100
```

```
C     SUBROUTINE UERTST (IER,NAME)                                      UERT0010
C                                                                       UERT0020
C-UEFTST---------------LIBRARY 1----------------------------------------UERT0030
C                                                                       UERT0040
C   FUNCTION            - ERROR MESSAGE GENERATION                      UERT0050
C   USAGE               - CALL UERTST(IER,NAME)                         UERT0060
C   PARAMETERS   IER    - ERROR PARAMETER. TYPE + N  WHERE              UERT0070
C                           TYPE= 128 IMPLIES TERMINAL ERROR            UERT0080
C                                  64 IMPLIES WARNING WITH FIX          UERT0090
C                                  32 IMPLIES WARNING                   UERT0100
C                           N    = ERROR CODE RELEVANT TO CALLING ROUTINEUERT0110
C                NAME   - INPUT VECTOR CONTAINING THE NAME OF THE       UERT0120
C                           CALLING ROUTINE AS A SIX CHARACTER LITERAL  UERT0130
C                           STRING.                                     UERT0140
C   LANGUAGE            - FORTRAN                                       UERT0150
C-----------------------------------------------------------------------UERT0160
C   LATEST REVISION     - JANUARY 18, 1974                              UERT0170
C                                                                       UERT0180
      SUBROUTINE UERTST(IER,NAME)                                       UERT0190
C                                                                       UERT0200
      DIMENSION          ITYP(5,4),IBIT(4)                              UERT0210
      INTEGER*2          NAME(3)                                        UERT0220
      INTEGER            WARN,WARF,TERM,PRINTR                          UERT0230
      EQUIVALENCE        (IBIT(1),WARN),(IBIT(2),WARF),(IBIT(3),TERM)   UERT0240
      DATA      ITYP     /'WARN','ING ','    ','    ','    ',            UERT0250
     *                    'WARN','ING(','WITH',' FIX',')   ',            UERT0260
     *                    'TERM','INAL','    ','    ','    ',            UERT0270
     *                    'NON-','DEFI','NED ','    ','    '/,           UERT0280
     *          IBIT     / 32,64,128,0/                                 UERT0290
      DATA      PRINTR   / 6/                                           UERT0300
      IER2=IER                                                          UERT0310
      IF (IER2 .GE. WARN) GO TO 5                                       UERT0320
C                                  NON-DEFINED                          UERT0330
      IER1=4                                                            UERT0340
      GO TO 20                                                          UERT0350
    5 IF (IER2 .LT. TERM) GO TO 10                                      UERT0360
C                                  TERMINAL                             UERT0370
      IER1=3                                                            UERT0380
      GO TO 20                                                          UERT0390
   10 IF (IER2 .LT. WARF) GO TO 15                                      UERT0400
C                                  WARNING(WITH FIX)                    UERT0410
      IER1=2                                                            UERT0420
      GO TO 20                                                          UERT0430
C                                  WARNING                              UERT0440
   15 IER1=1                                                            UERT0450
C                                  EXTRACT 'N'                          UERT0460
   20 IER2=IER2-IBIT(IER1)                                              UERT0470
C                                  PRINT ERROR MESSAGE                  UERT0480
```

```
      WRITE (PRINTR,25) (ITYP(I,IER1),I=1,5),NAME,IER2,IER                UERT0490
   25 FORMAT(' *** I M S L(UERTST) ***  ',5A4,4X,3A2,4X,I2,               UERT0500
     *   ' (IER = ',I3,')')                                                UERT0510
      RETURN                                                              UERT0520
      END                                                                 UERT0530
```

LIST OF REFERENCES

1. Harrison, W. F., On the Stability of Poiseuille Flow, Aeronautical Engineer's Thesis, Naval Postgraduate School, 1975.

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c.1 A program for the stability analysis of pipe Poiseuille flow.