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PROPAGATION OF MULTIWAVELENGTH LASER RADIATION THROUGH ATMOSPHERIC TURBULENCE

Oregon Graduate Center

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PROPAGATION OF MULTIWAVELENGTH LASER RADIATION THROUGH ATMOSPHERIC TURBULENCE

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Recent activity on several topics in propagation through atmospheric turbulence is described in this report. The statistics of target or receiver irradiance, conditioned on a prior measurement of the instantaneous value, are considered under conditions of moderately strong fluctuations, and it is found experimentally that such scintillations are not described by a bivariate log normal distribution. Hence as turbulence effects become stronger, predictions of the state of the turbulent channel at a given time delay following

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a reference measurement will be subject to more uncertainty. However, even though the exact joint distribution is not analytically known, approximate predictions based on the bivariate log normal are quite useful. Such predictions can comprise a basis for operating a system in which a low-power laser beam is used to determine a favorable time interval for firing a high-power beam in turbulence.

A study of computer simulation techniques is described which has the goal of making possible realistic models of propagation paths for the short-term, time-dependent statistics of scintillation, including the effects of localized or intermittent strength of turbulence. It is concluded that the random medium must be more completely described than in conventional simulation approaches, and that this may be accomplished through an empirical transformation of a gaussian distribution to a distribution which realistically describes the highly non-gaussian behavior of refractive-index fluctuations. An expression is given which accomplishes this goal, and an approach for including intermittency is described.

The important future topic on this program will be the experimental and analytical determination of scintillations from laser-illuminated diffuse and general targets. A preliminary analysis of scintillations from a diffuse target is given, which is valid in the near field of the target illumination area.

Summary

Recent activity on several topics in propagation through atmospheric turbulence is described in this report. The statistics of target or receiver irradiance, conditioned on a prior measurement of the instantaneous value, are considered under conditions of moderately strong fluctuations, and it is found experimentally that such scintillations are not described by a bivariate log normal distribution. Hence as turbulence effects become stronger, predictions of the state of the turbulent channel at a given time delay following a reference measurement will be subject to more uncertainty. However, even though the exact joint distribution is not analytically known, approximate predictions based on the bivariate log normal are quite useful. Such predictions can comprise a basis for operating a system in which a low-power laser beam is used to determine a favorable time interval for firing a high-power beam in turbulence.

A study of computer simulation techniques is described which has the goal of making possible realistic models of propagation paths for the short-term, time-dependent statistics of scintillation, including the effects of localized or intermittent strength of turbulence. It is concluded that the random medium must be more completely described than in conventional simulation approaches, and that this may be accomplished through an empirical transformation of a gaussian distribution to a distribution which realistically describes the highly non-gaussian behavior of refractive-index fluctuations. An expression is given which accomplishes this goal, and an approach for including intermittency is described.

The important future topic on this program will be the experimental and analytical determination of scintillations from laser-illuminated diffuse and general targets. A preliminary analysis of scintillations from a diffuse target is given, which is valid in the near field of the target illumination area.

TABLE OF CONTENTS

I. Introduction

This report describes recent activity on several topics in propagation through atmospheric turbulence. In Section II we discuss additional aspects of conditional fading statistics to supplement the treatment in an earlier report. In Section III we briefly describe work on the useful simulation of short-term scintillation and turbulence intermittency phenomena on a computer. An expanded treatment of the limiting behavior of the second moment of irradiance is given in Section IV. Finally, in Section V we present a preliminary analysis and discussion of scintillations from diffuse and general sources, which will comprise the primary topic for future experimental and analytical work on a follow-on program.

In addition to the work reported here, we have continued experiments on target irradiance behavior as a function of wander-cancellation-tracking and transmitter and turbulence parameters. In particular, this work has been extended to 10.6 microns. Data reduction and interpretation is being completed and a special report on this topic will be issued. We are also conducting long-path, strong-turbulence scintillation experiments with very small receiver apertures and wide bandwidths; this work, sponsored by the National Science Foundation, builds upon earlier efforts on the present program and will be reported later.

II. Further Results on Conditional Fading Statistics

In the preceding report on this program[1], we discussed the conditional statistics of scintillation or target irradiance fading at a time τ following the comparison of the instantaneous level with an arbitrary threshold. An approach was given which is readily implemented if the log irradiance fluctuations have a two-point or joint normal distribution, and which can be numerically extended to a general distribution.

It was found empirically that the bivariate normal distribution is accurate for weak irradiance fluctuations, and that the analytical treatment of the conditional statistics is successful in predicting experimental results. However, it was mentioned that under conditions of stronger

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[1] J. R. Kerr, et al, "Propagation of Multiwavelength Laser Radiation Through Atmospheric Turbulence", RADC Technical Report, April 1975.

fluctuations, a significant departure from bivariate normality occurs. The purpose of this section is to elaborate on the latter point.

We confine ourselves to strengths of fluctuations below the level of "saturated scintillations",[2] which will be treated later. It will in fact be shown that significant departure from bivariate normality occurs for moderate, unsaturated fluctuations.

We have measured the conditional distributions of the log irradiance fluctuations from a point source over a range of turbulence levels (Table 1). For each experimental run, distributions were calculated for a number of values of threshold (ℓ_o) and time interval τ, using 400,000 data points sampled at one millisecond intervals. Autocorrelation functions were also determined as in Figure 6 of Ref. 1.

TABLE 1. PARAMETERS FOR EXPERIMENTAL DATA ON CONDITIONAL FADING DISTRIBUTIONS OVER A 1.6 km PATH

Run	Mean Wind Speed (meters/sec)	Log Amplitude Variance (σ_χ^2)	Optical Wavelength (μ)
1	9	0.0015	10.6
2	3.5	0.010	10.6
3	9	0.076	0.488
4	1.5	0.14	10.6
5	1	0.307	10.6

As discussed in Ref. 1, the conditional probability distributions for Run #1 were essentially log normal for most values of threshold and time delay, and the observed mean and variance vs. τ for these distributions agreed with the analytical predictions based on this assumption.

Now let us consider a case of stronger scintillation, Run #4. Again the conditional distributions appeared qualitatively (log) normal, as supported by the skewness and kurtosis parameters shown in Table 2.

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[2] R. S. Lawrence and J. W. Strohbehn, Proc. IEEE 58, Oct. 1970, 1523-1545.

TABLE 2. SKEWNESS (S), AND KURTOSIS (K) OF Pr $[\chi(t + \tau)/\chi(t) = \ell_o]$ FOR RUN #4, WHERE χ IS THE LOG AMPLITUDE, AND σ_χ^2 IS ITS VARIANCE (=0.14).

S

$\frac{\ell_o}{\sigma_\chi}$ τ(ms)	2	4	8	16
-0.92	$9\text{x}10^{-4}$	0.075	0.193	0.13
-0.04	0.013	0.085	0.055	0.107
1.07	0.048	0.074	0.048	0.06
2.03	$-2.6\text{x}10^{-3}$	$-6.6\text{x}10^{-3}$	$-3.6\text{x}10^{-3}$	0.013

K

-0.92	3.06	3.81	2.4	4.04
-0.04	3.27	3.59	3.69	3.39
1.07	3.4	3.19	3.02	3.2
2.03	2.96	2.78	2.68	2.95

However, a comparison of observed variances and those predicted from the gaussian assumption (Table 3) shows large deviations; in fact, the rms error between the observed and predicted normalized variances is 13%. In Table 4 we show the results of a similar calculation for all runs, and it is seen that the stronger-fluctuation cases ($\sigma_\chi{}^2 > 0.1$) for this sample show significant deviations from joint normality. It is also clear that predictions such as those in Table 3 are quite sensitive to departures from joint normality.

A detailed error analysis confirms the non-gaussian behavior of the joint distributions in the stronger-fluctuation runs. This analysis shows that in the weak-fluctuation case the observed variances are within the allowed statistical error of that predicted by joint normality, while in the strong-fluctuation case, the deviations are outside this limit--especially for large values of ℓ_o.

A similar analysis of saturated log irradiance signals will be carried out in the future, including data from the long-path experiments mentioned in Section I.

TABLE 3. COMPARISON OF PREDICTED AND OBSERVED $\frac{\text{MEAN}}{\sigma_\chi}$ OF $P(\chi(t + \tau)/\chi(t) = \ell_o)$

$\frac{\ell_o}{\sigma_\chi}$	τ(ms)	2	4	8	16
- .92	Pred.	- .856	- .761	- .544	- .215
	Obs.	- .869	- .77	- .547	- .20
- .04	Pred.	.037	.033	.024	.0094
	Obs.	$4x10^{-3}$	.033	.051	.01
1.07	Pred.	.995	.885	.632	.250
	Obs.	1.00	.896	.626	.248
2.03	Pred.	1.89	1.67	1.19	.475
	Obs.	1.87	1.54	.962	.314

COMPARISON OF PREDICTED AND OBSERVED $\frac{\text{VARIANCE}}{\sigma_\chi^2}$ OF $P(\chi(t + \tau)/\chi(t) = \ell_o)$

$\frac{\ell_o}{\sigma_\chi}$	τ(ms)	2	4	8	16
- .92	Pred.	.134	.32	.651	.945
	Obs.	.101	.314	.679	.94
- .04	Pred.	.034	.32	.651	.945
	Obs.	.088	.255	.615	.96
1.07	Pred.	.134	.32	.651	.945
	Obs.	.19	.47	.80	1.00
2.03	Pred.	.134	.32	.651	.945
	Obs.	.268	.603	.95	.99

TABLE 4. RMS ERROR BETWEEN OBSERVED AND PREDICTED NORMALIZED VARIANCES OF THE CONDITIONAL LOG IRRADIANCE DISTRIBUTIONS

Run	σ_χ^2	RMS Error
1	0.0015	0.04
2	0.010	0.037
3	0.070	0.027
4	0.14	0.13
5	0.307	0.11

III. Computer Simulation for Short-Term Scintillation Statistics and Turbulence Intermittency

A. Introduction

The goal of this study is to devise computer simulation techniques which will make it possible to realistically model propagation paths for the short-term, time-dependent statistics of scintillation, including the effects of localized or intermittent strength of turbulence. This technique complements the analytical description of short-term effects given in preceding reports.[1,3]

Obviously the most ambitious approach is to model the complete refractive index field over the path, with a differential or integral formulation of the propagation mechanism which yields the detailed complex field behavior vs. time and space. Iteration is often utilized to correctly include the multiple scattering regime. There are two primary difficulties with this approach:

1. Computation requirements are prohibitive.
2. The refractive field, having nongaussian statistics, is not readily modelled in detail.

Let us consider concrete examples. Over a limited pathlength, perturbation analysis yields the following form for the complex phase with an elementary source:[4]

$$\Phi(x,y,L) = \int_0^L dz' \iint_{-\infty}^{\infty} dx'dy' \quad g(x-x',y-y',L-z')\varepsilon_1(x',y',z')$$

$$U_\phi(dK_x,dK_y,L)= \int_0^L dz' \; G(\underset{\sim}{K},L-z') \; U_\varepsilon(dK_x,dK_y,z') \tag{1}$$

where ε_1 is the (stochastic) dielectric perturbation, and the second

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[3] J. R. Kerr, et al, "Propagation of Multiwavelength Laser Radiation Through Atmospheric Turbulence", RADC Technical Report, November 1974.

[4] V.I. Tatarskii, The Effects of the Turbulent Atmosphere on Wave Propagation, available from National Technical Information Service (#TT-68-50464) 1971.

expression is simply the Fourier-Stieljes form of the first. In principle, ϵ_1 could be modelled in detail in three dimensions and with time evolution; the statistics of Φ would then be determined from a number of such realizations, i.e., a limited ensemble. Analytically, the solution is usually confined to a statistical quantity such as an autocorrelation function, which involves partial Fourier transforms and Kolmogorov spectra, so that the detailed knowledge of the dielectric or optical field for a particular case is not involved.

B. Use of Short-Path Phase Function

To model the system represented by Eq. (1), we must generate realizations of the three-dimensional dielectric field $\epsilon_1(x',y',z')$. However, this random variable is markedly nongaussian,[3] and therefore unsuitable for conventional simulation techniques. The approach generally used[5,6] is to derive a much simpler expression like Eq. (1), for the real phase only, as follows. Let U be the complex amplitude, and write

$$U(x,y,z) = e^{\Gamma(x,y,z)} w(x,y,z) \tag{2}$$

where Γ is the real phase and is assumed to have the short-path solution

$$\Gamma = \frac{ik}{2} \int_{z_o}^{z_o + \Delta z} dz \quad \epsilon_1(x,y,z) \quad . \tag{3}$$

When U from Eq. (2) is put back into the wave equation, it is found that w satisfies

$$\left[i2k \frac{\partial}{\partial z} + e^{-\Gamma} \nabla^2_{xy} \quad e^{\Gamma} \right] w = 0 \quad . \tag{4}$$

It is then assumed that a sufficient number of independent ϵ_1 regions are contained in the integral (3) to render Γ a gaussian random variable, and

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5. J. Herrman and J. C. Bradley, Numerical Calculation of Light Propagation, Lincoln Laboratory Rept. LTP-10, July 1971.
6. W. P. Brown, Jr., High Energy Laser Propagation, Midterm Technical Report, Contract N00014-73-C-0460, Hughes Research Laboratories, September 1973.

the modelling is limited to $\Gamma(x,y)$ realizations with independent (uncorrelated) iterations for each pathlength increment (Δz). Although Eq. (3) is valid only over very limited pathlengths, the "return to the differential equation" (4) at every Δz iteration leads to results which are valid for arbitrary pathlengths and which therefore include the saturation of scintillations.[2]

Finally, the generation of particular $\Gamma(x,y,z_0)$ realizations with the required transverse correlation function is accomplished through the following principle: a multidimensional gaussian random variable $f(\underset{\sim}{r})$ may be generated by convolving a gaussian white noise function $h(\underset{\sim}{r})$ with a deterministic function $e(\underset{\sim}{r})$ having the desired autocorrelation for f:

$$f(\underset{\sim}{r}) = e(\underset{\sim}{r}) * h(\underset{\sim}{r}) \tag{5}$$

In practice, one usually employs the spectral form of this expression:

$$f(\underset{\sim}{r}) = \int_{-\infty}^{\infty} d\underset{\sim}{K} \quad \sqrt{E(\underset{\sim}{K})} \quad H(\underset{\sim}{K}) \quad e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}} \tag{6}$$

where $E(\underset{\sim}{K})$ is the spatial power spectrum desired for $f(\underset{\sim}{r})$. In practice, the problem is discretized, so that H represents a grid of gaussian random numbers; $E(\underset{\sim}{K})$ is the familiar Kolmogorov spectrum[2] and $\Gamma(x_n,y_n)$ has the usual transverse phase structure function for propagation through turbulence.

Finally, the above approach may be extended to time dependent behavior by invoking the frozen-in turbulence or Taylor hypothesis[2,4] and having the randomly-generated grid move with the postulated wind vector.[6]

C. Possible Extensions of the Above Technique

The detailed nature of the ε_1 field is dropped out of the above technique by integrating over short, discrete paths and assuming that the result is gaussian and uncorrelated in adjacent z-segments, even though the integrand is nongaussian. For Δz much longer than the outer scale of turbulence[2,4] but short enough for Eq. (2) to be valid, this should certainly give accurate statistical results over a reasonable ensemble of U realizations.

In our study of short-term statistics, including turbulence intermittency, we have been interested in the detailed statistics of the turbulence field and in the detailed time behavior of the scintillating irradiance, including joint distributions and conditional statistics as in Section II. Such details are obviously lost by integrating over pathlength segments before generating a realization.

In order to extend the above or any other coding technique to include the detailed ϵ_1 statistics, we suggest the use of a versatile "Johnson" transformation[7] which we have found can adequately represent the nongaussian variable ϵ_1 in terms of a gaussian variable η_1:

$$\epsilon_1 = C_1 \sinh \frac{\eta_1}{C_2} \tag{7}$$

where C_1 and C_2 are determined by the first two momemts of ϵ_1. This technique simply comprises an empirical fit of the probability distribution of a nongaussian process to the gaussian distribution, obviously with the loss of independent information on higher moments. The autocorrelation or spectrum (E) of η_1 must be calculated from the known (isotropic) form for ϵ_1.

We thus write the generating expression for Γ, from Eqs. (3), (5), and (7), as

$$\Gamma(x,y) = \frac{ik}{2} \int_{z_o}^{z_o + \Delta z} dz\, C_1 \sinh \left(\frac{1}{C_2} \iiint_{-\infty}^{\infty} dx'dy'dz'\ h(x-x',y-y',z-z') \cdot e(x',y',z') \right) \tag{8}$$

The segments Δz should now be much less than the outer scale, in order to preserve maximum detail. This is also true of the transverse discretizing or grid scale. The transformed refractive field may again be translated

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[7] G. F. Hahn and S. F. Shapiro, Statistical Models in Engineering, Wiley, New York, 1967.

with the wind, in order to determine time behavior.

In principle, this technique may permit the determination of the detailed statistics of scintillation, including two-point or conditional behavior, higher moments (degree of fit to a log normal), autocorrelation functions, integral scales, and averaging time effects on spread in measured quantities such as log amplitude variance. However, it is clear that the required computer time may be prohibitive. This is currently under investigation for a two-dimensional representation (x,z).

A second extension of the technique described in Sec. III-B is to postulate large-scale random variations of turbulence strength (envelope of the ε_1 fluctuations), representing intermittency of turbulence. The nature of such variations has been discussed in Ref. 3. The Johnson transformation and attendant detail is not required for the basic results of interest, and in fact, as discussed in Ref. 3, the intermittency is only manifested in higher moments of ε_1 which are lost in the empirical transformation process.

IV. Asymptotic Expansion of the Second Moment of Irradiance

Using physical reasoning, we have argued in recent reports that the normalized variance of irradiance (σ_I^2) should approach unity at large values of transmitter diameter (D) over coherence length (ρ_0), where ρ_0 is determined over the reciprocal (target-to-transmitter) path through turbulence. This was shown directly from the analytic expression for the fourth moment of amplitude in Ref. 3.

Efforts have continued to numerically or analytically derive further details of the curve of σ_I^2 vs. (D/ρ_0), in order to complement the phenomenological treatment which successfully predicts the essential features.[8] The same problem has been treated by Banakh, et al,[9] using certain approximations and a Monte Carlo technique, with results which are highly useful but which have certain limitations as discussed in Ref. 8.

The purpose of this section is to extend the asymptotic analysis of Ref. 3, using an inverse expansion of (D/ρ_0). We begin with the integral

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[8] J. R. Kerr, et al, "Propagation of Multiwavelength Laser Radiation Through Atmospheric Turbulence", RADC Technical Report, May 1974.

[9] V. A. Banakh, et al, J. Opt. Soc. Am. 64, 516-518, April 1974.

expression for the fourth amplitude moment (second moment of irradiance):[8]

$$< I^2 > = \frac{k^4}{(2\pi L)^4} \iiiint d\underline{\rho}_1 d\underline{\rho}_2 d\underline{\rho}_3 d\underline{\rho}_4 \; \exp\left\{ - \frac{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2}{D^2/2} \right\}$$

$$\cdot \exp\left\{ - \left(\frac{D}{\rho_c}\right)^{5/3} \frac{1}{D^{5/3}} \left[|\underline{\rho}_1 - \underline{\rho}_2|^{5/3} + |\underline{\rho}_3 - \underline{\rho}_4|^{5/3} \right.\right.$$

$$\left.\left. + |\underline{\rho}_1 - \underline{\rho}_4|^{5/3} + |\underline{\rho}_2 - \underline{\rho}_3|^{5/3} - |\underline{\rho}_2 - \underline{\rho}_4|^{5/3} - |\underline{\rho}_1 - \underline{\rho}_2|^{5/3} \right] \right\} \quad (9)$$

We let $\underline{x}_1 = \underline{\rho}_{1/D}$ and $\beta = \left(D/\rho_o\right)^{5/3}$, and rewrite (9) as

$$< I^2 > = \left(\frac{kD^2}{2\pi L}\right)^4 \iiiint d\underline{x}_1 d\underline{x}_2 d\underline{x}_3 d\underline{x}_4 \; \exp\left\{ - 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) \right\}$$

$$\cdot \exp\left\{ - \beta \left[|\underline{x}_1 - \underline{x}_2|^{5/3} + |\underline{x}_3 - \underline{x}_4|^{5/3} + |\underline{x}_1 - \underline{x}_4|^{5/3} \right.\right.$$

$$\left.\left. + |\underline{x}_2 - \underline{x}_3|^{5/3} - |\underline{x}_2 - \underline{x}_4|^{5/3} - |\underline{x}_1 - \underline{x}_3|^{5/3} \right] \right\} \quad (10)$$

For large β the integrand is significantly different from zero only near the points $\underline{x}_1 = \underline{x}_2$ and $\underline{x}_1 = \underline{x}_4$. Consequently, one should be able to obtain approximations to the integral by considering contributions from near these two points. We note further that the integrand is symmetric with respect to interchange of $\underline{x}_2$ and $\underline{x}_4$ which means that we need only to calculate the contributions to the integral from $\underline{x}_1 \cong \underline{x}_2$ and multiply this result by two.

The following changes of variable simplify the calculation. Let

$$\underline{r} = \underline{x}_1 - \underline{x}_2$$

$$\underline{s} = \underline{x}_3 - \underline{x}_4$$

$$\underline{t} = \underline{x}_2 - \underline{x}_3 \qquad (11)$$

so that $\underline{x}_1-\underline{x}_4 = \underline{r}+\underline{s}+\underline{t}$; $\underline{x}_2-\underline{x}_4 = \underline{s}+\underline{t}$; and $\underline{x}_1-\underline{x}_3 = \underline{r}+\underline{t}$. These substitutions transform the integral to

$$< I^2 > = \left(\frac{kD^2}{2\pi L}\right)^4 \iiiint d\underline{x}_2 d\underline{t} d\underline{s} d\underline{r} \exp \left\{- 8x_2^2 + 8x_2 t \cos\theta_t - 4t^2 \right.$$

$$\left. - 2r^2 - 2s^2 - 4x_2 r \cos\theta_r + 4x_2 s \cos\theta_s - 4st \cos(\theta_s-\theta_t)\right\}$$

$$\cdot \exp\left\{- \beta r^{5/3} - \beta s^{5/3} - \beta t^{5/3} - \beta|\underline{r}+\underline{s}+\underline{t}|^{5/3} + \beta|\underline{r}+\underline{t}|^{5/3}\right.$$

$$\left. + \beta|s+t|^{5/3}\right\} \qquad (12)$$

where $\underline{x}_2$ has been taken to be the polar axis and θ_r, θ_s, and θ_t are the angles between $\underline{x}_2$ and $\underline{r},\underline{s},\underline{t}$ respectively.

We note that for $\beta>>1$ the major contribution to the integral comes from near the points $r = 0$, $s = 0$. This corresponds to the points $\underline{x}_1 = \underline{x}_2$, $\underline{x}_3 = \underline{x}_4$. The factor $\exp\left\{\beta t^{5/3} - \beta|\underline{r}+\underline{s}+\underline{t}|^{5/3} + \beta|\underline{r}+\underline{t}|^{5/3} + \beta|s+t|^{5/3}\right\}$ is bounded and does not extend the effective range of the r and s integrations. This effective range is $0(\beta^{-3/5})$. On the other hand the t integration is not so restricted since the above factor approaches unity as $t \to \infty$ for finite r and s. The range of the t integration is 0(1) due to the factor e^{-4t^2}.

We will now exploit the narrow range of the r and s integrations by expanding the integrand, apart from the factors $re^{-\beta r^{5/3}}$ and $se^{-\beta s^{5/3}}$, in a Taylor series about $r = s = 0$. Since

$$\int_{0}^{\infty} dr\ r^{n} e^{-\beta r^{5/3}} = \frac{3}{5}\,\Gamma\left(\frac{3n+3}{5}\right)\,\beta^{\frac{-3n+3}{5}} \tag{13}$$

This will result in an expansion in inverse powers of β. We thus write

$$< I^{2} > \;= \left(\frac{kD^{2}}{2\pi L}\right)^{4} \iint d\underline{x}_{2} d\underline{t} \quad \exp\left\{-\,8x_{2}^{2} + 8x_{2}t\,\cos\theta_{t} - 4t^{2}\right\}\; Q(\underline{x}_{2},\underline{t}) \tag{14}$$

where

$$Q(\underline{x}_{2},\underline{t}) = \iint d\underline{r}d\underline{s}\quad \exp(-\beta r^{5/3}-\beta s^{5/3})\ \exp\left\{-2r^{2}-2s^{2}-4x_{2}r\,\cos\theta_{r} + 4x_{2}s\,\cos\theta_{s} - 4st\,\cos(\theta_{s}-\theta_{t})\right\}\ \exp\left(-\beta f(r,s,t)\right) \tag{15}$$

and

$$f(r,s,t) = t^{5/3} + |\underline{r}+\underline{s}+\underline{t}|^{5/3} - |\underline{r}+\underline{t}|^{5/3} - |\underline{s}+\underline{t}|^{5/3} \quad .$$

Now expanding $f(r,s,t)$ in a Taylor's series about $r = s = 0$, we have

$$f(r,s,t) = f(0,0,t) + r\left.\frac{\partial f}{\partial r}\right|_{r=s=0} + s\left.\frac{\partial f}{\partial s}\right|_{r=s=0} + \frac{r^{2}}{2!}\left.\frac{\partial^{2} f}{\partial r^{2}}\right|_{r=s=0} + \frac{2rs}{2!}\left.\frac{\partial^{2} f}{\partial r\partial s}\right|_{r=s=0} + \frac{s^{2}}{s!}\left.\frac{\partial^{2} f}{\partial s^{2}}\right|_{r=s=0} + \ \ldots \tag{16}$$

We then note the following equalities:

$f(0,s,t) = f(r,0,t) = 0$

$$\left.\frac{\partial f}{\partial r}\right|_{r=s=0} = \left.\frac{\partial f}{\partial s}\right|_{r=s=0} = 0$$

$$\left.\frac{\partial^2 f}{\partial r^2}\right|_{r=s=0} = \left.\frac{\partial^2 f}{\partial s^2}\right|_{r=s=0} = 0$$

$$\left.\frac{\partial^2 f}{\partial r \partial s}\right|_{r=s=0} = \frac{5}{3}\, t^{-1/3} \left(\cos(\theta_r-\theta_s) - \frac{1}{3}\cos(\theta_r-\theta_t)\cos(\theta_s-\theta_t)\right)$$

$$\left.\frac{\partial^3 f}{\partial r^3}\right|_{r=s=0} = 0 = \left.\frac{\partial^3 f}{\partial s^3}\right|_{r=s=0}$$

$$\left.\frac{\partial^3 f}{\partial r^2 \partial s}\right|_{r=s=0} = -\frac{10}{9}\, t^{-4/3}\cos(\theta_r-\theta_t)\cos(\theta_r-\theta_s) - \frac{5}{9}\, t^{-4/3}\cos(\theta_s-\theta_t)$$
$$+ \frac{35}{27}\, t^{-4/3}\cos^2(\theta_r-\theta_t)\cos(\theta_s-\theta_t)$$

and

$$\left.\frac{\partial^3 f}{\partial r \partial^2 s}\right|_{r=s=0} = -\frac{10}{9}\, t^{-4/3}\cos(\theta_s-\theta_t)\cos(\theta_r-\theta_s) - \frac{5}{9}\, t^{-4/3}\cos(\theta_r-\theta_t)$$
$$+ \frac{35}{27}\, t^{-4/3}\cos^2(\theta_s-\theta_t)\cos(\theta_r-\theta_t) \qquad (17)$$

Therefore

$$\exp(-\beta f(r,s,t)) = 1 - \frac{5}{3}\,\beta r s t^{-1/3}\left(\cos(\theta_r-\theta_s) - \frac{1}{3}\cos(\theta_r-\theta_t)\cos(\theta_s-\theta_t)\right)$$
$$- \frac{3}{3!}\,\beta r^2 s t^{-4/3}\left[-\frac{10}{9}\cos(\theta_r-\theta_t)\cos(\theta_r-\theta_s) - \frac{5}{9}\cos(\theta_s-\theta_t)\right.$$
$$\left. + \frac{35}{27}\cos^2(\theta_r-\theta_t)\cos(\theta_s-\theta_t)\right]$$
$$- \frac{3}{3!}\,\beta r s^2 t^{-4/3}\left[-\frac{10}{9}\cos(\theta_s-\theta_t)\cos(\theta_r-\theta_s) - \frac{5}{9}\cos(\theta_r-\theta_t)\right.$$
$$\left. + \frac{35}{27}\cos^2(\theta_s-\theta_t)\cos(\theta_r-\theta_t)\right] - \ldots$$

$$+\frac{1}{2!}\,\frac{25}{9}\,\beta^2 r^2 s^2 t^{-2/3}\Big[\cos(\theta_r-\theta_s)-\frac{1}{3}\cos(\theta_r-\theta_t)$$

$$\cdot\cos(\theta_s-\theta_t)\Big]^2$$

$$+\frac{1}{4!}\left(\frac{5}{3}\right)^4\beta^4 r^4 s^4 t^{-4/3}\Big[\cos(\theta_r-\theta_s)-\frac{1}{3}\cos(\theta_r-\theta_t)$$

$$\cdot\cos(\theta_s-\theta_t)\Big]^4$$

$$+\frac{1}{6!}\left(\frac{5}{3}\right)^6\beta^6 r^6 s^6 t^{-2}\quad\Big[\cos(\theta_r-\theta_s)-\frac{1}{3}\cos(\theta_r-\theta_t)$$

$$\cdot\cos(\theta_s-\theta_t)\Big]^6$$

$$+\ .\ .\ . \tag{18}$$

Although (18) does not include all terms up to $O(r^6 s^6)$, those neglected either contribute zero under the $d\theta_r$ and $d\theta_s$ integrations (see below) or result in terms of lower order in β. The last term listed is proportional to t^{-2} and hence the t integration of this term diverges. This is because our expansion emphasizes the region near $t = 0$. The contribution of this term turns out to be $O(\beta^{-18/5})$ so that there is no point in including higher order terms.

As mentioned above, the $d\theta_r$ amd $d\theta_s$ integrations drive many terms in the expansion to zero, namely all those which are periodic with period 2π. Keeping only those terms which are non-zero and of $O(\beta^{-14/5})$ and greater:

$$Q(\underline{x}_2,\underline{t}) = 2\int_0^{2\pi} d\theta_r\int_0^{2\pi} d\theta_s\int_0^{\infty} drr\int_0^{\infty} dss\ \exp(-\beta r^{5/3}-\beta s^{5/3})$$

$$\cdot\left\{\ 1+\frac{1}{2}\left(\frac{5}{3}\right)^2\beta^2 r^2 s^2 t^{-2/3}\Big[\cos(\theta_r-\theta_s)-\frac{1}{3}\cos(\theta_r-\theta_t)\right.$$

$$\cdot\cos(\theta_s-\theta_t)\Big]^2$$

$$+\frac{1}{4!}\left(\frac{5}{3}\right)^{4}\beta^{4}r^{4}s^{4}t^{-4/3}\left[\cos(\theta_r-\theta_s)-\frac{1}{3}\cos(\theta_r-\theta_t)\right.$$

$$\left.\cdot\cos(\theta_s-\theta_t)\right]^{4}\Big\}$$

$$+\,O(\beta^{-18/5}) \qquad (19)$$

The factor 2 appears since we are considering only contributions to $< I^2 >$ from near $\underline{x}_1 = \underline{x}_2$ $(r = 0)$ and not from $\underline{x}_1 = \underline{x}_4$ $(\underline{r} + \underline{s} + \underline{t} = 0)$.

Performing the angle integrations gives

$$Q(\underline{x}_2,\underline{t}) = 2\int_0^\infty drr\int_0^\infty dss\,\exp(-\beta r^{5/3}-\beta s^{5/3})\Big\{4\pi^2+\frac{13\pi^2}{9}$$

$$\cdot\frac{1}{2}\left(\frac{5}{3}\right)^{2}\beta^{2}r^{2}s^{2}t^{-2/3}+\frac{121}{144}\pi^{2}\cdot\frac{1}{4!}\left(\frac{5}{3}\right)^{4}\beta^{4}r^{4}s^{4}t^{-4/3}\Big\}$$

$$+\,O(\beta^{-18/5})$$

$$=8\pi^2\cdot\left(\frac{3}{5}\right)^{2}\Gamma^{2}\left(\frac{6}{5}\right)\beta^{-12/5}+\frac{13\pi^2}{9}\Gamma^{2}\left(\frac{12}{5}\right)\beta^{-14/5}t^{-2/3}$$

$$+\frac{121}{1728}\pi^{2}\left(\frac{5}{3}\right)^{2}\Gamma^{2}\left(\frac{18}{5}\right)\beta^{-16/5}t^{-4/3}+O(\beta^{-18/5}) \qquad (20)$$

Since to this order $Q(\underline{x}_2,\underline{t})$ is independent of $\underline{x}_2$ and $d\theta_t$, we may make the substitution $\underline{v} = 2\underline{x}_2-\underline{t}$ so that

$$v^2 = 4x_2^2 - 4x_2t\cos\theta_t + t^2$$

$$\text{and}\quad d\underline{x}_2 = \frac{1}{4}\,d\underline{v} = \frac{1}{4}\,vdvd\theta_v \qquad (21)$$

resulting in

$$< I^2 > = \left(\frac{kD^2}{2\pi L}\right)^4 \frac{1}{4} \int_0^{2\pi} d\theta_t \int_0^\infty dt t \int_0^\infty d\theta_v \int_0^\infty dv\ v\ \exp(-2v^2-2t^2)$$

$$\cdot\ Q(\underline{x}_2,\underline{t})$$

$$= \left(\frac{kD^2}{2\pi L}\right)^4 \frac{\pi^2}{4} \int_0^\infty dt\ t\ \exp(-2t^2)\ Q(\underline{x}_2,\underline{t})$$

$$= \left(\frac{kD^2}{2\pi L}\right)^4 \frac{\pi^2}{4} \int_0^\infty dt\ t\ \exp(-2t^2) \left\{8\pi^2 \left(\frac{3}{5}\right)^2 \Gamma^2\left(\frac{6}{5}\right) \beta^{-12/5}\right.$$

$$+ \frac{13\pi^2}{9} \Gamma^2\left(\frac{12}{5}\right) \beta^{-14/5} t^{-2/3} + \frac{121}{1728} \pi^2 \left(\frac{5}{3}\right)^2 \Gamma^2\left(\frac{18}{5}\right) \beta^{-16/5}$$

$$\left.\cdot\ t^{-4/3}\right\} + 0(\beta^{-18/5})$$

$$= \left(\frac{kD^2}{2\pi L}\right)^4 \left\{\frac{\pi^4}{2} \left(\frac{3}{5}\right)^2 \Gamma^2\left(\frac{6}{5}\right) \beta^{-12/5} + \frac{13}{9} \cdot \frac{2^{-2/3}}{8} \pi^4\ \Gamma\left(\frac{2}{3}\right)\right.$$

$$\cdot\ \Gamma^2\left(\frac{12}{5}\right) \beta^{-14/5} + \frac{121}{1728} \frac{2^{-1/3}}{8} \pi^4\ \Gamma\left(\frac{1}{3}\right) \left(\frac{5}{3}\right)^2 \Gamma^2\left(\frac{18}{5}\right)$$

$$\left.\cdot\ \beta^{-16/5}\right\} + 0(\beta^{-18/5}) \tag{22}$$

The mean irradiance is

$$< I > = \left(\frac{k}{2\pi L}\right)^2 \iint d\underline{\rho}_1 d\underline{\rho}_2\ \exp\left\{-\frac{\rho_1^2 + \rho_2^2}{D^2/2}\right\} \exp\left\{-\left(\frac{D}{\rho_o}\right)^{5/3}\right.$$

$$\cdot \frac{1}{D^{5/3}} |\underline{\rho}_1 - \underline{\rho}_2|^{5/3}$$

$$= \left(\frac{kD^2}{2\pi L}\right)^2 \iint d\underline{x}_1 d\underline{x}_2 \ \exp(-2x_1^2 - 2x_2^2 - \beta|\underline{x}_1 - \underline{x}_2|^{5/3} \Big\}$$

$$= \left(\frac{kD^2}{2\pi L}\right)^2 \int d\underline{x}_2 \int d\underline{r} \ \exp(-4x_2^2 - 4x_2 r \cos\theta - 2r^2) \exp(-\beta r^{5/3}) \quad (23)$$

As above, we let $\underline{v} = 2\underline{x}_2 + \underline{r}$, giving

$$< I > = \left(\frac{kD^2}{2\pi L}\right)^2 \frac{\pi}{4} \int d\underline{r} \ \exp(-r^2 - \beta r^{5/3})$$

$$= \left(\frac{kD^2}{2\pi L}\right)^2 \frac{\pi^2}{2} \int_0^\infty dr \ r \ \exp(-\beta r^{5/3}) \left\{1 - r^2 + \frac{r^4}{2!} - \ldots\right\}$$

$$= \left(\frac{kD^2}{2\pi L}\right)^2 \frac{\pi^2}{2} \ \frac{3}{5} \ \Gamma\left(\frac{6}{5}\right) \beta^{-6/5} - 0(\beta^{-12/5}) \quad (24)$$

The second central moment of irradiance normalized to the mean irradiance is then

$$\sigma_I^2 \equiv \frac{<I^2> - <I>^2}{<I>^2} = 1 + \frac{13}{9} \ \frac{2^{-2/3}}{8} \ \Gamma\left(\frac{2}{3}\right) \ \Gamma^2\left(\frac{12}{5}\right) \cdot 4 \ \left(\frac{5}{3}\right)^2$$

$$\cdot \ \Gamma^{-2}\left(\frac{6}{5}\right) \beta^{-2/5} + \frac{121}{1728} \ 2^{-4/3} \ \Gamma\left(\frac{1}{3}\right) \left(\frac{5}{3}\right)^4 \ \Gamma^2\left(\frac{12}{5}\right) \ \Gamma^{-2}\left(\frac{6}{5}\right)$$

$$\cdot \beta^{-4/5} + 0(\beta^{-6/5})$$

$$= 1 + 3.132\beta^{-2/5} + 9.414\beta^{-4/5} + 0(\beta^{-6/5}) \quad (25)$$

We may thus write

$$^{(1)}\sigma_I^2 = 1 + 3.132\beta^{-2/5} \tag{26a}$$

$$^{(2)}\sigma_I^2 = 1 + 3.132\beta^{-2/5} + 9.414\beta^{-4/5} \tag{26b}$$

as the one- and two-term asymptotic approximation to σ_I^2. The one-term approximation agrees exactly with the result obtained by a different method by Gochelashvily.[10]

$^{(1)}\sigma_I^2$ and $^{(2)}\sigma_I^2$ are tabulated for a range of values of β in Table 5, along with D/ρ_0 and D_S used by the Russian investigators. Curves are shown vs. $\sqrt{D_S}$ in Figure 1, along with the result from Ref. 9. We note that our result, which should be accurate for large β (D_S), does not agree well with the Monte-Carlo (M-C) integration for $\sqrt{D_S} > 20$. We surmise therefore that our results represent a useful modification at large D_S.

The M-C calculation should be more accurate at small D_S or β, and it agrees with the one-term expansion for $8 < \sqrt{D_S} < 20$. However, the two-term expansion appears to fit the data somewhat better.

Two final points can be made. First, the structure function $(\sim r^{5/3})$ is valid only for path lengths (L) such that $L_c << L << L_i$, where L_c is the propagation distance at which the near field decays to e^{-1} of its value at the aperture and L_i is the propagation distance at which the coherence length is of the order of the inner scale.[11] For example, if $\ell_o \overset{\sim}{\sim} \rho_o \overset{\sim}{\sim} 1$ mm and $D \overset{\sim}{\sim} 10$ cm, we have $D/\rho_o = 100$, $\beta = 2150$, and $\sqrt{D_S} = 66$. The above analysis is then valid for $\beta << 2150$, while for $\beta >> 2150$ the structure function is proportional to r^2 and the resulting new integral can be evaluated by the same method used above.

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10. Prokhorov, A.M., et al, Proc. IEEE 63, 790-811, May 1975.
11. Lutomirski, R.F. and Yura, H.T., J. Opt. Soc. Am. 61, 482-487, April 1971.

TABLE 5. TABULATION OF NORMALIZED VARIANCE OF IRRADIANCE FOR ONE- AND TWO-TERM EXPANSIONS

$\sqrt{D_S}$	D/ρ_o	β	$\beta^{-2/5}$	$(1)\sigma_I^2$	$(2)\sigma_I^2$
4	3.48	8	0.435	2.363	4.147
6	5.66	18	0.3147	1.986	2.918
8	8.00	32	0.250	1.783	2.371
10	10.45	50	0.2091	1.655	2.067
12	13.01	72	0.1807	1.566	1.874
14	15.66	98	0.1598	1.500	1.741
16	18.38	128	0.1436	1.450	1.644
18	21.17	162	0.1307	1.409	1.570
20	24.02	200	0.1201	1.376	1.512
22	26.93	242	0.1113	1.349	1.465
24	29.90	288	0.1038	1.325	1.427
26	32.91	338	0.09737	1.305	1.394
28	35.97	392	0.09177	1.287	1.367
30	39.08	450	0.08684	1.272	1.343
32	42.22	512	0.08247	1.258	1.322
34	45.41	578	0.07856	1.246	1.304
36	48.63	648	0.07505	1.235	1.288
38	51.89	722	0.07188	1.225	1.274
40	55.19	800	0.06899	1.216	1.261
780	1949.4	304200	0.00641	1.020	1.020

Note: $\beta=(D/\rho_o)^{5/3} = \frac{1}{2} D_S$

$D_S=1.095C_n^2k^2L(D)^{5/3}$

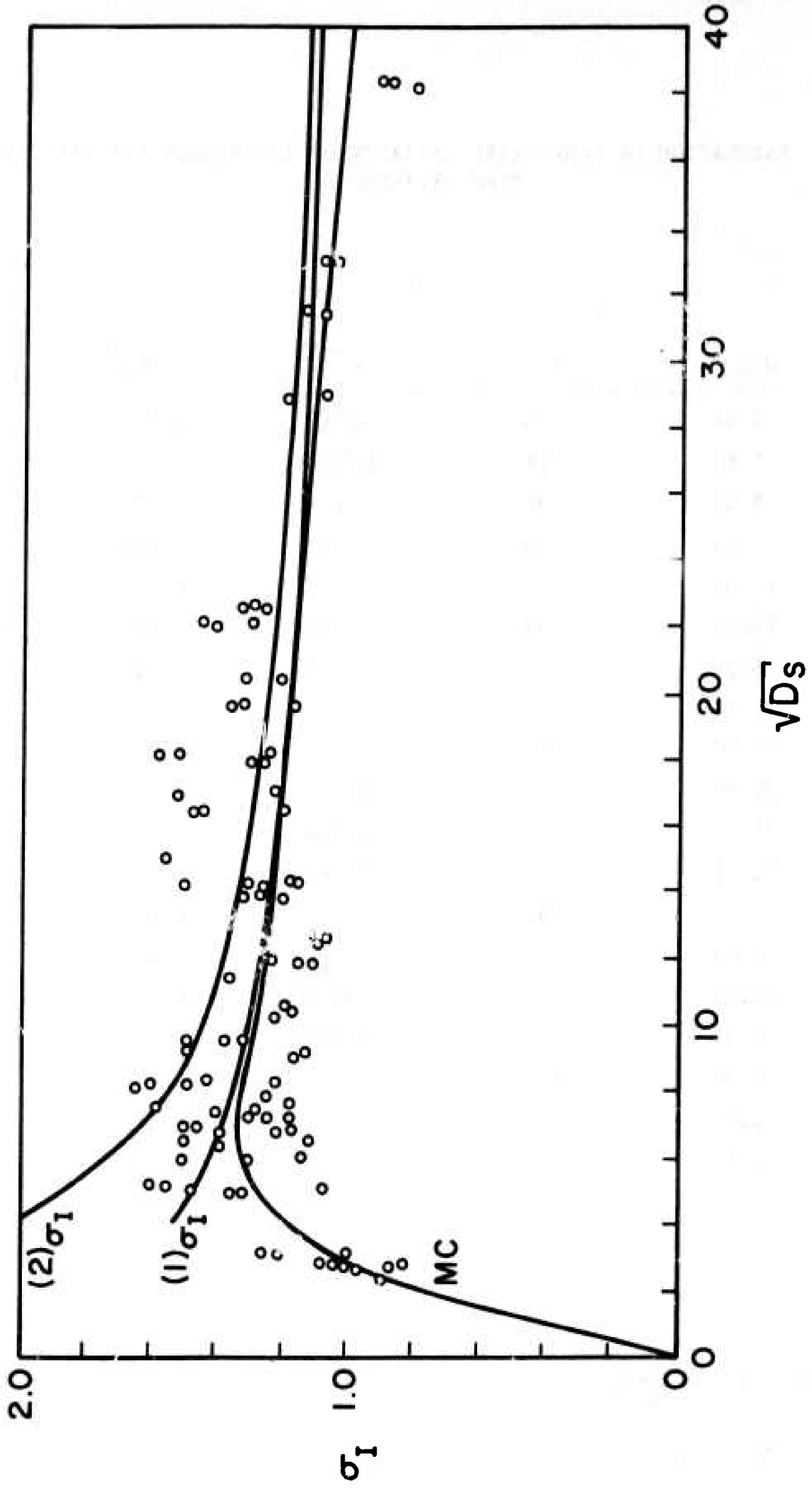

Figure 1. Comparison of one- and two-term expansions with experimental and Monte-Carlo results from Ref. 9.

The second point is that the reason our asymptotic series breaks down after the second term is that expanding quantities like $|\underline{r}+\underline{t}|^{5/3}$ as $t^{5/3}$ times a series in powers of r/t is valid only if r<<t. The effective range of the r integration is $O(\beta^{-3/5})$ which means that if $t>>\beta^{-3/5}$, the expansions are valid. Presumably we could avoid the divergence caused by the factors of $\frac{1}{t}$ as $t \to 0$ by cutting off the integration at e.g. $t= \beta^{-3/5}$. This introduces an error of $O(\beta^{-6/5})$, so that terms up to $\beta^{-4/5}$ should be acceptable. The t = 0 region corresponds to the region where $\underline{x}_2 \approx \underline{x}_4$, i.e., where the two maxima of the integrand coincide. It is not possible to handle the integration accurately in this case, and as β gets smaller this error becomes more significant.

V. Scintillations from Diffuse and General Sources--A Preliminary Analysis and Discussion

The major new area of investigation on this program is that of turbulence effects on radiation from diffuse and partially coherent sources, with particular attention to laser-illuminated targets and the performance of coherent optical adaptive (COAT) systems. This represents a natural extension of the recent work done on focused wander-tracking transmitters. The experimental effort will involve the determination of scintillation statistics from diffuse targets (speckles) and glints, including infrared wavelengths of primary interest.

The purpose of this section is to present a preliminary analysis for the case of a diffuse target, coherently illuminated, with the point of observation in the near field of the target illumination. This treatment includes the effects of turbulence between the illuminator and target and on the return path to the observer. Generalizations of the illuminator and target characteristics and geometry are underway and will be reported in the future.

A. Mean Irradiance at Receiver

The source, target, and receiver configuration is shown in Figure 2. The present analysis is confined to the case of a TEM_{00} laser illuminator. The source and target are assumed to be much smaller than the path length (L), and the distance between the receiver and source is greater than the

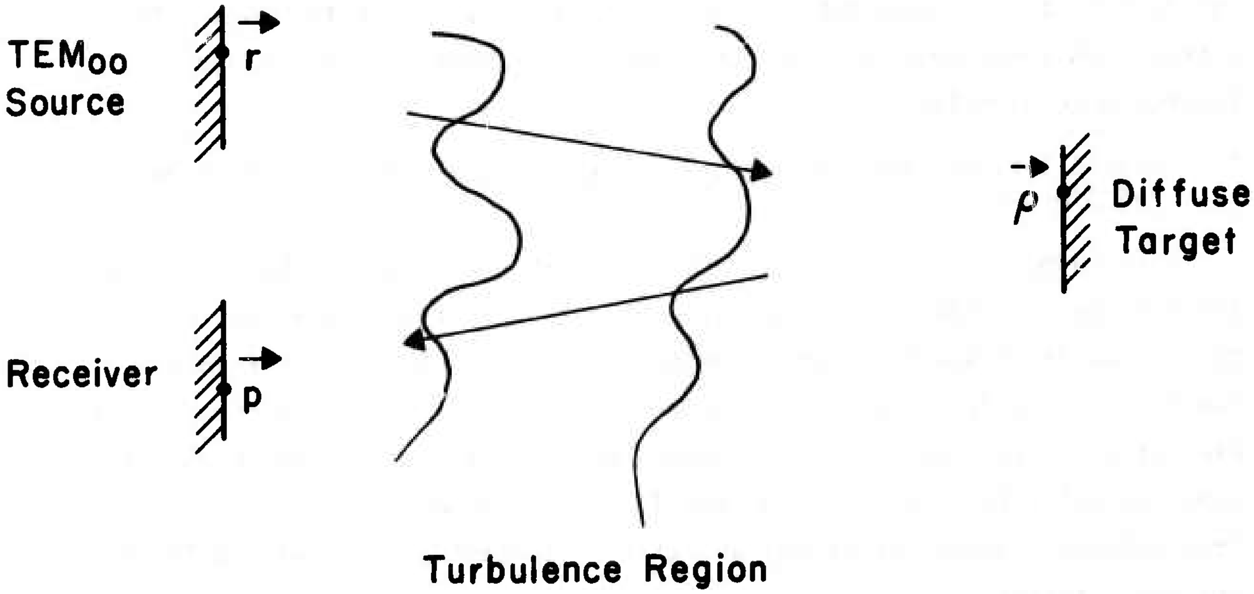

Figure 2. Target-Illuminator Configuration

source size and much smaller than the path length. These geometric conditions confine the problem to small angles and ensures that the outgoing and returning radiation experience independent turbulence regions; the latter limitation is thought to be inessential owing to the diffuse target characteristics and will be relaxed in subsequent work.

We write the source amplitude distribution as

$$U_o(\underline{r}) = U_o \exp\left(-\frac{r^2}{2\alpha_o^2} - \frac{ikr^2}{2F}\right) \qquad (27)$$

where α_o and F are the characteristics beam radius and focal length, respectively. The field at the target is written from the extended Huygens-Fresnel principle[12,13] as

$$U(\underline{\rho}) = \frac{ke^{ikL}}{2\pi iL} \int U_o(\underline{r}) \quad \exp\left[\frac{ik(\underline{\rho}-\underline{r})^2}{2L} + \psi_1(\underline{\rho},\underline{r})\right] d\underline{r} \qquad (28)$$

where ψ_1 describes the effects of the inhomogeneous medium on the propagation of a spherical wave. Combining (27) and (28), we have

$$U(\underline{\rho}) = \frac{ke^{ik\left[L + \frac{\rho^2}{2L}\right]}U_o}{2\pi iL} \int \exp\left[-\frac{r^2}{2\alpha_o^2} + \frac{ik}{2L}\left(1 - \frac{L}{F}\right) r^2 - \frac{ik}{L}\underline{\rho}\cdot\underline{r} + \psi_1(\underline{\rho},\underline{r})\right] d\underline{r} \qquad (29)$$

In particular, this applies to the special cases of a focused (L=F) collimated (F → ∞) beam respectively.

The field at the receiver is written by reapplying the Huygens-Fresnel principle to the field at the target:

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[12] Lutomirski, R.F. and Yura, H.T., Applied Optics 10, 1652-1658, July 1971.
[13] Yura, H., Applied Optics 11, 1399-1406, June 1972.

$$U(\underline{p}) = \frac{k e^{ik\left[L + \frac{p^2}{2L}\right]}}{2\pi i L} \int U'(\underline{\rho}) \ \exp\left[\frac{ik}{2L}\left(\rho^2 - 2\underline{p}\cdot\underline{\rho}\right) + \psi_2(\underline{p},\underline{\rho})\right] d\underline{\rho} \qquad (30)$$

where $U'(\underline{\rho})$ is the field solution after the reflection from the target, and ψ_2 represents the turbulence effect from the target to the receiver. The mean intensity at the receiver is then

$$\langle I(\underline{p})\rangle = \langle |U(\underline{p})|^2\rangle = \left(\frac{k}{2\pi L}\right)^2 \iint d\underline{\rho}_1 d\underline{\rho}_2 \ \langle U'(\underline{\rho}_1)U'^*(\underline{\rho}_2)\rangle$$

$$\cdot \exp\left[\frac{ik}{2L}\left((\rho_1^2 - \rho_2^2) - 2\underline{p}\cdot(\underline{\rho}_1 - \underline{\rho}_2)\right)\right]$$

$$\langle \exp\left[\psi_2(\underline{p},\underline{\rho}_1) + \psi_2^*(\underline{p},\underline{\rho}_2)\right]\rangle \qquad (31)$$

Through the assumption of a diffuse target, the reflected beam suffers a random phase delay from point-to-point over the target, and we write

$$\langle U'(\underline{\rho}_1)U'^*(\underline{\rho}_2)\rangle = \langle I(\underline{\rho}_1)\rangle \ \delta(\underline{\rho}_1 - \underline{\rho}_2) \qquad (32)$$

Using this in Eq. (31), the mean intensity becomes

$$\langle I(\underline{p})\rangle = \left(\frac{k}{2\pi L}\right)^2 \int d\underline{\rho}_1 \ \langle |U(\underline{\rho}_1)|^2\rangle \ \langle \exp\left[\psi_2(\underline{p},\underline{\rho}_1) + \psi_2^*(\underline{p},\underline{\rho}_1)\right]\rangle \qquad (33)$$

where the mean exponential term is unity from considerations of energy conservation.[13]

The resultant mean intensity at the receiver is then simply

$$<I(\underline{p})> = \left(\frac{k}{2\pi L}\right)^2 \int d\underline{\rho} \; <|U(\underline{\rho})|^2> \tag{34}$$

To complete the solution, we use Eq. (29) with Eq. (34). We note that the structure function^2 gives us

$$<\exp\left[\psi_1(\underline{\rho},\underline{r}_1) + \psi_1{}^*(\underline{\rho},\underline{r}_2)\right]> \; = e^{-\left(\frac{r}{\rho_o}\right)^{5/3}} \tag{35}$$

For the focused beam, we then have ($r \equiv |\underline{r}_1-\underline{r}_2|$):

$$<|U(\underline{\rho})|^2> = \left(\frac{k}{2\pi L}\right)^2 U_o{}^2 \iint d\underline{r}_1 d\underline{r}_2 \exp\left[-\frac{r_1^2+r_2^2}{2\alpha_o^2} - \frac{ik}{L}\,\underline{\rho}\cdot(\underline{r}_1-\underline{r}_2) - \left(\frac{r}{\rho_o}\right)^{5/3}\right] \tag{36}$$

Carrying out the integration indicated in (34), involving the Fourier-Bessel integral, we have finally

$$<I(\underline{p})> = \frac{1}{2\pi}\left(\frac{k}{L}\right)^2 U_o{}^2 \; \frac{\alpha_o^2}{2} \tag{37}$$

The result for the collimated beam is identical, and in fact could be deduced for an arbitrary beam focus (Eq. 27) through conservation of energy:

$$<I(\underline{p})> = \left(\frac{k}{2\pi L}\right)^2 \int d\underline{r} \; <|U(\underline{r})|^2>$$

$$= \left(\frac{k}{2\pi L}\right)^2 \; 2\pi \; U_o^2 \int_o^\infty r e^{-r^2/\alpha_o^2} \, dr$$

$$= \frac{1}{2\pi} \left(\frac{k}{L}\right)^2 U_o^2 \; \frac{\alpha_o^2}{2} \tag{38}$$

Thus the mean irradiance at the receiver (illuminator) plane is uniform and independent of turbulence level.

B. Covariance and Variance of Irradiance at Receiver

The correlation function of the irradiance at the receiver is

$$B_I(\underline{p}_1,\underline{p}_2) = \langle I(\underline{p}_1) I(\underline{p}_2)\rangle = \langle U(\underline{p}_1)U^*(\underline{p}_1)U(\underline{p}_2)U^*(\underline{p}_2)\rangle \tag{39}$$

We now point out that the statistics of the incoherent reflection from a diffuse object are gaussian, from the Central Limit Theorem, with a phase which is uncorrelated with amplitude and uniformly distributed from 0 to 2π. In the near field of the target illumination, we may expect the gaussian nature of this field to apply after perturbation by turbulence. A determination of the exact conditions necessary for this assumption to hold, and the generalization of the solution to cases where it does not, will be reported in future work. The gaussian assumption yields

$$B_I = \langle U(\underline{p}_1)U^*(\underline{p}_1)\rangle\langle U(\underline{p}_2)U^*(\underline{p}_2)\rangle + \langle U(\underline{p}_1)U^*(\underline{p}_2)\rangle\langle U^*(\underline{p}_1)U(\underline{p}_2)\rangle$$

$$\equiv \langle I(\underline{p}_1)\rangle\langle I(\underline{p}_2)\rangle + |\Gamma(\underline{p}_1,\underline{p}_2)|^2 \tag{40}$$

where Γ is the mutual coherence function at the receiver. It follows that the covariance of intensity given by

$$C_I(\underline{p}_1,\underline{p}_2) = B_I(\underline{p}_1,\underline{p}_2) - \langle I(\underline{p}_1)\rangle\langle I(\underline{p}_2)\rangle = |\Gamma(\underline{p}_1,\underline{p}_2)|^2 \tag{41}$$

The mutual coherence function at the receiver can be described from the extended Huygens-Fresnel formula.[13]

$$\Gamma(\underline{p}_1,\underline{p}_2) = \left(\frac{k}{2\pi L}\right)^2 \iint d\underline{\rho}_1 d\underline{\rho}_2 \ \langle \ U(\underline{\rho}_1)\overset{*}{U}(\underline{\rho}_2) \ \rangle \ \exp \ \Big\{ ik\big[R_1(\underline{\rho}_1,\underline{p}_1)$$

$$- R_2(\underline{\rho}_2,\underline{p}_2)\big]\Big\} \langle \exp \left[\psi_2(\underline{p}_1,\underline{\rho}_1) + \psi_2{}^*(\underline{p}_2,\underline{\rho}_2)\right]\rangle \tag{42}$$

where $R_1(\underline{\rho}_1,\underline{p}_1)$, $R_2(\underline{\rho}_2,\underline{p}_2)$ are the distances from $\underline{\rho}_1$ to $\underline{p}_1$ and $\underline{\rho}_2$ to $\underline{p}_2$ respectively.

By the Fresnel approximation

$$R_1(\underline{\rho}_1,\underline{p}_1) - R_2(\underline{\rho}_2,\underline{p}_2) \cong \frac{p_1^2-p_2^2+\rho_1^2-\rho_2^2}{2L} - \frac{\underline{p}_1.\underline{\rho}_1-\underline{p}_2.\underline{\rho}_2}{L} \tag{43}$$

Finally, from equations (43) and (44):

$$\Gamma(\underline{p}_1\underline{p}_2) = \left(\frac{k}{2\pi L}\right)^2 \exp\left[\frac{ik(p_1^2-p_2^2)}{2L}\right] \iint d\underline{\rho}_1 d\underline{\rho}_2 \ \langle U(\underline{\rho}_1)U^*(\underline{\rho}_2)\rangle$$

$$\cdot \ \exp \ \left\{ ik\left(\frac{\rho_1^2-\rho_2^2}{2L} - \frac{\underline{p}_1\cdot\underline{\rho}_1-\underline{p}_2\cdot\underline{\rho}_2}{L}\right)\right\}\langle\exp\left[\psi_2(\underline{p}_1,\underline{\rho}_1)+\psi_2{}^*(\underline{p}_2,\underline{\rho}_2)\right]\rangle \tag{44}$$

Since the wave is incoherent after reflection from the diffuse target, the coherence function at that plane can be represented by the Dirac delta function:

$$<U(\underline{\rho}_1)U^*(\underline{\rho}_2)> \;=\; <I(\underline{\rho}_1)>\delta(\underline{p}_1-\underline{p}_2) \qquad (45)$$

Using this in Eq. (44), $\Gamma(\underline{p}_1,\underline{p}_2)$ can be simplified:

$$\Gamma(\underline{p}_1,\underline{p}_2) = \left(\frac{k}{2\pi L}\right)^2 \exp\left[\frac{ik(p_1^2-p_2^2)}{2L}\right]\iint d\underline{\rho}\;<I(\underline{\rho})>\exp\left\{-\frac{ik}{L}(\underline{p}_1-\underline{p}_2)\cdot\underline{\rho}\right\} \cdot e^{-\left(\frac{\underline{p}}{\rho_o}\right)^{5/3}} \qquad (46)$$

In the absence of turbulence, this equation is entirely identical to the Van Cittert-Zernike theorem of coherence theory,[14] which is identical to a result obtained by Goodman for the mutual coherence function of a pulsed optical radar.[15]

To complete the solution, we utilize the mean intensity at the target for the focussed case, Eqs. (34) and (36):

$$<I(\underline{\rho})>_f = \left(\frac{k}{L}\right)^2 |U_o|^2 \frac{\alpha_o^2}{2}\int_0^\infty r\,dr\,J_o\left(\frac{k}{L}\rho r\right)e^{-\frac{rL}{4\alpha_o^2}-\left(\frac{r}{\rho_o}\right)^{5/3}} \qquad (47)$$

We thus have:

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[14] Born, M. and Wolf, E., Principles of Optics, Pergamon Press, N.Y., 1975.

[15] Goodman, J.W., Proc. IEEE 53, 1688-1700, Nov. , 1965.

$$\Gamma(p_1,p_2) = \left(\frac{k}{2\pi L}\right)^2 \left(\frac{k}{L}\right)^2 |U_o|^2 \frac{\alpha_o^2}{2} \int d\underline{\rho} \int_o^\infty r\,dr\, J_o\left(\frac{k}{L}\rho r\right)$$

$$\cdot\, e^{-\frac{r^2}{4\alpha_o^2} - \left(\frac{r}{\rho_o}\right)^{5/3}} \exp\left[-\frac{ik}{L}\underline{\rho}\cdot\underline{p} - \left(\frac{\underline{p}}{\rho_o}\right)^{5/3}\right]$$

$$\cdot \exp\left[\frac{ik(p_1^2-p_2^2)}{2L}\right]$$

$$= \frac{1}{2\pi}\left(\frac{k}{L}\right)^4 |U_o|^2 \frac{\alpha_o^2}{2} \int_o^\infty \rho\,d\rho \int_o^\infty r\,dr\, J_o\left(\frac{k}{L}\rho r\right) J_o\left(\frac{k}{L}\rho p\right)$$

$$\cdot\, e^{-\frac{r^2}{4\alpha_o^2} - \left(\frac{r}{\rho_o}\right)^{5/3} - \left(\frac{\underline{p}}{\rho_o}\right)^{5/3} + \frac{ik}{2L}\left(p_1^2-p_2^2\right)} \tag{48}$$

where $p = |\underline{p}_1-\underline{p}_2|$. From the Fourier-Bessel integral formula,

$$r\int_o^\infty \rho J_o\left(\frac{k}{L}r\rho\right) J_o\left(\frac{k}{L}p\rho\right) d\rho = \left(\frac{L}{k}\right)^2 \delta(r-p) \tag{49}$$

Eq. (48) can then be simplified:

$$\Gamma(\underline{p}_1,\underline{p}_2) = \frac{1}{2\pi}\left(\frac{k}{L}\right)^2 |U_o|^2 \frac{\alpha_o^2}{2}\, e^{-\frac{p^2}{4\alpha_o^2} - 2\left(\frac{p}{\rho_o}\right)^{5/3} + \frac{ik}{2L}\left(p_1^2-p_2^2\right)}$$

$$= \langle I(p)\rangle\, e^{-\frac{p^2}{4\alpha_o^2} - 2\left(\frac{p}{\rho_o}\right)^{5/3} + \frac{ik}{2L}\left(p_1^2-p_2^2\right)} \tag{50}$$

Finally, the normalized covariance function of irradiance for the focussed case can thus be written:

$$C_{I_N}(\underline{p})_f = e^{-\frac{\underline{p}^2}{2\alpha_o^2} - 4\left(\frac{\underline{p}}{\rho_o}\right)^{5/3}} \tag{51}$$

and the normalized variance is

$$\sigma_{I_f}^2 = 1. \tag{52}$$

For the collimated case, the same variance is obtained, and the covariance is

$$C_{I_N}(\underline{p})_C = e^{-4\left(\frac{\underline{p}}{\rho_o}\right)^{5/3} - 2\left[\left(\frac{1}{2\alpha_o}\right)^2 + \left(\frac{k\alpha_o}{4L}\right)^2\right] p^2} \tag{53}$$

In general, the normalized irradiance variance for an incoherent wave is known to be unity. The probability distribution for the irradiance is the exponential distribution, and for the amplitude it fits the Rayleigh distribution.

The covariance scale lengths, for the focussed case (Eq. 51), are $\sim\alpha_o$ and ρ_o, although the present analysis is clearly applicable only when the α_o term dominates ($\rho_o > \alpha_o$). For the collimated case, the dominating scale is $\sim \frac{L}{k\alpha_o}$ ($<\alpha_o$). These terms represent the "speckle" sizes for focussed and collimated beams respectively.

These results may be readily extended to power spectra and receiver aperture-smoothing analyses, which will be given in a future report.

VI. References

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