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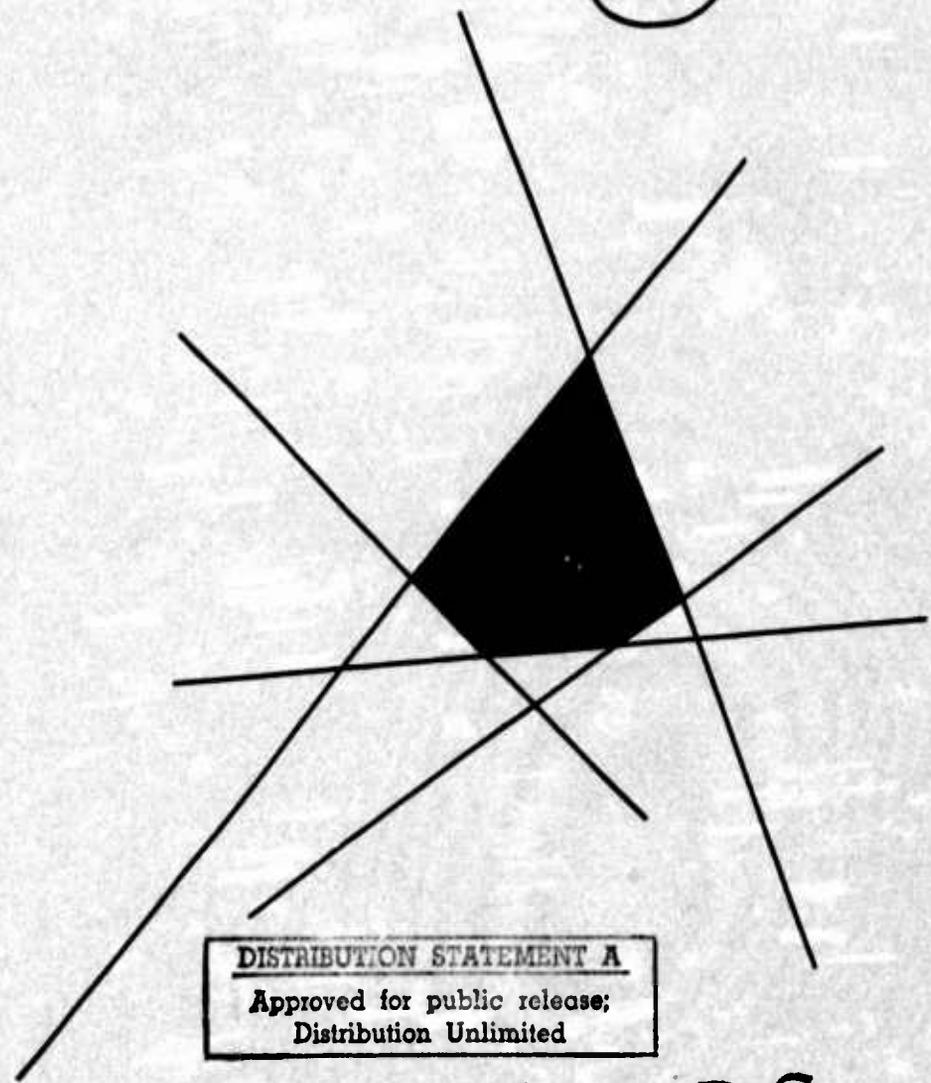
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BAYESIAN LIFE TESTING USING THE TOTAL Q ON TEST

by
WILLIAM S. JEWELL

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BAYESIAN LIFE TESTING USING THE TOTAL Q ON TEST[†]

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William S. Jewell

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13

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ABSTRACT

Suppose the basic shape of the cumulative failure (hazard) function has been identified for a certain component, and that an unknown parameter θ for a new production run of similar components is to be estimated. In particular, suppose that the failure function is of *proportional* type, $R(x) = \theta Q(x)$, where Q is the known shape function, and that θ is sampled from a prior gamma density. By using a new statistic, called the *total Q on test* (TQT), it is possible to perform Bayesian updating during a variety of lifetime testing programs in a manner similar to total time on test plots. This statistic can also be used with complete lifetime data, extending over several product runs, to identify the failure form Q , and to estimate the gamma hyperparameters. Extensions include the use of several TQT statistics to estimate the relative strength of competing hazard functions.

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BAYESIAN LIFE TESTING USING THE TOTAL Q ON TEST

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William S. Jewell

1. INTRODUCTION

Suppose that the random lifetime, \tilde{x} , of a certain component type has a distribution function $P(x | \theta)$, where θ is an unknown parameter that may vary from one production run to another, but is assumed constant within each run. We assume a prior distribution of interbatch variation, $P(\theta)$, is available.

There are two main types of estimation problems within the Bayesian framework:

- (1) *Model identification*, in which the forms of $P(x | \theta)$ and $P(\theta)$ are to be inferred from previous production runs;
- (2) *Parameter estimation*, in which selected components from a given batch are operated under normal or accelerated test conditions to infer the particular value of θ for that batch.

These two problems are interrelated, since the choice of model is often made to simplify parameter estimation, and the success of Bayesian updating depends upon how close the assumed models are to the true ones. However, a third factor which influences both of these tasks, and which is often overlooked, is the test design.

In the following, we shall examine Bayesian estimation under normal, but incomplete test conditions. We first show that a convenient modeling family is the proportional hazard class (with generalization to competing

hazard families), and then, with a certain natural conjugate prior, parameter estimation is easily carried out through a test statistic we call "total Q on test," a generalization of the concept of "total time on test." Furthermore, model identification can be simplified by using special total Q on test plots of the sample data.

2. INCOMPLETE LIFE TESTS

The classical manner in which lifetime distributions are inferred is to place N components, assumed to have the same values of θ , in a test environment which matches normal operating conditions for T hours, or until all components have all failed, if earlier. Normally, the economics of testing are such that the (random) number which have failed, $\tilde{C}(T)$, is much less than N . (See, e.g., [1] where this scheme is called a truncated data test, and [5], where it is a Type I censored test). The outcomes of the incomplete test can be described in terms of a data set D of the ordered random lifetimes $\{\tilde{x}_{1:N}\}$ and the random $\tilde{C}(T)$ as follows:

$$(2.1) \quad D = \left\{ \begin{array}{l} [\tilde{x}_{1:N} = x_1] \leq [\tilde{x}_{2:N} = x_2] \leq \cdots \leq [\tilde{x}_{C:N} = x_C] \leq T ; \\ \tilde{x}_{C+1:N} > T ; \cdots \tilde{x}_{N:N} > T \end{array} \right\}.$$

If $p(x | \theta)$ and $P^c(x | \theta)$ are the density and complementary (tail) distribution, respectively, corresponding to $P(x | \theta)$, then the likelihood of the data set D is:

$$(2.2) \quad \begin{aligned} L(D | \theta) &= p(x_1, x_2, \dots, x_N ; C(T) | \theta) \\ &= \frac{N!}{(N-C)!} \prod_{i=1}^C p(x_i | \theta) [P^c(T | \theta)]^{N-C} \prod_{i=C+1}^N \delta(x_i - T) \\ &\quad (x_1 \leq x_2 \leq \cdots \leq x_C \leq T) \end{aligned}$$

where unit impulse (degenerate) densities $\delta(x)$ are used for convenience

so that $\sum_{C=0}^N \iint L(D | \theta) dx_1 dx_2 \cdots dx_N = 1$.

An alternate testing scheme (called a censored data test in [1] and a Type II censored test in [5]), fixes the number of failures, C , in advance, and thus the duration of the test is a random variable,

$\tilde{T}(C) = \tilde{x}_{C:N}$. The likelihood (2.2) becomes a density

$$\begin{aligned}
 L(D | \theta) &= p(x_1, x_2, \dots, x_N | \theta) \\
 (2.3) \quad &= \frac{N!}{(N-C)!} \prod_{i=1}^C p(x_i | \theta) \left[P^c(x_C | \theta) \right]^{N-C} \prod_{i=C+1}^N \delta(x_i - x_C) \cdot \\
 &\quad (x_1 \leq x_2 \leq \dots \leq x_C)
 \end{aligned}$$

with $\iint L(D | \theta) dx_1, dx_2 \dots dx_N = 1$.

3. PROPORTIONAL HAZARD FAMILY: TOTAL Q ON TEST AND MAXIMUM LIKELIHOOD ESTIMATORS

Although (2.2) and (2.3) could, in principle, be calculated for any underlying distribution, it is desirable to pick a family which is both useful from a modeling point of view, and simple computationally.

The Koopman-Pitman-Darmois exponential family has extremely useful properties in regular Bayesian forecasting [2] [4]; but, it has no convenient integral form and does not seem useful in life testing applications.

However, every complementary distribution function can be written $P^c(x | \theta) = \exp(-R(x | \theta))$, where $R(x | \theta)$ is the *cumulative failure (hazard) function*, given θ ; a convenient way to parametrize this form is to suppose that $R(x | \theta) = \theta Q(x)$. Since $Q(x)$ is a monotone nondecreasing function which contain all of the failure shape information (thus preserving the properties of IFR, IFRA, etc. [1] for all θ), we could call it the *unit- or prototype-failure function*, and refer to the density and cumulative distributions:

$$(3.1) \quad P^c(x | \theta) = e^{-\theta Q(x)} \quad ; \quad p(x | \theta) = \theta q(x) e^{-\theta Q(x)} \quad ;$$

as the *proportional hazard family* ($q(x) = dQ(x)/dx$). We note the following properties:

- (a) The family includes the exponential, Weibull with given shape parameter, and Gumbel distributions;
- (b) A random lifetime, \tilde{x} , with distribution (3.1) and θ an integer can be interpreted as $\tilde{x} = \min(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_\theta)$, where $\Pr\{\tilde{y}_i > y\} = \exp(-Q(y))$, for all $i = 1, 2, \dots, \theta$, which suggests a certain physical failure model for this family;

- (c) It may be reasonable that production of different lots under varying conditions maintains the same shape for the hazard function and changes only the relative intensity of failures.

Most importantly, this family greatly simplifies estimation in life-testing, since the likelihood (2.2) becomes

$$\begin{aligned}
 & p(x_1, x_2, \dots, x_N; C | \theta) \\
 (3.2) \quad & = \frac{N!}{(N-C)!} \prod_{i=1}^C q(x_i) \prod_{i=C+1}^N \delta(x_i - T) \left\{ \theta^C e^{-\theta [TQT(x_1, x_2, \dots, x_C; C)]} \right\}, \\
 & \qquad \qquad \qquad (x_1 \leq x_2 \leq \dots \leq x_C)
 \end{aligned}$$

where we call

$$(3.3) \quad TQT(x_1, x_2, \dots, x_C; C = C(T)) = \sum_{i=1}^C Q(x_i) + (N-C)Q(T)$$

the *total Q on test*, a statistic of varying dimensions that is a natural generalization of the total time on test concept for the exponential density, $Q(x) = x$ [1]. Corresponding to (2.3) and the second testing scheme, we obtain a similar, but fixed-dimensional statistic:

$$(3.4) \quad TQT(x_1, x_2, \dots, x_C) = \sum_{i=1}^C Q(x_i) + (N-C)Q(x_C),$$

and a likelihood similar to (3.2).

Since only the term in braces in (3.2) is a function of θ , it is easy to show that the maximum likelihood estimators of $\tilde{\theta}$ and $1/\tilde{\theta}$ are:

$$(3.5) \quad \hat{\theta}^{-1} = \text{MLE}(\tilde{\theta}^{-1}) = \frac{\text{TQT}(\mathcal{D})}{c} ; \hat{\theta} = \text{MLE}(\tilde{\theta}) = \frac{c}{\text{TQT}(\mathcal{D})}$$

which coincides with known results for the exponential density under the two tests [1] [5].

4. OTHER LIFE TESTS

A variety of other life testing schemes also use appropriate generalizations of the total Q on test statistic. For example, if N units are placed on test, but a set of fixed, but possibly different test durations $\{T_i\}$ are given for all units, then we observe $\tilde{y}_i = \min\{\tilde{x}_i, T_i\}$. The likelihood of this unordered data is:

$$(4.1) \quad p(y_1, y_2, \dots, y_N; C | \theta) = \prod_{i \in C} q(y_i) \prod_{j \notin C} \delta(y_j - T_j) \left\{ \theta^C e^{-TQT(y_1, y_2, \dots, y_N; C)} \right\} \quad (0 \leq y_i \leq \infty)$$

where $C = \{i | \tilde{y}_i < T_i, i = 1, 2, \dots, N\}$ and $C = ||C||$. The total Q on test is

$$(4.2) \quad TQT(y_1, y_2, \dots, y_N; C) = \sum_{i=1}^N Q(y_i) = \sum_{i \in C} Q(x_i) + \sum_{j \notin C} Q(T_j).$$

This is called testing with withdrawals in [1].

Another testing procedure might be called renewal testing. Suppose a component is started on a single test stand at time zero; after the first failure, a new item is started; and so on, until the procedure terminates at a fixed time T , with the $(\tilde{C} + 1)^{st}$ item still operative.

The likelihood associated with $\mathcal{D} = \left\{ \tilde{x}_1 = x_1, \tilde{x}_2 = x_2, \dots, \tilde{x}_C = x_C; \right.$

$\left. \tilde{x}_{\tilde{C}+1} > T - \sum_{i=1}^C x_i \right\}$ is just:

$$(4.3) \quad p(x_1, x_2, \dots, x_C; C | \theta) = \prod_{i=1}^C q(x_i) \left\{ \theta^C e^{-TQT(x_1, x_2, \dots, x_C; C)} \right\}$$

over the simplex $0 \leq \sum_{i=1}^C x_i \leq T$, with total Q on test

$$(4.4) \quad \text{TQT}(x_1, x_2, \dots, x_C; C) = \sum_{i=1}^C Q(x_i) + Q\left(T - \sum_{i=1}^C x_i\right).$$

The generalization to stopping with the C^{th} failure, multiple test stands, and other combined censoring-truncating-withdrawal-renewal test schemes should now be obvious.

In fact, with the proportional hazard family it is clear that, no matter what the test set-up, one needs only to monitor the actual (complete or incomplete) lifetimes $\{y_i\}$ for each item, and the (fixed or stochastic) number, C , of completed tests. Then $\text{TQT} = \sum_{\text{all } i} Q(y_i)$ and (3.5) and the Bayesian formulae of the next section always hold.

5. BAYESIAN UPDATING UNDER LIFE TESTING; MIXED DISTRIBUTIONS

In a Bayesian formulation, we do not use (3.5) directly because we have prior information on the possible values of θ , perhaps through cumulative data on previous production runs, or, if one is so persuaded, from sampling expert opinion. To simplify the application of Bayes' law, it is convenient to pick a *natural conjugate prior*, whose form matches that in the braces in (3.2); this is just the gamma density

$$(5.1) \quad p(\theta) = \frac{Q_0 (Q_0 \theta)^{C_0 - 1} e^{-\theta Q_0}}{\Gamma(C_0)} \quad (0 \leq \theta \leq \infty)$$

with hyperparameters C_0 , Q_0 . The usefulness of (5.1) in modeling unimodal densities over $[0, \infty)$ is well-known. It is easy to see that Bayes' law then gives a posterior-to-data density of θ , $p(\theta | \mathcal{D})$, which is also gamma, but with updated hyperparameters:

$$(5.2) \quad \begin{aligned} C_0 + C + C, \\ Q_0 + Q_0 + TQT(\mathcal{D}). \end{aligned}$$

In place of the classical estimators (3.5), we have the posterior-to-data expectations

$$(5.3) \quad E\{\tilde{\theta}^{-1} | \mathcal{D}\} = (1 - z_1)E\{\tilde{\theta}^{-1}\} + z_1 \hat{\theta}^{-1}(\mathcal{D})$$

$$(5.4) \quad [E\{\tilde{\theta} | \mathcal{D}\}]^{-1} = (1 - z_2)[E\{\tilde{\theta}\}]^{-1} + z_2 [\hat{\theta}(\mathcal{D})]^{-1}$$

with "credibility factors"

$$(5.5) \quad z_1 = C/(C_0 - 1 + C) \quad ; \quad z_2 = C/(C_0 + C) .$$

This means that if the test gives a large number of complete observations, relative to C_0 , then the Bayesian and MLE estimators coincide. However, for relatively incomplete tests, more weight is given the prior expectations, $E(\theta^{-1})$ or $E(\theta)$. Linear mixing formulae of this type are well-known in Bayesian forecasting [2] [3] [4].

It is also of interest to examine the form of the *mixed density* of lifetimes, obtained by averaging over many batches. From (3.1) and (5.1)

$$(5.6) \quad P^C(x) = EP^C(x | \theta) = \{1 + [Q(x)/Q_0]\}^{-C_0};$$

$$p(x) = [C_0 q(x) | Q_0] \{1 + [Q(x)/Q_0]\}^{-C_0 - 1};$$

which is a generalization of a shifted Pareto. We see that if the prototype failure function is Gumbel, we get exponential tails (for large x) in the mixed distribution, while if the underlying failures are Weibull, we get the "more dangerous" algebraic tails. If life test data \mathcal{D} are drawn from a batch with fixed $\theta = \theta_1$, it follows that the forecast distribution for the remaining components from this batch, $P^C(x | \mathcal{D})$, is of form (5.6) with updated parameters (5.2). Furthermore, one can show that if the test data set \mathcal{D} is large enough, $P^C(x | \mathcal{D})$ approaches $\exp\{-\theta_1 Q(x)\}$, the true distribution for this batch, with probability one.

Generally, the failure function of the mixed distribution increases less rapidly than that of the prototype hazard:

$$(5.7) \quad R_{\text{MIXED}}(x) = -\ln[P^C(x)] = C_0 \ln[1 + (Q(x)/Q_0)]$$

and may be DFR even if $q(x)$ is increasing. (5.7) should not be confused with the average behavior of the individual failure functions:

$$(5.8) \quad R(x) = ER(x \mid \tilde{\theta}) = c_0 Q(x)/Q_0 ,$$

which always has the shape $Q(x)$.

6. TOTAL Q ON TEST TRANSFORM

We turn now to the problem of identifying the prototype failure function, $Q(x)$, and the hyperparameters C_0, Q_0 . First, let us examine the effect of plotting TQT with a Q which is not the necessarily correct one.

Paralleling Barlow and Campo [1], we define the *total Q on test transform* (TQTT) of any distribution F with respect to Q as:

$$(6.1) \quad H^{-1}(t) = H_{F/Q}^{-1}(t) = \int_0^{F^{-1}(t)} [1 - F(u)] dQ(u) \quad t \in [0,1].$$

This awkward notation is used because $H_{F/Q}$ (the inverse of $H_{F/Q}^{-1}$) is a distribution on $[0, \mu]$, where

$$(6.2) \quad \mu = H^{-1}(1) = \int_0^{F^{-1}(1)} F^c(u) dQ(u) \leq \infty,$$

and is the mean of F if $Q(x) = x$, the usual total time on test transform.

Now if $F^c(x) = \exp\{-\theta Q(x)\}$, it is easy to verify that

$$(6.3) \quad H^{-1}(t) = t/\theta, \quad t \in [0,1]$$

while, for a general failure function, $F^c(x) = \exp\{-R_F(x)\}$,

$$(6.4) \quad \left. \frac{dH^{-1}(t)}{dt} \right|_{x = F^{-1}(t)} = \frac{q(x)}{r_F(x)}$$

where $r_F(x) = dR_F(x)/dx$.

Thus, if we could plot the TQTT for any Q , we could isolate those areas where $r(x)$ was larger [smaller] than $q(x)$ as those regions where the TQTT has slope less [greater] than unity. Constants of proportionality can be eliminated by plotting the scaled total Q on test transform (STQTT), $H^{-1}(t)/H^{-1}(1)$, thus giving us a continuous increasing function on the unit square which intersects $(0,0)$ and $(1,1)$. And the closer $Q(x)$ is to the form of $R(x)$, the closer the STQTT is to the straight line (obtained for the exponential distribution with the conventional total time on test transform).

The empirical usefulness of this for identification can be seen from the following. Suppose we have observed a censored life test for a sample of N items from the proportional hazard family with fixed, but unknown θ , say $F(x) = P(x | \theta)$, and there are C failures. The empirical life-time distribution, $F_C(x)$, can be obtained from the ordered complete samples $\{x_{1:N} \leq x_{2:N} \cdots x_{C:N}\}$

$$(6.5) \quad F_C(x) = \begin{cases} 0 & 0 \leq x < x_{1:N} \\ 1/C & x_{1:N} \leq x \leq x_{i+1:N} ; \\ 1 & x_{C:N} \leq x \end{cases}$$

however this is unlikely to be of direct use because of the small size of C .

Suppose we plot instead the *empirical Q on test ratio*

$$(6.6) \quad \text{EQTR}(i | D) = \frac{\text{TQT}(x_{1:N}, x_{2:N}, \dots, x_{i:N})}{\text{TQT}(x_{1:N}, x_{2:N}, \dots, x_{C:N})}$$

versus $(i | C)$ ($i = 1, 2, \dots, C$), where TQT is the statistic defined in (3.4). Since

$$(6.7) \quad \frac{1}{N} \text{TQT}(x_{1:N}, x_{2:N}, \dots, x_{C:N}) = \int_0^{F_C^{-1}(C/N)} [1 - F_C(u)] dQ(u),$$

one can show, following [1], that

$$(6.8) \quad \text{EQTR}(1 | D) \rightarrow \frac{H^{-1}(t)}{H^{-1}(u)} \quad (0 \leq t \leq u \leq 1)$$

uniformly as $N \rightarrow \infty$, and $1/N \rightarrow t$, $C/N \rightarrow u$. In other words, the empirical Q on test ratio tends to look like the scaled total Q on test transform as the number of data points increases, and thus can be used for model identification merely by trying different Q functions, until a satisfactory straight line approximation is obtained. Naturally, for any finite sample, there will be fluctuations in the ratio (6.6); Barlow and Campo [1] give results for the exponential case using total time on test, which should also be useful once an approximate form Q has been found.

Perhaps it is appropriate at this point to compare our procedure for nonexponential model identification with that set forth in Barlow and Campo, assuming always there is sufficient data. In their approach, they would plot empirical ratios corresponding to the scaled total time on test (e.g. (6.6) and (3.4) with $Q(x) = x$); departure from the exponential would show up as a departure from a straight line. Special characteristics, such as IFR, IFRA, NBU, etc. would be apparent from inspection, and then through the use of families of transparent overlays on the unit square, the final distribution would be chosen visually.

In our approach, we propose to try successive forms for Q , plotting the empirical ratio (6.6) each time. Regions of departure from a straight line, together with (6.4), are supposed to suggest new empirical modifications in Q , which are carried out until the EQTR is apparently a straight line.

7. EXAMPLE

Figures 1 through 5 show various empirical Q on test ratios, based on 107 samples for the right rear brake of a D9G-66A caterpillar tractor given by Barlow and Campo [1]. Three prototype failure functions, $Q(x) = x^1$, $x^{1.5}$, and x^2 are given in each plot, from top to bottom respectively.

Figures 1, 2, and 3 illustrate the effect of varying the number of complete samples, with Figure 1 showing the original 107 lifetimes, and 2 and 3 showing the curves with only 54 and 27 lifetimes, selected at random. Clearly the variability increases with decreasing data, although $Q(x) = x^{1.5}$ is always the best of the functions chosen. Figure 1 suggests, via (6.4), that a slight modification of the failure function between 0-0.05 and 0.65-1.00 would give a better fit.

Figures 4 and 5 show the effect of incomplete life data by censoring the original 107 samples at the 54th and 27th lifetimes, respectively. Again, the model comparison is remarkably consistent, even with small duration tests, although the discrimination (separation between the plots) is less with censored data.

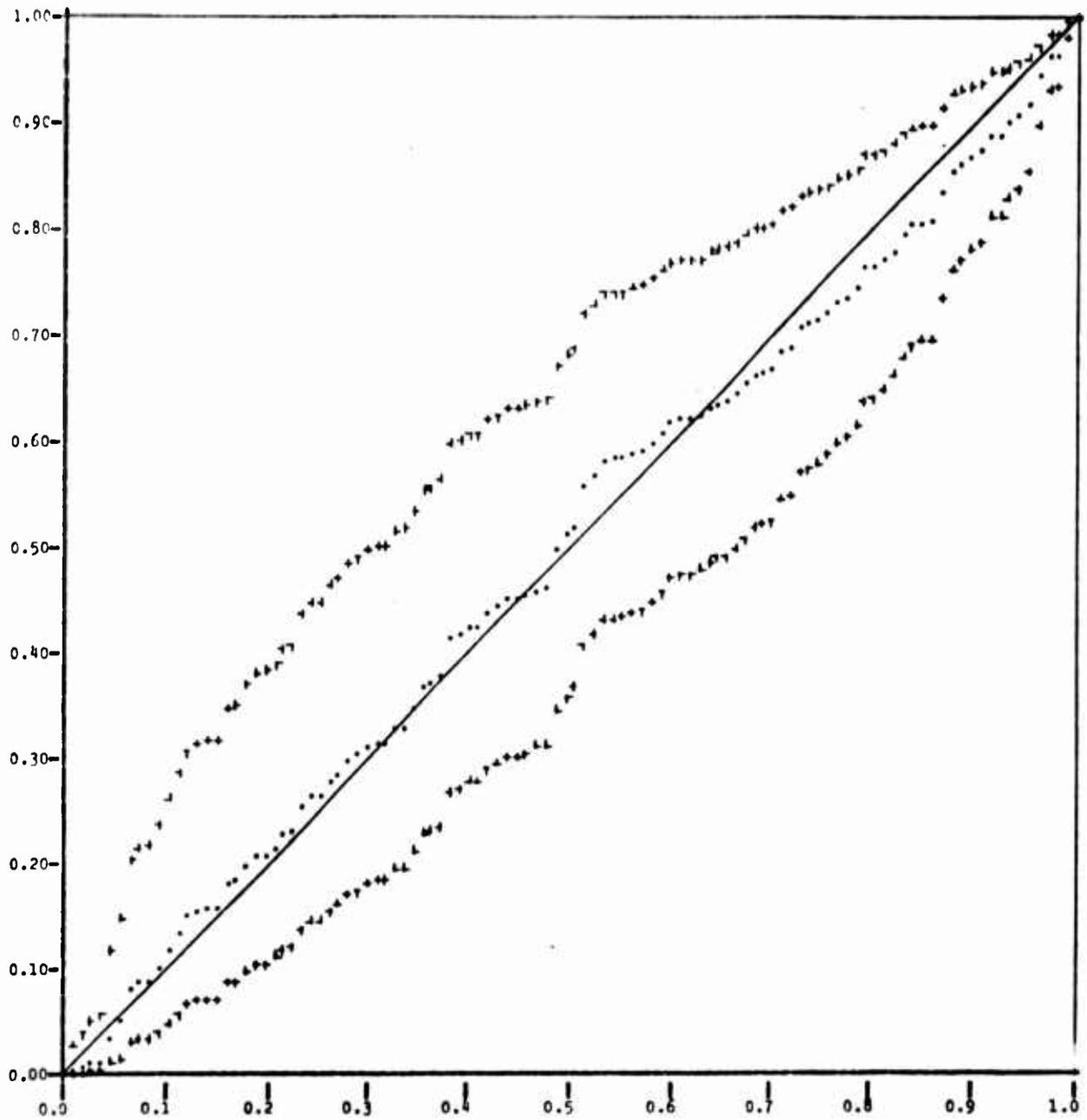


FIGURE 1: EMPIRICAL Q ON TEST RATIO. 107 COMPLETE SAMPLES, RIGHT REAR BRAKE [BARLOW AND CAMPO]. $Q(x) = x^1$, $x^{1.5}$, AND x^2 .

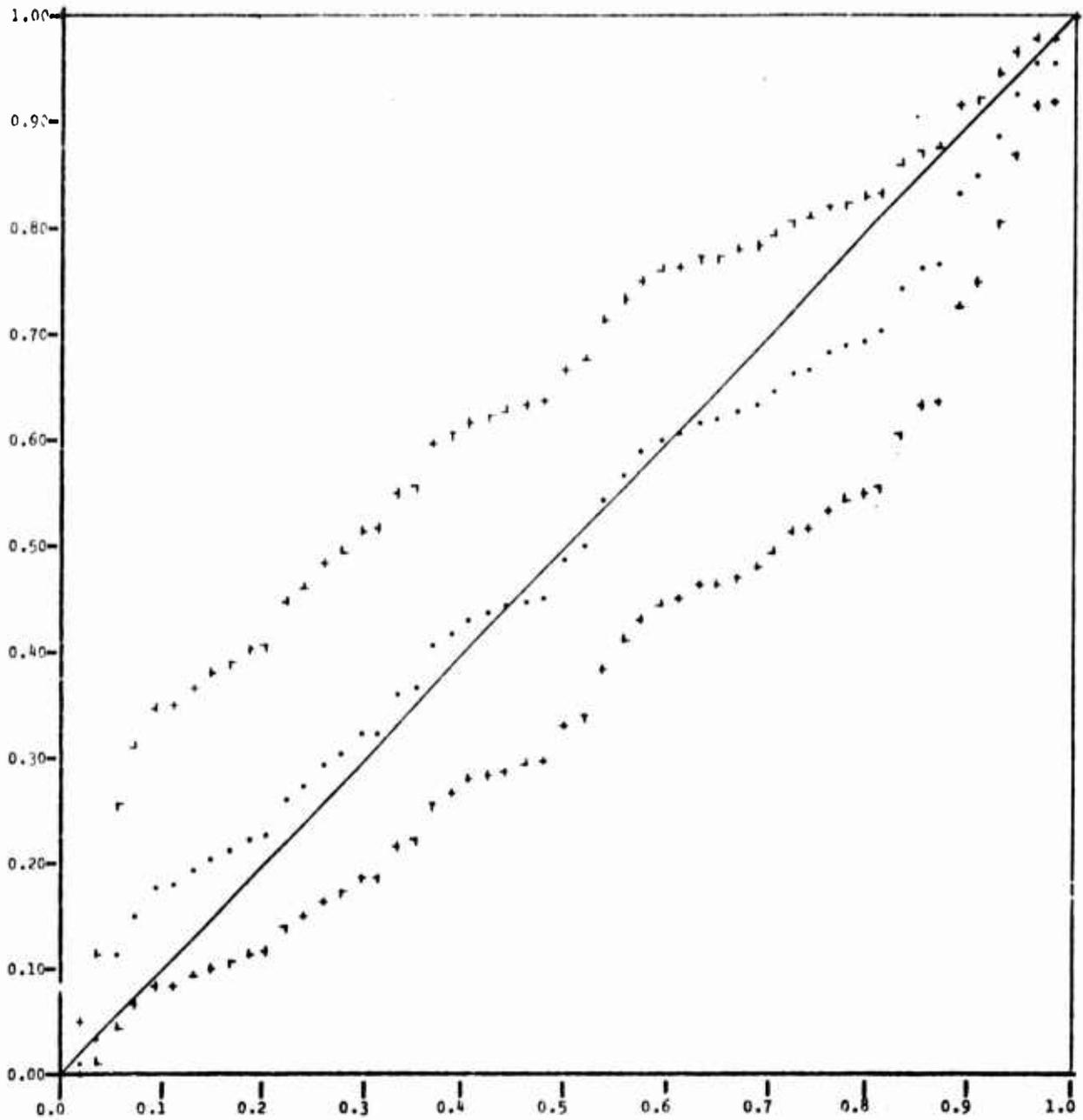


FIGURE 2: EMPIRICAL Q ON TEST RATIO. 54 COMPLETE SAMPLES, RIGHT REAR BRAKE [BARLOW AND CAMPO]. $Q(x) = x^1$, $x^{1.5}$, AND x^2 .

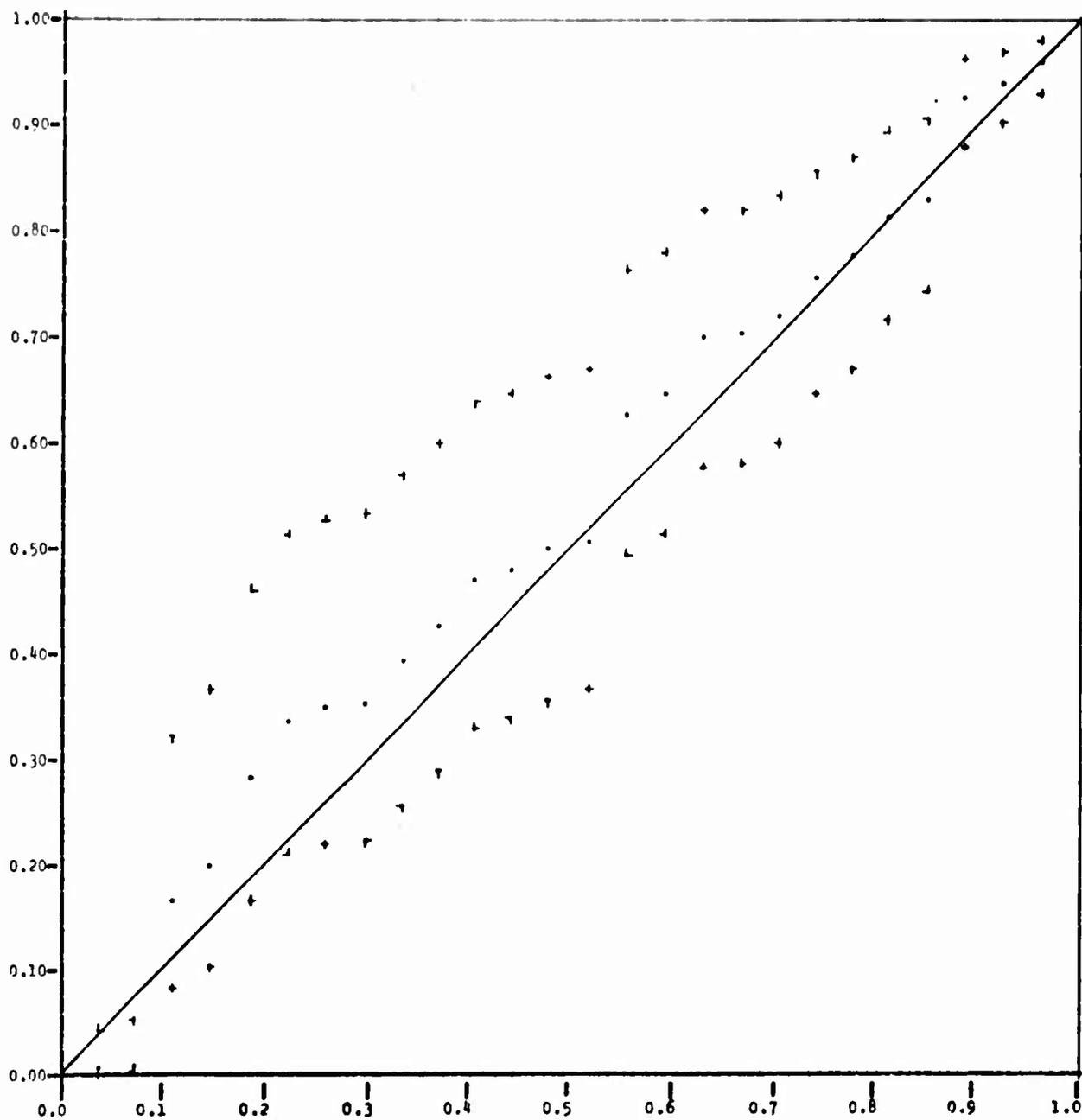


FIGURE 3: EMPIRICAL Q ON TEST RATIO. 27 COMPLETE SAMPLES, RIGHT REAR BRAKE [BARLOW AND CAMPO]. $Q(x) = x^1$, $x^{1.5}$, AND x^2 .

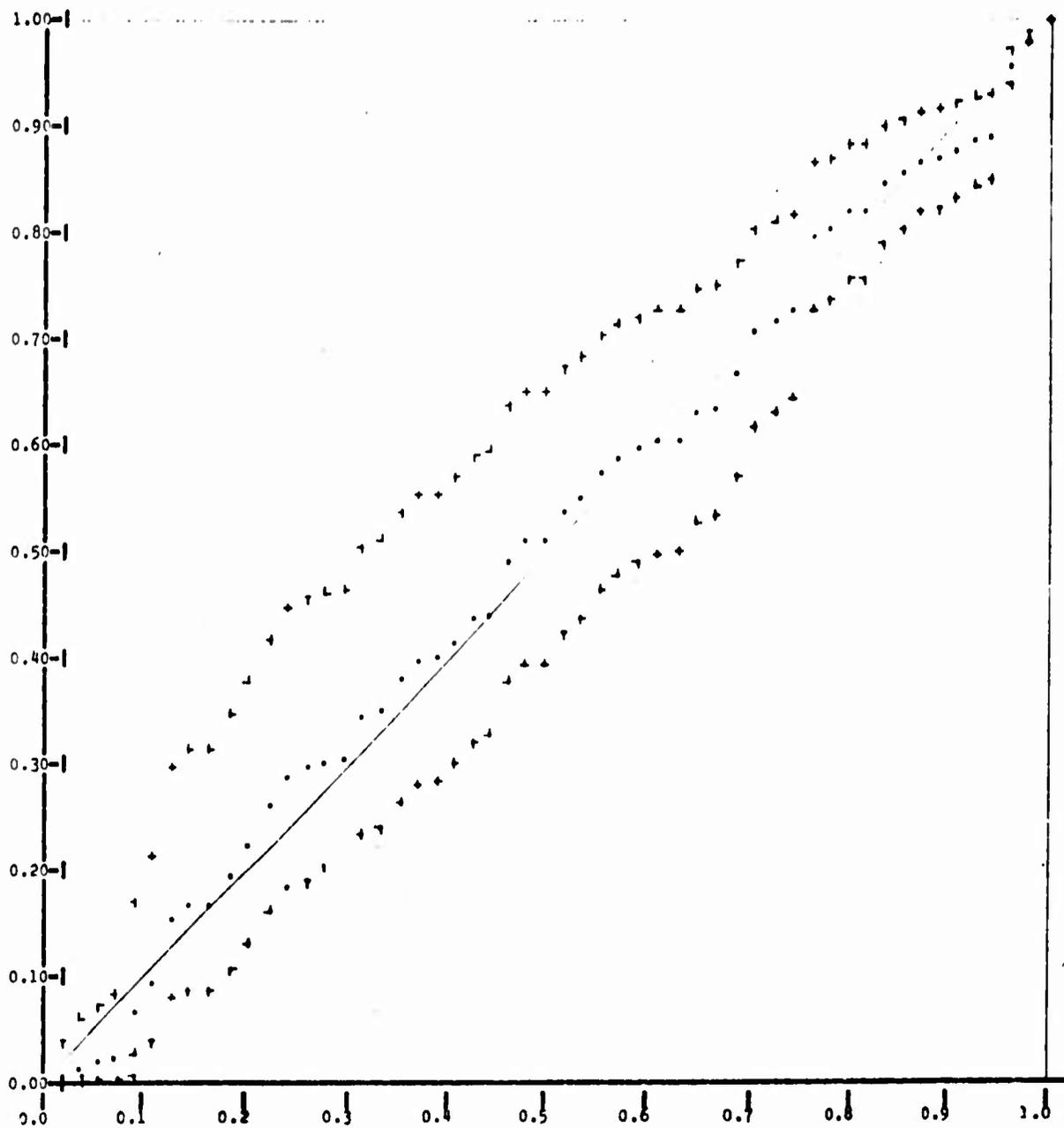


FIGURE 4: EMPIRICAL Q ON TEST RATIO. 107 SAMPLES, CENSORED AT 54TH SAMPLE, RIGHT REAR BRAKE [BARLOW AND CAMPO]. $Q(x) = x^1$, $x^{1.5}$, AND x^2 .

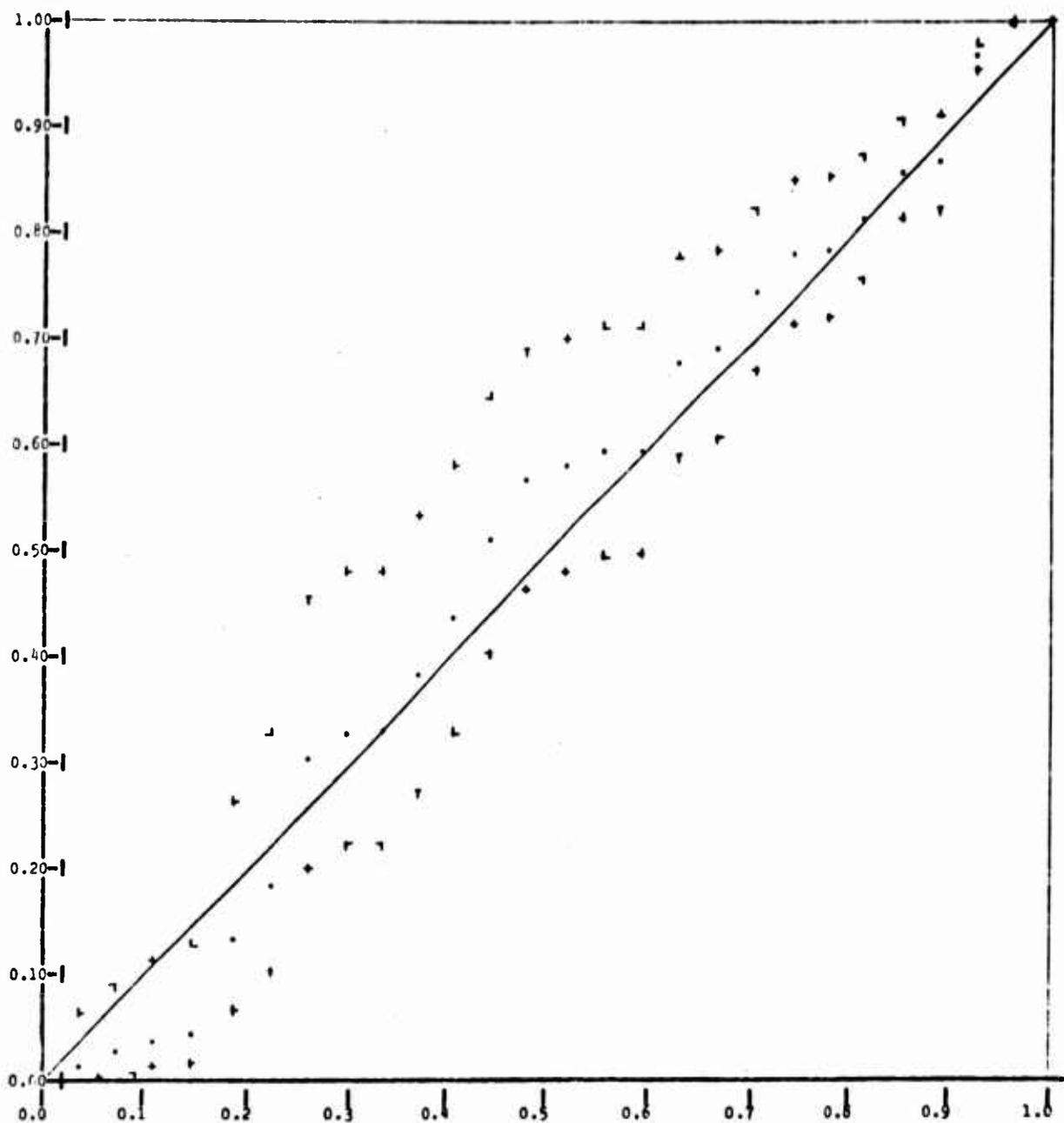


FIGURE 5: EMPIRICAL Q ON TEST RATIO. 107 SAMPLES, CENSORED AT 27TH SAMPLE, RIGHT REAR BRAKE [BARLOW AND CAMPO]. $Q(x) = x^1$, $x^{1.5}$, AND x^2 .

8. DATA FROM SEVERAL BATCHES; ESTIMATION OF HYPERPARAMETERS AND MODEL IDENTIFICATION

The use of (6.6) to identify Q depends on having sufficient complete lifetimes for a fixed value of θ . However, a more usual situation would be that we have a moderate number of samples from several production runs, each with different values of θ .

If we pool the data, then the resulting mixed distribution will be given by (5.6), from which we see that it is difficult to extricate the identification of Q from the estimation of C_0 and Q_0 . In fact, the TQTT of $P(x)$ with respect to the correct $Q(x)$ is *not* a straight line, but is

$$(8.1) \quad H^{-1}(t) = \frac{Q_0}{C_0 - 1} \left\{ 1 - (1 - t)^{C_0 - 1/C_0} \right\}$$

which is straight only for C_0 large. A normalized plot of this function is shown in Figure 6.

However, maximum likelihood estimates \hat{C}_0 , \hat{Q}_0 can be reliably obtained from a large amount M of pooled data, given Q . First, \hat{Q}_0 is found so that the following two sums are equal,

$$(8.2) \quad \sum(\hat{Q}_0) = \sum_{i=1}^M \left[\frac{Q(x_i)}{\hat{Q}_0 + Q(x_i)} \right] = \sum_{i=1}^M \ln \left[1 + \frac{Q(x_i)}{\hat{Q}_0} \right],$$

and then:

$$(8.3) \quad \hat{C}_0 = M / \sum(\hat{Q}_0) .$$

There remains the problem of identifying Q . We suggest the following:

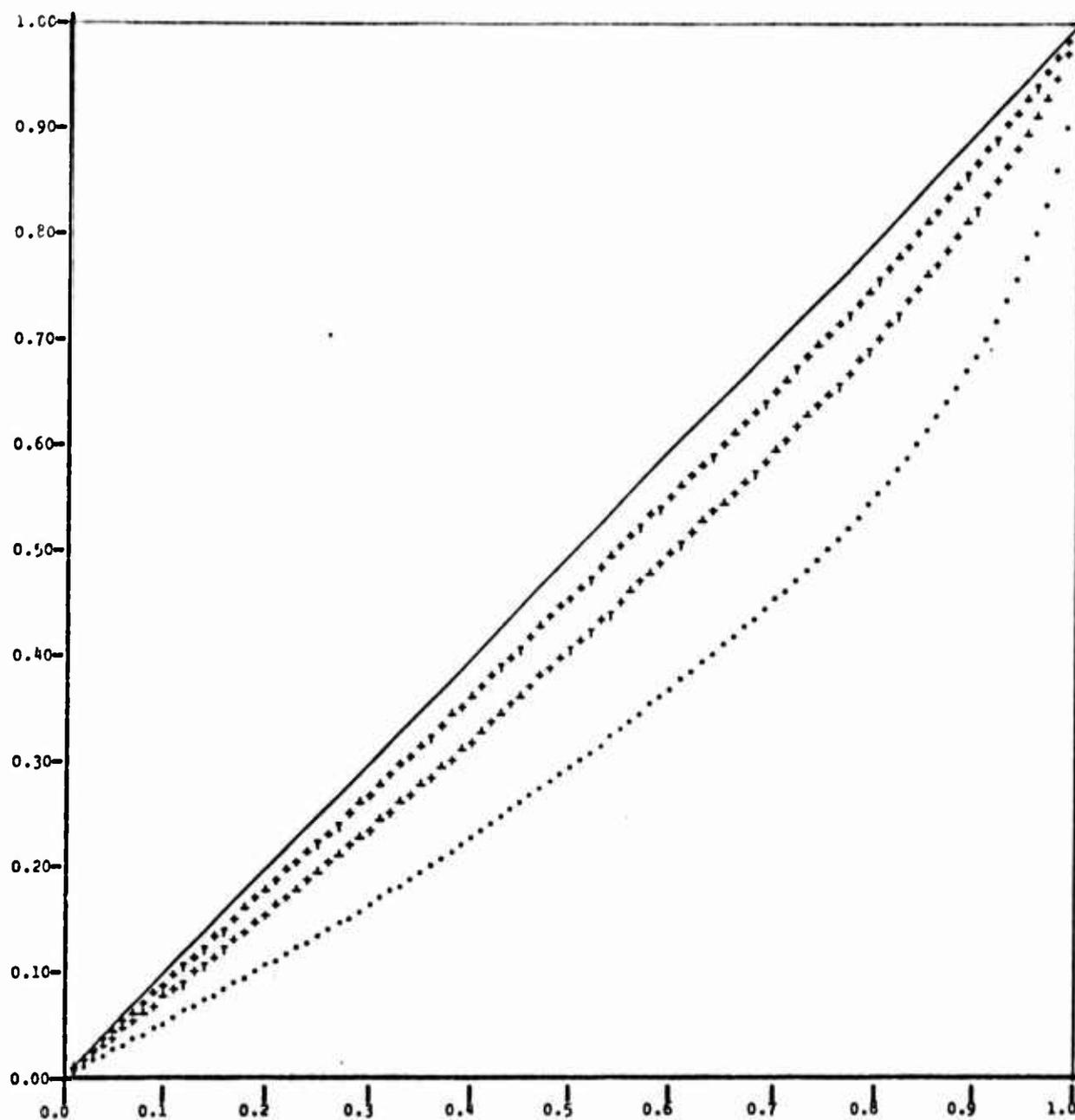


FIGURE 6: NORMALIZED TOTAL Q ON TEST TRANSFORM FOR THE GAMMA-MIXED PROPORTIONAL HAZARD FAMILY WITH THE CORRECT $Q(x)$, FOR VALUES OF THE HYPERPARAMETER $c_0 = 2, 4, 8$

- (1) For each value of θ (production run), plot the empirical ratio (6.5) for the appropriate number of completed lifetimes;
- (2) Connect these points by straight lines, or other reasonable curves;
- (3) For various values of the abscissa, find the average value of the ordinate over all runs.

Even though the individual curves may vary because of the few samples per batch, the average STQTT should more nearly reflect the theoretical transform curve. One then varies the form of Q , as before, until a satisfactory fit is obtained.

Figure 7 shows this procedure for 5 runs, with 10 complete samples in each run, when $R(Q^{-1}(x)) = x^{1.5}$. The raw data is shown unconnected, the average values over all runs for $t = 0(0.1)1.0$ are connected by straight line segments, while the theoretical STQTT is the curvilinear line. Limited computational experience with other simulated data gives similar results, surprisingly smooth when averaged over all runs.

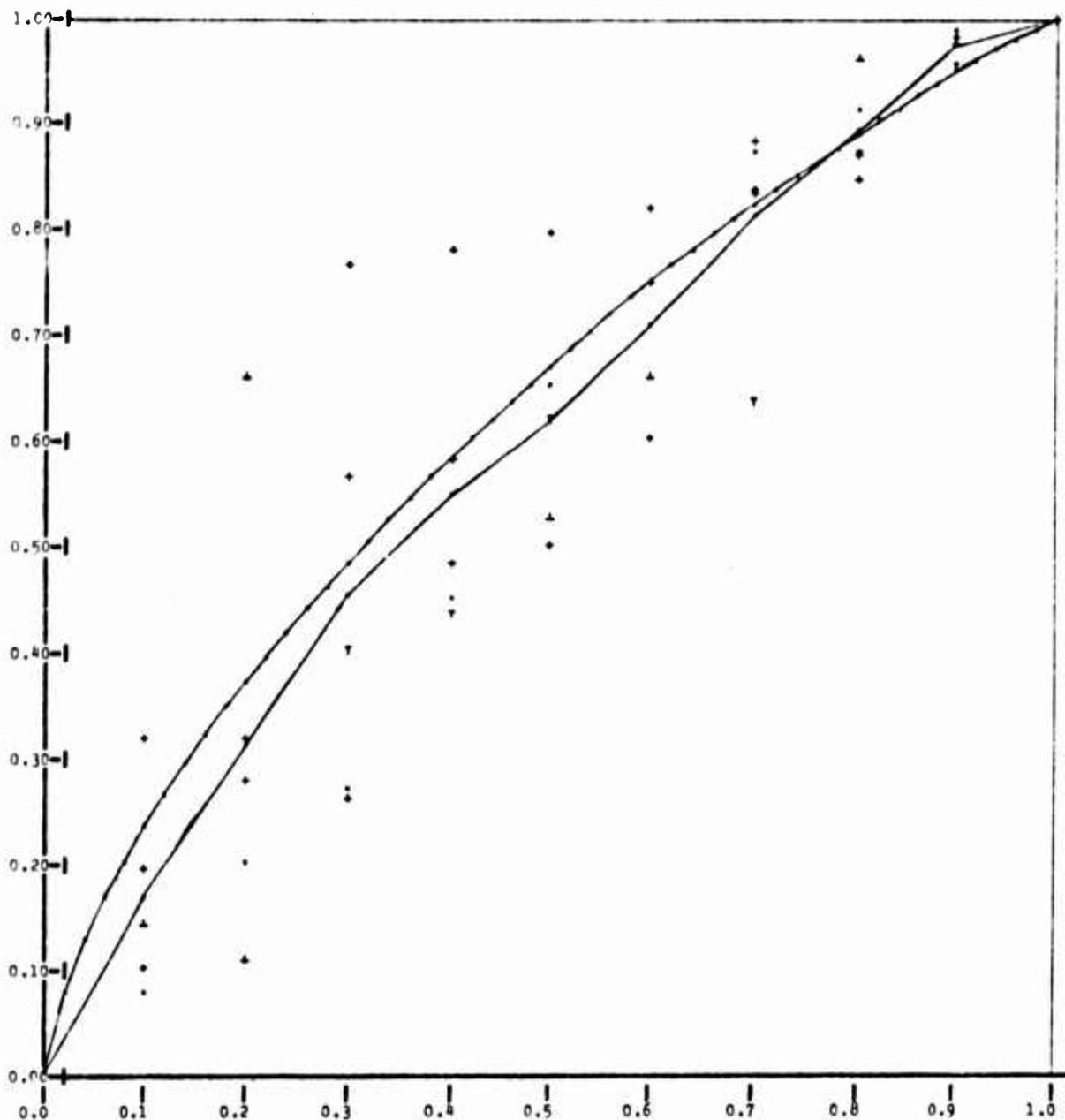


FIGURE 7: SIMULATED TQT DATA FROM PROPORTIONAL HAZARD FAMILY WITH $R(Q^{-1}(x)) = x^{1.5}$. FIVE RUNS, 10 COMPLETE SAMPLES FROM EACH RUN, COMPARING RAW DATA, AVERAGE CURVE, AND THEORETICAL CURVE.

9. EXTENSIONS

A natural extension to (3.1) is to consider the *competing hazard family*

$$(9.1) \quad P^C(x | \theta) = \exp \left[- \sum_{i=1}^m \theta_i Q_i(x) \right] ;$$

$$p(x | \theta) = \left[\sum_{i=1}^m \theta_i q_i(x) \right] \exp \left[- \sum_{i=1}^m \theta_i Q_i(x) \right] .$$

Here the Q_i , and their derivatives, q_i , are prototype failure constituents which are competing in an unknown mixture specified by parameters $\theta_1, \theta_2, \dots, \theta_m$, subject to a known m -dimensional prior density. For instance, with $m = 2$, one constituent might be random, $q_1(x) = 1$, and the other IFRA, $Q_2(x)/x \uparrow$ in x . Unfortunately, the likelihood is now more complex; corresponding to (3.2), we have for $m = 2$

$$(9.2) \quad p(x_1, x_2, \dots, x_N ; C | \theta_1, \theta_2) = \frac{N!}{(N-C)!} \prod_{i=C+1}^N \delta(x_i - T) \cdot$$

$$\left\{ \prod_{i=1}^C [\theta_1 q_1(x_i) + \theta_2 q_2(x_i)] \right\} e^{-\theta_1 T Q_1 T(D) - \theta_2 T Q_2 T(D)}$$

where TQ_1T and TQ_2T refer to the two possible test statistics. In place of θ^C , we have C terms corresponding to the possible powers of θ_1 and θ_2 . The term in inner braces is proportional to

$$\sum_{i=1}^C p_i(D) \theta_1^i \theta_2^{C-i}$$

where

$$(9.3) \quad p_1(\mathcal{D}) = K \cdot \prod_{\substack{\text{permutations} \\ \text{of } x_1, x_2, \dots, x_C}} \underbrace{q_1(x_{j_1}) q_1(x_{j_2}) \cdots q_1(x_{j_1})}_{i \text{ terms}} \underbrace{q_2(x_{j_{i+1}}) \cdots q_2(x_{j_C})}_{(C-1) \text{ terms}}$$

and K is adjusted so that $\sum p_1(\mathcal{D}) = 1$. The complexity for general m is easily visualized.

A convenient prior consists of independent gammas:

$$(9.4) \quad p(\theta_1, \theta_2 \mid C_{01}, Q_{01}; C_{02}, Q_{02}) \propto \left(\theta^{C_{01}-1} e^{-Q_{01}\theta} \right) \left(\theta^{C_{02}-1} e^{-Q_{02}\theta} \right);$$

we see that the posterior-to-data density is a mixture of C terms:

$$(9.5) \quad p(\theta_1, \theta_2 \mid \mathcal{D}) = \sum_{i=1}^C p_i(\mathcal{D}) \cdot p(\theta_1, \theta_2 \mid C_{01}+i, Q_{01}+TQ_1T(\mathcal{D}); C_{02}+TQ_2T(\mathcal{D}))$$

from which, for example, the updated means can be obtained, e.g.,

$$(9.6) \quad E(\tilde{\theta}_1 \mid \mathcal{D}) = \sum_{i=1}^C p_i(\mathcal{D}) \frac{C_{01} + i}{C_{01} [E\{\tilde{\theta}_1\}]^{-1} + TQ_1T(\mathcal{D})}.$$

Although these Bayesian calculations are more complicated than before, they are easily computerized for $m = 2$ or 3 .

Another possible extension is to a *competing lifetime model family*

$$(9.7) \quad P^c(\mathbf{x} \mid \theta) = \sum_{i=1}^m \pi_i \exp[-\theta_i Q_i(\mathbf{x})];$$

$$p(\mathbf{x} \mid \theta) = \sum_{i=1}^m \pi_i \theta_i q_i(\mathbf{x}) \exp[-\theta_i Q_i(\mathbf{x})].$$

Here the $\pi_i \geq 0$ ($i = 1, 2, \dots, m$), $\sum_{j=1}^m \pi_j = 1$, are random variables, with, say, a Dirichlet prior distribution, and the θ_i may be fixed or subject to a joint or independent prior. What happens, in general, is that, as data is received, the posterior estimates of the π_i change (along with the θ_i), thus tending to select the "correct" model. Unfortunately, even for $m = 2$, this approach leads to terms in which varying numbers of the Q_i are summed, eliminating the usefulness of the TQT statistic.

However, it is possible to extend the class of priors somewhat by using a *competing prior model family*

$$(9.8) \quad p(\theta) = \sum_{i=1}^m \pi_i p_i(\theta) ; P^c(x | \theta) = \exp \{-\theta Q(x)\} ;$$

where the π_i are either fixed, or have their own prior. Here the size of the computation remains fixed, and the TQT changes the weights π_i as a function of the data. For instance, if the $p_i(\theta) = p(\theta | C_{oi}, Q_{oi})$ are gamma, and the initial π_i are fixed, then posterior to the data we have from (3.2):

$$(9.9) \quad \pi_i(D) = K \cdot \pi_i \frac{\Gamma(C_{oi} + C)}{\Gamma(C_{oi})} \frac{Q_{oi}^{C_{oi}}}{[Q_{oi} + TQT(D)]^{C_{oi} + C}} ,$$

where K is chosen so that $\sum \pi_i(D) = 1$. Of course, this approach requires much more effort in estimating the hyperparameters.

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