Asymptotic Properties of the Mie Coefficients

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Analytic expressions for the asymptotic forms of the Mie coefficients are derived. The asymptotic regions are defined as those where one of the dimensionless parameters x or n is much larger than the other, x being the Mie size parameter and n the coefficient index. The accuracy and range of applicability of the results is investigated and some important special cases are treated in detail. The utility of the results in numerical computations is also discussed.

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1. INTRODUCTION

In recent times the problem of electromagnetic radiation scattering by aerosols has become quite important because of the widespread use of optical wavelength sources in various applications. An extensive theory has been developed over the years for the analysis of this phenomenon and it has been applied and discussed in a number of papers and texts. The core of this theory is the exact solution to the problem of scattering of a plane wave of arbitrary wavelength by a single spherical particle of arbitrary size and index of refraction. This solution was first published by Mie[1] in 1908 and has been reworked in several modern treatments.[2]

Since the details of the scattering problem are not of major interest here, it is sufficient to state that Mie's analysis shows that the field far from the sphere can be completely described in terms of two "amplitude functions," S_1 and S_2, given by

$$S_1(\theta,x,m) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n(x,m)\, \pi_n(\theta) + b_n(x,m)\, \tau_n(\theta)\right]$$

$$S_2(\theta,x,m) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n(x,m)\, \tau_n(\theta) + b_n(x,m)\, \pi_n(\theta)\right]$$

where we have used the notation of Van de Hulst.[2] These results apply to a sphere of radius r and index of refraction m_o (which may be complex) embedded in a transparent medium of real refractive index n_o and having the same magnetic permeability as the sphere. The incident radiation is a plane wave of wavelength λ (in the transparent medium). The variables in the above expressions are defined in terms of these fundamental parameters:

$x = 2\pi r/\lambda$, the Mie size parameter

$m = m_o/n_o$.

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[1] *G. Mie, Ann. Physik, 30 (1908), 377.*

[2] *H. Van de Hulst, Light Scattering by Small Particles, John Wiley and Sons (1962).*

The quantity θ is the scattering angle, and the "angular coefficents" π_n and τ_n are defined in terms of the Legendre polynomials,

$$\pi_n(\theta) = \frac{dP_n(\cos\theta)}{d\cos\theta} ,$$

$$\tau_n(\theta) = \cos\theta \, \pi_n(\theta) - \sin^2\theta \frac{d\pi_n(\theta)}{d\cos\theta} .$$

The first few expressions for these functions are

$$\pi_1(\theta) = 1 \qquad \tau_1(\theta) = \cos\theta$$

$$\pi_2(\theta) = 3\cos\theta \qquad \tau_2(\theta) = 3\cos 2\theta$$

The quantities a_n and b_n are known as the "Mie coefficients" and are the main topic of this report. They will be defined in detail in section 2. These quantities are important because a number of parameters that are important in scattering studies can be expressed in terms of the a_n and b_n. For example, the particle's scattering cross section is

$$Q_{ext}(x,m) = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n+1) \,\mathrm{Re}\, (a_n + b_n),$$

or, as shown in Van de Hulst,[2]

$$Q_{ext} = \frac{4}{x^2} \,\mathrm{Re}\, [S(0)],$$

where $S(0) = S_1(0,x,m) = S_2(0,x,m)$.

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[2] *H. Van de Hulst, Light Scattering by Small Particles, John Wiley and Sons (1962).*

Other scattering functions exist that involve $a_n + b_n$, $|a_n| + |b_n|$, or various other linear combinations. Additional quantities that are of great importance involve more complicated combinations arising from terms like $|S|^2$.

It should also be pointed out that the amplitude functions are related to two orthogonal electric field components in the transverse plane of the scattered field. The reference plane for these components is called the scattering plane and is defined by the propagation vectors of the incident and scattered waves. Because of this correspondence, the polarization state of the scattered field can be determined from the amplitude functions and the incident field. For example, the scattered light is linearly polarized when S_1 or S_2 is zero regardless of the nature of the incident light. If the incident field is linearly polarized, the scattered field is generally elliptically polarized since S_1 and S_2 generally differ in phase. In the special case where S_1 is the negative of S_2, polarization is preserved (i.e., the polarization state of the scattered light is the same as that of the incident).

When one considers the more practical case of an aerosol composed of many spherical particles, the Mie coefficients retain their importance because the scattering properties of the conglomerate are determined by those of the individual particles. For example, in a fog containing N particles per unit volume with n(r) dr particles with radii between r and r + dr per unit volume, the extinction coefficient, α, is given by

$$\alpha = \int_0^\infty Q_{ext}(x)\, n(r)\, \pi r^2\, dr$$

where $Q_{ext}(x)$ is, of course, a function of r because of the definition of the Mie size parameter. This extinction coefficient describes the decrease in the intensity of the propagating wave over a path length ℓ through Bouguer's law,

$$\frac{I(\ell)}{I(0)} = e^{-\alpha\ell}.$$

In view of the above discussion it is obvious that once the character of an aerosol is known (by specifying n(r) and m) one needs the capability of numerically computing the a_n and b_n and many (sometimes complicated) functions of these parameters. These computations must be performed at each of the wavelengths and scattering angles where one wishes to investigate the scattering.

Generally, the complexity of the scattering functions requires the computations to be carried out by an automated procedure. This is especially true for the Mie coefficients themselves because they involve (as we shall see in section 2) the Ricatti-Bessel functions of complex argument and their derivatives.

Unfortunately, the evaluation of these functions in certain ranges of n or mx creates many programming difficulties or computational inaccuracies. In addition, there are many cases where the available tabulated values of the required mathematical functions are in a form that makes reliable manual checking of the computer results difficult or impossible. Therefore, one would like to know the properties of the a_n and b_n to have some idea, at least (or more accurate information, if possible), of what to expect from the computations. The results discussed in this report address this need to some extent and may also offer a means to reduce computation time.

The properties of the Mie coefficients have been studied and some useful results are presented by Van de Hulst.[2] In this report the asymptotic properties of these functions are considered. That is, the behavior when one of the parameters x or n is much larger than the other will be examined. The case where x >> n has been treated recently by Chylek[3] and at an earlier time by Stratton.[4] Their results, which include scattering by large spheres and can also apply to some perplexing questions[3] that arise in scattering theory, will be discussed and somewhat expanded. For n >> x, the discussion in this report will be detailed. The latter parameter range deserves our attention since one is usually required to compute the Mie coeffients to values of n significantly greater than x to achieve a reasonable computational accuracy in any of the scattering functions. This parameter range,

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[2]*H. Van de Hulst, Light Scattering by Small Particles, John Wiley and Sons (1962).*

[3]*P. Chylek, Large-Sphere Limits of the Mie-scattering functions, J. Opt. Soc. Am., 63, 6 (June 1973), 699.*

[4]*J. Stratton, Electromagnetic Theory, McGraw Hill Book Company (1941).*

especially if x is large, is a difficult one computationally; hence, a knowledge of the coefficient's properties could be most helpful. This range also includes the elementary one of Rayleigh scattering.

The remainder of this report contains the derivation and analysis of the desired asymptotic formulas. The accuracy and range applicability of these expressions are also discussed and some points of special interest are treated in detail.

2. THE CASE OF n >> x

2.1 Calculations and Some Basic Properties

The Mie coefficients discussed in the previous section are given by

$$a_n(x,m) = \frac{\psi_n(x)\psi_n'(mx)-m\psi_n'(x)\psi_n(mx)}{\xi_n(x)\psi_n'(mx)-m\psi_n(mx)\xi_n'(x)} \quad (1)$$

and

$$b_n(x,m) = \frac{m\psi_n(x)\psi_n'(mx)-\psi_n(mx)\psi_n'(x)}{m\xi_n(x)\psi_n'(mx)-\psi_n(mx)\xi_n'(x)} \quad (2)$$

where $\psi_n(x)$ and $\xi_n(x)$ are the Ricatti-Bessel functions and the prime denotes the first derivative of the function with respect to its argument. The Ricatti-Bessel functions can be expressed in terms of more well-known functions by

$$\psi_n(x) = xj_n(x)$$

and

$$\xi_n(x) = xh_n^{(2)}(x) = x\,[j_n(x)-iy_n(x)]$$

where $j_n(x)$, $y_n(x)$ and $h_n^{(2)}(x)$ are the spherical Bessel functions of the first, second, and third kind, respectively. The last function is also known as the Hankel function of the second kind.

The first property to be demonstrated is that the a_n and b_n approach zero for sufficiently large values of n. An examination of any of a number of the properties of the forward scattered field shows that this statement must be true, because they involve summations to $n = \infty$ of an increasing function of n multiplied by $a_n + b_n$, or the real part of that sum, or the sum of the magnitudes, etc. The physically objectionable situation where these summations increase without bound results if the a_n and b_n do not vanish. A more rigorous proof of this property can be obtained with the aid of the asymptotic expansions of the spherical Bessel functions.

The forms that are of most use at this point are obtained from the ascending series representation of these functions. When conditions are such that the first terms of these series are accurate representations of the functions themselves,

$$j_n(Z) \approx Z^n/(2n+1)!! \tag{3}$$

and

$$y_n(Z) \approx -(2n-1)!!/Z^{n+1} \tag{4}$$

where K!! signifies the product $K \cdot (K-2) \cdot (K-4) --- \cdot 3 \cdot 1$, for K an odd integer.

Equation (3) is accurate when

$$\frac{|Z|^2}{2(2n+3)} << 1 \tag{5}$$

and equation (4) requires

$$\frac{|Z|^2}{2(2n-1)} << 1. \tag{6}$$

These conditions give a more refined definition of the asymptotic region that was discussed previously (i.e., the region where $n >> x$ and $n >> |m|x$), and they will be used again in discussing the accuracy of our results.

The use of equations (3) and (4) gives

$$\psi_n(x) \approx x^{n+1}/(2n+1)!! \tag{7}$$

and

$$\xi_n(x) \approx \frac{x^{n+1}}{(2n+1)!!} + \frac{i(2n-1)!!}{x^n} . \tag{8}$$

When these approximations* are used in definitions (1) and (2) the results are

$$a_n \approx \frac{1}{1 - iA_n} \tag{9}$$

and

$$b_n \approx \frac{1}{1 + iB_n} \tag{10}$$

where

$$A_n(x,m) \approx \left(\frac{2n+1}{n+1}\right) \frac{[(2n-1)!!]^2}{x^{2n+1}} \left(\frac{n+nm^2+1}{m^2-1}\right) \tag{11}$$

$$B_n(x,m) \approx [(2n-1)!!]^2 \frac{(2n+1)(2n+3)}{x^{2n+1}} \left[\frac{1}{2} - \frac{2n+1}{x^2(m^2-1)}\right] . \tag{12}$$

**The calculation of* b_n *requires the use of the next higher order term (in x/n) in the series expansion of* $j_n(x)$ *and* $y_n(x)$ *as the first term gives* $b_n = 0$ *to first order.*

In view of conditions (5) and (6) it is clear that whenever the parameters are such that our approximations are valid it will also be true that the bracketed factor in equation (12) will be dominated by the second term. Thus,

$$B_n(x,m) \approx \frac{-[(2n+1)!!]^2 \;(2n+3)}{(m^2-1)\;\; x^{2n+3}} \,. \qquad (13)$$

From equations (11) and (13) it should be clear that in the asymptotic case the functions A_n and B_n increase in magnitude with increasing n; and hence we conclude that for any value of x the a_n and b_n tend toward zero for n sufficiently large. This behavior of A_n and B_n may not be obvious from equations (11) and (13), but can easily be verified by considering the ratio of consecutive terms. For example,

$$\frac{A_{n+1}}{A_n} \approx \frac{(2n+1)^2}{x^2} \, \frac{(2n+2)}{(2n+1)} \, \frac{(n+1)}{(n+2)} \left[\frac{n+(n+1)m^2+2}{n+nm^2+1} \right] .$$

Where n is large (i.e., n >> 1) this expression can be reduced to

$$\frac{A_{n+1}}{A_n} \rightarrow 4 \left(\frac{n}{x}\right)^2 \qquad (14)$$

which demonstrates the point clearly.

This increasing trend of $|A_n|$ and $|B_n|$ allows us to deduce an additional property of the Mie coefficients. For n sufficiently large,

$$|A_n| >> 1$$

and

$$|B_n| >> 1$$

so, for the case of real m (a lossless sphere), equations (9) and (10) may be approximated by

$$a_n \approx \frac{1+i\ A_n}{A_n^2} = \frac{1}{A_n^2} + \frac{i}{A_n} \tag{15}$$

$$b_n \approx \frac{1-i\ B_n}{B_n^2} = \frac{1}{B_n^2} - \frac{i}{B_n}\ . \tag{16}$$

Thus, the Mie coefficients possess the property

$$\mathrm{Re}(a_n) \approx [(I_m(a_n)]^2 \tag{17}$$

$$\mathrm{Re}(b_n) \approx [(I_m(b_n)]^2 \tag{18}$$

under the stated conditions.

The accuracy of these approximations and their range applicability will be discussed later on. We shall first apply the approximations to several special cases.

2.2 Rayleigh Scattering

Let us now attempt to apply our results to the special case of Rayleigh scattering. In this situation $x \ll 1$ and so $n \gg |m|x$ for n as small as unity. Furthermore, because of relation (14) the $a_n + b_n$ decrease very rapidly with increasing n. Thus, it is not necessary to consider n greater than one or two. From equations (15) and (16), including equations (11) and (13),

$$\mathrm{Im}\ a_1 \approx (2/3)\ x^3 \left(\frac{m^2-1}{m^2+2}\right),\ \mathrm{Re}\ a_1 \approx (\mathrm{Im}\ a_1)^2 \tag{19}$$

$$\text{Im } a_2 \approx (1/15)x^5 \left[\frac{m^2-1}{2m^2+3} \right], \text{ Re } a_2 \approx (\text{Im } a_2)^2 \qquad (20)$$

$$\text{Im } b_1 \approx \frac{x^5(m^2-1)}{45}, \text{ Re } b_1 \approx (\text{Im } b_1)^2 . \qquad (21)$$

From these results it follows that the amplitude functions are

$$S_1(\theta) \approx i\, x^3 \left(\frac{m^2-1}{m^2+2} \right) \qquad (22)$$

$$S_2(\theta) \approx i\, x^3 \left(\frac{m^2-1}{m^2+1} \right) \cos \theta \qquad (23)$$

when terms of order greater than x^3 are neglected. This is the dominant form of the scattering, and is perfectly symmetric with the light scattered at right angles ($\theta = \pi/2,\ 3\pi/2$) and in the backward ($\theta = \pi$) and forward ($\theta = 0$) directions being linearly polarized.

If one includes the x^5 terms the result is

$$S_1(\theta) \approx i\, x^3 \left(\frac{m^2-1}{m^2+1} \right) \left\{ 1 + x^2 \left[\frac{m^2+2}{30} + \frac{1}{6} \left(\frac{m^2+2}{2m^2+3} \right) \right] \cos \theta \right\} \qquad (24)$$

$$S_2(\theta) \approx i\, x^3 \left(\frac{m^2-1}{m^2+2} \right) \left\{ \cos \theta + x^2 \left[\frac{m^2+2}{30} + \frac{m^2+2}{2m^2+3} \left(\frac{\cos 2\theta}{6} \right) \right] \right\}. \qquad (25)$$

These expressions* can be used to determine the requirements for the scattering to deviate from the ideal Rayleigh case. This deviation

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**There should actually be a small correction term added to the bracketed multiplier of x^2 in equations (24) and (25). This is necessary because one should include corrections from higher-order terms in the series expansion of the spherical Bessel functions to a_1 when considering S to terms in x^5. The additional term is $0.6(m^2 - 2)/m^2 + 2$.*

obviously occurs when the neglected terms in x^5 begin to become significant. Note that when this occurs the right angle scattering is no longer composed of a single component, as

$$S_1(\pi/2) \approx i\, x^3 \left(\frac{m^2-1}{m^2+2} \right)$$

and

$$S_2(\pi/2) \approx i\, x^3 \left(\frac{m^2-1}{m^2+2} \right) \left\{ x^2 \left[\frac{m^2+2}{30} - \frac{1}{6}\left(\frac{m^2+2}{2m^2+3} \right) \right] \right\}$$

$$= i\, \frac{x^5}{15} \left[\frac{(m^2-1)^2}{2m^2+3} \right].$$

Thus, the first deviation from Rayleigh scattering at right angles takes the form of an additional light component polarized in the direction normal to that of the dominant scatter.

Since the intensity of the light components is proportional to $|S/k|^2$ (k is the propagation constant), the cross-polarized light at $\pi/2$ or $3\pi/2$ is x^4 smaller than the dominant scatter and thus has a λ^{-8} instead of the well-known λ^{-4} wavelength dependence of Rayleigh scattering. Note that at $\theta = \pi$ we have

$$S_1(\pi) = -\, S_2(\pi)$$

from either equations (22) and (23) or (24) and (25). This relation can easily be shown to be exact from the definitions of S_1 and S_2. The consequence of this property is that the light scattered at π radians has the same polarization as the incident radiation.

2.3 Scattering by Perfectly Conducting Spheres

For a perfectly conducting material the imaginary part of m is infinite. Thus, the results of the previous sections cannot be expected to apply since they require $n >> |m|x$. However, notice that if m is considered large, but finite, equation (11) becomes independent of m. That is,

$$A_n(x,m) \approx \left(\frac{2n+1}{n+1} \right) \left(\frac{[(2n+1)!!]^2}{x^{2n+1}} \right) n \qquad (26)$$

and this result can certainly be made valid by selecting n large enough. (That is, $n \gg |m|x$.) We shall find that this result is valid when m is actually infinite and n finite.

In attempting to apply the above procedure to the expression for B_n we encounter some difficulty. When equation (13) is investigated we find that B_n cannot be simplified in any way and is still m-dependent. This can be seen by rewriting equation (13) as the product of two factors,

$$B_n \approx - \left\{ \frac{[(2n+1)!!]^2}{x^{2n+1}} \right\} \left[\frac{2n+3}{x^2(m^2-1)} \right],$$

the first of which is independent of m and the second of which is always greater than unity no matter how large we make m. This occurs because n must be selected large enough for equation (13) to be valid, as expressed by equation (5). This approach therefore yields no information.

The desired results can be obtained by returning to the definitions of a_n and b_n, equations (1) and (2). Examination of these expressions shows that

$$a_n(x,\infty) = \frac{\psi_n'(x)}{\xi_n'(x)} \tag{27}$$

$$b_n(x,\infty) = \frac{\psi_n(x)}{\xi_n(x)} . \tag{28}$$

For† $n \gg x$ equations (3) and (4) can be employed to obtain

$$a_n(x,\infty) \approx \frac{1}{1 - i \dfrac{(2n+1)}{n+1} \dfrac{n[(2n-1)!!]^2}{x^{2n+1}}} \tag{29}$$

---

†*As in the general case, $n \gg x$ is only a rough indicator of validity. The more accurate requirement is $2(2n+3)/x^2 \gg 1$.*

$$b_n(x,\infty) \approx \frac{1}{1 + i(2n+1)\frac{[(2n-1)!!]^2}{x^{2n+1}}} \tag{30}$$

Note that equation (29) is consistent with (26). These results indicate that for m = ∞ the coefficients still possess the property

$$\begin{aligned} Re(a_n) &= I_m^2(a_n) \\ Re(b_n) &= I_m^2(a_n) \end{aligned} \tag{31}$$

and also that

$$a_n \rightarrow b_n^* \tag{32}$$

for large n (i.e., for $(n+1)/n \rightarrow 1$).

One should not be surprised that the results derived for finite values of m, equations (11) and (13), cannot be extended to conducting spheres, equations (29) and (30). After all, there are basic differences in the physical scattering mechanisms: when m is infinite there is no penetration of the field into the sphere; in addition, surface currents exist. These differences can be most easily demonstrated for small spheres. When m is finite, x can always be selected small enough for the previously derived Rayleigh equations to apply. When m is infinite, scattering of this type is not observed, no matter how small x is made. Consider equations (29) and (30) for small values of x. Again, we need only consider a_1 and b_1 for the dominant form of the scattering. These are

$$a_1 \approx \frac{1}{1 - \frac{3i}{2x^3}} \simeq i\,\frac{2x^3}{3} \tag{33}$$

$$b_1 \approx \frac{1}{1 + i\,\frac{3}{x^3}} \simeq -i\,\frac{x^3}{3} \tag{34}$$

and so the amplitude functions are

$$S_1(\theta) \approx i\, x^3 \,(1 - \frac{\cos\theta}{2})$$

$$S_2(\theta) = i\, x^3 \,(\cos\theta - \tfrac{1}{2}) .$$

This result is not at all similar to that for Rayleigh scattering.

In fact, the intensity of the backward ($\theta = \pi$) scattered light is nine times that of the forward ($\theta = 0$) scattered light. In the Rayleigh case these intensities are equal. (If one considers the higher-order terms in equations (22) and (23) it can be seen that the forward scattering exceeds the backward.) It has been shown by Van de Hulst[2] that these differences in small particle scattering are due to the presence of electric dipole (due to surface charge) and magnetic dipole (due to the surface currents) radiation from the conducting sphere, as compared to only electric dipole radiation (due to the induced volume polarization) for lossless spheres.

2.4 Accuracy of the Approximation

At this point we would like to investigate the accuracy of the approximations and the conditions under which they apply.

The simplest starting point is the case of m infinite, since exact and relatively simple expressions for a_n and b_n are available here. With the definitions of ψ_n and ξ_n in equations (27) and (28) we obtain

$$a_n = \frac{\psi_n'(x)}{\xi_n'(x)} = \frac{(x\, j_n(x))'}{(x h_n^{(2)}(x))'} \tag{35}$$

and

$$b_n = \frac{\psi_n'(x)}{\xi_n'(x)} = \frac{1}{1 - i\,\frac{y_n(x)}{j_n(x)}} \tag{36}$$

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[2] *H. Van de Hulst, Light Scattering by Small Particles, John Wiley and Sons (1962).*

Tables I and II contain values of the imaginary parts of $a_n(x)$ and $b_n(x)$ over a range of x's for various values of n computed from both the approximation formulas, equations (29) and (30), and from the exact relations given above. Equation (36) can be evaluated with the aid of the extensive tables of spherical Bessel functions given by Abramowitz and Stegum[5] and the National Bureau of Standards.[6]*

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[5] *M. Abramowitz and L. Stegum, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, 55, Sixth Printing (November 1967).*

[6] *National Bureau of Standards Mathematical Tables Project, Tables of Spherical Bessel Functions, Vols I and II, Columbia University Press (1947).*

The tables of ref 6 give $j_n(x)$ for values of x from zero to 25 (in steps of 0.1) for n from zero to 14. These tables also give $j_{-n}(x)$, which can be used to compute $y_n(x)$ by means of the relation (ref 5)

$$y_n(x) = (-1)^{n+1} j_{n+1}(x)$$

The tables of ref 5 give $j_n(x)$ and $y_n(x)$ for values of x from zero to 10 (in steps of 0.1) for n from zero to 10, for x from zero to 25 (in steps of 0.5) for n of 20 and 21, and for n from zero to 20, 30, 40, 50, and 100 for x of 1, 2, 5, 10, 50, and 100.

TABLE I. VALUES OF $Im(a_n)$, $m = \infty$

x	n	n/x	$\frac{2(2n+3)}{x^2}$	$Im(a_n)$ (actual)	$Im(a_n)$ (approx)	Relative error.* ε
0.1	1	10	1000	6.67×10^{-4}	6.65×10^{-4}	3×10^{-3}
1	2	2	14	3.04×10^{-2}	3.33×10^{-2}	9.5×10^{-2}
1	5	5	26	1.12×10^{-7}	1.22×10^{-7}	8.9×10^{-2}
1	10	10	46	1.17×10^{-19}	1.22×10^{-19}	4.2×10^{-2}
5	10	2	1.84	1.21×10^{-5}	5.83×10^{-5}	3.82
5	50	5	8.24	4.18×10^{-89}	5.36×10^{-89}	2.8×10^{-1}
10	20	2	0.86	1.84×10^{-9}	2.50×10^{-8}	12.6
10	50	5	2.06	5.02×10^{-59}	1.36×10^{-58}	1.71

*ε = (|actual value - approximate value|)|actual value|.

TABLE II. VALUES OF $Im(b_n)$, $m = \infty$

x	n	n/x	$\frac{2(2n+3)}{x^2}$	$Im(b_n)$ (actual)	$Im(b_n)$ (approx)	Relative error, ε
0.1	1	10	1000	-3.31×10^{-4}	-3.33×10^{-4}	6×10^{-3}
1	2	2	14	-1.72×10^{-2}	-2.22×10^{-2}	2.9×10^{-1}
1	5	5	26	-9.26×10^{-8}	-1.02×10^{-7}	10^{-1}
1	10	10	46	-1.06×10^{-19}	-1.11×10^{-19}	4.7×10^{-2}
5	10	2	1.84	-1.53×10^{-5}	-5.30×10^{-5}	2.46
5	50	10	8.24	-4.10×10^{-89}	-5.26×10^{-89}	2.8×10^{-1}
10	20	2	0.86	-1.91×10^{-9}	-2.38×10^{-8}	11.5
10	50	5	2.06	-4.93×10^{-59}	-1.33×10^{-58}	1.70

The evaluation of equation (35) is a more difficult task because one must use

$$\psi_n'(x) = xj_n'(x) + j_n(x)$$

and

$$\xi_n'(x) = (xj_n'(x) + j_n(x)) - i\,(xy_n'(x) + y_n(x))$$

with

$$f_n'(x) = \frac{nf_n(x)}{x} - f_{n+1}(x)$$

where $f_n(x)$ is $j_n(x)$ or $y_n(x)$.

Examination of the tables shows a strong relationship between the percentage error and the parameter $2(2n+3)/x^2$. In all the tabulated ranges of x, the error decreases as that parameter increases. This decrease appears to be more rapid in the case of b_n, but the accuracy itself appears better for a_n for a given value of $2(2n+3)/x^2$. Of course, as n becomes large the accuracy of a_n approaches that for b_n, since they become conjugates of each other.

We have shown only the imaginary parts of a_n and b_n in the table; obviously the relative error in the real part is greater than ε. Because of relation (31), the relative error in the real part of a_n and b_n, δ, is given by

$$\delta = \varepsilon(2-\varepsilon) . \tag{37}$$

The approximation errors can be seen to be very small for the case of small x. As x becomes larger the value of n must be increased to validate the formulas, but a value of $2(2n+3)/x^2$ as small as about 25 makes the approximation useful.

When m is finite the full relations (1) and (2) must be employed to compute the actual values of a_n and b_n. Consider first the case where m is real. Since we are primarily concerned with Mie scattering in clouds or fog, the obvious choice for a typical value of m is that of liquid water at a typical operational wavelength. Fortunately, a real value of about 1.33 is sufficiently accurate for our purpose for all visible and infrared wavelengths up to about 1 or 2 μ. The exact and approximate values of a_n and b_n for this index are shown in tables III and IV for several values of x and n. Again, a strong relationship exists between the accuracy of the approximation and the parameter $2(2n+3)/m^2x^2$. These tables are not as extensive as I and II because of the difficulties in computing the values, but it is clear that the case of real finite m does not behave differently from the case of conducting spheres, i.e., for $2(2n+3)/m^2x^2$ sufficiently large the approximate formulas are accurate enough to be useful. In this case, however, a value of about 10 may be acceptable for this parameter. This point is verified in tables V and VI, which also give the actual and approximate values of the coefficients for a real value of m. The tables show errors of only about 7 and 3 percent in a_n and b_n for $2(2n+3)/m^2x^2$ as low as 6.5. (An oddity in table V is that of a_n for n=2. The error here is smaller than expected and is inconsistent with the behavior in all other tables.)

TABLE III. VALUES OF $Im(a_n)$, m = 1.33

x	n	n/x	$\frac{2(2n+3)}{m^2x^2}$	$Im(a_n)$ (actual)	$Im(a_n)$ (approx)	Relative error, ε
0.1	1	10	565	1.36×10^{-4}	1.36×10^{-4}	0.00
1	2	2	7.9	7.10×10^{-3}	7.84×10^{-3}	1.00×10^{-1}
1	5	5	14.7	2.95×10^{-8}	3.16×10^{-8}	7.10×10^{-2}
5	10	2	1	5.43×10^{-6}	1.56×10^{-5}	1.87

TABLE IV. VALUES OF $Im(b_n)$, m = 1.33

x	n	n/x	$\frac{2(2n+3)}{m^2x^2}$	$Im(b_n)$ (actual)	$Im(b_n)$ (approx)	Relative error, ε
0.1	1	10	565	1.71×10^{-7}	1.71×10^{-7}	0.00
1	2	2	7.9	4.54×10^{-4}	4.88×10^{-4}	7.50×10^{-2}
1	5	5	14.7	5.17×10^{-10}	5.47×10^{-10}	5.80×10^{-2}
5	10	2	1	8.38×10^{-7}	2.11×10^{-6}	1.50

TABLE V. VALUES OF $Im(a_n)$, m = 2

x	n	n/x	$\frac{2(2n+3)}{m^2x^2}$	$Im(a_n)$ (actual)	$Im(a_n)$ (approx)	Relative error, ε
0.1	1	10	250	3.34×10^{-4}	3.33×10^{-4}	3.0×10^{-3}
1	2	2	3.5	1.75×10^{-2}	1.82×10^{-2}	3.4×10^{-2}
1	5	5	6.5	6.61×10^{-8}	7.05×10^{-8}	6.7×10^{-2}
5	10	2	0.46	1.31×10^{-5}	3.43×10^{-5}	1.6

TABLE VI. VALUES OF $Im(b_n)$, m = 2

x	n	n/x	$\frac{2(2n+3)}{m^2x^2}$	$Im(b_n)$ (actual)	$Im(b_n)$ (approx)	Relative error, ε
0.1	1	10	250	6.68×10^{-7}	6.67×10^{-7}	1.5×10^{-3}
1	2	2	3.5	2.01×10^{-3}	2.67×10^{-3}	3.3×10^{-1}
1	5	5	6.5	2.08×10^{-9}	2.14×10^{-9}	2.9×10^{-2}
5	10	2	0.46	4.75×10^{-6}	8.23×10^{-6}	7.3×10^{-1}

Finally, where m is complex, equations (15) and (16) do not apply and (9) and (8) must be employed. For liquid water drops a complex index is applicable at longer wavelengths,[7,8] such as the particularly important 10.6 μm and 4 μm regions. Deirmendjian[7] and

[7] *D. Deirmendjian, Electromagnetic Scattering on Spherical Polydispersions, American Elsevier Publishing Company (1969).*

[8] *F. S. Harris, Calculated Mie Scattering Properties in the Visible and Infrared of Measured Los Angeles Aerosol Size Distribution, Applied Optics, 11 (November 1972), 2697.*

Harris[8] give the real part of m a value in the range of 1.2 to 1.5 in this longer wavelength region, and an imaginary part ranging from a low value of 0.006 to an upper limit of 0.094. Since the computations for complex m are quite tedious, we shall only consider some typical values of m and only some important values of x. The most important of these values are arrived at by considering the typical mean particle radius in a cloud to be 3 µm, which implies an x of about 5 at 4 µm (with m = 1.33 - i.06) and of about 2 at 10.6 µm (with m = 1.33 - i.10). The exact and approximate results are shown in tables VII and VIII. These tables indicate that, as when m is real, a value of about 10 or greater for the parameter $2(2n+3)/(|m|^2x^2)$ makes the approximations quite acceptable. When m is complex, however, the relationship between the fractional error in the real (δ) and imaginary (ε) parts of a_n and b_n is not as obvious as in the case of real m.

In summary, we have seen that the accuracy of the approximate formulas is controlled by the value of the parameter K, where

$$K = \begin{cases} \dfrac{2(2n+3)}{|m|^2x^2} & m \text{ finite} \quad (38a) \\[2ex] \dfrac{2(2n+3)}{x^2} & m \text{ infinite} \; . \quad (38b) \end{cases}$$

The formulas are quite accurate (about 5 to 10 percent or less error) for K as small as 10 (25 for m = ∞). This kind of accuracy is certainly good enough for a check of computed data points and for analysis of the asymptotic behavior of various scattering parameters. This level is not, however, good enough for numerical calculation of scattering parameters that involve sums over n of functions of a_n and b_n. Of

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[8] *F. S. Harris, Calculated Mie Scattering Properties in the Visible and Infrared of Measured Los Angeles Aerosol Size Distribution, Applied Optics, 11 (November 1972), 2697.*

TABLE VII. VALUES OF a_n FOR COMPLEX m

x	n	m	$\frac{2(2n+3)}{\|m\|^2 x^2}$	$Re(a_n)$ (exact) (approx)	$Im(a_n)$ (exact) (approx)	ε	δ
1	2	1.33 -.06i	7.9	1.23 1.30 x 10^{-3}	7.10 7.89 x 10^{-3}	0.11	0.06
2	6	"	4.2	2.16 2.73 x 10^{-7}	1.39 1.80 x 10^{-6}	0.29	0.26
5	10	"	1.0	0.871 2.37 x 10^{-6}	0.545 1.58 x 10^{-5}	1.7	1.9
0.1	1	1.33 -0.01i	562	3.75 3.75 x 10^{-5}	1.37 1.37 x 10^{-4}	0	0
1	2	"	7.9	2.01 2.13 x 10^{-3}	7.13 7.90 x 10^{-3}	0.06	0.11
1	5	"	14.6	7.58 8.10 x 10^{-9}	2.98 3.21 x 10^{-8}	0.07	0.08
5	10	"	1.0	1.45 4.90 x 10^{-6}	0.549 2.59 x 10^{-5}	2.4	3.7

course, the accuracy can be improved without limit at the expense of increasing n, but (except for small x) this step may render the whole approximation useless because the (accurate) values of a_n and b_n may be so small that the nth and higher terms in the summation contribute nothing to the total summation.

Before moving to the case of x >> n, we should point out an apparent paradox in our results. According to equation (38a), larger and larger values of n are required to validate the approximation as m increases. However, when m becomes infinite equation (38b) indicates that a smaller value of n is suddenly acceptable. This apparent inconsistency is easily resolved when one considers the fact that equations (27) and (28) are not only valid for infinite m, but also

TABLE VIII. VALUES OF b_n FOR COMPLEX m

x	n	m	$\frac{2(2n+3)}{\|m\|^2 x^2}$	$Re(b_n)$ (exact) (approx)	$Im(b_n)$ (exact) (approx)	ε	δ
2	6	1.33 -.06i	4.2	1.60 1.91 $\times 10^{-8}$	7.44 9.21 $\times 10^{-8}$	0.19	0.24
5	10	"	1	1.91 4.39 $\times 10^{-7}$	0.830 2.11 $\times 10^{-6}$	1.5	1.3
0.1	1	1.33 x.1i	562	5.91 5.91 $\times 10^{-8}$	1.69 1.69 $\times 10^{-7}$	0	0
1	2	"	7.9	1.63 1.69 $\times 10^{-4}$	4.46 4.82 $\times 10^{-4}$	0.08	0.04
1	5	"	14.6	1.81 1.89 $\times 10^{-10}$	5.10 5.40 $\times 10^{-10}$	0.06	0.05
5	10	"	1	3.18 7.29 $\times 10^{-7}$	0.815 2.08 $\times 10^{-6}$	1.6	1.3

apply for finite but sufficiently large values. Though it is impossible, or at least very difficult, to estimate the value of m required for this to occur, examination of the general expressions a_n and b_n clearly verifies the correctness of this statement. The implication is that as m increases a gradual transition in the requirements on n occurs. This transition transforms the requirements defined by equation (38a) into those of (38b) while the forms of a_n and b_n given in equations (27) and (28) become valid (even though m is finite).

Thus, it is clear that when m is much smaller than some unspecified value the minimum required n is given by equation (38a) and when m is much larger than this value the requirement is given by equation (38b). Fortunately, we need not be concerned with "intermediate" values of m where neither equation (38a) nor (38b) applies. In situations of practical interest where m is finite, its magnitude is so small that equations (27) and (28) cannot possibly be valid and hence equation (38a) applies. Also, the most interesting case involving a large index is that of m infinite and therefore equation (38b) applies.

3. THE CASE OF x >> n

3.1 Calculations and Some Basic Properties

Let us now turn our attention to the case of mx >> n. This region was treated by Chylek[3] as the limiting one for very large spheres (large x), but the results here require only that mx be large compared to n. Results similar to those of Chylek were obtained by Stratton[4] in an earlier work.

The approach taken here will be somewhat different than that used in the previous papers, as we start with the asymptotic form of the Bessel functions,[4]

$$j_n(Z) \approx \frac{\sin\left(Z - \frac{n\pi}{2}\right)}{Z} \tag{39}$$

$$y_n(Z) \approx \frac{-\cos\left(Z - \frac{n\pi}{2}\right)}{Z} \tag{40}$$

for $|Z| >> n$. These expressions give

$$\psi_n(Z) \approx \sin\left(Z - \frac{n\pi}{2}\right)$$

$$\xi_n(Z) \approx \sin\left(Z - \frac{n\pi}{2}\right) + i\cos\left(Z - \frac{n\pi}{2}\right)$$

$$\psi_n'(Z) \approx \cos\left(Z - \frac{n\pi}{2}\right)$$

$$\xi_n'(Z) \approx \cos\left(Z - \frac{n\pi}{2}\right) - i\sin\left(Z - \frac{n\pi}{2}\right),$$

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[3] *P. Chylek, Large-Sphere Limits of the Mie-scattering functions, J. Opt. Soc. Am.,* <u>63</u>, *6 (June 1973), 699.*

[4] *J. Stratton, Electromagnetic Theory, McGraw Hill Book Company (1941).*

Examination of these expressions and equations (1) and (2) reveals some useful properties of the Mie coefficients. First, we notice that a change in the index n of two steps in the above expressions simply results in a change of sign on the right side. That is,

$$\psi_{n+2}(Z) \approx \sin\left(Z - \frac{(n+2)\pi}{2}\right) = -\sin\left(Z - \frac{n\pi}{2}\right) \approx -\psi_n(Z),$$

and,

$$\psi'_{n+2}(Z) \approx -\psi'_n(Z)$$

$$\xi_{n+2}(Z) \approx -\xi_n(Z)$$

$$\xi'_{n+2}(Z) \approx \xi'_n(Z) \quad .$$

Since the expressions for a_n and b_n involve only the product of various pair combinations of the functions ψ_n, ψ'_n, ξ_n, ξ'_n, we see that the sign changes introduced by an index shift of two steps cancel, hence

$$a_{n+2}(x,m) \approx a_n(x,m) \tag{41}$$

$$b_{n+2}(x,m) \approx b_n(x,m) \, . \tag{42}$$

Next, we see that an index shift of one step in equations (39) and (40) transforms ψ_n and ξ_n into their derivatives. That is,

$$\psi_{n+1}(Z) \approx \sin\left(Z - \frac{[n+1]\pi}{2}\right) = -\cos\left(Z - \frac{n\pi}{2}\right) \approx -\psi'_n(Z)$$

and similarly,

$$\xi_{n+1}(Z) \approx \xi'_n(Z) \, .$$

In the same way one can also determine that

$$\psi'_{n+1}(Z) \approx \psi_n(Z)$$

and

$$\xi'_{n+1}(Z) \approx \xi_n(Z) \; .$$

When these relations are used in equations (1) and (2) we find that

$$a_{n+1}(x,m) \approx b_n(x,m) \tag{43}$$

and

$$b_{n+1}(x,m) \approx a_n(x,m) \; . \tag{44}$$

The results of equations (41), (42), (43), and (44) can be summarized as follows for $|m|x >> n$,

$$a_1(m,x) \approx b_2(m,x) \approx a_3(m,x) \approx b_4(m,x) \approx \ldots \tag{45}$$

$$b_1(m,x) \approx a_2(m,x) \approx b_3(m,x) \approx a_4(m,x) \approx \ldots \tag{46}$$

The results obtained so far are valid for any value of m (including infinity), but to proceed further we shall consider real, complex, and infinite values separately.

3.2 Perfectly Conducting Spheres ($m = \infty$)

Here we may use the approximations in equations (27) and (28) to obtain

$$a_n = \frac{\cos \varphi_n}{\cos \varphi_n - i \sin \varphi_n} = \frac{1+e^{2i\varphi_n}}{2}$$

$$b_n = \frac{\sin \varphi_n}{\sin \varphi_n + i \cos \varphi_n} = \frac{1-e^{2i\varphi_n}}{2}$$

where ϕ_n is equal to $xn\pi/2$. With the aid of these expressions and some simple trigonometric identities one can obtain the following results:

$$\left.\begin{aligned} a_n(x) &= a_n(x \pm k\pi) \\ b_n(x) &= b_n(x \pm k\pi) \end{aligned}\right\} \quad k = 0,1,2\ldots\ldots\ldots \tag{47}$$

$$\left.\begin{aligned} b_n(x) &= a_n(x \pm \tfrac{k\pi}{2}) = a_{n\pm k}(x) \\ a_n(x) &= b_n(x \pm \tfrac{k\pi}{2}) = b_{n\pm k}(x) \end{aligned}\right\} \quad k = 1,3,5\ldots\ldots\ldots \tag{48}$$

$$a_n(x) + b_n(x) = 1 + 0i \tag{49}$$

$$|a_n(x) - b_n(x)| = 1 \tag{50}$$

$$|a_n(x)|^2 + |b_n(x)|^2 = 1 \tag{51}$$

$$a_n b_n^* = i \sin \phi_n \cos \phi_n = -a_n^* b_n . \tag{52}$$

Additional relationships can also be constructed by using equations (45) and (46) in the above expressions. Furthermore, the coefficients can be expressed as

$$a_n = \cos^2 \varphi_n + i \sin \varphi_n \cos \varphi_n \tag{53}$$

$$b_n = \sin^2 \varphi_n - i \sin \varphi_n \cos \varphi_n \tag{54}$$

from which one can easily determine the relations between their various parts. For example,

$$\mathrm{Re}\,[a_n(x + \pi/4)] = \tfrac{1}{2} - \mathrm{Im}\,[a_n(x)] \tag{55}$$

and

$$\mathrm{Im}\,[a_n(x + \pi/4) = \mathrm{Re}\,[a_n(x)] - \tfrac{1}{2} . \tag{56}$$

Also, by rewriting equation (53) (or the above exponential representation of a_n) as

$$a_n = (\tfrac{1}{2} + \tfrac{1}{2} \cos 2\phi_n) + \tfrac{1}{2} i \sin 2\phi_n$$

we see that

$$|a_n - \tfrac{1}{2}| = \tfrac{1}{2}. \qquad (57)$$

Similarly,

$$|b_n - \tfrac{1}{2}| = \tfrac{1}{2}. \qquad (58)$$

These last results are useful for checking numerical results, and it has been shown by Van de Hulst that they are exact expressions for any value of x, provided that m is real. We shall see later that equations (57) and (58) are also special cases of the general result for complex m in the large x approximation.

One can estimate the requirement on the ratio of x to n for our results to be accurate by examining some particular examples. Figures 1 and 2 are plots of $\text{Re}\left(a_n + b_n\right)$ and $\text{Im}\left(a_n + b_n\right)$ versus x for various values of n. If we set some arbitrary criteria for judging the accuracy of equation (49), say

$$|1 - \text{Re}(a_n + b_n)| \leq 10^{-2} \qquad (59)$$

and

$$|\text{Im}(a_n + b_n)| \leq 10^{-2} \qquad (60)$$

then a minimum acceptable value of x can be determined for each n. Examination of the figures indicates that the above criteria are met for all $x \geq x_o$ where x_o is approximately given by

$$x_o = 2(n+1) . \qquad (61)$$

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[2] *H. Van de Hulst, Light Scattering by Small Particles, John Wiley and Sons (1962).*

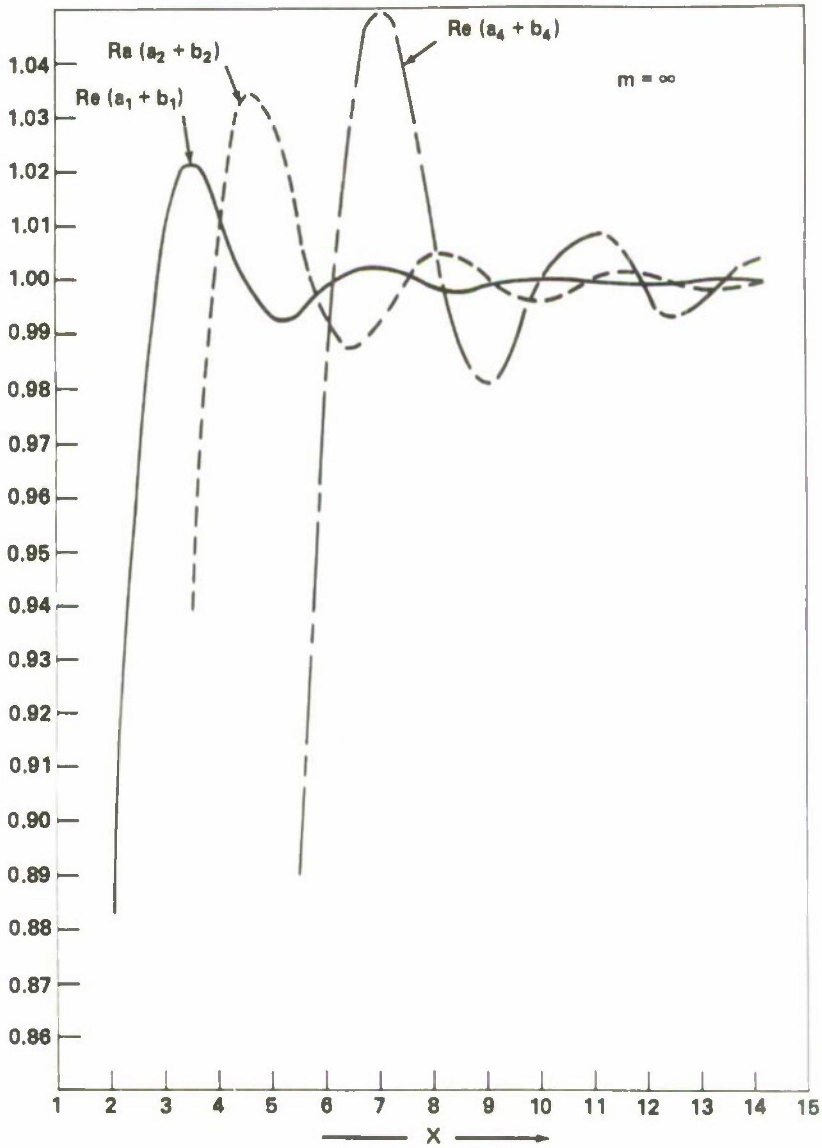

Figure 1. Real $(a_n + b_n)$ versus x for $m = \infty$.

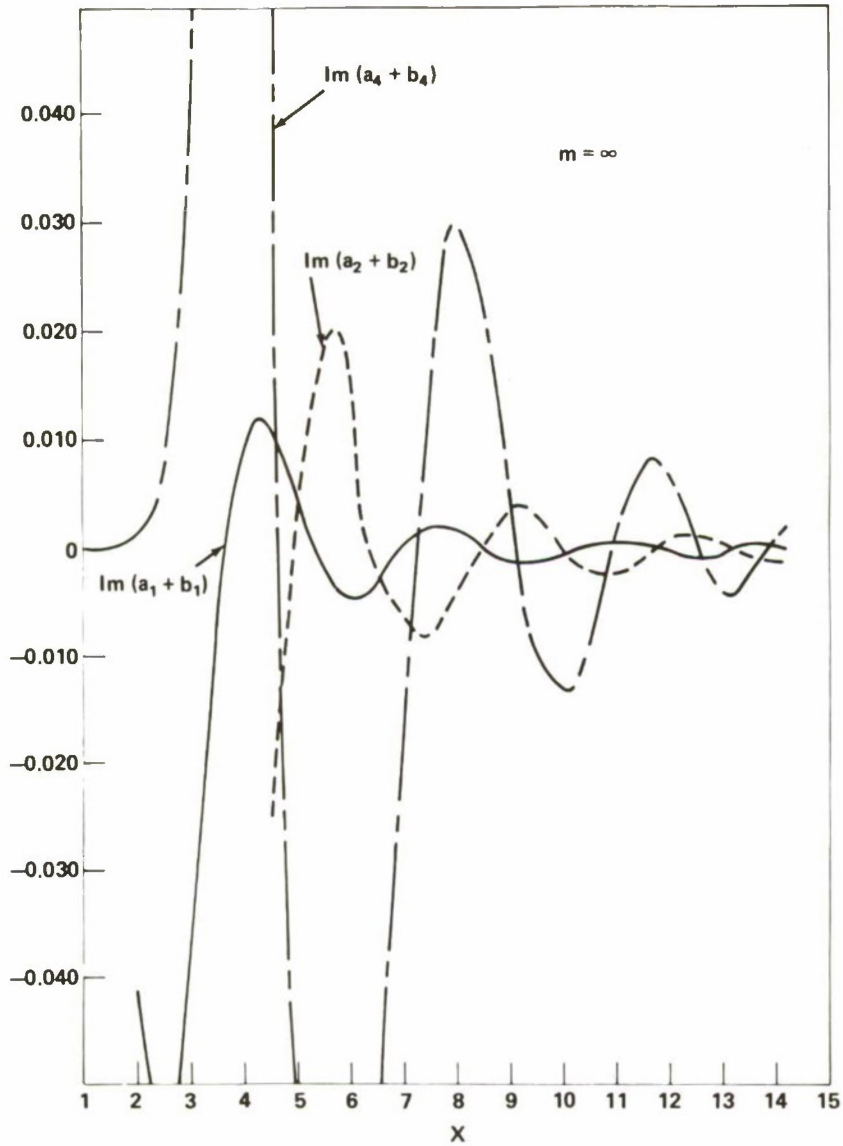

Figure 2. Imaginary $(a_n + b_n)$ versus x for $m = \infty$.

This result can be used as a guide to the applicability of all our results concerning the linear combinations of the a_n and b_n and of their parts. This estimate is similar to one made by Chylek[3] $\left(x_o = 2n+1\right)$ in connection with the accuracy of the relations between the a_n and b_n themselves [(45) and (46)]. However, it appears that equation (61) is not sufficiently large for the applicability of these equations. For x greater than the x_o given above, the real and imaginary parts of these functions are different for each n. This phase shift can cause gross differences in the quantities that are supposedly equal by equations (45) and (46). As x increases, this phase shift decreases slowly until at some sufficiently large value equations (45) and (46) apply to within a required tolerance up to some maximum n.

3.3 Sphere with Real, Finite Index of Refraction

For real values of m, equations (39) and (40) can be used to obtain

$$a_1 = \frac{-\cos x \sin mx + m \sin x \cos mx}{\sin mx \left(i \sin x - \cos x\right) + m \cos mx \left(\sin x + i \cos x\right)}$$

$$b_1 = \frac{\cos mx \sin x - m \cos x \sin mx}{m \sin mx \left(i \sin x - \cos x\right) + \cos mx \left(\sin x + i \cos x\right)}$$

These may be rewritten as

$$a_1 = \frac{f_1^2 (m,x) + i f_1 (m,x) f_2 (m,x)}{f_3(m,x)} \tag{62}$$

$$b_1 = \frac{g_1^2 (m,x) + i g_1 (m,x) g_2 (m,x)}{g_3(m,x)} \tag{63}$$

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[3] *P. Chylek, Large-Sphere Limits of the Mie-scattering functions, J. Opt. Soc. Am., 63, 6 (June 1973), 699.*

where

$$f_1 = \sin mx \cos x - m \cos mx \sin x$$

$$f_2 = \sin mx \sin x + m \cos mx \cos x$$

$$f_3 = \sin^2 mx + m^2 \cos^2 mx$$

$$g_1 = m \sin mx \cos x - \cos mx \sin x$$

$$g_2 = m \sin mx \sin x + \cos mx \cos x$$

$$g_3 = m^2 \sin^2 mx + \cos^2 mx .$$

Examination of these expressions seems to indicate that no further simplifications can be made, nor can one detect any special properties of a_n or b_n or their various linear combinations. This last observation, that quantities like $\left(a_n + b_n\right)$ or $|a_n|^2 + |b_n|^2$, etc., do not approach a limiting value for large x, is itself significant and is illustrated graphically by Chylek.[3]

3.4 Sphere with Complex Index

For complex values of m one may substitute $m = u + iv$ into equations (62) and (63) and use the expressions

$$\sin [(u + iv)x] = \sin ux \cosh vx + i \sinh vx \cosh ux$$

and

$$\cos [(u + iv)x] = \cos ux \cosh vx - i \sin ux \sinh vx$$

to determine a_1 and b_1. This operation is simplified when the approximations

[3] *P. Chylek, Large-Sphere Limits of the Mie-scattering functions, J. Opt. Soc. Am., 63, 6 (June 1973), 699.*

$$\sinh vx \approx \begin{cases} \dfrac{e^{vx}}{2}, & v > 0 \\ \dfrac{-e^{-vx}}{2}, & v < 0 \end{cases}$$

and

$$\cosh vx \approx \frac{e^{|v|x}}{2}$$

are employed. This is a valid procedure if

$$e^{|v|x} >> 1 .$$

With these approximations, one finds that after a good deal of manipulation,

$$a_1 + b_1 \approx 1 + 0i \tag{64}$$

$$|a_1|^2 + |b_1|^2 \approx \tfrac{1}{2}\left(1 + \left|\frac{m-1}{m+1}\right|^2\right) \tag{65}$$

$$|a_1 - b_1| \approx \left|\frac{m-1}{m+1}\right| \tag{66}$$

$$|a_1 - \tfrac{1}{2}| \approx |b_1 - \tfrac{1}{2}| \approx \tfrac{1}{2}\left|\frac{m-1}{m+1}\right| \tag{67}$$

$$\mathrm{Re}(a_1 b_1^*) \approx \frac{u}{|m+1|^2} \frac{\mathrm{Re}(m)}{|m+1|^2} . \tag{68}$$

Though derived for n=1 these results of course apply for any value of n because of relations (45) and (46). Note that if the imaginary part of m is allowed to approach ∞, equations (65), (66), (67) and (68) reduce to our previous results for a perfectly conducting sphere, equations (50), (51), (52) and (57).

It should also be pointed out that for given values of x and n these asymptotic forms are more accurate for larger values of v, i.e., Im(m), because of our approximations for the hyperbolic functions. Alternatively, one can say that the asymptotic limit is approached more rapidly (lower value of x) for a given value of n at larger values of v.

This point can be seen quite clearly in figure 3 by examining the plots of Re $(a_1 + b_1)$ for two values of v, -0.06 and -0.1; the latter value settles to the asymptotic value at a lower value of x. If we apply our previous criteria, equation (59), for determining a value of x where the limit is reached, that is

$$|1 - \text{Re}\ (a_1 + b_1)| \leq 10^{-2}$$

for all $x \geq x_o$, we find by examination of the figure that

$$x_o \sim 21 \text{ for } v = -0.1$$

$$x_o \sim 35 \text{ for } v = 0.06 .$$

Note that both these values fit the relation

$$x_o \simeq \left|\frac{2}{v}\right| \qquad (69)$$

quite well. The form of this expression is not surprising when we consider the previously established requirement of

$$e^{|v|x} >> 1.$$

Apparently, a value of e^2 is sufficiently greater than unity for our criteria to be met.

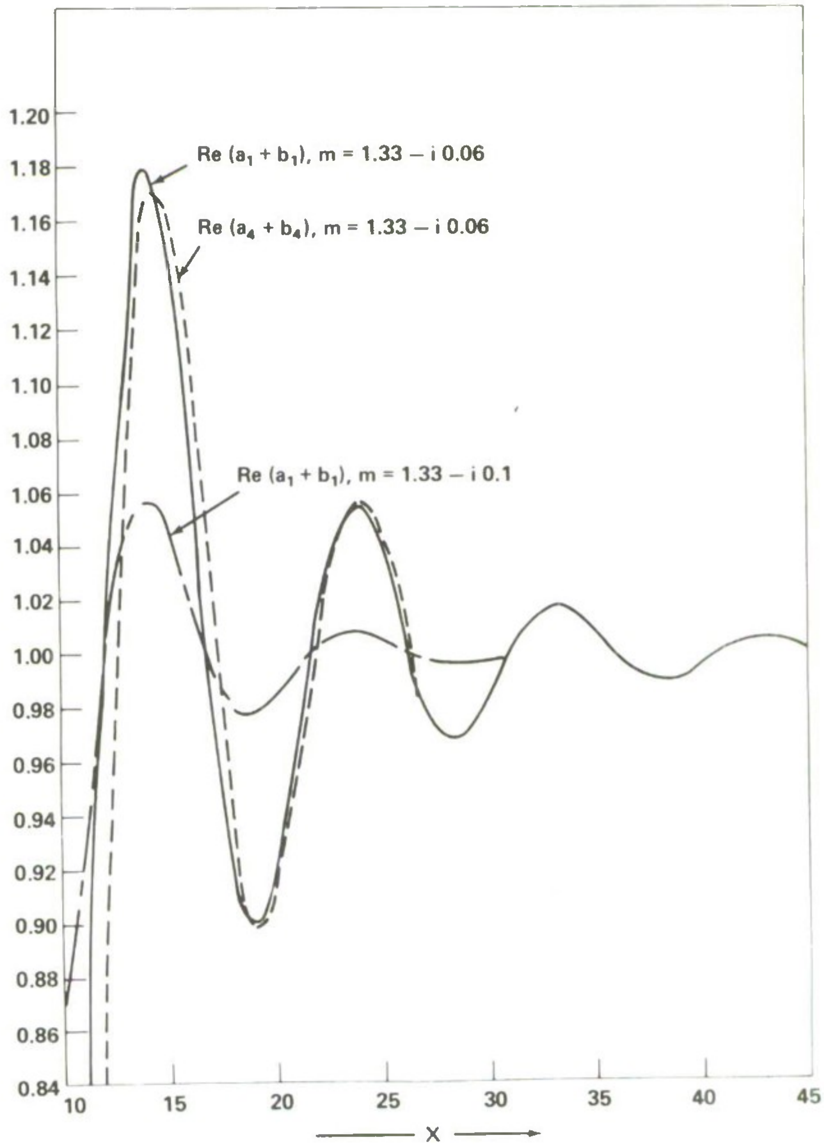

Figure 3. Real $(a_n + b_n)$ versus x for complex m.

The values of x_o given by equation (69) in these examples are much greater than those given by equation (61) for the conducting sphere. This occurs because the v dependence of the approximations dominates at the low v considered so far (as compared to the basic requirement that $m|x|>>n$). As v increases, we can expect the value of x required for accuracy to decrease to the limiting value established in the case of infinite conductivity ($v = \infty$). Thus, at a given value of x greater than 2/v, the limit in n for our results to apply is probably set by equation (61) (with a modification for finite m),

$$x > \frac{2(n + 1)}{|m|} \quad .$$

Since the minimum x required is 2/v, we have

$$n < \frac{|m|}{v} - 1 \quad . \tag{70}$$

To verify this result the data should be plotted for higher values of n, values greater than given by equation (70), in figure 3. This has not been done, but an examination of some data seems to indicate that equation (70) may be overly generous in n by a factor of about 2.

4. CONCLUSION

The desired asymptotic forms of the Mie coefficients have been derived and discussed. The accuracy of our approximations and their ranges of applicability have also been investigated. These results can be of great value in verifying computer calculations and could possibly be employed to simplify programs and reduce computation time. In addition, the results are a necessity for analytical investigations of scattering by spheres whenever the parameters are in the asymptotic regions. This last point has been illustrated by our detailed discussions of several cases of particular importance.

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