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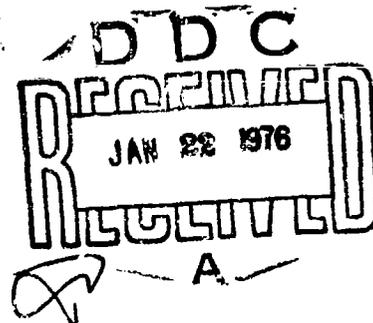
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In this report, we have calculated the degradation in the focal-plane irradiance distribution due to atmospheric turbulence, and the potential improvement realizable by employing a wavefront tilt-correcting aperture. It has been shown that, when the aperture diameter is of the order of the outer scale of turbulence, virtually no improvement is realized relative to the uncompensated case. For the case when the long-term coherence length is small compared with the diameter, there is considerable improvement

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20. ABSTRACT (cont'd)

over the long-term case; however, the focal-point intensity can still be several decibels down from its vacuum value, implying no better than several times diffraction-limited performance. When the coherence length is not much smaller than the diameter, close to diffraction-limited performance can be expected.

Comparisons have also been made of the reduction of on-axis intensity with no compensation, tilt-correction, and a full phase-compensating aperture It is shown that the effective coherence length of the compensated aperture due to the residual amplitude fluctuations is greater than the long-term coherence length by a factor of the square-root of the Fresnel number of the aperture; for high Fresnel number systems, this larger coherence length results in considerable increases in on-axis intensity.

SUMMARY

In this report, we have calculated the degradation in the focal-plane irradiance distribution due to atmospheric turbulence, and the potential improvement realizable by employing a wavefront tilt-correcting aperture. It has been shown that, when the aperture diameter is of the order of the outer scale of turbulence, virtually no improvement is realized relative to the uncompensated case. For the case when the long-term coherence length is small compared with the diameter, there is considerable improvement over the long-term case; however, the focal-point intensity can still be several decibels down from its vacuum value, implying no better than several times diffraction-limited performance. When the coherence length is not much smaller than the diameter, close to diffraction-limited performance can be expected.

Comparisons have also been made of the reduction of on-axis intensity with no compensation, tilt-correction, and a full phase-compensating aperture. It is shown that the effective coherence length of the compensated aperture due to the residual amplitude fluctuations is greater than the long-term coherence length by a factor of the square-root of the Fresnel number of the aperture; for high Fresnel number systems, this larger coherence length results in considerable increases in on-axis intensity.

CONTENTS

1. INTRODUCTION

To overcome the degradation in beam quality of the far-field—or focal-plane—intensity pattern induced by atmospheric turbulence, it is necessary to measure and compensate for the phase distortions generated in the wavefront. The simplest form of aperture compensation is the correction for the linear—or tilt—component of wavefront distortion. This component is identical to the effective angle-of-arrival which would be induced in imaging the target from the transmitter position, and the concept has been demonstrated, in principle, by Kerr, *et al.* [1].

The improvement in beam quality so obtained represents the lowest-order correction of a class of possible higher-order distortions, up to utilizing a complete phase-compensated or "adaptive" aperture [2,3]. Because of its relative simplicity, it is important to quantify and compare the anticipated improvement in beam quality with that of the full adaptive aperture for which, in the presence of turbulence alone (i.e., no thermal blooming), essentially diffraction-limited performance can be expected, at least in principle. In this report, we compute the ensemble-averaged focal-plane pattern for a gaussian beam for both an uncompensated and a tilt-corrected aperture, and the relative on-axis intensities for the uncompensated, tilt-corrected, and fully compensated apertures.

In Sec. 2, we calculate the tilt-corrected phase-structure function and the MTF of the atmosphere-transmitter combination. This MTF is compared with that derived by Fried, and it is shown that Fried's MTF overestimates the atmospheric transmission of high spatial frequencies. In Sec. 3, we examine the effect of this tilt-corrected MTF on the

1

average irradiance pattern in the focal plane; in Sec. 4, we derive
analytic forms for the M. for a gaussian beam propagating through
Kolmogorov-type turbulence. The MTFs are then used to compute the
average focal-plane irradiance as degraded by the turbulence, but
compensated by the tilt-corrected aperture.

Of particular interest is the effect of retaining an outer-scale
dependence in the turbulence spectrum. The outer scale is the approxi-
mate dimension of the refractive-index correlation length in the medium.
Now, tilt or angle-of-arrival fluctuations are produced by refraction
through the central portion of a single eddy; hence, we expect negligible
tilt-components over dimensions larger than the outer scale of turbulence.
The consequence of this effect is that tilt compensation for aperture
dimensions of the order or greater than the outer scale of turbulence
produces insignificant improvement in beam quality. This is particularly
important for applications employing apertures greater than $\approx$20 cm near
the ground, where the outer scale is typically tens of centimeters.

For the usually treated case when the aperture is small compared
with the outer scale of turbulence, we compare the atmospheric MTF so
derived with the approximate "short-term" MTF derived by D. Fried [4].
It is shown that Fried's assumptions result in unrealistically high
values of the atmospheric MTF for high spatial frequencies. However,
when his approximate MTF is multiplied by that of the aperture, the
errors in Fried's approach are of the order 10-percent optimistic.
This results in approximately 10-percent optimistic results in the
focal-plane intensity. Finally, we compute the focal-plane patterns
and the ratio of the on-axis intensity to its vacuum value in terms of

the ratio of the gaussian beam radius w_o to the spherical-wave coherence length ρ_o. The results show that, even with tilt compensation, several-decibel loss can be expected when w_o/ρ_o is much greater than unity, although the beam quality is still better by several decibels than without compensation.

To compare these results with the improvement realizable from a full phase-compensating aperture, in Appendix B we generate an atmospheric MTF due to the turbulence-induced amplitude fluctuations which remain after phase fluctuations have been compensated. It is shown that amplitude-induced MTF has a coherence length larger than ρ_o by a factor equal to the square root of the Fresnel number of the aperture, N_F. Hence, effects due to phase-compensated apertures scale as $w_o/\rho_o\sqrt{N_F}$. For the large Fresnel numbers of interest in high-energy laser applications for producing well focused spots, the residual amplitude effects will be quite small. Specific examples are given in the text for 10.6-μm propagation.

2. THE TILT-CORRECTED PHASE-STRUCTURE FUNCTION AND MTF

The most general expression for the phase corresponding to a tilted wavefront is given by $\underline{a} \cdot \underline{r}$, where $\underline{r}$ is an arbitrary point in the aperture. It can be shown that $2\pi\underline{a}/\lambda = \underline{\theta}$ is the angle that the plane of the wavefront makes with the aperture plane (where λ is the wavelength). The algorithm for choosing the vector $\underline{a}$ is that $\underline{a} \cdot \underline{r}$ represent the "best" linear approximation to the phase over the aperture, and is given in Eq. (9).

With $\underline{a}$ appropriately chosen, the tilt-corrected phase at the aperture point $\underline{r}$, $\phi'(\underline{r})$, is given by

$$\phi'(\underline{r}) = \phi(\underline{r}) - \underline{a} \cdot \underline{r} \quad . \tag{1}$$

The tilt-corrected phase difference between any two points $\underline{r}_1, \underline{r}_2$ in the aperture is then

$$\phi'(\underline{r}_2) - \phi'(\underline{r}_1) = \phi(\underline{r}_2) - \phi(\underline{r}_1) - \underline{a} \cdot \underline{\rho} \quad , \tag{2}$$

where $\underline{\rho} = \underline{r}_2 - \underline{r}_1$, and the tilt-corrected phase-structure function is given by

$$D_{\phi'}(\underline{r}_2, \underline{r}_1) \equiv \left\langle [\phi'(\underline{r}_2) - \phi'(\underline{r}_1)]^2 \right\rangle$$

$$= D_\phi(\rho) + \left\langle (\underline{a} \cdot \underline{\rho})^2 \right\rangle - 2\left\langle \underline{a} \cdot \underline{\rho}[\phi(\underline{r}_2) - \phi(\underline{r}_1)] \right\rangle$$

$$= D_\phi(\rho) - D_t(\underline{\rho}, \underline{R}) \quad . \tag{3}$$

Here,

$$D_\phi(\rho) = \left\langle [\phi(\underline{r}_2) - \phi(\underline{r}_1)]^2 \right\rangle \tag{4}$$

is the long-term phase-structure function, and

$$D_t(\underline{r}_1, \underline{r}_2) = 2\left\langle \underline{a} \cdot \underline{\rho}[\phi(\underline{r}_2) - \phi(\underline{r}_1)] \right\rangle - \left\langle (\underline{a} \cdot \underline{\rho})^2 \right\rangle \tag{5}$$

is the change in the phase-structure function due to tilt correction.[*]

The effect of tilt correction on the (long-term) average is to "improve" the atmospheric MTF from its uncompensated value

$$M_A^{\ell t} = \exp\left[-\frac{1}{2} D_\phi(\underline{r}_1 - \underline{r}_2)\right] \tag{6a}$$

to the tilt-corrected value

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[*]Fried [4] has added and subtracted $2(\underline{a} \cdot \underline{\rho})^2$ to Eq. (3) to write it in the form

$$D_\phi'(\rho) = D_\phi(\rho) - (\underline{a} \cdot \underline{\rho})^2 - 2\left\langle \underline{a} \cdot \underline{\rho}\left\{[\phi(r_2) - \underline{a} \cdot \underline{r}_2] - [\phi(r_1) - \underline{a} \cdot \underline{r}_1]\right\} \right\rangle . \tag{3'}$$

He has then argued that for any given tilt, as measured by $\underline{a}$, the deviations of the phase relative to the tilt, $\phi'(\underline{r}) = \phi(r) - \underline{a} \cdot \underline{r}$, is independent of the tilt. From essentially this statement, the last term in Eq. (3') is set equal to zero. While one can argue about the reality of this assumption, the most direct counterargument follows from actually calculating the quantity D_ϕ' as given in Eq. (3), and comparing the result with that obtained from Fried's assumption. A second powerful counter-argument follows from the demonstration that D_ϕ', as calculated from the first two terms in Eq. (3'), can actually be negative, which is impossible from its definition. Fried has actually calculated a structure function which can be negative, given by the difference of Eqs. (5.3) and (5.8) in Ref. 4:

$$D_\phi'(\rho) = 6.88(\rho/\rho_0)^{5/3} \left[1 - 1.026(\rho/D)^{1/3}\right] \qquad \text{(Fried)} \quad ,$$

but "approximates" it by his Eq. (5.9a) where the term 1.026 is set equal to unity. We have carried out precise numerical calculations, and shown that the 1.026 factor is indeed correct, indicating the inconsistency of the assumption.

$$M_A^{tc} = \exp\left\{-\frac{1}{2}\left[D_\phi(\underline{r}_1-\underline{r}_2) - D_t(\underline{r}_1,\underline{r}_2)\right]\right\} \quad . \tag{6b}$$

For a spherical wave propagating through inhomogeneous (but isotropic) turbulence, the phase-correlation function is given by the expression [5]

$$B_\phi(\rho) = \left\langle\phi(\underline{r}_1)\phi(\underline{r}_2)\right\rangle = 4\pi^2 k^2 \delta \int_0^L ds \int_0^\infty dK \, K\Phi_n(K,s) \, J_0(K\rho s/L) \quad , \tag{7a}$$

and the phase-structure function by

$$D_\phi(\rho) = 8\pi^2 k^2 \delta \int_0^L ds \int_0^\infty dK \, K\Phi_n(K,s)[1-J_0(K\rho s/L)] \quad , \tag{7b}$$

where $\phi_n(K,s)$ is the spectrum of refractive-index fluctuations at the point s along the path, L is the path length, and k ($=2\pi/\lambda$) is the wave number of the light. The factor δ varies from 1 to 1/2 as the range L increases from the near field to the far field of the receiver.

Fried [4] has suggested that the instantaneous tilt vector $\underline{a}$ be chosen to give the best fit to $\phi(\underline{r})$ in the least-square sense over the aperture (at each instant of time). For an aperture weighted by an arbitrary weighting function $W(\underline{r})$, this implies that we choose $\underline{a}$ such that

$$\frac{\partial}{\partial a_i} \int [\phi(\underline{r})-\underline{a}\cdot\underline{r}]^2 W(\underline{r}) \, d^2\underline{r} = 0 \quad . \tag{8}$$

For symmetric weighting functions, it can easily be shown that the
required value of $\underline{a}$ is

$$\underline{a} = \frac{\int \phi(\underline{r})\underline{r} \, W(r) \, d^2\underline{r}}{\pi \int_0^\infty W(r) \, r^3 \, dr} \quad . \tag{9}$$

In Appendix A, we substitute the expression for $\underline{a}$ in Eq. (9) into
Eq. (5), and carry out the indicated averaging over the turbulence
fluctuations. The result is

$$D_t(\underline{r}_1,\underline{r}_2) = 8\pi^2 k^2 \delta \int_0^L ds \int_0^\infty dK \, K\Phi_n(K,s) \left\{ \frac{2\int_0^\infty W(r) J_1\left(\frac{Ksr}{L}\right) r^2 \, dr}{\int_0^\infty W(r) r^3 \, dr} \right.$$

$$\cdot \left[J_1\left(\frac{Ksr_2}{L}\right)[r_2 - r_1 \cos(r_1,r_2)] + J_1\left(\frac{Ksr_1}{L}\right)[r_1 - r_2 \cos(r_2,r_1)] \right]$$

$$\left. - \left[\frac{2\int_0^\infty W(r) J_1\left(\frac{Ksr}{L}\right) r^2 \, dr}{\int_0^\infty W(r) r^3 \, dr} \right]^2 (r_2 - r_1)^2 \right\} \quad . \tag{10}$$

Combining Eq. (10) with the expression for $D_\phi(r_2 - r_1)$ given by Eq.
(7b) and with Eq. (6b) yields the tilt-corrected mutual coherence function
for the field at the aperture points $\underline{r}_1, \underline{r}_2$ from a spherical-wave source.

3. THE EFFECT OF TILT CORRECTION ON THE LONG-TERM AVERAGE IRRADIANCE PATTERN

We now interpret the phase ϕ of the preceding section as $\phi(\underline{r},\underline{p})$—the turbulence-induced phase shift along the path from the transmitter point $(\underline{r},0)$ to the target point at range z, $(\underline{p},z)$. Then, neglecting amplitude fluctuations, the total field at the target point is given by the extended Huygens-Fresnel Principle [6] as

$$U(\underline{p},z) \cong - \frac{ik}{2\pi z} \int U_A(\underline{r}) \, e^{ik(\underline{p}-\underline{r})^2/2z} \, e^{i\phi(\underline{p},r)} \, d^2\underline{r} \quad . \qquad (11)$$

The quantity $U_A(\underline{r})$ is the field in the transmitting aperture, and $k = 2\pi/\lambda$ is the wave number of the radiation. For example, if we assume the aperture field to be a nontruncated gaussian of transmitted power P, e^{-1} intensity radius w_o, and focused at range f, then the aperture field is

$$U_A(\underline{r}) = \sqrt{\frac{P}{\pi w_o^2}} \exp\left[- \frac{r^2}{2} \left(\frac{1}{w_o^2} + \frac{ik}{f} \right) \right] \quad . \qquad (12)$$

The focal-plane or far-field intensity pattern is then given by

$$I(\underline{p},f) = |U(\underline{p},f)|^2 = \frac{P}{\pi w_o^2} N_F^2 \left| \frac{1}{2\pi} \int e^{-r^2/2w_o^2} \, e^{-\frac{ik}{f}\underline{p}\cdot\underline{r}} \, e^{i\phi(\underline{p},\underline{r})} \, d^2\underline{r} \right|^2 \quad ,$$

$$(13)$$

where $\underline{p}$ is the transverse distance of the target point from the optic axis, and $N_F = kw_o^2/f$ is the Fresnel number of the aperture.

In the absence of turbulence, there are no phase fluctuations—$\phi = 0$—and Eq. (13) produces the vacuum intensity in the focal plane:

$$I_v(\underline{p},f) = \frac{P}{\pi w_o^2} N_F^2 \, e^{-p^2/(w_o/N_F)^2} \quad . \tag{14}$$

Let us now assume that the dominant phase distortion induced over the plane of the transmitter $\underline{r}$, from a point source located at the focal point, is a pure tilt of the wavefront, or angle-of-arrival fluctuations. It can then be seen that

$$\phi_t = k(\underline{\theta} \cdot \underline{r}) \quad , \tag{15}$$

where $\underline{\theta}$ is the instantaneous arrival angle. If the arrival angle is independent of the location of the point $\underline{p}$, then substituting Eq. (15) into Eq. (13) yields

$$I(\underline{p},f) = \frac{P}{\pi w_o^2} N_F^2 \, e^{-(\underline{p}-\underline{\theta}f)^2/(w_o/N_F)^2} \equiv I_v(\underline{p}-\underline{\theta}f,f) \quad . \tag{16}$$

Comparing Eq. (16) with Eq. (13), it follows that if 1) amplitude fluctuations are negligible, 2) the phase distortion from a point in the target plane is pure wavefront tilting, and 3) the same tilt angle would be generated by all points in the target spot dimension (i.e., the spot lies in an isoplanatic region), then the degradation takes the form of a diffraction-limited spot that dances with time in the focal plane.

Now, is the above three conditions are met, we can introduce a compensating tilt in the transmitted wavefront by changing the aperture field from $U_A(\underline{r})$ to $U_A(r) \, e^{-ik(\underline{\theta} \cdot \underline{r})}$. The cancellation of the phase terms in Eq. (1) then implies complete compensation for beam dancing in the target plane. To compute the effect of tilt compensation for arbitrary $\phi(\underline{p},\underline{r})$ on the focal plane, or far-field pattern, we write Eq. (13) as a double integral:

$$I(\underline{p},f) = \left(\frac{k}{2\pi f}\right)^2 \iint U_A(\underline{r}_1) U_A^*(\underline{r}_2) \, e^{-\frac{ik}{f} \, \underline{p} \cdot (\underline{r}_1 - \underline{r}_2)}$$

$$\cdot \, e^{i[\phi(\underline{p},\underline{r}_1) - \phi(\underline{p},\underline{r}_2)]} \, d^2\underline{r}_1 \, d^2\underline{r}_2 \quad . \qquad (17)$$

The average turbulence-degraded intensity pattern is found by averaging Eq. (17) and using the relation

$$\left\langle e^{i[\phi(\underline{p},\underline{r}_1) - \phi(\underline{p},\underline{r}_2)]} \right\rangle = M_s(\underline{r}_1 - \underline{r}_2) = \exp\left[-\frac{1}{2} D_\phi(\underline{r}_1 - \underline{r}_2)\right] \quad , \qquad (18)$$

where M_s is the spherical-wave mutual coherence function, and D_ϕ is the phase-structure function. For the long-term average, D_ϕ is a function of the difference coordinate $\underline{\rho} = \underline{r}_1 - \underline{r}_2$; substituting in Eq. (17) and letting $\underline{R} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$ yields for the average intensity

$$\left\langle I(\underline{p},f) \right\rangle = \left(\frac{k}{2\pi f}\right)^2 \int d^2\underline{\rho} \, e^{-\frac{ik}{f} \, \underline{p} \cdot \underline{\rho}} \, M_T(\underline{\rho}) \quad , \qquad (19)$$

where the total long-term MTF is

$$M_T^{\ell t}(\underline{\rho}) = e^{-\frac{1}{2} D_\phi(\rho)} \int U_A\left(\underline{R}+\frac{1}{2}\,\underline{\rho}\right) U_A^*\left(\underline{R}-\frac{1}{2}\,\underline{\rho}\right) e^{\frac{ik}{f}\,\underline{\rho}\cdot\underline{R}}\, d^2\underline{R} \quad .$$

(20a)

Aside from a constant factor, the integral in Eq. (17) may be recognized as the MTF of the aperture distribution; hence, $M_T^{\ell t}$ is the product of the atmospheric MTF and the MTF of the transmitter. For the tilt-compensated aperture, the effective phase distortion at the aperture is reduced from $\phi(\underline{r})$ to $\phi'(\underline{r})$; as shown in the next section, the resulting phase-structure function is a function of $\underline{r}_1$ and $\underline{r}_2$, or both $\underline{\rho}$ and $\underline{R}$. Then, for the tilt-corrected aperture, the average intensity is again given by Eq. (19), but with M_T given by

$$M_T^{tc} = \int U_A\left(\underline{R}+\frac{1}{2}\,\underline{\rho}\right) U_A^*\left(\underline{R}-\frac{1}{2}\,\underline{\rho}\right) e^{\frac{ik}{f}\,\underline{\rho}\cdot\underline{R}}\, e^{-\frac{1}{2} D_{\phi'}(\underline{\rho},\underline{R})}\, d^2\underline{R}$$

$$= e^{-\frac{1}{2} D_\phi(\rho)} \int U_A\left(\underline{R}+\frac{1}{2}\,\underline{\rho}\right) U_A^*\left(\underline{R}-\frac{1}{2}\,\underline{\rho}\right) e^{\frac{ik}{f}\,\underline{\rho}\cdot\underline{R}}\, e^{\frac{1}{2} D_t(\underline{\rho},\underline{R})}\, d^2\underline{R} \quad .$$

(20b)

In contrast with the long-term MTF case, because the tilt-corrected phase-structure function is not a function of the difference coordinate $\underline{\rho}$ alone, the total tilt-corrected MTF cannot be written as the product of an atmospheric MTF and an aperture MTF. However, the total distortion can still be calculated from Eq. (20b).

4. THE GAUSSIAN BEAM AND KOLMOGOROV TURBULENCE

4.1 The Tilt-Corrected Structure Function

For the gaussian beam of Eq. (.2), the weighting function over the aperture (aside from a normalization factor) is

$$W(r) = |U_A(\underline{r})|^2 = e^{-r^2/w_o^2} \quad . \tag{21}$$

Substituting in Eq. (10) and letting $x = \frac{sa}{2L} K$ yields for the tilt-correction to the structure function

$$D_t(\underline{p},\underline{r}) = 8\pi^2 k^2 \delta \int_0^\infty ds \left(\frac{2L}{sw_o}\right)^2 \int_0^\infty dx \ x\phi_n\left(\frac{2L}{sa} x,s\right) H(x,\underline{p},\underline{R}) \quad , \tag{22}$$

where

$$H(x,\underline{p},\underline{R}) = 2f(x) \ G(x,\underline{p},\underline{R}) - f^2(x) \ (\rho/w_o)^2 \quad . \tag{23}$$

Here,

$$f(x) = \frac{\int_0^\infty W(y) \ J_1(2xy)y^2 \ dy}{W(y)y^3 \ dy} = x \ e^{-x^2} \tag{24}$$

for $W(y) = e^{-y^2}$, and (noting that $\underline{r}_1 = \underline{R} + \underline{\rho}/2$, $\underline{r}_2 = \underline{R} - \underline{\rho}/2$)

$$G = \frac{J_1(2xr_1/w_o)}{(r_1/w_o)} \left(\frac{r_1^2 - \underline{r}_1 \cdot \underline{r}_2}{w_o^2} \right) + \frac{J_1(2xr_2/w_o)}{(r_2/w_o)} \left(\frac{r_2^2 - \underline{r}_2 \cdot \underline{r}_1}{w_o^2} \right) \quad , \qquad (25)$$

where J_1 is the Bessel function of order 1. We have shown numerically that the Bessel function may be approximated by the first two terms in the series, yielding[*]

$$H \simeq (\rho/w_o)^2 [2xf(x) - f^2(x)] - \left[\frac{\rho^2}{4w_o^2} + \frac{R^2}{w_o^2} (1 + 2\cos^2\psi) \right] x^3 f(x) \quad , \qquad (26)$$

where ψ is the angle between $\underline{\rho}$ and $\underline{R}$, and $f(x)$ is given by Eq. (24).

For a Kolmogorov turbulence spectrum, we let

$$\Phi_n(K,s) = \frac{0.033c_n^2}{\left[K^2 + K_o^2 \right]^{11/6}} \quad , \qquad (27)$$

where C_n is the index structure constant, and $K_o = L_o^{-1}$ is the spatial frequency corresponding to the outer scale L_o. Allowing both C_n and L_o to vary along the propagation path and substituting Eqs. (27) and (2?) into Eq. (22) yields

$$D_t(\underline{\rho},\underline{r}) = 0.82k^2 \delta w_o^{5/3} (\rho/w_o)^2 \int_0^\infty C_n^2(s)(s/L)^{5/3} B(s) \quad , \qquad (28)$$

where

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[*]Under Fried's assumption, H would be given by $H_F = f^2(x)(\rho/w_o)^2$.

$$B(s) = A_1(s) - A_2(s) \left[\frac{\rho^2}{4w_0^2} + \frac{R^2}{w_0^2} (1+2\cos^2\psi) \right] \qquad . \qquad (29)$$

Here,

$$A_1(s) = \frac{1}{2} \int_0^\infty \frac{y\, e^{-y}\, (2-e^{-y})}{[y+y_0(s)]^{11/6}}\, dy = e^{y_0} \left[\Gamma\left(\frac{1}{6}, y_0\right) - y_0 \Gamma\left(-\frac{5}{6}, y_0\right) \right]$$

$$- 2^{-7/6}\, e^{2y_0} \left[\Gamma\left(\frac{1}{6}, 2y_0\right) - 2y_0 \Gamma\left(-\frac{5}{6}, 2y_0\right) \right] \qquad (30a)$$

and

$$A_2(s) = \frac{1}{2} \int_0^\infty \frac{y^2\, e^{-y}}{[y+y_0(s)]^{11/6}}\, dy$$

$$= \frac{1}{2}\, e^{y_0} \left[\Gamma\left(\frac{7}{6}, y_0\right) - 2y_c \Gamma\left(\frac{1}{6}, y_0\right) + y_0^2 \Gamma\left(-\frac{5}{6}, y_0\right) \right] \qquad , \qquad (30b)$$

where

$$y_0 = y_0(s) = \left(\frac{w_0 s}{2L_0 L} \right)^2 \qquad (31)$$

and

$$\Gamma(x,y) = \int_x^\infty e^{-t}\, t^{y-1}\, dt$$

is the incomplete gamma function.

From Eqs. (28) through (31), we see that D_t takes the form

$$\frac{1}{2} D_t(\underline{\rho},r) = (\rho/w_o)^2 \left\{ B_1 - B_2\left[\frac{\rho^2}{4w_o^2} + \frac{r^2}{w_o^2}(1+2\cos^2\psi)\right]\right\} \quad , \tag{32}$$

where

$$B_{1,2} = 0.41k^2\delta w_o^{5/3} \int_0^L C_n^2(s)(s/L)^{5/3} A_{1,2}(s) \, ds \quad . \tag{33}$$

4.2   The Tilt-Corrected MTF

We now compute the overlap integral of Eq. (20), using Eq. (32) for D_t and assuming a gaussian beam. Thus,

$$M_T^{tc} = e^{-\frac{1}{2}D_\phi(\rho)} \int U_A\left(\underline{R}+\frac{1}{2}\underline{\rho}\right) U_A^*\left(\underline{R}-\frac{1}{2}\underline{\rho}\right) e^{\frac{ik}{f}\underline{\rho}\cdot\underline{R}} e^{\frac{1}{2}D_t(\underline{\rho},\underline{R})} d^2\underline{R}$$

$$= e^{-\frac{1}{2}D_\phi(\rho)} \int e^{-\rho^2/4w_o^2} \; e^{-r^2/w_o^2} \exp\left\{\frac{B_1\rho^2}{w_o^2} - \frac{B_2\rho^2}{w_o^2}\left[\frac{\rho^2}{4w_o^2}+\frac{r^2}{w_o^2}(1+2\cos^2\psi)\right]\right\} d^2\underline{r}$$

$$= e^{-\frac{1}{2}D_\phi(\rho)} \pi w_o^2 \, e^{-\rho^2/w_o^2} \, e^{B_1\rho^2/2w_o^2} \, e^{-B_2\rho^4/4w_o^4}$$

$$\cdot \int_0^\infty I_o\left(\frac{B_2\rho^2 x}{w_o^2}\right) e^{-x(1+2B_2\rho^2/w_o^2)} x \, dx$$

$$= e^{-\frac{1}{2}D_\phi(\rho)} \pi w_o^2 \, e^{-\rho^2/w_o^2} \, e^{B_1\rho^2/w_o^2} \, e^{-B_2\rho^4/4w_o^4} \left[\left(1+2B_2\frac{\rho^2}{w_o^2}\right)^2 - \left(\frac{B_2\rho^2}{w_o^2}\right)^2\right]^{-1/2} \quad .$$

$$\tag{34}$$

Note that the term $e^{-\frac{1}{2}D_\phi(\rho)} \pi w_o^2 e^{-\rho^2/w_o^2}$ is the result that would have been obtained for the overlap integral $M_T^{\ell t}(\rho)$ of Eq. (20a). Hence, we obtain for the effective tilt-corrected atmospheric MTF:

$$M_A^{tc} = e^{-\frac{1}{2}D_\phi(\rho)} \; e^{B_1\rho^2/w_o^2} \; e^{-B_2\rho^4/4w_o^4} \left[1+4B_2\left(\frac{\rho}{w_o}\right)^2+3B_2^2\left(\frac{\rho}{w_o}\right)^4\right]^{-1/2} \quad , \quad (35)$$

where $B_{1,2}$ is given in terms of $A_{1,2}$ by Eq. (34).

4.3 Approximate Solutions

$w_o \ll L_o$

We first examine the frequently considered case where the aperture dimension is small compared with the outer scale of turbulence. In this case, the long-term structure function is given by

$$D_\phi(\rho) = 2.91k^2\delta\rho^{5/3} \int_0^R C_n^2(s)(s/L)^{5/3} \, ds \equiv 2(\rho/\rho_o)^{5/3} \quad , \quad (36)$$

where ρ_o is the e^{-1} point of the long-term MTF. Then, for $w_o/L_o \to 0$, Eq. (30) yields $A_1 \simeq \Gamma(1/6)(1-2^{-7/6}) = 3.08$ and $A_2 \simeq \frac{1}{2}\Gamma(7/6) = 0.46$; from Eq. (33), $B_{1,2} \to 0.28(w_o/\rho_o)^{5/3} A_{1,2}$ and $B_{1,2} = (0.86, 0.13)(w_o/\rho_o)^{5/3}$. Substituting in Eq. (35) then yields

$$M_A^{tc} = \left[1+0.52\left(\frac{w_o}{\rho_o}\right)^{5/3} - \left(\frac{\rho}{w_o}\right)^2 + 0.05\left(\frac{w_o}{\rho_o}\right)^{10/3}\left(\frac{\rho}{w_o}\right)^4\right]^{-1/2}$$

$$\cdot \exp\left\{-(\rho/\rho_o)^{5/3}\left[1 - 0.86(\rho/w_o)^{1/3} + 0.13(\rho/w_o)^{7/3}\right]\right\} \quad , \quad (37)$$

and the total tilt-corrected MTF of the atmosphere and lens combination is given by

$$M_T^{tc} = e^{-\rho^2/w_o^2} M_A^{tc} \quad . \tag{38}$$

For comparison, the long-term MTF is given by

$$M_T^{\ell t} = e^{-\rho^2/w_o^2} e^{-(\rho/\rho_o)^{5/3}} \quad . \tag{39}$$

From Eq. (39), we obtain the usual result that turbulence only degrades the beam quality when $\rho_o \lesssim w_o$; i.e., when the coherence length is less than the aperture dimension. In Fig. 1, we plot the tilt-corrected MTF given by Eq. (37) for various values of the parameter (ρ_o/w_o).

To compare these results with those that would be obtained using Fried's assumption, we use the fact that Fried would have obtained Eq. (22) with $H(x,\underline{\rho},R)$ replaced by $H_F = f^2(s)(\rho/w_o)^2$. Then, Eqs. (28) and (29) remain valid with $A_2^F = 0$, and

$$A_1^F = \frac{1}{2} \int_0^\infty \frac{y \, e^{-2y}}{(y+y_o)^{11/6}} \, dy \xrightarrow[y_o \to 0]{} \frac{1}{2^{7/6}} \Gamma(1.6) = 2.48 \quad .$$

Then, $B_1^F = 0.28(w_o/\rho_o)^{5/3} A_1^F = 0.70(w_o/\rho_o)^{5/3}$ and, hence, Fried's atmospheric MTF becomes

$$M_A^F = \exp\left\{ -(\rho/\rho_o)^{5/3} [1-0.70(\rho/w_o)^{1/3}] \right\} \tag{40}$$
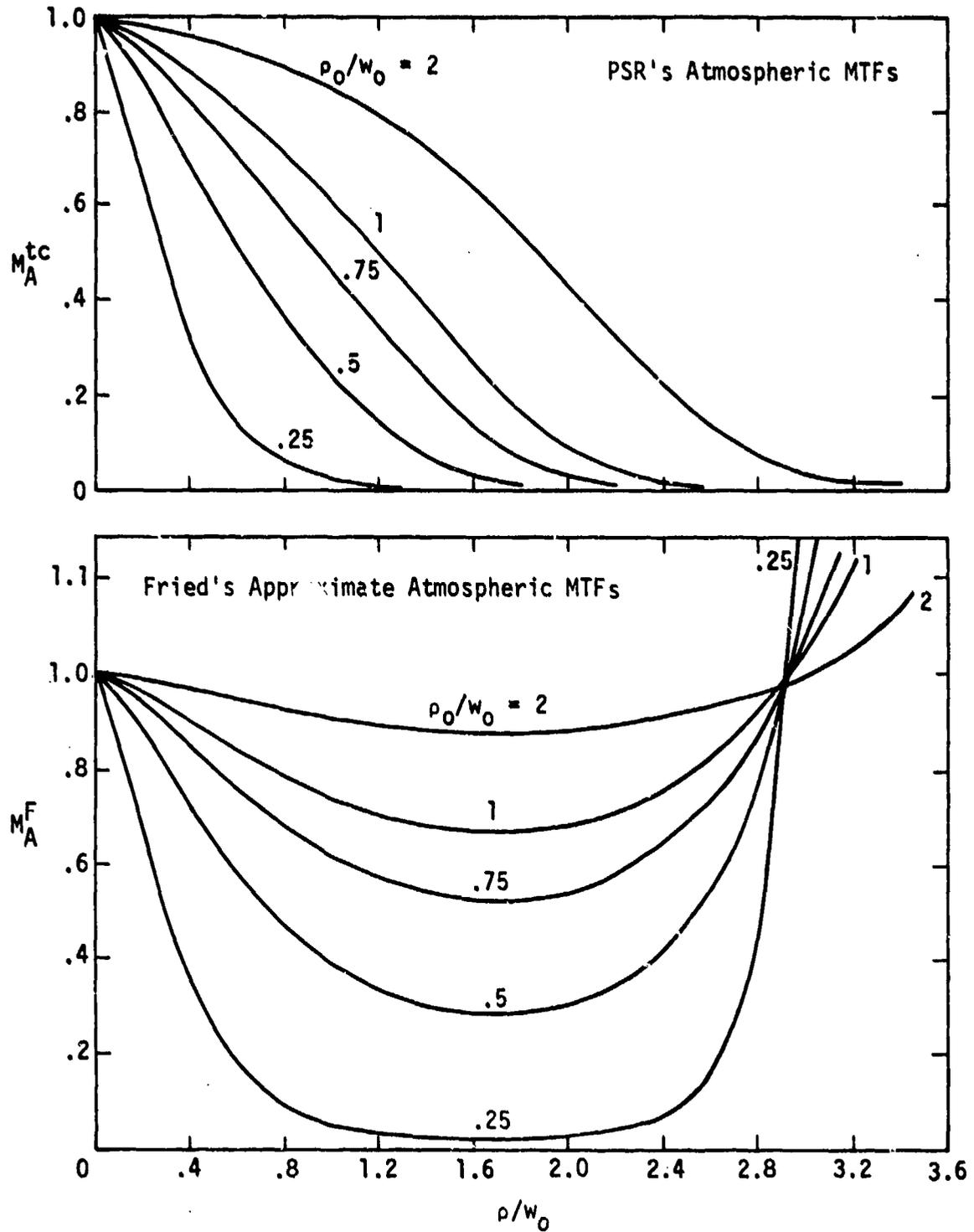
Transverse dimension normalized to gaussian radius

Fig. 1--Tilt-corrected atmospheric MTFs

and Fried's total MTF becomes

$$M_T^F = e^{-(\rho/w_o)^2} M_A^F = \exp\left[-\left(\frac{\rho}{w_o}\right)^2 - \left(\frac{\rho}{\rho_o}\right)^{5/3} + \frac{\rho^2}{w_o^{5/3}\rho_o^{1/3}} \right] \qquad . \qquad (41)$$

In general, while the tilt-corrected atmospheric MTFs predicted by our more precise theory are very different from those generated with the Fried approximation (as indicated in Fig. 1), when we form the composite MTF of the aperture atmosphere system, which is the only measurable quantity, the results are not greatly divergent. Generally, Fried's assumptions produce considerable errors for the higher spatial frequencies of the atmospheric MTF, in the region where the MTF of the aperture is small, which reduces the effective error. As a result, the total tilt-corrected MTFs--plotted in Fig. 2--differ primarily in the region for which the transverse dimension ρ is greater than the gaussian radius of the aperture, w_o.

We compute the focal-plane irradiance patterns for the respective cases by taking the Fourier transform of the total MTF. From Eq. (19), for M_T independent of the direction of $\underline{\rho}$, we obtain

$$\frac{\langle I(\alpha,f) \rangle}{I_{vac}(0,f)} = 2 \int_0^\infty M_T(x w_o/\rho_o) \, J_o(2\alpha x) x \, dx \qquad (42)$$

for the average intensity normalized to the on-axis vacuum pattern. The parameter $\alpha = k\rho w_o/2f = \pi p w_o/\lambda f$ is the ratio of the angle subtended by the point p at the aperture to the vacuum diffraction angle.
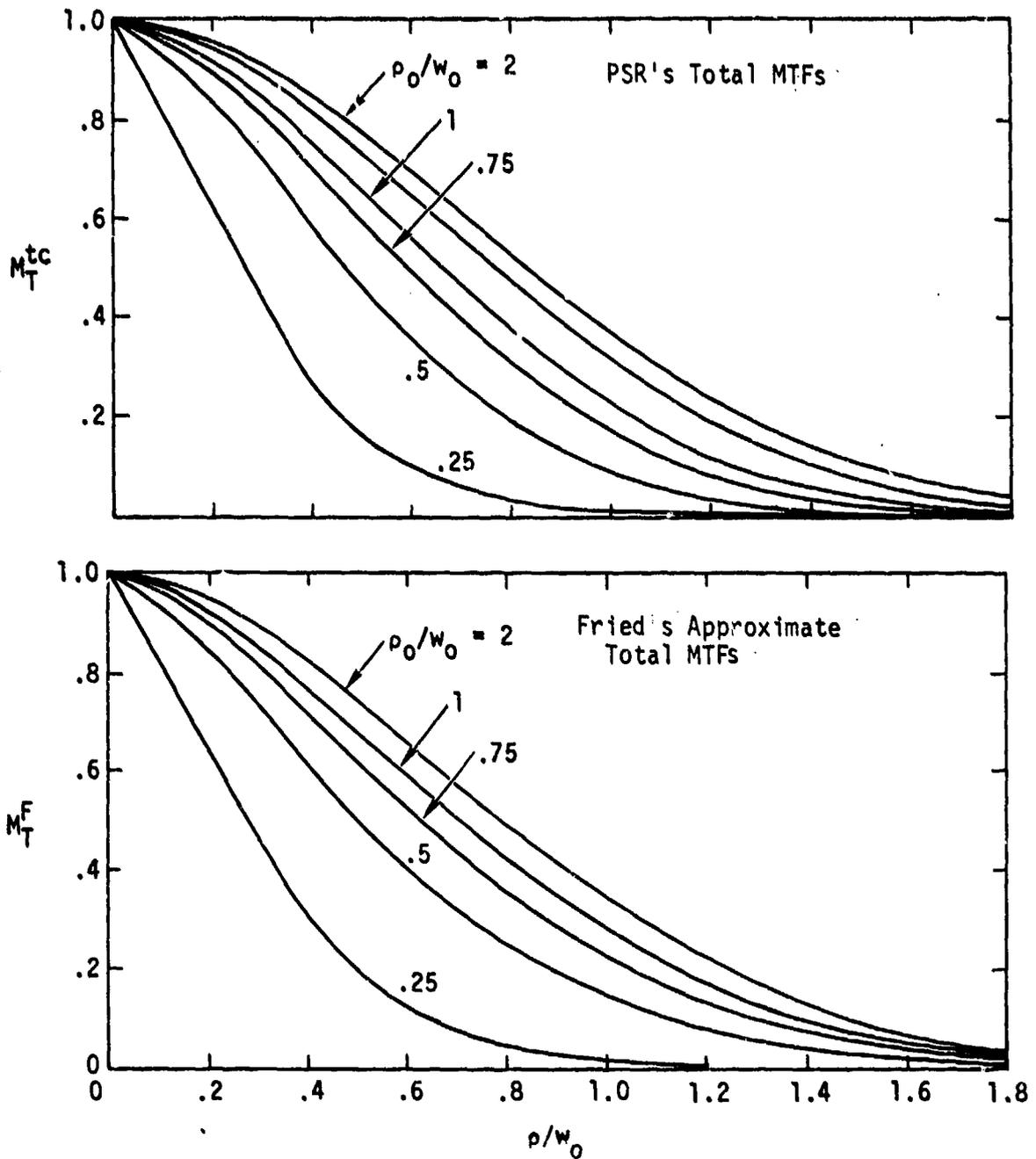
Transverse dimension normalized to gaussian radius

Fig. 2--Tilt-corrected total MTFs

In Fig. 3, we plot the irradiance pattern of Eq. (42) using the total tilt-corrected MTF and comparing the result with Fried's. The numerical results yield an approximately 10-percent difference in the on-axis intensities, with Fried's being optimistic due to the higher spatial frequency content of his MTF. In Fig. 4, we plot the reduction in focal-plane intensity relative to the vacuum value versus w_o/ρ_o-- the ratio of the beam radius to the coherence length ρ_o. From Eq. (42), this reduction is given by

$$\frac{\langle I(0,f)\rangle}{I_{vac}(0,f)} = 2 \int_0^\infty M_T(x,w_o/\rho_J)x \, dx \quad . \tag{43}$$

To estimate the improvement anticipated with a full compensating aperture, we derive, in Appendix B, an approximate MTF which contains the effects of the amplitude-induced distortion remaining after phase compensation has occurred. It is shown that, in contrast with the phase result where turbulence degrades the pattern for $\rho_o \lesssim w_o$, amplitude effects appear when $\rho_o \gtrsim w_o/2\sqrt{N_F}$, where N_F is the Fresnel number of the lens. For the large Fresnel numbers of interest in high-power applications, the amplitude effects are indeed much smaller than the phase effects, indicating considerable increase in on-axis intensity with a full compensating aperture. For illustrative purposes, we have chosen a small Fresnel number of 4.75, corresponding to $w_o = 20$ cm, $\lambda = 10.6$ μm, and $f = 5$ km. The results for all of the cases are shown in Fig. 4, indicating the increase in on-axis intensity for the various configurations. It is seen that the Fried solution considerably overestimates the improvement.

$$2 \int_0^\infty M(x, w_0/\rho_0) \, J_0(2\alpha x) \, x dx$$
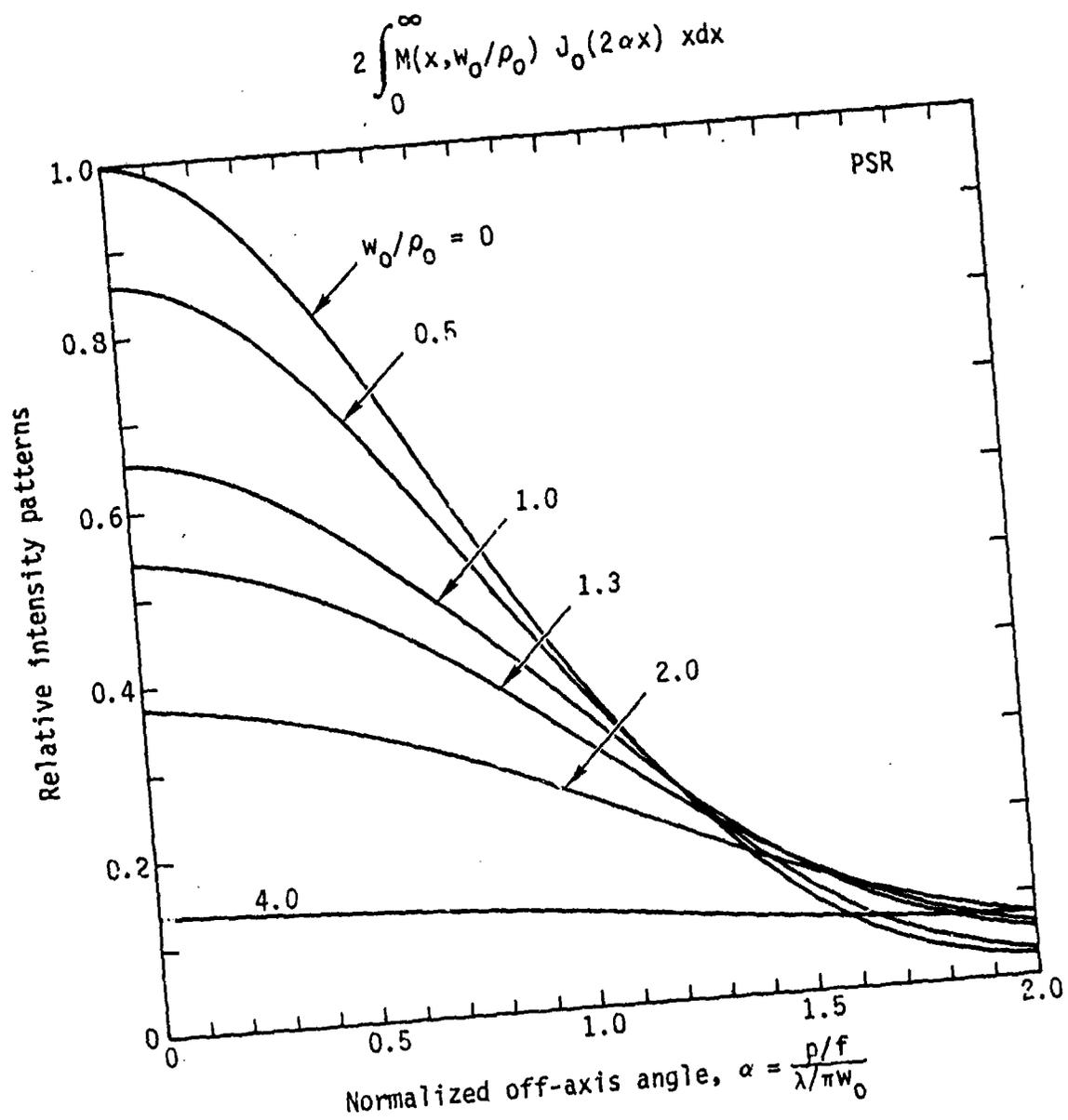
Fig. 3a--Relative intensity patterns for tilt-corrected apertures (PSR)

$$2 \int_0^\infty M(x, w_0/\rho_0) \, J_0(2\alpha x) \, x \, dx$$
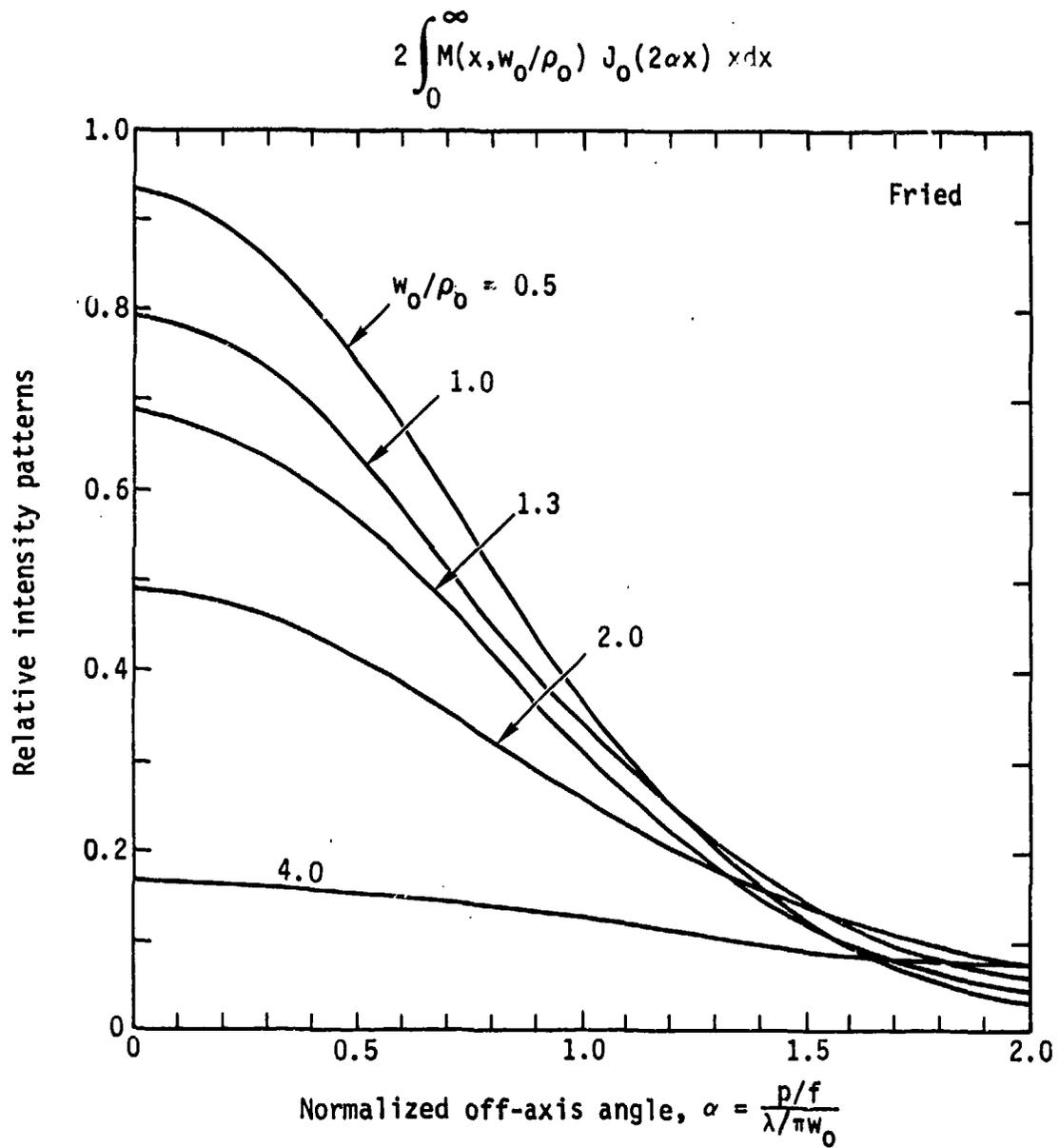
Fig. 3b--Relative intensity patterns for tilt-corrected apertures (Fried)

$$2 \int_0^\infty M(x, w_0/\rho_0) \, x \, dx$$

phase compensated
($N_F = 4.75$)

tilt corrected (Fried)

tilt corrected (PSR)

long term

On-axis intensity relative to vacuum

Ratio of gaussian beam radius to coherence length, w_0/ρ_0
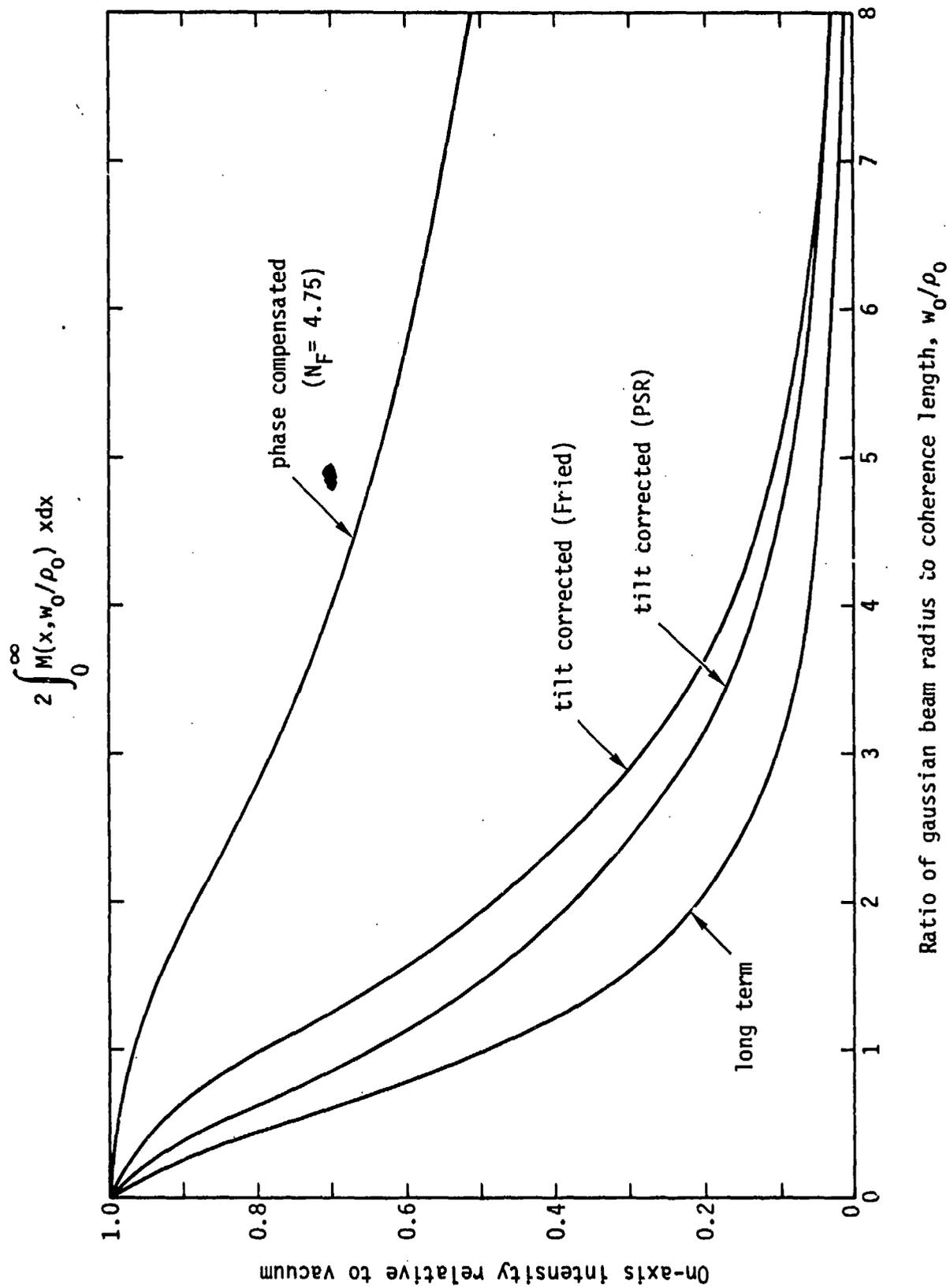
Fig. 4--On-axis intensities relative to vacuum for uncompensated, tilt-corrected, and full-compensated apertures

$\underline{w_o \gg L_o}$

For the case where the outer scale of turbulence is large compared with the aperture dimension, the long-term phase-structure function can be well approximated by

$$D_\phi(\rho) \simeq 2\langle\phi^2\rangle \left[1 - e^{-\rho^2/L_o^2}\right] \tag{44}$$

where, for uniform turbulence,

$$\langle\phi^2\rangle = 0.39 k^2 C_n^2 L_o^{5/3} \Delta L$$

is the phase variance at a point. We then evaluate the functions $A_{1,2}$ of Eq. (30) using the asymptotic expansion

$$\Gamma(a, y_o) \simeq y^{a-1} e^{-y_o} \qquad , \qquad y_o \gg 1 \qquad .$$

From Eq. (30), we obtain $A_1 \simeq (7/8) y_o^{-11/6}$, $A_2 \simeq y_o^{-11/6}$.

The condition $y_o \gg 1$ follows from the assumption $w_o \gg L_o$, assuming that the turbulence is contained not closer than a distance L from the focal plane. where $L_1 \gtrsim LL_o/w_o$. Then, from Eq. (33),

$$B_{1,2} = (0.36, 0.41) k^2 \delta w_o^{5/3} \left(\frac{2L_o}{w_o}\right)^{11/3} \int_{L_1}^{L} C_n^2(s)(L/s)^2 \, ds \tag{45}$$

and, for C_n constant between L_1 and L,

$$B_{1,2} \simeq (0.36, 0.41) k^2 \delta w_o^{5/3} \left(\frac{2L_o}{w_o}\right)^{11/3} C_n^2 (L - L_1) \quad , \quad L_1 \simeq L$$

$$\simeq (11.7, 13.4) \delta \left\langle \phi^2 \right\rangle (L_o/w_o)^2 \quad .$$

Substituting in Eq. (35) then yields

$$M_A^{tc} = \exp\left\{-\phi_o^2 \left[1 - e^{-(\rho/L_o)^{5/3}} - 11.7\delta\left(\frac{L_o}{w_o}\right)^2 \left(\frac{\rho}{w_o}\right)^2 + 1 - 1.1\left(\frac{\rho^2}{w_o^2}\right)\right]\right\}$$

$$\cdot \left[1 + 53.6\delta \left\langle \phi^2 \right\rangle \frac{L_o^2}{w_o^2} + 53.9\delta^2 \left\langle \phi^2 \right\rangle^2 (L_o/w_o)^4\right]^{-1/2} \quad . \tag{46}$$

Hence, we obtain

$$M_A^{tc} \simeq \exp\left\{-\phi_o^2 \left[1 - e^{-(\rho/L_o)^2}\right]\right\} \simeq M_A^{\ell t} \quad , \quad L_o \gg w_o \quad , \tag{47}$$

and the tilt-corrected MTF becomes identical with the long-term MTF—

i.e., we realize negligible improvement in tilt compensation when the

aperture is greater than the outer scale. This result is anticipated

on physical grounds, as we expect the higher-order phase distortions

to dominate the linear-phase fluctuations for apertures of these

dimensions, implying little improvement when they are removed.

Appendix A

CALCULATION OF THE TILT-CORRECTED STRUCTURE FUNCTION

In this appendix, we evaluate the second and third terms of Eq. (3) (p. 4) for an arbitrary symmetric aperture weighting function $W(r)$.

Evaluation of $\left\langle (\underline{a} \cdot \underline{\rho})^2 \right\rangle$

The vector $\underline{a}$ defined by Eq. (9) of the main text lies in the aperture plane, and for isotropic turbulence will be uniformly distributed in angle over the plane of the aperture. For isotropic turbulence, the expected value $\left\langle \cos^2(\underline{a},\underline{\rho}) \right\rangle = 1/2$, where $(\underline{a},\underline{\rho})$ denotes the angle between $\underline{a}$ and $\underline{\rho}$, and we obtain[*]

$$\left\langle (\underline{a} \cdot \underline{\rho})^2 \right\rangle = \frac{1}{2} \left\langle a^2 \right\rangle \rho^2 \quad . \tag{A-1}$$

We write Eq. (9) as a double integral, which yields

$$\left\langle a^2 \right\rangle = \left\langle \underline{a} \cdot \underline{a} \right\rangle = A^2 \iint d^2\underline{r}_1 \, d^2\underline{r}_2 \, W(\underline{r}_1)W(\underline{r}_2)(\underline{r}_1 \cdot \underline{r}_2) \left\langle \phi(\underline{r}_1)\phi(\underline{r}_2) \right\rangle \quad . \tag{A-2}$$

where

$$A = \left[\pi \int_0^\infty W(r)r^3 \, dr \right]^{-1} \quad .$$

The term $\left\langle \phi(\underline{r}_1)\phi(\underline{r}_2) \right\rangle$ in Eq. (A-2) is the correlation function, $B_\phi(\underline{r}_1 - \underline{r}_2)$, between the phases at the points $\underline{r}_1$ and $\underline{r}_2$. Substituting

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[*]The same result can be derived directly from the definition of $\underline{a}$ given by Eq. (9).

the expression for a spherical wave source propagating through inhomogeneous turbulence--Eq. (7a)--and inverting the order of integration, we obtain

$$\langle a^2 \rangle = 4\pi^2 k^2 \delta A^2 \int_0^L ds \int_0^\infty dK\, K\phi_n(K,s) \iint d^2\underline{r}_1\, d^2\underline{r}_2\, W(\underline{r}_1)W(\underline{r}_2)(\underline{r}_1 \cdot \underline{r}_2)$$

$$\cdot J_0\left(\frac{Ks}{L}\, |\underline{r}_1 - \underline{r}_2|\right) \quad , \tag{A-3}$$

into which we substitute the identity

$$J_0\left(\frac{Ks}{L}\, |\underline{r}_1 - \underline{r}_2|\right) = \sum \epsilon_n \cos[n(\underline{r}_1,\underline{r}_2)]\, J_n\left(\frac{Ks}{L}\, r_1\right) J_n\left(\frac{Ks}{L}\, r_2\right) \quad , \tag{A-4}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$, $n = 1, 2, \ldots$. The term $\underline{r}_1 \cdot \underline{r}_2$ contains the term $\cos(\underline{r}_1, \underline{r}_2)$ and, because of the angular orthogonality, the only nonvanishing contribution comes from $n = 1$. Thus, we obtain

$$\langle a^2 \rangle = 4\pi^2 k^2 \delta \epsilon_1 A \int_0^L ds \int_0^\infty dK\, K\phi_n(K,s) \iint d^2\underline{r}_1\, d^2\underline{r}_2\, W(r_1)W(r_2)r_1 r_2$$

$$\cdot \cos^2(\underline{r}_1,\underline{r}_2)\, J_1\left(\frac{Ks}{L}\, r_1\right) J_1\left(\frac{Ks}{L}\, r_2\right) \quad . \tag{A-5}$$

Expanding $\cos^2(r_1,r_2)$ in terms of the double-angle formula, only the "1/2" remains after integration, and we obtain

$$\frac{1}{2}\left\langle a^2 \right\rangle \rho^2 = 2\pi^2 k^2 \delta\rho^2 \int_0^\infty ds \int_0^\infty dK \, K\phi_n(K,s)$$

$$\cdot \left[2\pi A \int_0^\infty W(r) \, J_1\!\left(\frac{Ks}{L}\, r\right) r^2 \, dr \right]^2 \tag{A-6}$$

for symmetric $W(r)$.

Evaluation of $\psi = 2\left\langle \underline{a}\cdot\underline{\rho}[\phi(\underline{r}_2)-\phi(\underline{r}_1)]\right\rangle

With $\underline{a}$ defined by

$$\underline{a} = A\int W(r)\phi(\underline{r})\underline{r}\, d^2\underline{r} \quad , \tag{A-7}$$

we have

$$\psi = 2A\int d^2\underline{r}\; W(r)\left[(\underline{r}\cdot\underline{r}_2)-(\underline{r}\cdot\underline{r}_1)\right]\left\langle\phi(\underline{r})\phi(\underline{r}_2)-\phi(\underline{r})\phi(\underline{r}_1)\right\rangle$$

$$= 2A\int d^2\underline{r}\; W(r)\left[(\underline{r}\cdot\underline{r}_2)B_\phi(\underline{r}-\underline{r}_2) + (\underline{r}\cdot\underline{r}_1)B_\phi(\underline{r}-\underline{r}_1)\right.$$

$$\left. - (\underline{r}\cdot\underline{r}_1)B_\phi(\underline{r}-\underline{r}_2) - (\underline{r}\cdot\underline{r}_2)B_\phi(\underline{r}-\underline{r}_1)\right] \quad . \tag{A-8}$$

Substituting Eq. (7a) for B_ϕ into Eq. (A-8) and interchanging the order of integration, we obtain

$$\psi = 8\pi^2 k^2 \delta \int_0^L ds \int_0^\infty dK\; K\phi_n(K,s)\left[\Gamma(\underline{r}_1,\underline{r}_1) + \Gamma(\underline{r}_2,\underline{r}_2) - \Gamma(\underline{r}_1,\underline{r}_2) - \Gamma(\underline{r}_2,\underline{r}_1)\right] \quad ,$$

$$\tag{A-9}$$

where

$$\Gamma(\underline{r}_i, \underline{r}_j) = Ar_i \int_0^\infty dr \ r^2 \ W(r) \int_0^{2\pi} d\theta \ \cos(\underline{r}, \underline{r}_i) \ J_o\left(\frac{Ks}{L}\ |\underline{r} - \underline{r}_j|\right) \ . \qquad (A-10)$$

If we substitute Eq. (A-4) in Eq. (A-10) and expand the cosine term as $\cos[(\underline{r}, r_j) - (\underline{r}_i, r_j)]$, we obtain

$$\Gamma(\underline{r}_i, \underline{r}_j) = 2\pi Ar_i \cos(r_i, r_j) \ J_1\left(\frac{Ks}{L}\ r_j\right) \int_0^\infty W(r) \ J_1\left(\frac{Ksr}{L}\right) r^2 \ dr \ . \qquad (A-11)$$

Collecting terms, we thus have

$$\psi = 8\pi^2 k^2 \delta \int_0^L ds \int_0^\infty dK \ K\Phi_n(K,s) \cdot 2\pi A \int_0^\infty W(r) \ J_1\left(\frac{Ksr}{L}\right) r^2 \ dr$$

$$\cdot \left\{ J_1\left(\frac{Ksr_2}{L}\right)[r_2 - r_1 \ \cos(\underline{r}_1, \underline{r}_2)] + J_1\left(-\frac{Ksr_1}{L}\right)[r_1 - r_2 \ \cos(\underline{r}_1, \underline{r}_2)] \right\} \ . \qquad (A-12)$$

Appendix B

THE PHASE-COMPENSATED MTF

With the compensation of phase fluctuations, the MTF is given by the amplitude term

$$M_\chi = e^{-\frac{1}{2}D_\chi} = e^{-\frac{1}{2}\left\langle [\chi(\underline{r}_2)-\chi(\underline{r}_1)]^2\right\rangle} = e^{-\left\langle\chi^2\right\rangle[1-B_\chi(\rho)]} \quad , \quad \text{(B-1)}$$

where $\left\langle\chi^2(\underline{r}_1)\right\rangle = \left\langle\chi^2(\underline{r}_2)\right\rangle = \left\langle\chi^2\right\rangle$ is the variance of log amplitude, and B_χ is the amplitude conduction function. Now, it is known that for $\left\langle\chi^2\right\rangle << 1.$

$$\left\langle\chi^2\right\rangle \approx \left\langle\chi_1^2\right\rangle = 0.13k^{7/6}C_n^2 z^{11/6} \simeq 0.25\left(z/k\rho_o^2\right)^{5/6}$$

$$= 0.25\left(z/kw_o^2\right)^{5/6}(w_o/\rho_o)^{5/3} = \frac{(w_o/\rho_o)^{5/3}}{4N_F^{5/6}} \quad , \quad \left\langle\chi^2\right\rangle << 1 . \quad \text{(B-2)}$$

where

$$\rho_o = \left(0.5k^2 C_n^2 z\right)^{-3/5} \quad\quad\quad \text{(B-3)}$$

is the spherical-wave coherence length, and $N_F = kw_o^2/z$ is the Fresnel number of the lens. For $\left\langle\chi^2\right\rangle \gtrsim 1$, the variance of log amplitude saturates to a value of order unity; hence, we have, as an approximate formula,

$$\left\langle \chi^2 \right\rangle \propto \frac{\left\langle \chi_1^2 \right\rangle}{1 + \left\langle \chi_1^2 \right\rangle} = \frac{1}{1 + \frac{4N_F^{5/6}}{(w_o/\rho_o)^{5/3}}} \qquad . \qquad (B-4)$$

From Eq. (B-4), it follows that $\left\langle \chi^2 \right\rangle$ becomes appreciable when

$$\rho_o \lesssim \frac{w_o}{2\sqrt{N_F}} \qquad .$$

In contrast with the phase effects, which become important when $\rho_o \lesssim w_o$, much stronger turbulence is requried before amplitude effects appear.

The MTF will only differ appreciably from unity in the saturation regime, when $\left\langle \chi_1^2 \right\rangle \gtrsim 1$. In this regime, it has been shown [7] that the correlation length of the amplitude fluctuations is ρ_o itself, and we approximate the correlation function by $\rho_o^2/(\rho^2 + \rho_o^2)$. Hence, an approximate MTF which retains the essential dependence on the parameters of the problem is

$$M_\chi = \exp\left\{-\frac{1}{1 + \frac{4N_F^{5/6}}{(w_o/\rho_o)^{5/3}}} \frac{\rho^2}{\rho^2 + \rho_o^2}\right\}$$

$$= \exp\left\{-\frac{1}{1 + \frac{4N_F^{5/6}}{(w_o/\rho_o)^{5/3}}} \frac{[(w_o/\rho_o)x]^2}{1 + [(w_o/\rho_o)x]^2}\right\} \qquad , \qquad (B-5)$$

where $x = \rho/w_o$.

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