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ENERGY SOLUTIONS FOR PREDICTING DEFORMATIONS IN
BLAST-LOADED STRUCTURES

P. S. Westine, et al

Southwest Research Institute

Prepared for:

Edgewood Arsenal

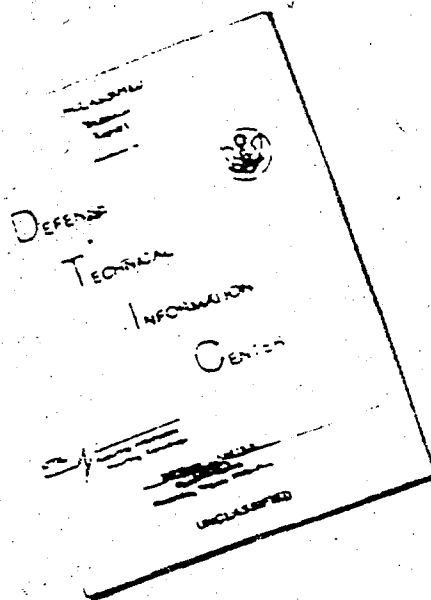
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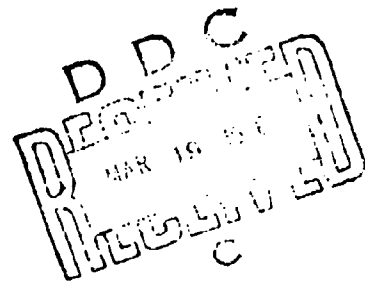
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ENERGY SOLUTIONS FOR PREDICTING DEFORMATIONS IN BLAST-LOADED STRUCTURES

by

P. S. Westine

W. E. Baker



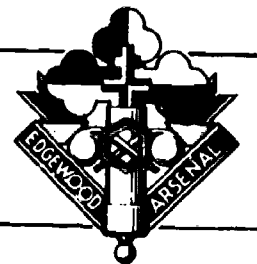
November 1975

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Contract No. DAAA15-75-0083
DAAD05-74-C-0751



DEPARTMENT OF THE ARMY
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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER EM-CR-76027	2. GOVT ACCESSION NO	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ENERGY SOLUTIONS FOR PREDICTING DEFORMATIONS IN BLAST-LOADED STRUCTURES		5. TYPE OF REPORT & PERIOD COVERED Technical Report, Jan. 1974 to Sept. 1974
		6. PERFORMING ORG. REPORT NUMBER Report No. 6
7. AUTHOR(s) P. S. Westine W. E. Baker		8. CONTRACT OR GRAND NUMBER(s) DAAD05-74-C-0751 DAAA15-75-C-0083
9. PERFORMING ORGANIZATION NAME AND ADDRESSES Southwest Research Institute P. O. Drawer 28510 San Antonio, Texas 78284		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PA, A 4932 Project 5751264
11. CONTROLLING OFFICE NAME AND ADDRESS Commander, Edgewood Arsenal Attn: SAREA-TS-R Aberdeen Proving Ground, MD 21010		12. REPORT DATE November 1975
		13. NUMBER OF PAGES 33
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Commander, Edgewood Arsenal Attn: SAREA-MT-H Aberdeen Proving Ground, MD 21010		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Energy solutions Plates Impulsive response Simple structures Quasi-static response Plastic deformation Beams		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Equations are developed for predicting permanent deformations of structural elements, based on energy solutions. The equations apply to either impulsive or quasi-static response of simple structures, beams and plates. Some solutions are compared to literature data.		

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SUMMARY

Energy solutions are excellent analysis procedures for predicting residual strains or deformations in structural elements when transient behavior is of little interest. In this report, we demonstrate how energy procedures can be used in rigid-plastic structural solutions when members are loaded either impulsively or quasi-statically by blast waves.

In the impulsive loading realm, the kinetic energy imparted to a structural member is equated to the plastic strain energy, whereas in the quasi-static loading realm, the work performed in deforming a structural member is equated to the plastic strain energy. An assumed first mode structural deformation pattern works well when calculating plastic strain energy in either of the loading realms. Experimental test data on deformed simply-supported and cantilever beams, clamped circular plates, and clamped rectangular plates demonstrate the validity of these solutions.

The test data on different types of structural elements are important, as the beam data involve only bending behavior, the circular plates have both bending and extensional action, and the rectangular plates introduce shearing behavior into the strain energy calculations. Because all solutions are closed-form ones, design formulae result which can be used to evaluate plastic deformation in blast loaded structural members.

This report is a reprint of a paper presented at the 16th Explosive Safety Seminar, Hollywood Beach, Florida, September 1974.

PREFACE

The investigation described in this report was authorized under PA, A 4932, Project 5751264. The work was performed at Southwest Research Institute under Contracts DAAD05-74-C-0751 and DAAAI5-75-C-0083.

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ACKNOWLEDGEMENTS

Some of the solutions presented here were developed for Mr. John Hennessey under Contract DAAE07-74-C-0192 with the U.S. Army Tank-Automotive Command. His cooperation and the help of other personnel at SwRI are appreciated.

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ENERGY SOLUTIONS FOR PREDICTING DEFORMATIONS IN BLAST-LOADED STRUCTURES

I. INTRODUCTION

Energy solutions are excellent analysis procedures for predicting residual strains or deformations in structural components whenever transient (time-dependent) behavior is of little interest. Although schools readily teach energy procedures for obtaining elastic solutions, few investigators apply the approach to dynamic plasticity problems.

Dynamic rigid-plastic energy solutions began in the early 1950's when Lee and Symonds^[1] used the static plastic-hinge concept, considered beam inertia, and propagated a traveling hinge to analytically obtain the upper bound for permanent deformation in a beam under a transverse load. Their Brown University associates and graduates such as S. R. Bodner, W. Prager, N. Jones, J. B. Martin, R. M. Haythornthwaite and others then added refinements, illustrations of which are given in Refs. 2-6. It was J. E. Greenspon in the 1960's who pointed out that one could obtain solutions without going through the details of propagating a plastic hinge along structural members [Refs. 7-11]. Greenspon noted that the residual strain energy stored in a plastically deformed member could be calculated by assuming a final deformed shape. This strain energy was then equated to the energy flux in an explosive blast wave.^[11] We disagree with this last step, which made deformations independent of structural orientation relative to the enveloping blast wave, thus ignoring an important effect observed in many experiments. In addition, Greenspon's procedure forces pressures and impulses in the blast wave to obey the relationship $P/I = \text{constant}$, whose asymptotes for both pressure and impulse are $P = 0$ and $I = 0$. The response of real targets is related to non-zero P and I limits, so this conclusion is also unacceptable.

We would emphasize that Greenspon was correct in his strain energy estimation procedures. Estimates of structural deformation would have been correct had he equated strain energy to the kinetic energy imparted to the structure for short duration impulsive loads. When durations are long relative to the structural response time, the strain energy is equated to the work performed when the peak load moves through the distance that the structure deforms. Hence, two separate procedures are required, one to obtain the solution for the impulsive loading realm, and the other to obtain the solution for the quasi-static loading realm. We will proceed to illustrate these procedures by computing results and comparing the test data. Our first illustration is a rheological model whose exact solution can be obtained and compared to the answers given by energy procedures.

II. SINGLE DEGREE-OF-FREEDOM SYSTEMS

Consider first a single-degree-of-freedom, rigid-plastic system as in Figure 1a. The motion of the mass m is resisted by a Coulomb friction element f when the blast load $p(t)$ is applied to the structure. We will approximate the blast loading with an exponential decay as in Figure 1b

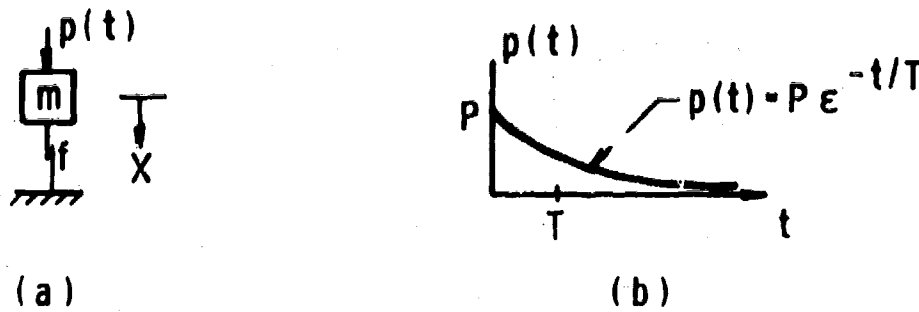


FIGURE 1. RIGID-PLASTIC, SINGLE-DEGREE-OF-FREEDOM DAMAGE MODEL

(where P is the maximum applied force and T is the time constant associated with the duration of loading). If $P/f \leq 1.0$, we have the trivial case where the residual deformation X equals 0 because the mass never moves. If $P/f \geq 1.0$, we can write the differential equation of motion:

$$P e^{-t/T} - f = m \frac{d^2 x}{dt^2} \quad (1)$$

By direct integration, we obtain for the case of zero initial velocity, the velocity relationship:

$$\frac{dx}{dt} = \frac{PT}{m} \left[1 - e^{-t/T} - \left(\frac{f}{P}\right) \left(\frac{t}{T}\right) \right] \quad (2)$$

Integrating again, we obtain for the case of zero initial displacement, the displacement equation:

$$x = \frac{PT^2}{m} \left[\frac{t}{T} + e^{-t/T} - \frac{1}{2} \left(\frac{f}{P}\right) \left(\frac{t}{T}\right)^2 - 1 \right] \quad (3)$$

Motion continues until the velocity, Eq. (2), equals zero or until:

$$e^{-t/T} + \left(\frac{f}{P}\right) \left(\frac{t}{T}\right) = 1.0 \quad (4)$$

We cannot explicitly solve for t/T in Eq. (4), as it is a transcendental equation; therefore, we assume values of P/f , solve for t/T , and substitute into the displacement equation [Eq. (3)] to obtain the maximum deformation X . Table 1 gives the results of such a calculation.

The maximum deformation X in the third column of Table 1 has been made nondimensional by dividing the left and right sides of Eq. (3) by $(PT^2)/m$. A solution can be presented for the maximum displacement by plotting $(Xm)/(PT^2)$ versus P/f . We have elected to divide P/f by $(Xm)/(PT^2)$ to form a new fourth column in Table 1 and to plot this new column $(PT)^2/(Xmf)$ versus P/f . The reason for this manipulation is that the product PT equals the applied total impulse I , and in this manner we create a scaled load-impulse or $P-I$ diagram. The solid line in Figure 2 is this scaled $P-I$ diagram for a simple rigid-plastic structure.

TABLE I

t/T	P/f	$(Xm)/(PT^2)$	$(PT)^2/(Xmf)$
	1.00	0	∞
0.09	1.033	0.001	1033
0.35	1.18	0.003	394
0.50	1.27	0.009	141
0.75	1.42	0.024	59.2
1.00	1.58	0.052	30.3
1.50	1.94	0.143	13.6
2.00	2.31	0.270	8.56
3.00	3.16	0.630	5.02
4.00	4.08	1.06	3.84
6.00	6.01	2.00	3.01
9.00	9.00	3.50	2.57
13.0	13.0	5.50	2.36
20.0	20.0	9.00	2.22
35.0	35.0	16.50	2.12
50.0	50.0	24.0	2.08
75.0	75.0	36.5	2.05
100.0	100.0	49.0	2.04
	∞		2.00

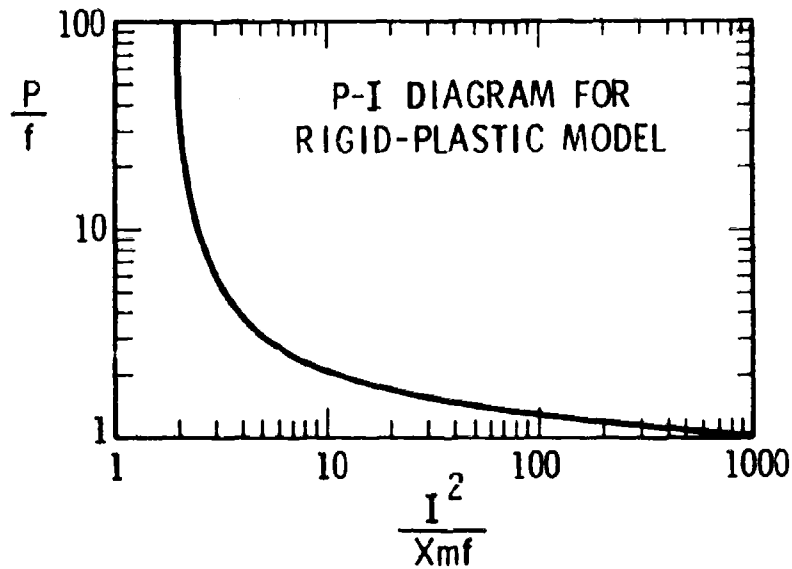


FIGURE 2. P-I DIAGRAM FOR RIGID-PLASTIC SYSTEM

Observe in Table 1 and Figure 2 that whenever $I^2/(Xmf)$ is greater than about 60, the duration of loading T is larger than the response time t and P/f equals 1.0. Similarly, whenever P/f is greater than about 20, durations of loading T are smaller than the response times t

of the responding structure, and $I^2/(Xmf)$ equals 2.0. The energy solutions which we will apply estimate both of these asymptotes.

The strain energy U stored in plastic deformation is given by:

$$U = fX \quad (5)$$

The kinetic energy KE imparted to the mass equals:

$$KE = \frac{1}{2} m V_e^2 = \frac{1}{2} m \left(\frac{I}{m} \right)^2 = \frac{I^2}{2m} \quad (6)$$

The work W done by the maximum force P acting through the distance X is:

$$W = PX \quad (7)$$

Equating the strain energy U , Eq. (5), to the kinetic energy KE , Eq. (6), yields the asymptote for the impulsive loading realm or:

$$\frac{I^2}{Xmf} = 2.0 \quad (\text{impulsive loading realm}) \quad (8)$$

The other asymptote is obtained by equating U , Eq. (5), to the work W , Eq. (7).

$$\frac{P}{f} = 1.0 \quad (\text{quasi-static loading realm}) \quad (9)$$

Had we wished to calculate maximum elastic deformations rather than plastic ones, the same procedures would apply. Replacing the Coulomb friction element with a linear elastic spring in Figure 1a would have yielded the analytical solid curved line shown in Figure 3. This solution also has asymptotes for the impulsive and quasi-static loading realms that can be obtained using energy procedures. The strain energy in an elastic system would be given by Eq. (10) rather than Eq. (5).

$$U = \frac{1}{2} kX^2 \quad (10)$$

In an elastic system the kinetic energy KE and work W are still given by Eqs. (6) and (7), respectively. For an elastic system, equating Eq. (10) to Eq. (6) yields the asymptote for the impulsive loading realm.

$$X = \frac{I}{\sqrt{km}} \quad (11)$$

or

$$\frac{X}{P/k} = \sqrt{\frac{k}{m}} T \quad (\text{impulsive loading realm}) \quad (12)$$

Similarly, equating Eq. (10) to Eq. (7) yields the quasi-static loading realm asymptote:

$$\frac{X}{P/k} = 2.0 \quad (\text{quasi-static loading realm}) \quad (13)$$

Both the impulsive loading realm asymptote and quasi-static loading realm asymptote are shown in Figure 3. Figure 3 illustrates that, if these asymptotes are known for an elastic as well as plastic system, the deformations can be predicted using energy procedures.

As has been illustrated in these simple models, the principles are as follows:

- (1) To estimate the impulsive loading realm structural deformation asymptote, estimate the strain energy in a deformed structure and equate this strain energy to the kinetic energy imparted to the structure.
- (2) To obtain the quasi-static structural deformation asymptote, equate the strain energy to the work performed by the peak force deflecting with the structure.

We are now prepared to illustrate these principles as they are applied to beam, plates, and similar more complex structural components which are loaded by blast waves or other transient pulses. The first structural components that we will study are cantilever beams.

III. BLAST LOADED CANTILEVER BEAMS

Consider a clamped cantilever beam of rectangular cross section as depicted in Figure 4. We will assume that the deformed shape for this structural component is given by

$$w = w_0 \left[1 - \cos \frac{\pi x}{2L} \right] \quad (14)$$

Notice that the assumed deformed shape has no deflection and no slope at $x = 0$ and no moment at $x = L$. The maximum deformation and maximum slope occur at $x = L$, and the maximum moment occurs at $x = 0$. These are the correct boundary conditions. A good solution does depend upon selecting an appropriate deformed shape.

Because no membrane action is developed, the strain energy is dissipated in bending. In an elastic member, the bending strain energy is given by:

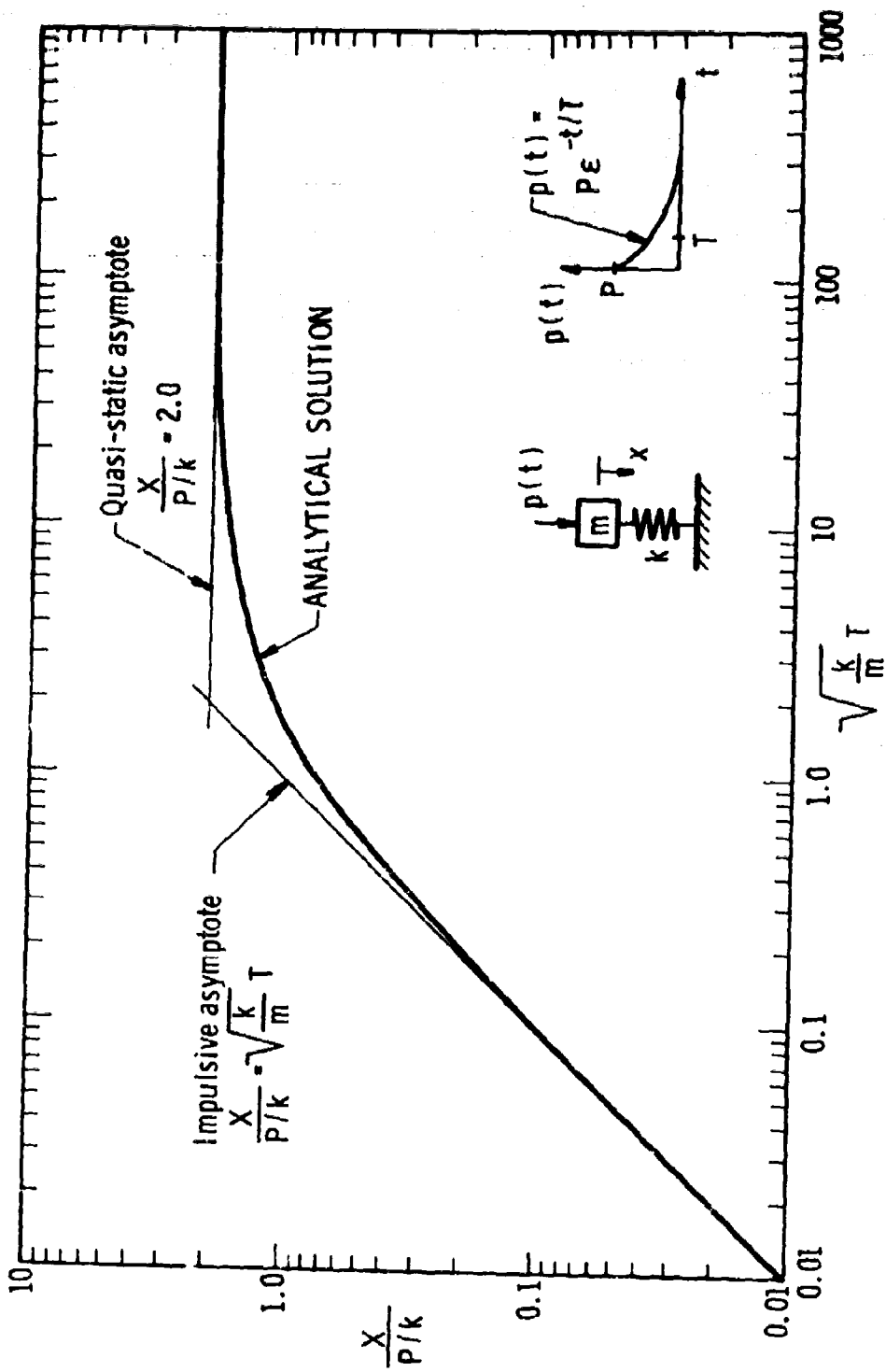


FIGURE 3. MAXIMUM RESPONSE TO FORCE PULSE

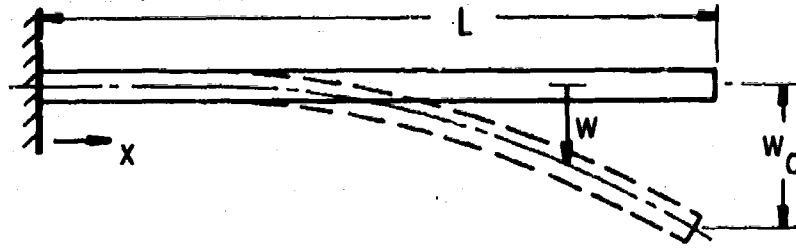


FIGURE 4. SCHEMATIC DRAWING OF CLAMPED CANTILEVER BEAM

$$U = \int_0^L \frac{M^2 dx}{2EI} \quad (15)$$

where the moment M is equal to $El(d^2w/dx^2)$. Differentiating Eq. (14) and substituting into Eq. (15) then yields:

$$U = \frac{\pi^4 Elw_0^2}{32L^4} \int_0^L \cos^2\left(\frac{\pi x}{2L}\right) dx \quad (16)$$

Or, upon integrating:

$$U = \frac{\pi^4 Elw_0^2}{64L^3} \quad (\text{bending in elastic cantilever beam}) \quad (17)$$

The kinetic energy imparted to the cantilever beam of mass density ρ , width b , and thickness h by a uniform specific impulse of intensity i is given by:

$$KE = \sum_{\text{beam}} \frac{1}{2} m v_0^2 \quad (18)$$

or

$$KE = \int_0^L \frac{1}{2} (\rho b h dx) \left(\frac{ib dx}{\rho b h dx} \right)^2 \quad (19)$$

Integration of the preceding quantity then yields:

$$KE = \frac{i^2 bL}{2\rho h} \quad (20)$$

The equation defining the impulsive loading realm asymptote for the elastic maximum tip deformation is then obtained by equating Eq. (20) to Eq. (17). The width b drops out of the solution when $bh^3/12$ is substituted for the second moment of area I in Eq. (17).

$$\frac{w_0}{L} = \frac{\sqrt{384}}{\pi^2} \left(\frac{L}{h}\right) \left(\frac{i}{h\sqrt{E\rho}}\right) \quad (21)$$

The quasi-static loading realm asymptote is predicted by estimating the work W associated with the peak drag load Q . The work is:

$$W = \sum_{\text{beam}} C_D Q b \, dx \, w \quad (22)$$

or

$$W = C_D Q b w_0 \int_0^L \left(1 - \cos \frac{\pi x}{2L}\right) dx \quad (23)$$

Integration of the preceding quantity then yields:

$$W = \left(1 - \frac{2}{\pi}\right) C_D Q b L w_0 \quad (24)$$

Equating Eq. (24) to Eq. (17) and substituting for the second moment of area then yields the quasi-static asymptote for the elastic maximum tip deformation.

$$\frac{w_0}{L} = \frac{875.5}{\pi^5} \left(\frac{C_D Q}{E}\right) \left(\frac{L}{h}\right)^3 \quad (25)$$

Maximum elastic strains at the root of the cantilever beam can also be calculated from the maximum tip deformations, either Eq. (21) or Eq. (25), dependent upon the loading realm. Substituting the second derivative of Eq. (14) into the moment curvature relationship, the moment into $\sigma = Mc/I$, the stress σ into $\epsilon = \sigma/E$, and $bh^3/12$ for the second moment of area I , yields the strain equation in terms of the maximum tip deformation. This equation is given by:

$$\epsilon = \frac{\pi^2 h w_0}{8L^2} \cos \frac{\pi x}{2L} \quad (26)$$

The strain ϵ will be a maximum at the root of the cantilever beam where $\cos \pi x/2L$ equals 1.0.

Substituting Eq. (21) for w_0 into Eq. (26) then yields the maximum strain equation for the impulsive loading realm.

$$\epsilon = 2.45 \frac{i}{h\sqrt{E\rho}} \quad (\text{impulsive loading realm}) \quad (27)$$

Equation (26) for the impulsive loading realm can be compared to experimental test data to demonstrate the validity of this analysis procedure. In 1958, Baker, et al.,^[12] detonated H.E. explosive charges in the vicinity of 6061-T6 aluminum cantilever beams. Plotted in Figure 5 is the maximum bending strain as a function of $i/(L\sqrt{\rho E})$ for beams with a length of 12 in. and thickness of 0.051 inch. Some uncertainty exists in computing the impulse imparted to the beams because of an air blast wave diffracting around the beams; hence, the test data are plotted as bars. As can be seen in Figure 5, Eq. (27) for the maximum elastic strain at the root of the cantilever beams accurately predicts experimentally observed results.

The strain at the root of cantilever beams in the quasi-static loading realm can be obtained by substituting Eq. (25) into Eq. (26) and setting the $\cos[(\pi\ell)/(2L)]$ equal to 1.0. This result yields:

$$\epsilon = 3.534 \left(\frac{L}{h}\right)^2 \left(\frac{C_D Q}{E}\right) \quad (\text{quasi-static realm}) \quad (28)$$

J. D. Day in unreported tests ran five blast loading experiments on clamped cantilever steel beams with strain gauges at the root of the beam. The steel beams were $6 \times 3/4 \times 3/4$ in. and were exposed to side-on overpressures P_s as summarized in Table 2. The drag pressure Q can be calculated from P_s , and assuming that C_D equaled 1.75, we obtain calculated strains as shown in the last column of Table 2. These calculated strains compare very favorably with experimental observed results.

Plastic response of cantilever beams can be estimated as easily as elastic response. If we assume a rigid-plastic model, the bending strain energy now becomes the integration of the moment-curvature relationships over the length of the beams. In other words:

$$U = \int_0^L M_y \frac{d^2 w}{dx^2} dx \quad (29)$$

Differentiating Eq. (14), the assumed deformed shape and substituting it into Eq. (29) gives:

$$U = \frac{\pi^2 M_y w_0}{8L^2} \int_0^L \cos \frac{\pi x}{2L} dx \quad (30)$$

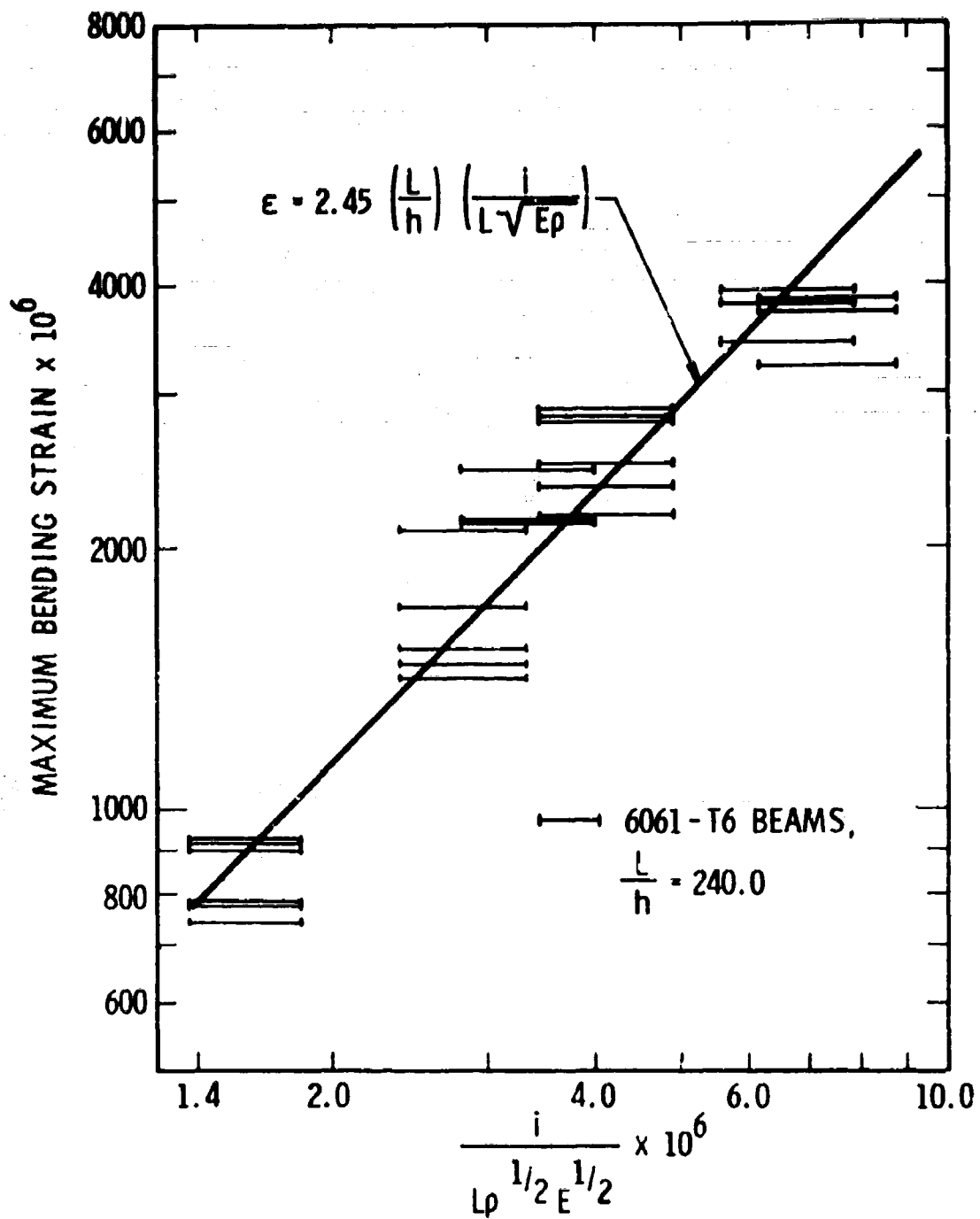


FIGURE 5. ELASTIC RESPONSE OF CANTILEVERS, IMPULSIVE LOADING REALM

TABLE 2. ELASTIC STRAINS IN CANTILEVERS,
QUASI-STATIC LOADING REALM

Test No.	P_y (psi)	Q (psi)	Experimental $\epsilon \times 10^{+6}$	Calculated $\epsilon \times 10^{+6}$
1	8.80	1.76	23.2	23.6
2	8.68	1.74	22.9	23.4
3	10.48	2.42	32.5	32.5
4	16.10	5.47	80.4	73.5
5	20.78	8.73	141.0	117.5

But in a rectangular member the yield moment $M_y = \sigma_y h^2 / 4$; hence, upon substituting for M_y and completing the integration, we obtain:

$$U = \frac{\pi \sigma_y b h^2 w_0}{16L} \quad (31)$$

Equating Eq. (31) to the kinetic energy, Eq. (20), then yields the asymptote for permanent residual tip deflection in the impulsive loading realm.

$$\frac{w_0}{L} = \frac{8}{\pi} \left(\frac{L}{h} \right) \left(\frac{i}{h \sqrt{\rho \sigma_y}} \right)^2 \quad (32)$$

Reference 12 also presents test data on permanent tip deflections for the same 6061-T6 aluminum cantilever beams as were used in the elastic strain. Figure 5, comparison. Plotted in Figure 6 are these scaled tip deflections as a function of $i/(L\sqrt{\rho E})$. Equation (32) was placed in the same format as the data in Figure 6 by multiplying and dividing Eq. (32) by the elastic modulus E . As can be seen in Figure 6, the agreement is relatively good. The disagreement that does arise at small values of scaled impulse $i/(L\sqrt{\rho E})$ is caused by our use of a rigid-plastic rather than elastic-plastic analysis.

The writers have no data for a cantilever beam permanent tip deflection comparison in the quasi-static loading realm. In the quasi-static loading realm, the displacement X becomes indeterminate when U , Eq. (31), is equated to WK , Eq. (24). The solution in the quasi-static loading realm is given by:

$$\frac{P}{\sigma_y} = 0.540 \left(\frac{h}{L} \right)^2 \quad (33)$$

The displacement w_0 in Eq. (33) is indeterminate just as X was indeterminate in the rigid-plastic rheological model, Eq. (9), for the quasi-static loading realm. This conclusion is correct for perfectly plastic nonhardening systems in the quasi-static loading realm. A hardening stress-strain law does result in finite displacements.

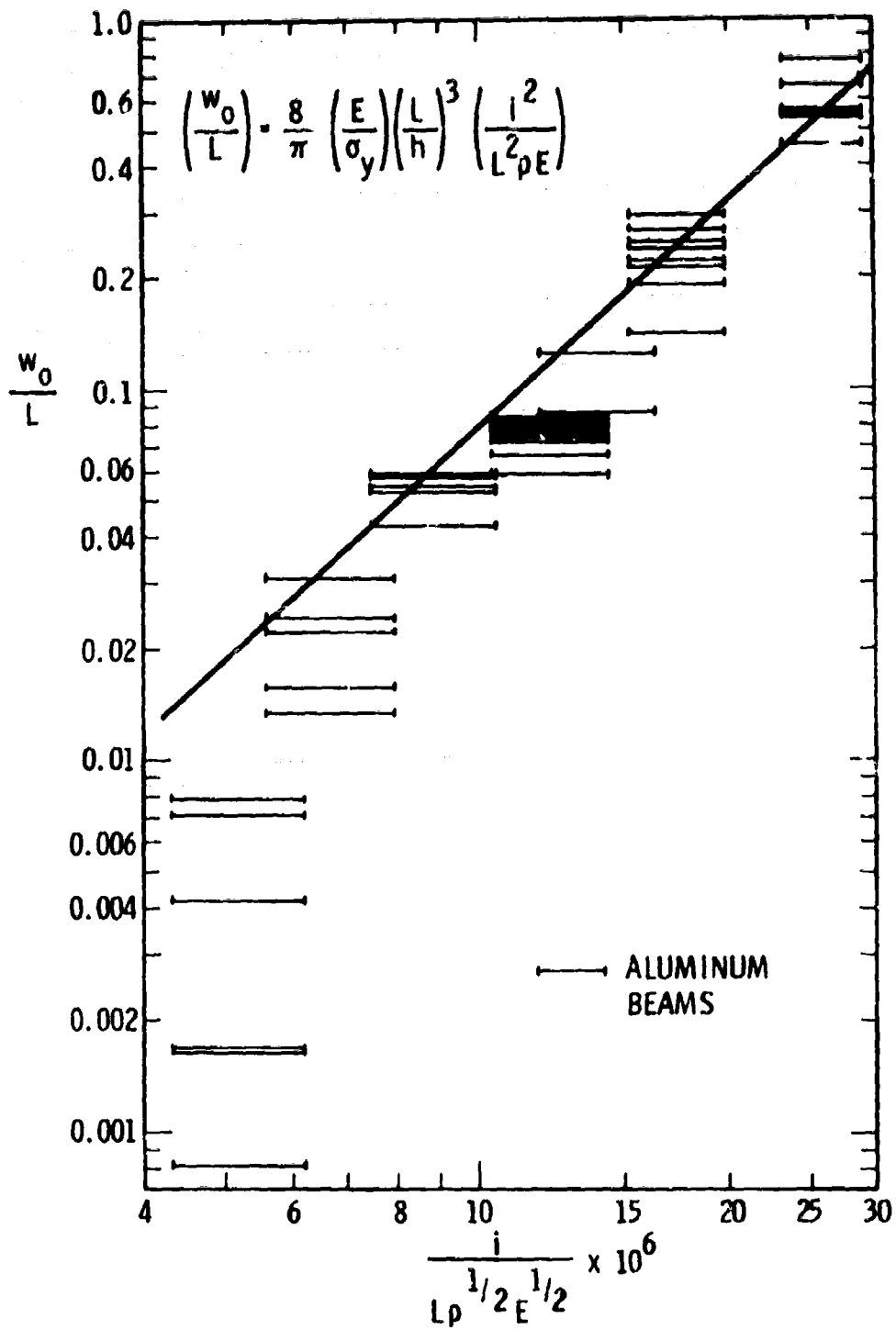


FIGURE 6. DAMAGE TO CANTILEVER BEAMS, IMPULSIVE LOADING REALM

IV. BENDING IN SIMPLY-SUPPORTED AND CLAMPED BEAMS

Our next illustration will be plastic bending in a simply-supported beam being loaded with a uniform load. Figure 7 shows the deformed shape of a bent simply-supported beam.

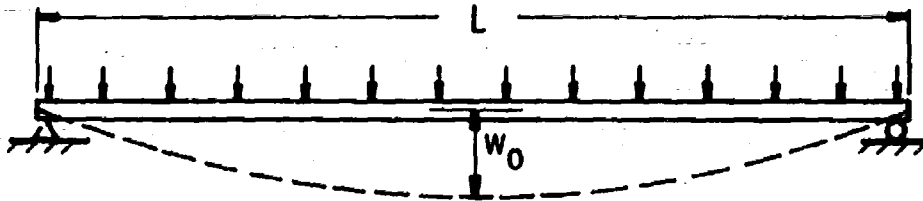


FIGURE 7. DEFORMED SHAPE OF SIMPLY-SUPPORTED BEAM

To calculate strain energy in this member, we must assume a deformed shape. Selecting:

$$w = w_0 \left(1 - \frac{4x^2}{L^2} \right) \quad (34)$$

as an appropriate deformed shape gives the appropriate boundary conditions. At $x = 0$, center of the beam, $w = w_0$ and the slope $(dw/dx) = 0$, while at the ends of the beam, $x = \pm L/2$, the deflections $w = 0$ and the slopes $(dw/dx) = -8xw_0/L^2$, a maximum value. The strain energy equals the plastic yield moment M_y for the beam cross-section times the change in angle of rotation integrated over the entire beam. Because the beam is symmetric and the change in angle of rotation with respect to x approximately equals $-d^2w/dx^2$, the strain energy U equals:

$$U = -2 \int_0^{L/2} M_y \frac{d^2w}{dx^2} dx \quad (35)$$

Differentiating Eq. (34), substituting it into Eq. (35) and integrating then yields:

$$U = \frac{8M_y w_0}{L} \quad (36)$$

The kinetic energy KE is obtained by summing the impulse squared divided by two times the incremental mass [see Eq. (6)] over the entire beam. If b is the width of the loaded member, ρ the density, A the cross-sectional area, and i the specific impulse, this summation yields the following integration.

$$KE = 2 \int_0^{L/2} \frac{i^2 b^2 (dx)^2}{2\rho A(dx)} \quad (37)$$

or

$$KE = \frac{i^2 b^2 L}{2\rho A} \quad (38)$$

Equating U , Eq. (36), to KE , Eq. (38), yields the asymptote for the impulsive loading realm.

$$\frac{i^2 b^2 L}{\rho M_y A} = 16 \left(\frac{w_0}{L} \right) \quad (\text{impulsive realm s.s. beam}) \quad (39)$$

The work W is obtained by integrating over the length of the beam, the forces times the distances through which they move. This operation is performed by integrating $pb \, dx$ times the assumed deformed shapes, Eq. (34).

$$W = 2 \int_0^{L/2} pbw_0 \left(1 - \frac{4x^2}{L^2} \right) dx \quad (40)$$

or

$$W = \frac{2}{3} pbLw_0 \quad (41)$$

Equating W , Eq. (41), to U , Eq. (36), yields the quasi-static asymptote,

$$\frac{pbL^2}{M_y} = 12 \quad (\text{quasi-static realm s.s. beam}) \quad (42)$$

So far these calculations have assumed that the beam is simply-supported and free to rotate at the ends. If the beam is clamped at the ends, no rotation occurs, but can move inwards so that no membrane action is developed; we can use many of the results which have already been developed. To do this, assume that a clamped beam is really two simply-supported beams that have been split and joined end to end as in Figure 8.

This new configuration implies that:

$$U = 2U_{s.s.} = \frac{16M_y w_0}{\rho} \quad (43)$$

and that

$$KE = 2KE_{s.s.} = \frac{i^2 b^2 L}{\rho A} \quad (44)$$

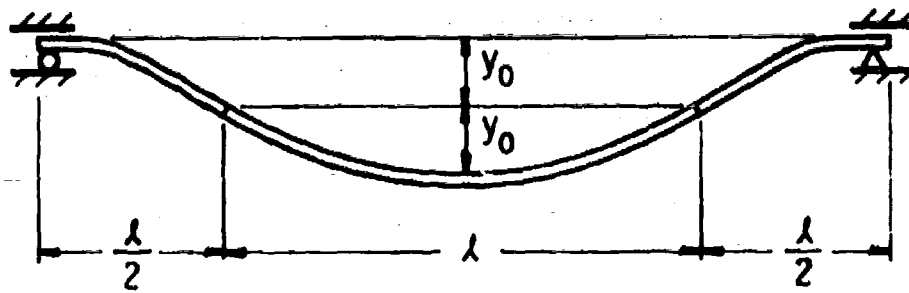


FIGURE 8. DEFORMED SHAPE OF CLAMPED BEAMS

but $\ell = L/2$ and $y_0 = w_0/2$, so substituting for ℓ and y_0 and equating U to KE yields:

$$\frac{i^2 h^2 L}{\rho A M_y} = 32 \frac{w_0}{L} \quad (\text{impulsive realm clamped beam}) \quad (45)$$

Because Eq. (45) for clamped beams is twice Eq. (39) for simply-supported beams, we can write

$$\frac{i^2 h^2 L}{N \rho A M_y} = 16 \frac{w_0}{L} \quad (\text{impulsive realm beam bending}) \quad (46)$$

where

$N = 1.0$ for simply-supported beams

$N = 2.0$ for clamped beams

To experimentally demonstrate the validity of this solution, we have plotted experimental data taken by Florence and Firth^[1,3] and compared these data to Eq. (46). Because Florence and Firth used beams with rectangular cross-sections, $M_y = \sigma_y b h^2 / 4$. Substituting for M_y , $b h$ for A , and 2ℓ for L (they used half spans), yields:

$$\frac{i^2}{N \rho \sigma_y h^2} = \left(\frac{h}{\ell} \right) \left(\frac{w_0}{\ell} \right) \quad (\text{impulsive realm rectangular beam bending}) \quad (47)$$

All of the beams tested by Florence and Firth had an ℓ/h ratio of 36, so this comparison is made by plotting $i/(h\sqrt{N\rho\sigma_y})$ versus w_0/ℓ . All beams are impulsively loaded using sheet explosive. Both clamped and pinned beams made of 2024-T4 aluminum, 6061-T6 aluminum, 1018 cold rolled steel, and 1018 annealed steel are included in this comparison. Figure 9 demonstrates the validity of Eq. (47) and this analysis procedure.

Because we have no data for clamped beams or simply-supported beams in the quasi static realm, we will not develop the equations, but will give the results. Because the clamped beam in bending is 8/3 times Eq. (42) for simply-supported bending, we can write:

$$\frac{pbL^2}{M_y} = 12N^{1.415} \quad (\text{quasi-static realm beam bending}) \quad (48)$$

where

$N = 1$ simply-supported beam

$N = 2$ clamped beam

V. CIRCULAR PLATE (BENDING AND EXTENSIONAL BEHAVIOR)

The next solution that we will evaluate is for the residual mid-point deformation in uniformly impulsed clamped circular plates. This problem adds an additional term to the strain energy expression, as both bending and extensional action will be present. A possible deformed shape for a clamped plastically deformed circular plate is:

$$w = \frac{w_0}{2} \left(1 + \cos \frac{\pi r}{R} \right) \quad (49)$$

The deformed shape is being described by a radial (radius r and angle θ) coordinate system with its origin at the center of the plate. Because of symmetry, the deformed shape is independent of the angle θ . Equation (49) meets the appropriate boundary conditions for a clamped circular plate in that at the center with $r = 0$, w is a maximum deformation of w_0 , and $(dw/dr) = 0$ while at the edge of the plate with $r = R$, $w = 0$ and $(dw/dr) = 0$. An inflection point occurs at $r = R/2$ when the curvature changes from negative to positive for increasing values of r .

Because no change in length occurs circumferentially, there is no circumferential strain and no circumferential strain energy in a clamped circular plate. The radial strain energy per unit volume equals the stress σ times the strain ϵ in any structural element. If we assume that the plate is yielding, the stress σ must equal the yield stress σ_y in a rigid plastic constitutive relationship, and if plane sections remain plane, the bending strain for small deformations is given by $-z(d^2w/dr^2)$ where z is the plate coordinate perpendicular to the r and θ coordinates. To compute the bending strain energy contribution U_b , create a circular differential torus of circumference $2\pi r$, thickness dz , and width dr . The differential bending strain energy dU_b equals this volume times σ_y times $-z(d^2w/dr^2)$. The total bending strain energy U_b is obtained by integrating the differential strain energy over the plate thickness and plate radius. Equation (50) describes this mathematical procedure.

$$U_b = \int_{-h/2}^{+h/2} \int_0^R (2\pi r \, dr \, dz) (\sigma_y) \left(-z \frac{d^2 w}{dr^2} \right) + \int_{h/2}^{+h/2} \int_{R/2}^R (2\pi r \, dr \, dz) (-\sigma_y) \left(-z \frac{d^2 w}{dr^2} \right) \quad (50)$$

Equation (50) must be integrated in parts because the sign of the bending stress changes at the inflection point which is located at $R/2$. Differentiating Eq. (49), substituting it into Eq. (50), and integrating over dz plus dr then yields:

$$U_b = \frac{\pi^2 \sigma_y h^2 w_0}{4} \quad (51)$$

Next we estimate the extensional strain energy U_e through a similar procedure. Extensional strains in a circular plate are generally given by $\epsilon = \sqrt{1.0 + (dw/dr)^2} - 1.0$. Using the binomial expansion and retaining only the first two terms leads to the approximate expression for extensional strains, $\epsilon \approx 0.5 (dw/dr)^2$. We will use the same differential torus to compute extensional strain energy as was used to compute bending strain energy. The total extensional strain energy U_e is obtained by integrating the differential volume times σ_y times $0.5 (dw/dr)^2$ over plate thickness and plate radius. Equation (52) mathematically describes this procedure for obtaining extensional strain energy.

$$U_e = \int_{-h/2}^{+h/2} \int_0^R (2\pi r \, dr \, dz) (\sigma_y) \left[\frac{1}{2} \left(\frac{dw}{dr} \right)^2 \right] \quad (52)$$

Differentiating Eq. (49), substituting it into Eq. (52), and integrating yields:

$$U_e = \frac{\pi^3 \sigma_y h w_0^2}{16} \quad (53)$$

The total strain energy U is the sum of U_e [Eq. (53)] and U_b [Eq. (51)], or:

$$U = \frac{\pi^2}{4} \sigma_y h^2 w_0 + \frac{\pi^3}{16} \sigma_y h w_0^2 \quad (54)$$

The kinetic energy KE for a uniformly applied impulsive load imparted to the plate is obtained just as it was for a beam, by summing up the impulse squared divided by two times the incremental mass over the surface of the entire plate. This summation leads to the following integration.

$$KE = \int_0^R \frac{i^2 (2\pi r)^2 (dr)^2}{2\rho h (2\pi r) (dr)} \quad (55)$$

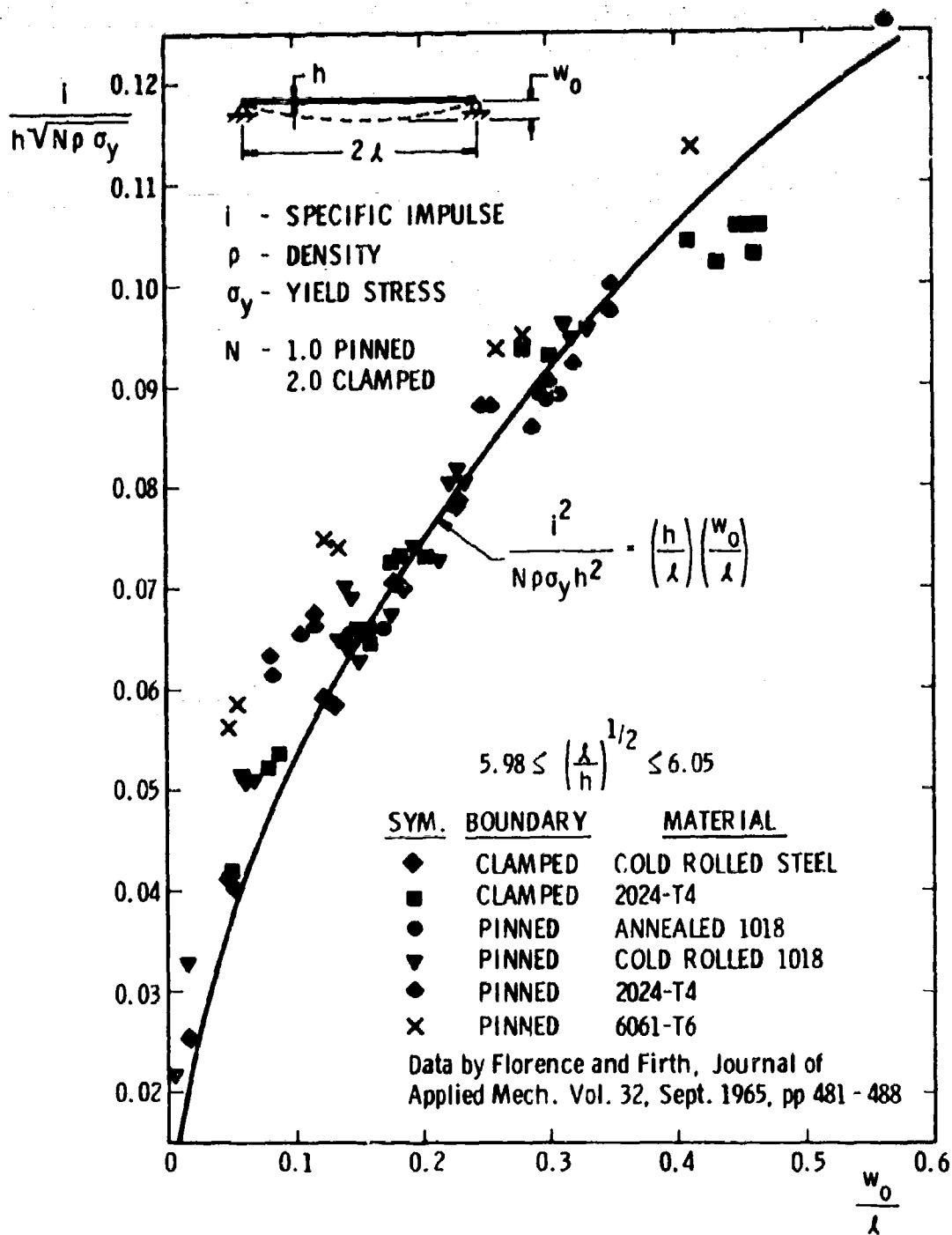


FIGURE 9. BEAM BENDING IN THE IMPULSIVE LOADING REALM

or:

$$KE = \frac{\pi^2 R^2}{2\rho h} \quad (56)$$

Equating U [Eq. (54)] to KE [Eq. (56)] yields the asymptote for the impulsive loading realm.

$$\left[\frac{iR}{\sqrt{\rho \sigma_v} h^2} \right]^2 = \frac{\pi}{2} \left(\frac{w_0}{h} \right) + \frac{\pi^2}{8} \left(\frac{w_0}{h} \right)^2 \quad (\text{impulsive realm clamped circular plate}) \quad (57)$$

A comparison between Eq. (57) for a uniform impulse on a clamped circular plate and experimental test data can be made using test results by Florence.^[14] Residual permanent mid-span deformations were measured on both clamped circular 6061-T6 aluminum plates and 1018-cold rolled steel plates that had been loaded uniformly with various layers of sheet explosive. The 22 aluminum data points, 20 steel data points, and Eq. (57) are all shown in Figure 10. Once again the validity of this solution and analysis procedure are substantiated. If there exists a systematic error, it is a tendency for the analytical curve to slightly underestimate deformations whenever w_0/h is large. This error is probably caused by the assumed deformed shape not yielding the minimum strain energy. For small values of w_0/h , the analytical curve overestimates deformations. This error is created because we assume that deformations extend over the entire span of the plate. When loads are small and deformations small, the deformed shape covers only a portion of the entire plate.

VI. RECTANGULAR PLATE (ADDITION OF SHEARING STRAINS)

The final solution that we will develop is for the plastic response of rectangular plates from uniformly applied transverse impulses. This solution introduces a new complication associated with a lack of radial symmetry and the bending plus extensional shearing forces that are created thereby. For a clamped-clamped uniformly loaded rectangular plate, we will assume a deformed shape given by:

$$w = \frac{w_0}{4} \left(1 + \cos \frac{\pi x}{X} \right) \left(1 + \cos \frac{\pi y}{Y} \right) \quad (58)$$

where

X and Y are half spans

x and y are the rectangular coordinate system with its origin in the center of the plate.

This assumed deformed shape meets the appropriate deflection and slope criteria in the middle of the plate and along all boundaries. Lines of inflection occur at $y = Y/2$ and $x = X/2$. The

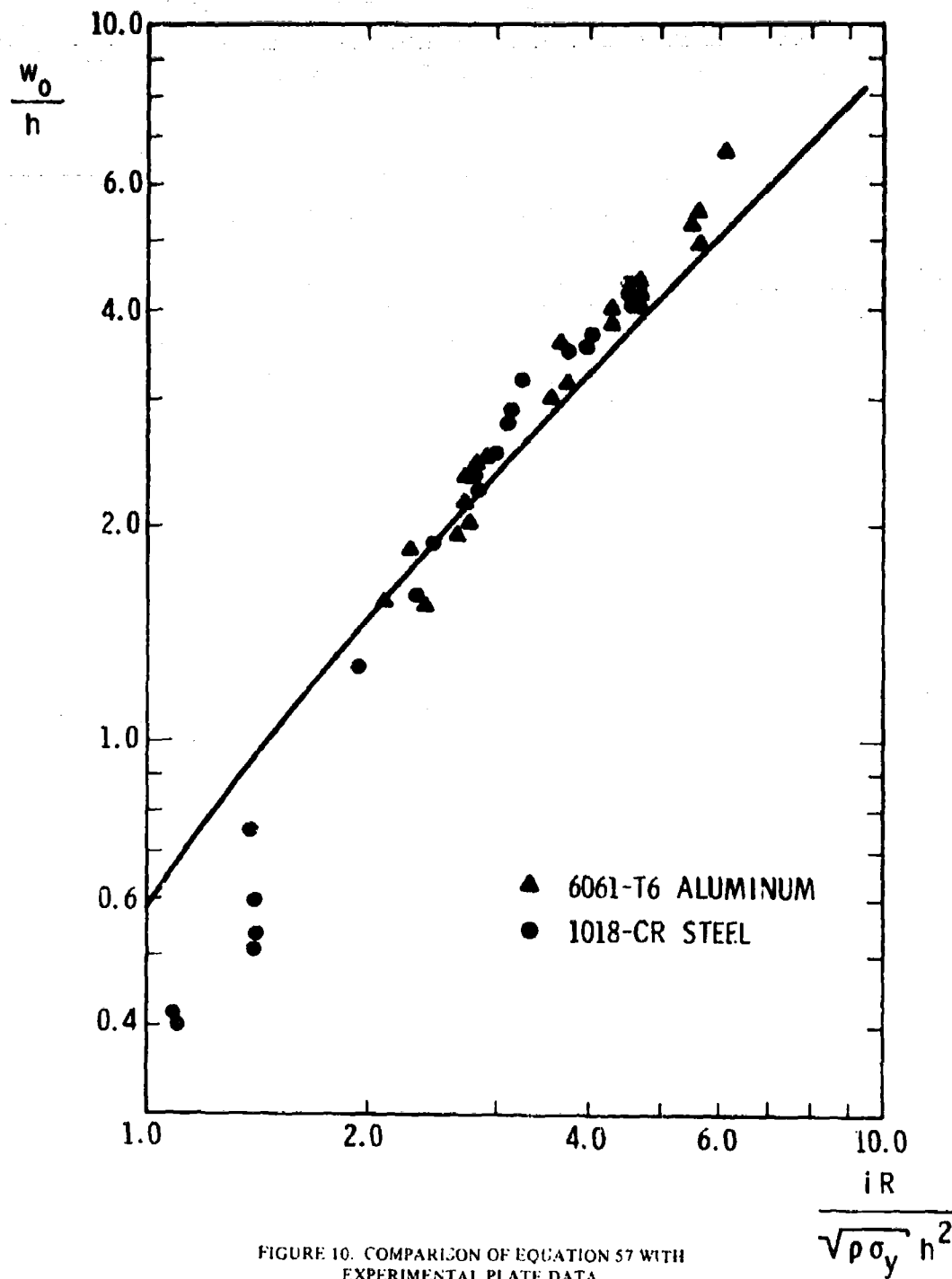


FIGURE 10. COMPARISON OF EQUATION 57 WITH EXPERIMENTAL PLATE DATA

strain energy per unit volume in a structural element under a biaxial state of stress is:

$$\frac{U}{\text{Vol.}} = \int_{\text{strains}} [\sigma_{xx} d\epsilon_{xx} + 2\sigma_{xy} d\epsilon_{xy} + \sigma_{yy} d\epsilon_{yy}] \quad (59)$$

Because we have yielding, we will assume that $\sigma_{xx} = \sigma_y$ and $\sigma_{yy} = \sigma_y$, but for the shearing stress we will use a Huber-Mises-Hencky distortion energy yield criteria of $\sigma_{xy} = \sigma_y/\sqrt{3}$. The normal bending strains are as in the circular plate problem with $\epsilon_{xx} = -z(\partial^2 w/\partial x^2)$ and $\epsilon_{yy} = -z(\partial^2 w/\partial y^2)$. The bending shearing strain $\epsilon_{xy} = 2z(\partial^2 w/\partial x\partial y)$. The bending strain energy U_b is then given by:

$$U_b = 8 \int_0^{h/2} dz \int_0^X dx \int_0^Y dy \sigma_y \left(-z \frac{\partial^2 w}{\partial x^2} - z \frac{\partial^2 w}{\partial y^2} \right) + 8 \int_0^{h/2} dz \int_0^X dx \int_0^Y dy \left(2 \frac{\sigma_y}{\sqrt{3}} \right) \left(2z \frac{\partial^2 w}{\partial x\partial y} \right) \quad (60)$$

The bending strain energy contributions from the normal stress and strains must be obtained by four part integration (from 0 to $X/2$ and 0 to $Y/2$, from $X/2$ to X and 0 to $Y/2$, from 0 to $X/2$ and $Y/2$ to Y , plus from $X/2$ to X and from $Y/2$ to Y) because the lines of inflection change the sign of the stress. Differentiating Eq. (58), substituting it into Eq. (60), and performing the triple integration then yields:

$$U_b = \frac{\pi}{2} \sigma_y h^2 w_0 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{4}{\sqrt{3}} \sigma_y h^2 w_0 \quad (61)$$

The first term on the right hand side of Eq. (61) is the contribution from the normal bending strains, and the second term is the contribution from the bending shear strains.

Equation (59) applies to the extensional response as well as bending response. The normal extensional strains are as in the circular plate problem with $\epsilon_{xx} = 0.5(\partial w/\partial x)^2$ and $\epsilon_{yy} = 0.5(\partial w/\partial y)^2$. The extensional shearing strain $\epsilon_{xy} = (\partial w/\partial x)(\partial w/\partial y)$ as a first approximation. These additional observations mean that the extensional strain energy U_e is given by:

$$U_e = 4 \int_0^X dx \int_0^Y dy \int_0^h dz \sigma_y \left[\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] + 4 \int_0^X dx \int_0^Y dy \int_0^h dz \left(\frac{2\sigma_y}{\sqrt{3}} \right) \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right) \quad (62)$$

Differentiating Eq. (58), substituting it into Eq. (62), and performing the triple integration yields:

$$U_e = \frac{3\pi^2}{32} \sigma_y h w_0^2 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{8}{\sqrt{3}} \sigma_y h w_0^2 \quad (63)$$

The total strain energy U is then the sum of U_e [Eq. (63)] and U_b [Eq. (61)], or:

$$U = \frac{\pi}{2} \sigma_y h^2 w_0 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{4}{\sqrt{3}} \sigma_y h^2 w_0 + \frac{3\pi^2}{32} \sigma_y h w_0^2 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{8}{\sqrt{3}} \sigma_y h w_0^2 \quad (\text{clamped plate}) \quad (64)$$

The strain energy in a simply-supported plate can also be estimated using the same procedure, with an assumed deformed shape described by:

$$w = w_0 \cos \frac{\pi x}{2X} \cos \frac{\pi y}{2Y} \quad (65)$$

The previous procedure gives as a value for the strain energy U ,

$$U = \sigma_y h^2 w_0 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{4}{\sqrt{3}} \sigma_y h^2 w_0 + \frac{\pi^2}{8} \sigma_y h w_0^2 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{4}{\sqrt{3}} \sigma_y h w_0^2 \quad (\text{s.s. plate}) \quad (66)$$

Notice that Eq. (66) for strain energy in a simply-supported plate is similar to Eq. (64) for strain energy in a clamped plate. The only difference in these equations is in the numerical values of the coefficients which accompany each term on the right hand side of these expressions. This observation means that if we insert a coefficient N that is equal to 1.0 for simply-supported plate and equal to 2.0 for clamped plates, a general strain energy equation can be written as in Eq. (67).

$$U = \frac{\pi^{N-1}}{N} \sigma_y h^2 w_0 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{4}{\sqrt{3}} \sigma_y h^2 w_0 + \frac{3^{N-1} \pi^2}{8N^2} \sigma_y h w_0^2 \left[\frac{Y}{X} + \frac{X}{Y} \right] + \frac{4N}{\sqrt{3}} \sigma_y h w_0^2 \quad (67)$$

The kinetic energy KE imparted to a plate is not dependent upon the deformed shape. For a uniformly applied impulsive load imparted to a plate, the kinetic energy is obtained

(as it was for a circular plate) by summing up the impulse squared divided by two times the incremental mass over the surface of the plate. The appropriate integration is given by:

$$KE = 4 \int_0^X \int_0^Y \frac{i^2(dx)^2(dy)^2}{2\rho h(dx)(dy)} \quad (68)$$

or

$$KE = \frac{2i^2 XY}{\rho h} \quad (69)$$

Equating Eq. (67) and Eq. (69) finally yields a general rectangular plate equation for deformations caused by uniform impulsive loads. The following equation is this relationship.

$$\left[\frac{iX}{\sqrt{\rho\sigma_y h^2}} \right]^2 = \frac{\pi^{N-1}}{2N} \left[1 + \left(\frac{X}{Y} \right)^2 \right] \left(\frac{w_0}{h} \right) + \frac{2}{\sqrt{3}} \left[\frac{X}{Y} \right] \left(\frac{w_0}{h} \right) + \frac{3^{N-1} \pi^2}{16N^2} \left[1 + \left(\frac{X}{Y} \right)^2 \right] \left(\frac{w_0}{h} \right)^2 + \frac{2N}{\sqrt{3}} \left[\frac{X}{Y} \right] \left(\frac{w_0}{h} \right)^2 \quad (70)$$

We will demonstrate the validity of this solution by using clamped ($N = 2$) rectangular plate data reported by N. Jones, T. O. Uran, and S. A. Tekin.^[15] Rectangular plates with an aspect ratio (Y/X) equal to 1.695 were loaded with sheet explosive in these experiments. Both hot-rolled mild steel plates and 6061-T6 aluminum plates were tested and can be seen plotted in Figure 11. Equation (69) is also shown in Figure 11; however, left and right sides of Eq. (69) were multiplied by $(Y/X)^2$ to cast it into the format of their data. Excellent correlation appears in Figure 11 when Eq. (70) is compared to experimental test results.

Unfortunately, no experimental quasi-static loading realm data exist on dynamically loaded, plastically deformed plates, simply-supported beams, or clamped beams. The analytical solution is easily developed for a plate. The work W performed on either simply-supported or clamped plates is obtained by integrating $\rho w dx dy$ and equals:

$$W = \frac{\pi^2(N-2)}{16N^2} \rho w_0 XY \quad (71)$$

When this work is equated to the general strain energy expression, Eq. (67), we obtain the solution for the quasi-static loading realm of rectangular plates.

$$\frac{\rho}{\sigma_y} \left(\frac{X}{h} \right)^2 = \frac{\pi^{2/N}}{2^{4/N}} \left[1 + \left(\frac{X}{Y} \right)^2 \right] + \frac{4^{(2N-3)}}{\sqrt{3}\pi^{2(N-2)}} \left[\frac{X}{Y} \right] + \frac{3^{(N-1)}4^{(N-2)}\pi^{2(3-N)}}{32} \left[1 + \left(\frac{X}{Y} \right)^2 \right] \left(\frac{w_0}{h} \right) + \frac{2^{(N-1)}4^{(2N-3)}}{\sqrt{3}\pi^{2(N-2)}} \left[\frac{X}{Y} \right] \left(\frac{w_0}{h} \right) \quad (72)$$

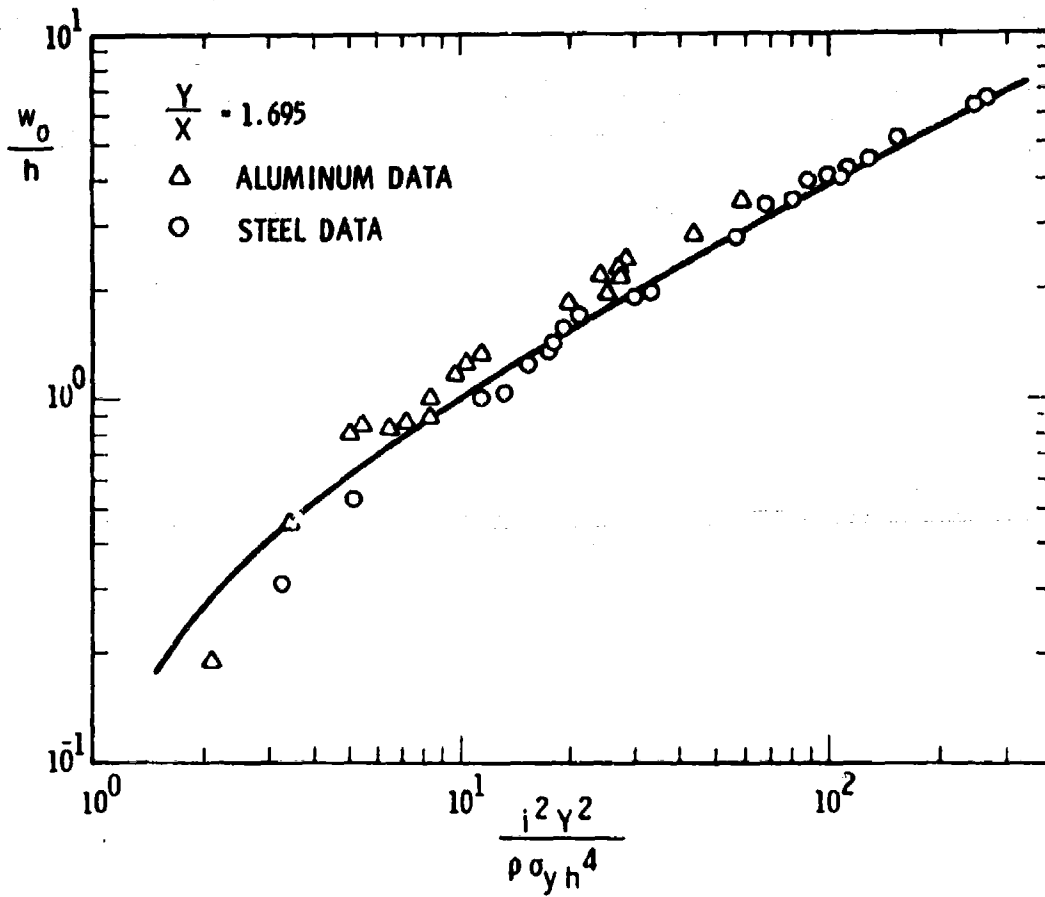


FIGURE 11. PREDICTED AND EXPERIMENTAL DEFORMATIONS IN UNIFORMLY LOADED RECTANGULAR PLATES

VII. DISCUSSION

These procedures and the resulting formulae comprise the structural analysis approach that was used by SwRI to control deformations in the design of a plastically-yielding, blast-suppressing structure. This analysis procedure yields explicit expressions for nondimensional deformations in terms of nondimensional loading parameters. If other deformed shapes were to be assumed, the same nondimensional parameters would occur; however, the numerical coefficients would have other values. These analyses can also be applied to shell type structures with no more difficulty than was demonstrated in the development of the plate solutions. The approach is attractive because no involved solutions to complex differential equations are required—only integrations of work, kinetic energy, and strain energy are needed for a given structural element.

The illustrations shown in this paper have included a single-degree-of-freedom plastic model, a simple elastic model, elastic cantilever beams, plastic simply-supported beams, plastic clamped beams, plastic clamped circular plates, plastic simply-supported rectangular plates, and plastic clamped rectangular plates. Experimental test data obtained by other investigators have been used to demonstrate the validity of these solutions, especially in the impulsive loading realm. It is interesting to note that all plastic beam solutions have the functional formats

$$\frac{i^2 L}{\rho \sigma_y h^3} = f\left(\frac{w_0}{L}\right) \quad (\text{impulsive realm any beam}) \quad (73)$$

and

$$\frac{p L^2}{\sigma_y h^2} = \text{constant} \quad (\text{quasi-static realm any beam}) \quad (74)$$

As extensional behavior is developed by elements such as plates, these general functional formats are:

$$\frac{i^2 X^2}{\rho \sigma_y h^4} = f\left(\frac{Y}{X}, \frac{w_0}{h}\right) \quad (\text{impulsive realm plates}) \quad (75)$$

and

$$\frac{p X^2}{\sigma_y h^2} = f\left(\frac{Y}{X}, \frac{w_0}{h}\right) \quad (\text{quasi-static realm plates})$$

One could use experimental test data that had been nondimensionalized to graphically present solutions following these formats.

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