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STATE-OF-THE-ART FOR PREDICTION OF PAVEMENT RESPONSE

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for

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16. Abstract

This report focuses on the finite element idealization as related to principles of pavement analysis, while other techniques and topics are introduced to provide a complete picture. The topics presented are organized to illustrate the similarities and consequences of using various prediction techniques. These topics include: the theoretical basis of the principal techniques, material models, comparison of analytical and measured results, and the selection of a prediction technique.

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PREFACE

The work reported herein was performed by the Civil Engineering Laboratory, Port Hueneme, California, for the U. S. Army Engineer Waterways Experiment Station (WES), Vicksburg, Mississippi, under MIPR No. A35200-5-0002 during the period 26 August 1974-12 September 1975. The work was jointly funded by the Office, Chief of Engineers, and the Federal Aviation Administration. The principal investigators were Messrs. John E. Crawford and Michael G. Katona.

WES personnel directly concerned with this project were Dr. W. R. Barker, project engineer and research civil engineer, Pavement Design Division, Soils and Pavements Laboratory (S&PL), and Mr. J. P. Sale, Chief, S&PL. COL G. H. Hilt, CE, was Director of WES and Mr. F. R. Brown was Technical Director during the project.

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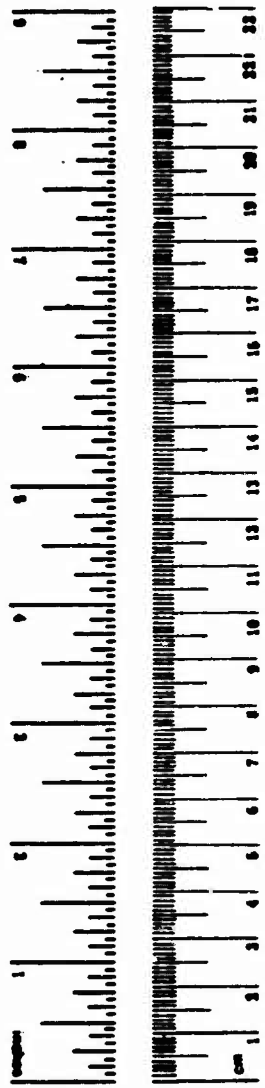
METRIC CONVERSION FACTORS

Approximate Conversions to Metric Measures

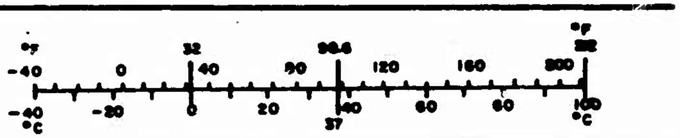
Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in ≡ 2.54 (exactly). For other exact conversions and more detailed tables, see NBS Misc. Publ. 286, Units of Weights and Measures, Price $2.25, SD Catalog No. C13.10:286.



Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	35	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



1 SCOPE

The objective of this paper is to describe as completely as is practical the present state-of-the-art for predicting the structural response of surface-loaded pavements. In scope the paper primarily focuses on the finite element techniques as related to pavement analysis. Other techniques and topics are introduced to provide a complete picture and offer a context for discussion.

Most of the information presented is based on the personal experiences of the authors and their colleagues. No attempt has been made to compile an exhaustive list of computer codes applicable to pavement analysis and design.* However, each area relating to state-of-the-art prediction of pavement response is discussed and summarized.

Common to all prediction techniques are the fundamental laws of mechanics as typified by the theory of elasticity. Therefore, to this extent all prediction techniques share a common origin. However, this commonality becomes obscured as soon as the assumed mathematical model is defined. That is, by specifying geometry, constitutive laws, and boundary conditions, a plethora of prediction techniques emerge ranging from simple linear plate theory to complex, nonlinear, three-dimensional continuum theory. Moreover, several techniques applied to a common situation may produce different results. This leads to confusion and provides a challenge for this paper to meet.

* A list of pavement analysis techniques has been compiled by Laboratoire Central des Ponts et Chaussees. These techniques are mostly elastic layer and finite element. Most of the techniques listed are in use outside the United States.

2 THEORETICAL BASIS OF THE PRINCIPAL ANALYTICAL TECHNIQUES

The techniques considered are Westergaard, elastic layer (i.e., the Burmister problem, including Boussinesq), and finite element. These are the principal techniques used for pavement analysis. Each of these methods is based on the fundamental laws of mechanics which require consideration of four distinct mechanistic concepts: (a) equilibrium equations, (b) kinematical relations, (c) constitutive laws, and (d) boundary conditions.

2.1 MATHEMATICAL CONCEPTS FOR ALL TECHNIQUES

All techniques require that the "real" pavement be idealized by a mathematical model. The use of mathematics requires definition of a domain to be simulated by the idealization. The "principal" techniques present two choices: the semi-infinite domain characteristic of Westergaard and elastic layer, and the finite domain or "soil island" typical of finite element. Usually pavement problems are most easily modeled as semi-infinite bodies, and the necessity for truncation of the "real" situation is an obstacle in the application of the finite element technique.

2.1.1 EQUILIBRIUM

Static equilibrium equations mathematically express the concept that the summation of forces at any point within the idealization and on the boundaries of the idealization must equal zero. For the classical techniques used to solve the Westergaard and elastic layer idealizations, this requires the solution of partial differential equations. These equations are usually complex enough that they must be solved by computer. While the equations are an "exact" expression for equilibrium, use of numerical procedures to solve them does introduce some level of approximation. On the other hand, equilibrium equations derived by using the finite element method are a linear algebraic set of equations that approximate the equilibrium condition for the structure. While the number of equations and coefficients is large, they are readily formed and solved by the computer.

While finite element techniques are generally not as exact as classical ones, their versatility and ease of derivation are vastly superior in most cases. For a limited range of problems (e.g., linear, layered pavements), classical solutions offer the most efficient way to calculate responses at a few points; however, for a large number of points, finite element techniques are much more efficient.[1] This follows from the requirement that for a classical idealization, the partial differential equations must be resolved at each location where responses are computed. Conversely, the finite element technique solves for all responses of the idealization at once.

In summation, study of the equilibrium concept shows that classical solutions in limited circumstances may be more computationally efficient than finite element schemes; however, a high price in versatility is paid for the use of classical solutions.

2.1.2 KINEMATICAL RELATIONS

Another part of any mathematical idealization of a pavement is the relation of displacement to strain and the variation of strain-displacement within the idealization. A linear strain-displacement relation for most airfield idealizations is appropriate (i.e., small displacement theory may be used). An exception to this occurs when significant deflections and membrane forces are present in a pavement layer; an example is shown in Section 4.5. A more significant aspect of kinematical relations is the consideration of discontinuous strain-displacement fields to model the phenomena of interlayer debonding and joints.

Implementation of complicated kinematical relations highlights the distinctive advantage of numerical procedures. While some complication in the numerical algorithm results, no significant increase in difficulty for the solution of the equilibrium equations occurs. Introduction of nonlinear discontinuous displacement fields in the partial differential equations of the classical techniques in all but the simplest circumstances produces an intractable problem. In summary, for kinematically complicated problems only numerical procedures such as finite element are of practical benefit.

2.1.3 CONSTITUTIVE LAWS

For the prediction of pavement response, specification of appropriate constitutive laws is the only significant challenge still to be solved conceptually. Constitutive equations mathematically represent the observed phenomenological relations between stress and strain in a given material and are, at best, approximations of the "real" material behavior. Materials are characterized using a variety of measures (for example, bulk, shear, and Young's modulus; Poisson's ratio; void ratio; and Westergaard's modulus of subgrade reaction), which are derived from a variety of tests (triaxial, shear, plate bearing, CBR, and penetrometer). Using this array of data, materials are either linearized by computation of appropriate constants, or a nonlinear time-dependent mathematical model is constructed from the material data and a behavioral hypothesis. Any practical application of nonlinear time-dependent behavior must be done within a numerical procedure. Further discussion of constitutive laws is presented in Section 3.

2.1.4 BOUNDARY CONDITIONS

Prescribing the boundary conditions for an idealization is generally broken into two parts: displacement and force. For the "principal" techniques this requires prescription of forces which simulate aircraft and gravitational loads. Additionally, because of the "soil island" nature of finite element schemes, displacements that attempt to simulate the actual continuous nature of soil must be prescribed along the edges of the idealization.

2.2 MATHEMATICS ASSOCIATED WITH THE WESTERGAARD IDEALIZATION

The Westergaard idealization transforms the airfield problem to the problem of a surface load applied to a plate on a Winkler foundation (Figure 1). As normally used,[2] the method considers the subgrade to

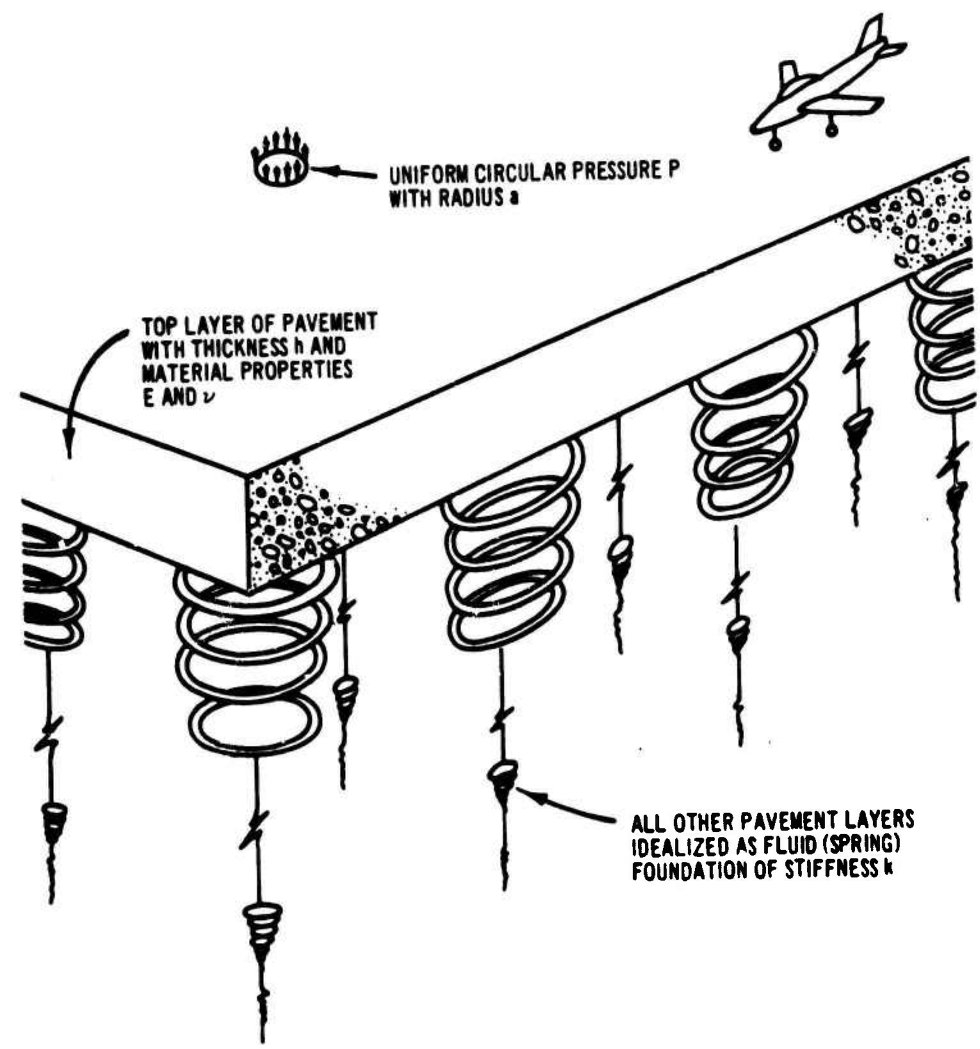


Figure 1. Westergaard pavement idealization

have a stiffness k ; the pavement to be characterized by Young's modulus E , Poisson's ratio ν , and pavement thickness h ; and the load to be circular and of uniform pressure P and radius a . For this situation the peak deflection is:

$$\delta = \frac{P}{8k\ell^2}\left[1 - \frac{a^2}{8\pi\ell^2}\log\left(\frac{Eh^3}{ka^4}\right) - \frac{3a^2}{8\pi\ell^2}\right] \tag{1}$$

where ℓ is the radius of relative stiffness and

$$\ell^4 = \frac{Eh^3}{12(1 - \nu^2)k}$$

and the maximum tensile stress in the pavement is:

$$\sigma = \frac{3P}{8\pi h^2}\left[(1 + \nu)\log\left(\frac{Eh^3}{ka^4}\right)\right] \tag{2}$$

These equations are a first-term approximation to the Westergaard idealization, and are generally adequate for computing the idealization's response when the top layer is very stiff compared to the other layers. The Westergaard first-term approximate solutions are also available for idealizations of loads applied at pavement edges and joints, and for elliptic tire prints.[2]

More rigorous solutions for the Westergaard idealization may be found in References 3 and 4. Reference 4 also presents solutions for this idealization based on the mathematics for the elastic layer idealization. Since the Westergaard idealization is a rather crude approximation of the real problem, more rigorous solutions of this idealization are seldom warranted. Comparisons of Westergaard and elastic layer analyses are presented in Section 4.1.

2.3 MATHEMATICS ASSOCIATED WITH THE ELASTIC LAYER IDEALIZATION

The elastic layer idealization transforms the airfield problem to the problem of a load applied to an elastic, horizontally uniform, layered system (Figure 2). As normally used, the method considers N

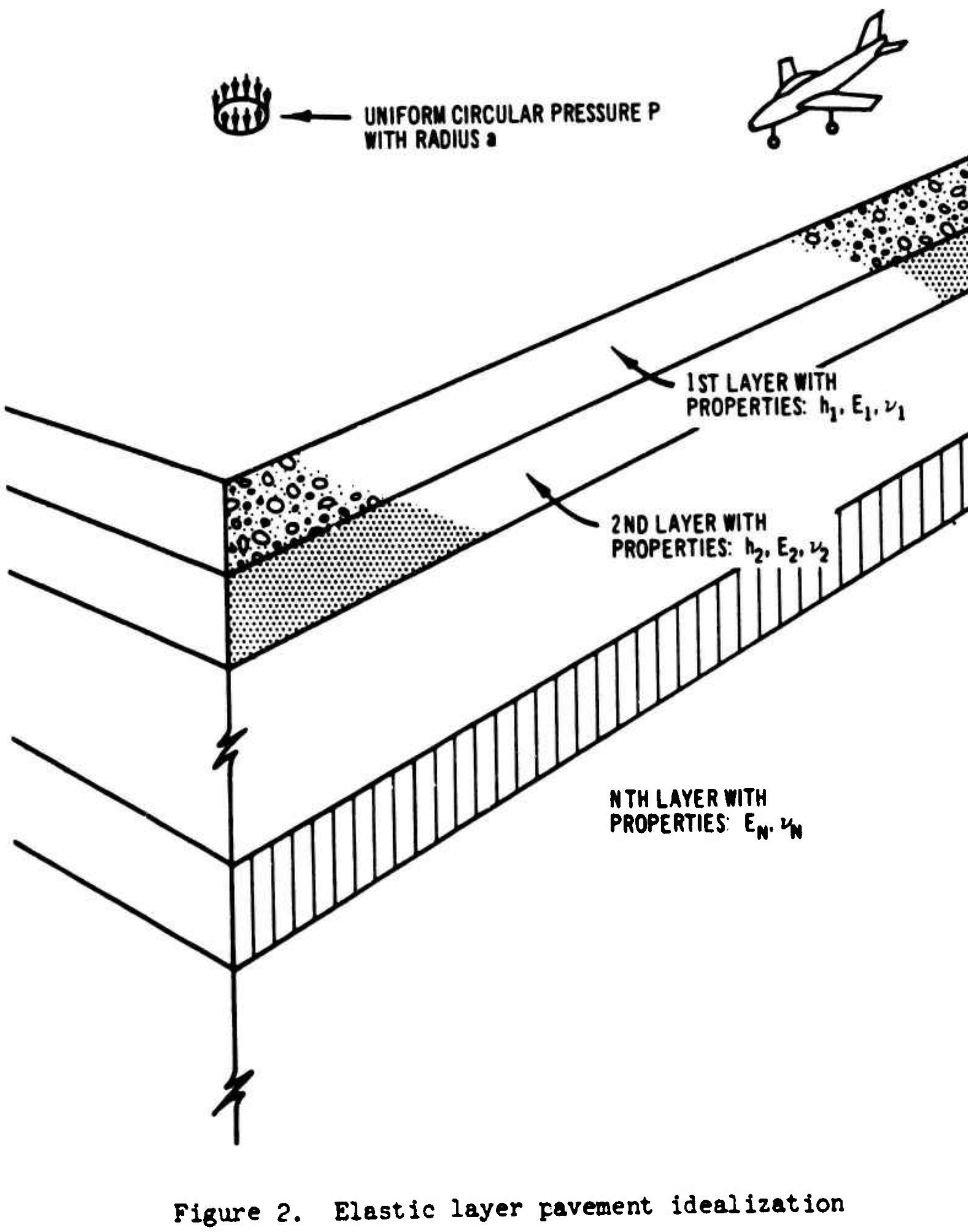


Figure 2. Elastic layer pavement idealization

bonded layers where each layer is defined by E , ν , and h . Each layer extends horizontally to infinity with the bottom layer also extending vertically to infinity. The applied load of uniform pressure P is circular with radius a . This technique requires solution of rather complex partial differential equations which are usually solved numerically with the assistance of a computer. To compute responses similar to the Westergaard equations (Equations 1 and 2) requires solution of the following integral equations for peak deflection:*

$$\delta = \frac{Pa(1 + \nu_1)}{E_1} \int_0^\infty [\alpha_1 + \alpha_2(4\nu_1 - 2)(\alpha_3 - \alpha_4)] \frac{J_1(x)}{x} dx \qquad (3)$$

and for maximum tensile stress in the top layer:

$$\sigma = \frac{-P}{2} \int_0^\infty [\alpha_5 - \alpha_2 + (\alpha_6 + \alpha_4)(4\nu_1 + 1) + ph_1(\alpha_6 - \alpha_4)] e^{-ph_1} J_1(x)\, dx \qquad (4)$$

The α constants are determined from boundary conditions while J_1 is a Bessel function, p is a transform parameter, and x is the product of p and a . The derivation for these expressions is given in Reference 4.

Several computer programs are available for computation of elastic layer responses (References 4-7). Most programs allow consideration of either bonded or frictionless layer interfaces; results of such considerations are reported in Reference 8. All programs require numerical quadrature of integral equations and evaluation of Bessel functions;

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* For a one-layer system (the Boussinesq problem), $\delta = \frac{2Pa(1 - \nu^2)}{E}$.

some programs perform these functions more accurately than others. The results obtained from this idealization for linear, elastic, horizontally uniform, layered pavement systems are "exact" within the context of the theory of elasticity, assuming numerical errors are negligible. Thus, elastic layer responses may be used to check other linear elastic techniques. Section 4 presents comparisons of elastic layer responses with those of Westergaard and finite element, as well as comparisons of computations derived from various elastic layer programs.

2.4 MATHEMATICS ASSOCIATED WITH THE FINITE ELEMENT IDEALIZATION

In the finite element analysis of airfield pavements, the continuous pavement system is idealized by a finite number of elements that are related to one another through common points, called nodes (Figure 3). Each element and node are assigned a number, and the bulk of the data preparation consists of inputting to the computer the coordinates of each node point and the node numbers and material type associated with each element. Additional data concerning material characterization (e.g., E and ν) for each material type, locations of loads, and displacement boundary conditions are also required. These data allow the computer to form a mathematical idealization of the pavement system. The accuracy of this idealization is dependent on a wide variety of factors including the type of spatial approximation and material characterization used, the experience of the engineer in preparation and interpretation of the results, and the type of computer hardware available. An effort will be made throughout the report to elucidate these factors.

While it is possible to obtain a reasonably good working knowledge of classical techniques through a limited number of references (e.g., 2-5), obtaining a good working knowledge of finite element techniques requires familiarity with significantly more references and considerable operational experience with the finite element programs themselves. The authors have found References 9-12 useful in providing basic knowledge of the finite element process. More detailed knowledge

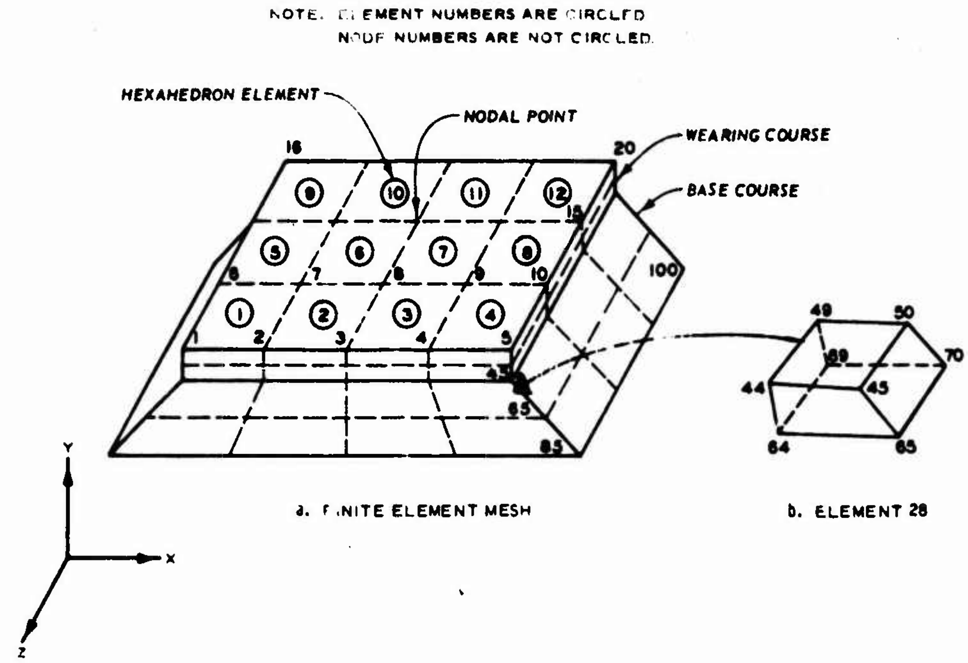

Figure 3. Pavement idealization using the finite element procedure

with respect to the basic procedure and airfield analysis is available in various reports and articles which are mentioned throughout this report. Unfortunately, all the material taken as a whole does not replace experience or the necessity for good "structural" sense. Even the simplest finite element idealizations are relatively easy to do poorly with respect to accuracy of results, efficient utilization of computer hardware, and magnitude of engineer/programmer effort required. While this situation is certainly cause for concern, it is possible to obtain finite element results easily that either closely duplicate those of classical techniques or that provide, to a high degree of confidence, responses that only the finite element procedure can compute.

2.4.1 SELECTION OF A FINITE ELEMENT IDEALIZATION

Four classes of finite element idealization are applicable to the analysis of pavement systems.* These classes are grouped according to

* Within each class many permutations of the finite element procedure are permissible, e.g., consideration of linear and/or nonlinear materials.

spatial limitations on the variables involved and are: (a) plane strain slice, (b) axisymmetric solid, (c) prismatic solid, and (d) three-dimensional solid. A brief description of each class follows:

a. A plane strain idealization considers a slice of the pavement in which the structure is limited to two dimensions (Figure 4). Only results within the slice are computed, and changes in load or material along the Z-axis are neglected. Because generally more appropriate classes of idealizations, i.e., (b), (c), and (d), have become available, usage of plane strain analysis outside research activities is almost nonexistent. A joint analysis using this class is shown in Section 4.5.

b. The axisymmetric solid idealization considers a layered, solid, conical frustrum or cylinder of pavement loaded circularly at its pole (Figure 5). A variation of the method is achieved by superimposing results of one analysis upon another to simulate a multiwheel load (Figure 6). This type of idealization has been widely employed in a number of different computer codes for pavement analysis (e.g., References 1, 13-17).* This technique usually requires the least cost and effort to use, but it lacks the capability to idealize other than layer systems and cannot perform nonlinear analysis of multiwheel aircraft.

c. The prismatic solid idealization considers a nearly general three-dimensional system. No variation of geometrical configuration is allowed from cross section to cross section (X-Y plane in Figure 7), which produces a prismatic solid along the Z-axis; however, loading is arbitrary. This class offers a degree of structural idealization sophisticated enough for a variety of real world situations. The ease of use is comparable to the axisymmetric analysis, while the cost of calculation is an order of magnitude greater. References 1 and 19-22 present results derived from the AFPAV computer program for prismatic analysis.

d. The three-dimensional idealization considers a complete general structure; material and load configurations are arbitrary (Figure 8). The difficulty of use is an order of magnitude greater than the other methods, and the cost of calculation is two orders of magnitude greater than the axisymmetric method. Because of cost limitations, the use of this idealization is limited to special applications. One of these is described in Reference 23.

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* Most of these codes are based on a general axisymmetric code written by E. L. Wilson, Reference 18.

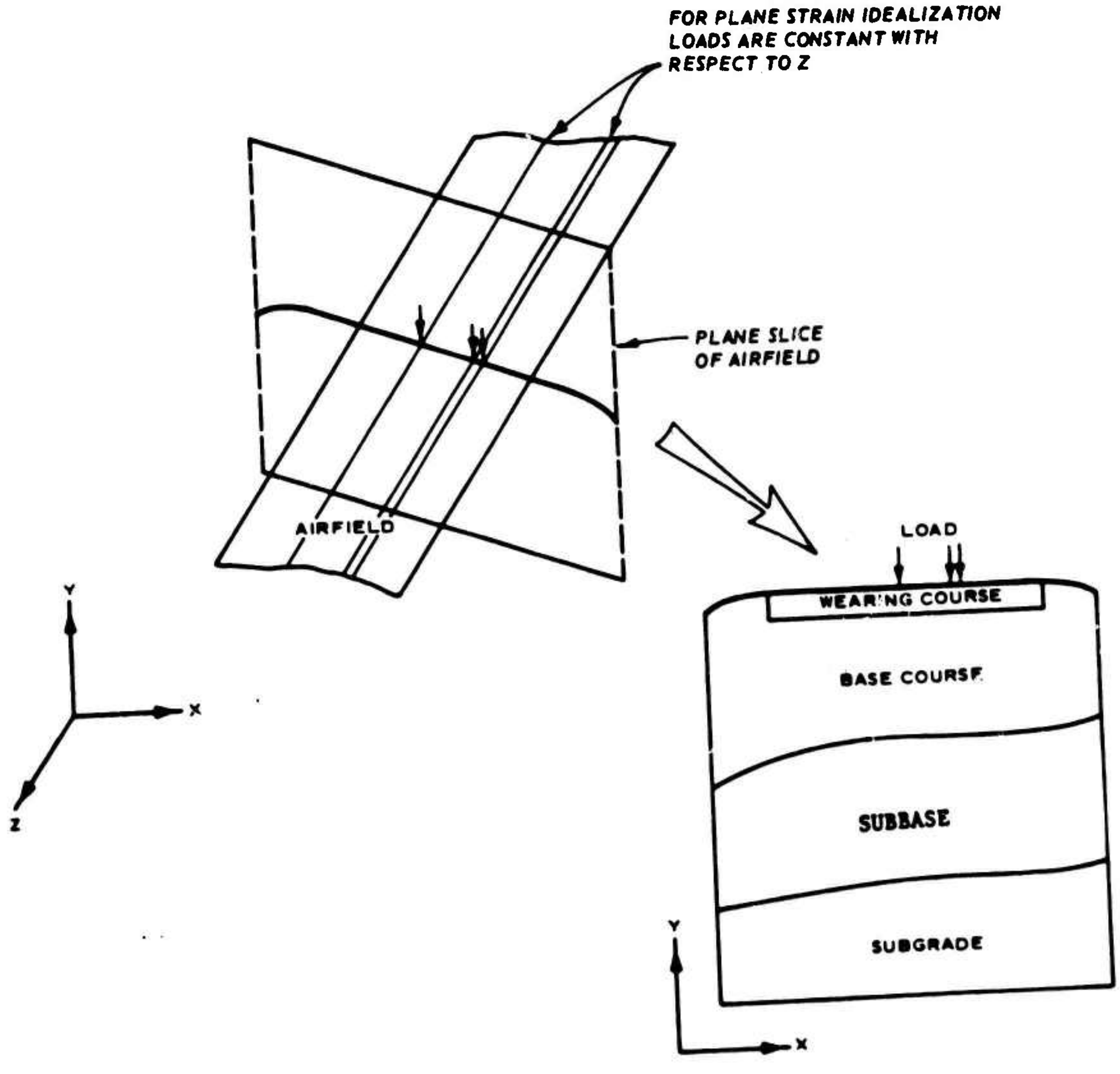


Figure 4. Plane strain idealization

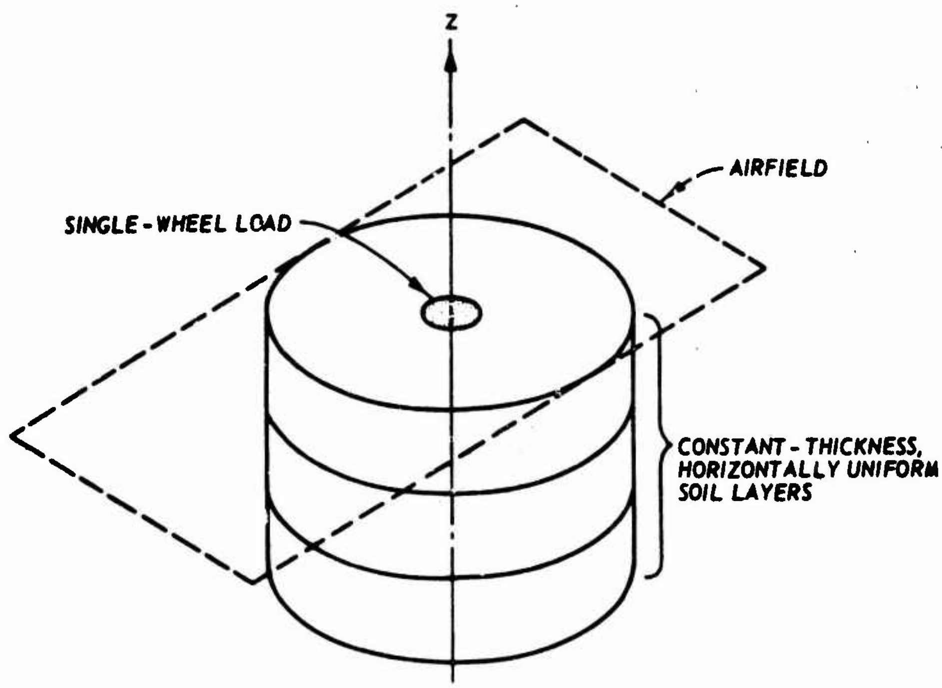


Figure 5. Axisymmetric idealization for single-wheel load

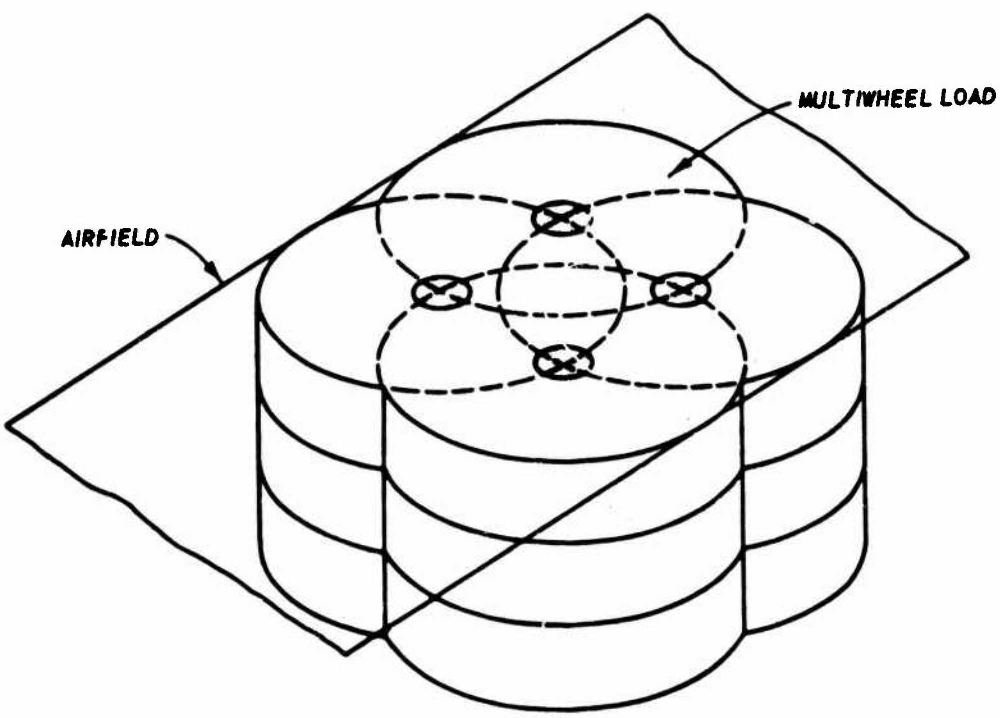


Figure 6. Axisymmetric idealization of multiwheel load using superposition

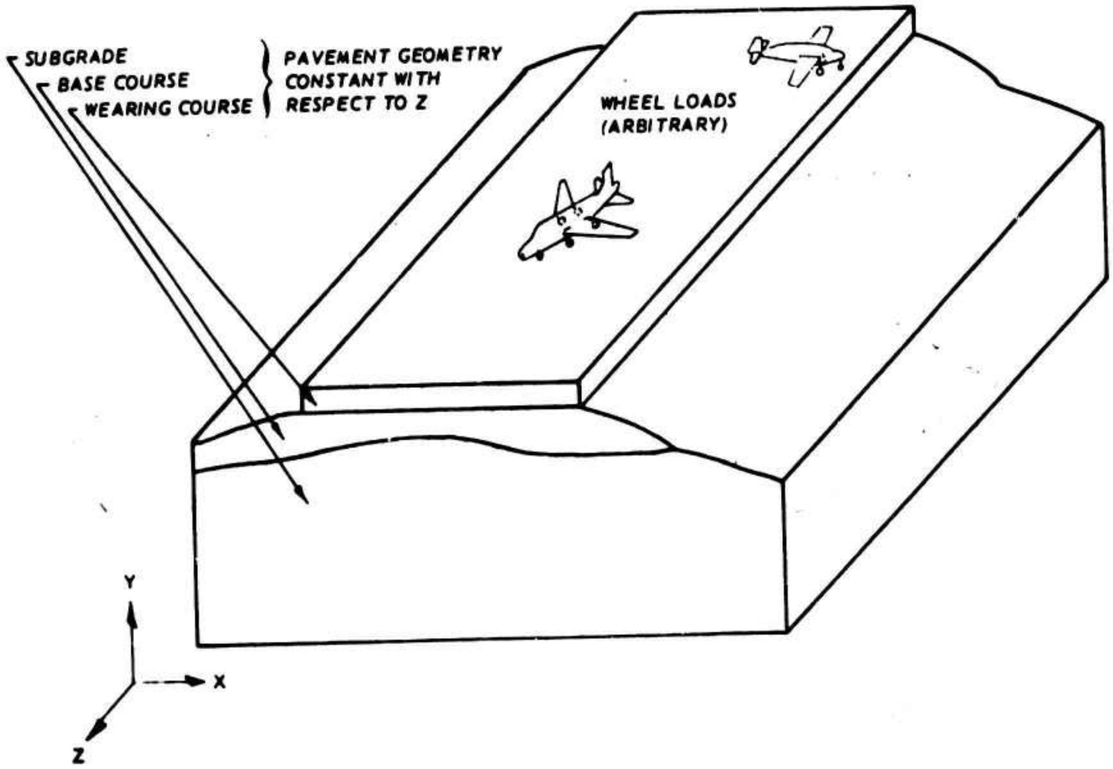
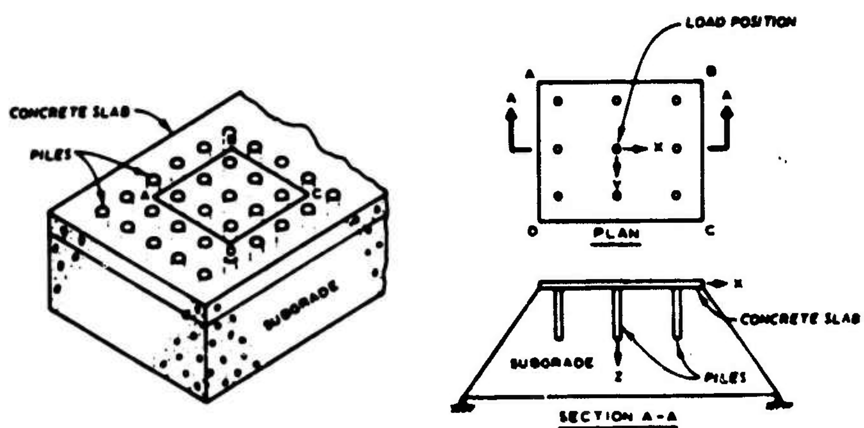


Figure 7. Prismatic idealization



a. PILE-REINFORCED PAVEMENT SYSTEM b. IDEALIZED PILE-REINFORCED PAVEMENT STRUCTURE

Figure 8. Three-dimensional idealization

2.4.2 MATHEMATICAL BASIS OF THE FINITE ELEMENT PROCEDURE

The following brief introduction to the mathematics associated with finite element analysis is included to provide context for subsequent discussions.

The theorem of minimum potential energy is used to derive the governing equations for the finite element method and may be written as:

$$\delta P = 0 \tag{5}$$

where

$$P = \sum_{i=1}^{I} P_i \tag{6}$$

and

$$P_i = \frac{1}{2} \int_V \{\varepsilon\}^T [C] \{\varepsilon\} \, dV - \int_S \{\delta\}^T \{\tau\} \, dS - \int_V \{\delta\}^T \{f\} \, dV \tag{7}$$

where

I = number of elements

P_i = potential energy for i^{th} element

V = volume of i^{th} element

$\{\varepsilon\}^T$ = transformed strain tensor

$[C]$ = stress-strain transformation (i.e., the constitutive equations)

$\{\varepsilon\}$ = strain tensor

S = surface area of i^{th} element

$\{\delta\}^T$ = transformed displacement field for the i^{th} element

$\{\tau\}$ = prescribed surface tractions

$\{f\}$ = prescribed body forces per unit volume

To evaluate these integrals requires that within an element the possible modes of deformation be restricted. The theorem further requires that the displacement field across element boundaries be continuous and that certain basic deformation modes be present in all elements regardless of class of idealization or element type.

An acceptable set of deformation modes for a two-dimensional element is shown in Figure 9. The displacement field produced from the combination of these modes is:

$$u = \alpha_1 + \alpha_2 s + \alpha_3 t + \alpha_4 st \tag{8}$$

$$v = \alpha_5 + \alpha_6 s + \alpha_7 t + \alpha_8 st \tag{9}$$

The α terms are constants which determine the amount of participation for each mode, s and t are the element's natural coordinates, and u and v are the displacements in the s and t directions, respectively. While Equations 8 and 9 are a convenient way to determine the types of deformation to which the element is limited, they are not convenient expressions for incorporation into the finite element solution procedure in that the generalized constant α has no physical meaning. A more appropriate description of the element's displacement field is obtained by manipulating Equations 8 and 9 so that the element's displacement field (u,v) is expressed as a function of the element's nodal displacements (u_n, v_n) and interpolation functions (h_n). Thus,

$$u = h_1 u_1 + h_2 u_2 + h_3 u_3 + h_4 u_4 \tag{10}$$

$$v = h_1 v_1 + h_2 v_2 + h_3 v_3 + h_4 v_4 \tag{11}$$

where

$$h_1 = 1/4(1 + s)(1 + t)$$
$$h_2 = 1/4(1 - s)(1 + t)$$
$$h_3 = 1/4(1 - s)(1 - t)$$
$$h_4 = 1/4(1 + s)(1 - t)$$

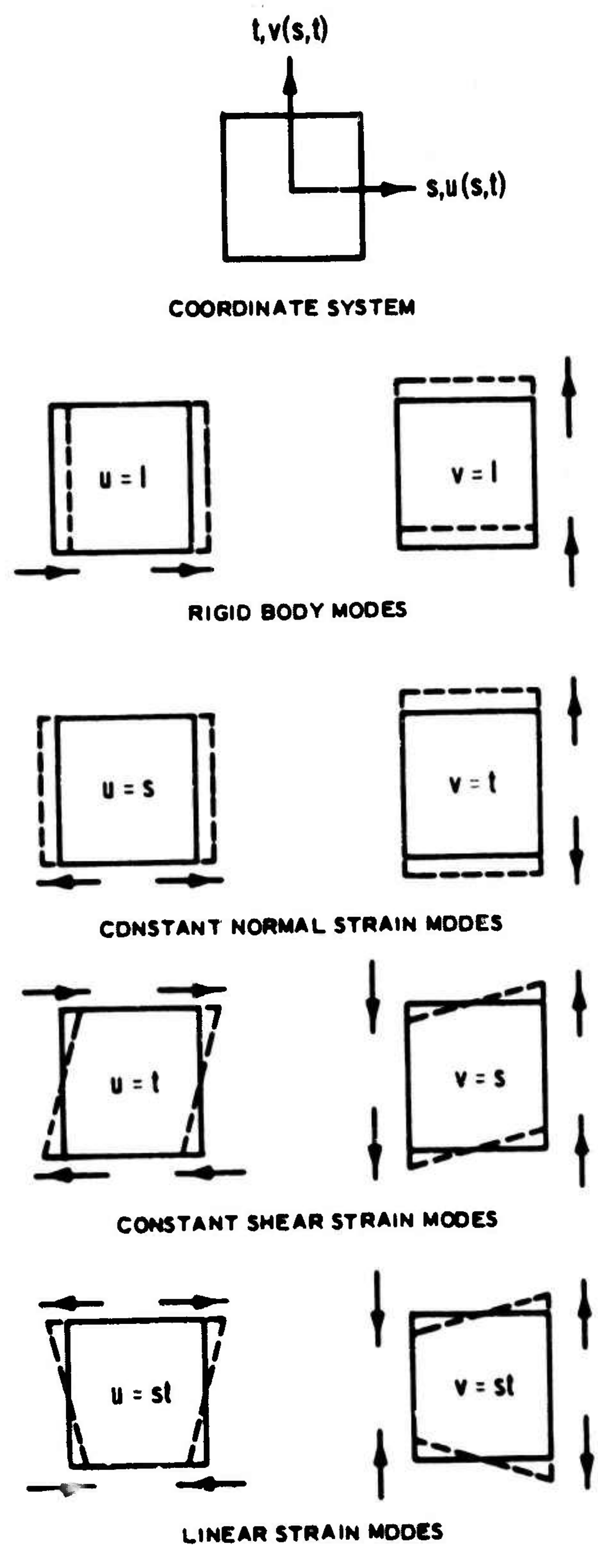


Figure 9. Basic modes of deformation for a plane strain element

The element shapes that are used with the four classes of finite element idealizations are shown in Figure 10; most programs use elements that have quadrilateral cross sections. While there are differences in the derivation of each element type based on the class of idealization, the basic deformation modes associated with an element type are the same regardless of class. The selection of these modes and the derivation of equations similar to 10 and 11 are based either on isoparametric concepts or on an assemblage of constant or linear strain triangles. Delving into element technology is beyond the scope of this report, but it is important to know that there are many ways to derive the basic element stiffness with concomitantly varying results. A number of basic element types are shown in Figure 11. References 10 and 24 provide a comprehensive presentation concerning selection of deformation modes.

Using the classical strain-displacement relations, the strain tensor is defined in terms of nodal displacement:

$$\{\varepsilon\} = [B]\{U_i\} \tag{12}$$

where

$[B]$ = strain displacement transformation
$\{U_i\}$ = nodal displacements for the i^{th} element (i.e., u_n, v_n)

Combining Equations 7 and 12 and taking the variation results in formation of the element stiffness:

$$[K_i]\{U_i\} = \{F_i\} \tag{13}$$

where

$[K_i]$ = the i^{th} element stiffness
$\{F_i\}$ = forces applied to the nodes of the i^{th} element

or

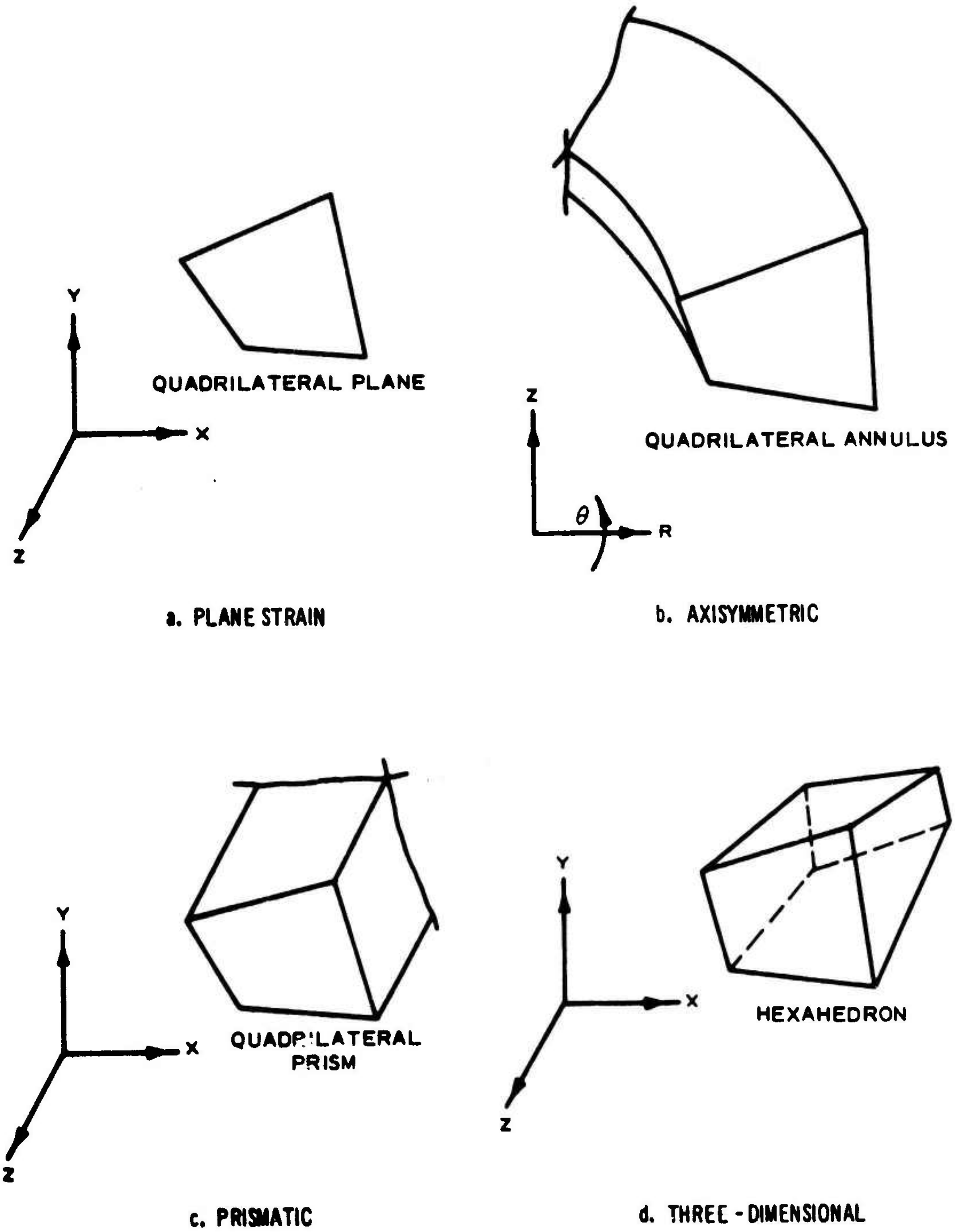


Figure 10. Element shapes

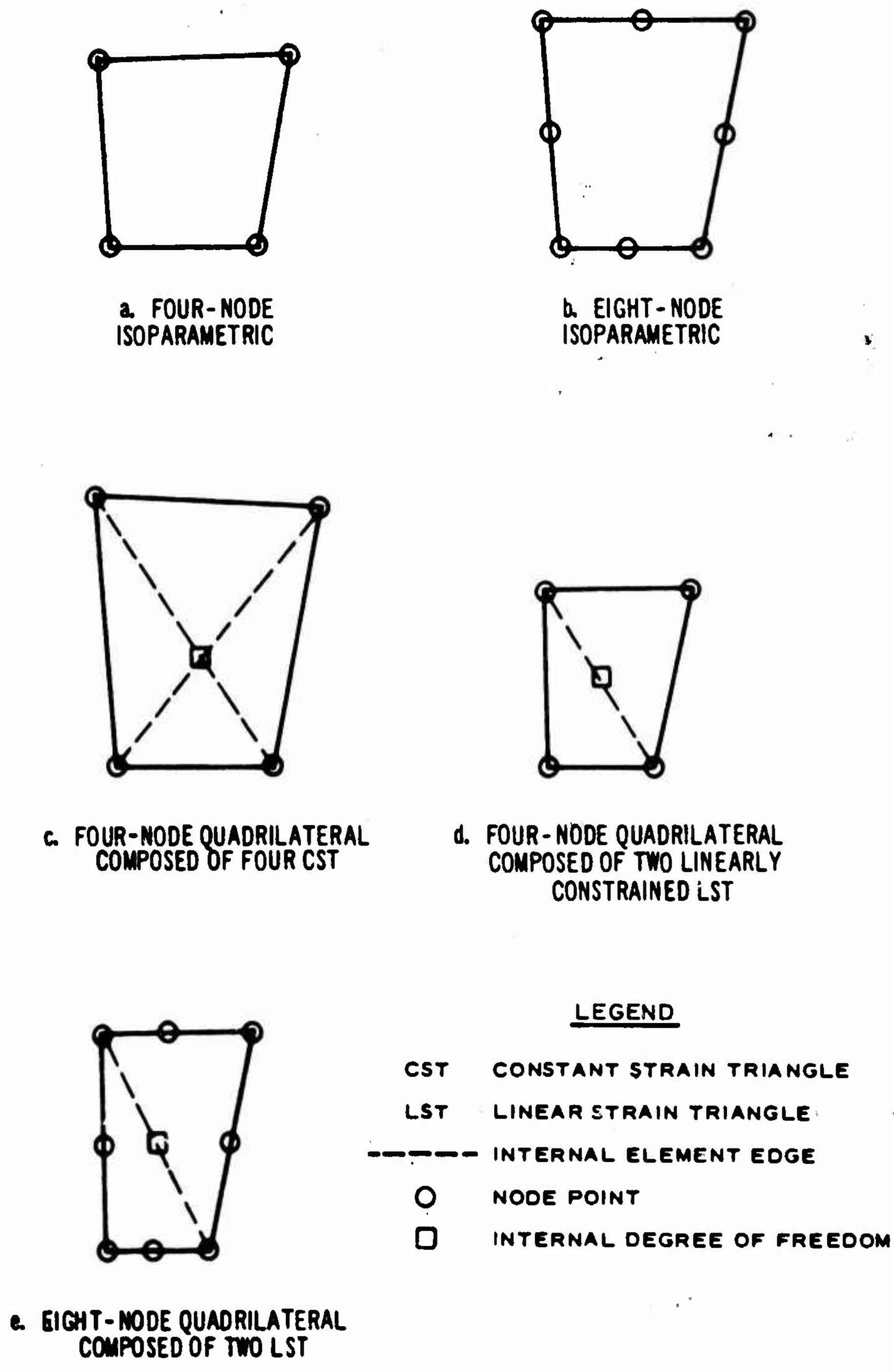


Figure 11. Standard element types

$$[K_i] = \int_V [B]^T [C] [B] \, dV$$

Adding together $[K_i]$ and $\{F_i\}$ for all I elements results in the formation of the relation between nodal forces and displacements for the whole pavement structure. That is, a set of algebraic equations is derived:

$$[K]\{U\} = \{F\} \tag{14}$$

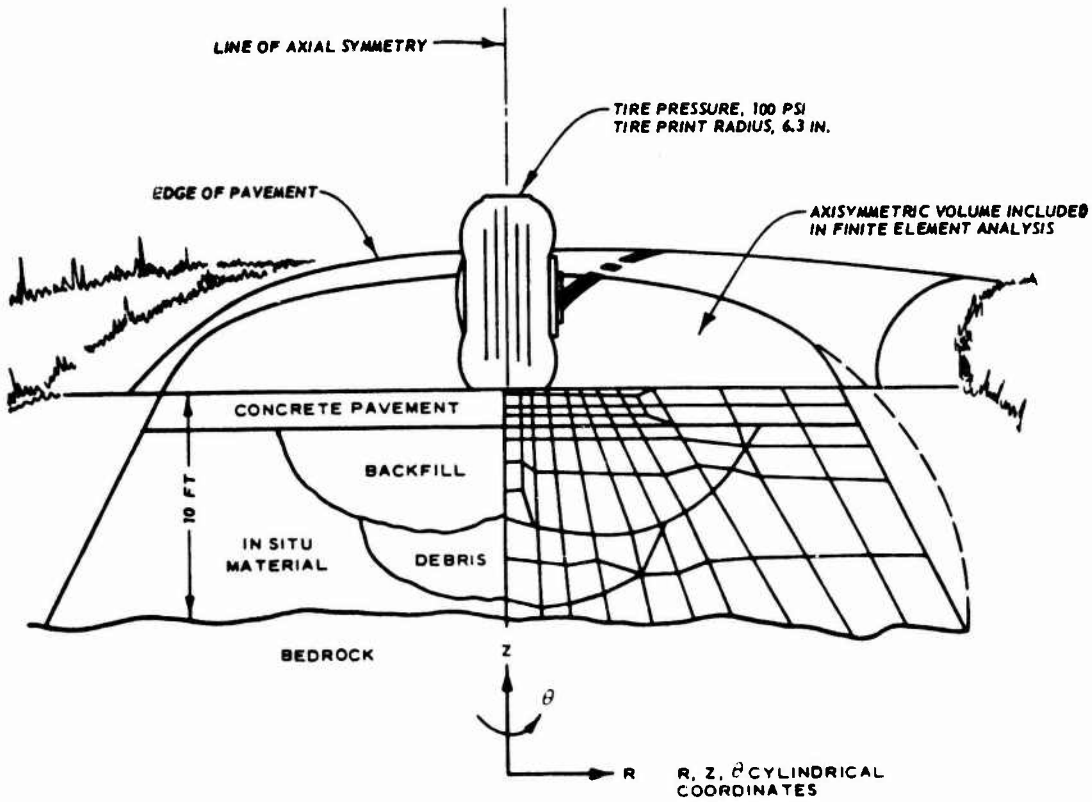
where $[K]$ is the stiffness for the whole pavement structure. The unknowns, $\{U\}$, are the nodal displacements while the constants, $\{F\}$, are the prescribed loads applied to each nodal point. After displacement boundary conditions are prescribed, the set of equations is solved using standard linear algebraic solution techniques. This yields values of displacement for each nodal point (i.e., $\{U\}$). By applying Equation 12 for each element, the element strains are calculated. The stresses are obtained from the strains.

For a linear structural idealization, $[K]$ remains constant irrespective of the magnitude of the idealization's displacement, strain, or stress. The nonlinear analysis of a structure primarily implies that $[K]$ changes in some manner as a function of displacement, stress, or strain. There are two widespread methods employed to perform nonlinear analysis. One is known as the initial strain method; the other is the tangent method. From a user viewpoint these two methods are not very different, though with reference to the computer operations, significant differences in efficiency and approach occur. Both methods are appropriate for pavement analysis and are used not only to model nonlinear materials, but to utilize large deformation theory, to simulate material cracking, and to allow debonding at element interfaces. Discussion of the merits of these methods in particular situations is beyond the scope of this report. References 25-27 are suggested for further study.

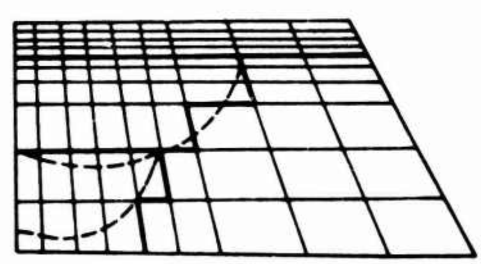
2.4.3 SELECTION OF A MESH

Selection of a mesh encompasses a variety of decisions: choice of element sizings and spacings, determination of optimal pattern for node and element numbering, selection of the mesh domain (i.e., size of soil island), and prescription of the boundary conditions. The mesh selected depends on the class of idealization (e.g., prismatic or axisymmetric) and the permutation involved (e.g., nonlinear or linear axisymmetric); the type of element employed (e.g., constant strain triangle or linear strain triangle); and the type of structure (e.g., flexible versus rigid pavement). Also important are the type of computer hardware available and the level of effort to be expended. Fortunately, for most applications, only the axisymmetric and prismatic solids are of practical interest; and most computer codes available in these classes use elements that have essentially linear displacement fields (shown in Figure 11a and 11c). Using these element types results in the stress-strain fields that are nearly constant within an element. This result is the primary factor in determining mesh selection: where the gradient of the stress-strain field is large, more elements are needed; and conversely, where the stress-strain is relatively constant, fewer elements are needed. The mesh is also affected by geometry and material boundaries.

Figure 12 depicts an axisymmetric finite element model of a repaired bomb crater in an airfield pavement; this study is reported in Reference 17. Small elements are placed directly under the tire where the stress-strain gradient is high. Large elements are used farther from the load because the stress-strain field is relatively constant. Various odd-shaped elements are used to match the material boundaries. The outside edge of the mesh is slanted outward to accommodate the spreading of the load. The depth of the mesh is fixed at a depth where no significant strain occurs in the soil. For this case, a trial-and-error scheme was employed in which element layers were successfully added to the bottom of the mesh (thus increasing its depth) until the change in peak displacement between successive runs was less than 10 percent. The width of the mesh is set by a similar process while the slope of the side



a. IRREGULAR MODEL



b. REGULAR MODEL

Figure 12. Mesh modeling of repaired crater

is usually taken as 30 deg,* based on results reported in Reference 28. The displacement boundary conditions chosen were fixed along the bottom and free or horizontally fixed along the sides.

An additional criterion in mesh selection is ease of data preparation. Figure 12b represents an alternative mesh to that of Figure 12a. To minimize preparation of element and nodal data, a regular lattice mesh has been constructed. The material interfaces within the soil mass are only approximately defined by changing the element material numbers at the appropriate places. Justification for this sacrifice of detail lies in the approximate nature of the crater's definition. To match this approximation exactly, as in Figure 12a, is unwarranted.

These criteria present a reasonable set of specifications for this pavement model; however, under unusual circumstances or for other types of models, different specifications may be necessary.

The difficulties most often associated with finite element analysis result from an engineer's "getting in over his head." Invariably an engineer will proceed directly to solve his problem, regardless of its nature. Complicated problems must be solved in stages; experience is extremely valuable in designing a competent mesh and producing satisfactory results.

In the authors' opinion, the best procedure for acquiring this experience consists of starting with a simple linear problem (e.g., that shown in Figure 13) and varying each mesh parameter (e.g., element distribution and number, depth and width of mesh). In some circumstances these variations will produce significant changes in response. The accuracy of results is assessed by comparison with an elastic layer solution which is "exact" for a linear, layered pavement. For the linear system, stress-strain should be in good agreement for the two prediction techniques while finite element displacements will generally be a few percent less than those of elastic layer.

* A table of factors for converting U. S. customary units of measurement to metric (SI) units is given on page 7.

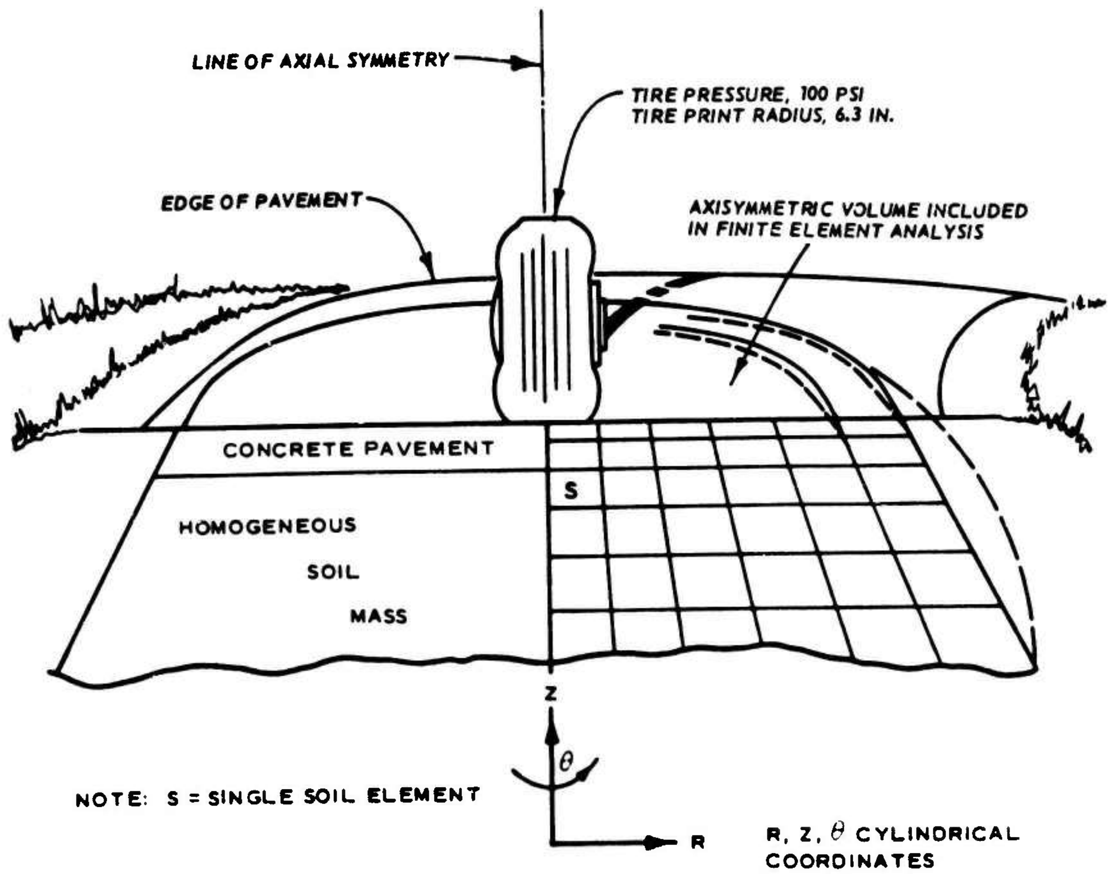


Figure 13. Finite element model of pavement system

For nonlinear and nonuniform-layer systems, a similar procedure to this one should be adopted. From known results or past computations, the system parameters should be gradually changed to achieve the desired result.

3 MATERIAL MODELS

The most important component of any prediction method is the material model utilized to obtain the structural responses. It is paradoxical that the material model is also the most elusive and misused relationship in pavement problems. That is, other components of the pavement problem such as geometry, boundary conditions, and loading are relatively easy to define accurately compared to the material model. Much of the difficulty related to material models is due to the plethora of proposed methods and models reported in the literature. Constitutive forms ranging from linear to highly nonlinear and time dependent have been suggested. In short, there is no unanimity of opinion on material models for pavements. Although divergent opinions provide an atmosphere for active research, they do little to help the plight of practicing engineers who desperately need guidelines on material models for pavement analysis. In the following sections, attempts will be made to provide general guidelines for selecting material models; however, it must be understood that there is no universal answer to the problem. Each pavement system is a unique problem; and the material model must be chosen commensurate with the knowledge of the materials, the availability of analytical tools, and the specified performance criteria.

Implementation of material models cannot be divorced from the analytical technique employed for the pavement analysis. To illustrate this assertion, a suggested finite element framework suitable for accommodating most material models is presented along with various solution strategies. After this framework is established, material models are examined and discussed.

3.1 SUGGESTED ANALYTICAL FRAMEWORK FOR FINITE ELEMENT CODES

To begin with, a finite element code must be modularized to accommodate new material laws easily as they become available or otherwise be doomed to extinction. Secondly, the code should be structured for incremental loading as opposed to one-step total loading since the one-step loading is inherently a subset of incremental loading and most all material models can be cast in an incremental form. In addition, algorithms and storage for iterating within the load step should be provided, i.e., the ability to apply a load step repeatedly until both equilibrium and the material model are in agreement. These notions are described in a schematic flow chart (Figure 14).

This general algorithm allows considerations of several popular solution strategies for treating nonlinear materials including the secant method, the tangent method, the modified tangent method, and the chord method.* Figure 15 illustrates the concepts associated with each of these methods. The secant method implies that the total load is applied in one step and the process is iterated to find a secant modulus satisfying both equilibrium and the material law (stress-strain curve). The tangent method implies that the load is applied in a series of steps. At the end of each step the tangent of the material law is evaluated at the accumulated stress-strain level to provide the modulus for the next load step. Note that the stress-strain responses calculated by this method increasingly diverge from the material law under monotonic loading. The modified tangent method avoids this divergence by iterating within the load step to determine a modulus which is an average of the material law tangent at the beginning and end of the load step. The chord method is the secant method applied in a step-by-step fashion; the chord method inscribes the material law.

The selection of one method over another is largely a question of cost versus accuracy. The secant method is acceptably accurate in cases of near proportional loading (i.e., principal stresses remain in a near constant ratio throughout the load path) and is generally the least costly procedure. The tangent method, which is comparable in cost, is more appropriate for nonproportional loading and also provides a history of the response of the structure. Both the modified tangent and chord methods provide improved accuracy over the secant and tangent methods; however, their cost is significantly greater.

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* These solution strategies are an integral part of (but different from) the nonlinear methods employed by the finite element program mentioned at the end of Section 2.4.2.

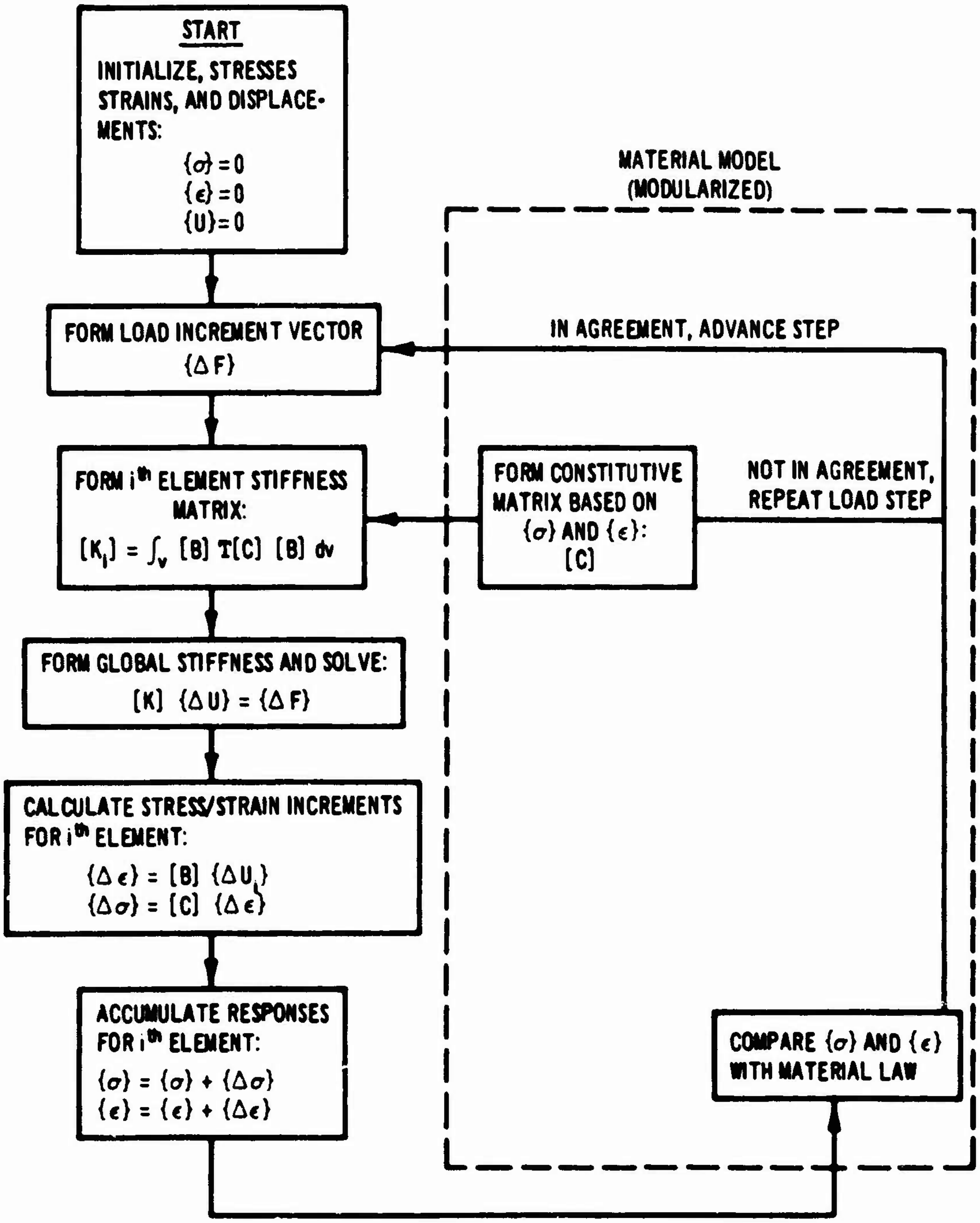


Figure 14. Schematic flow diagram for nonlinear materials

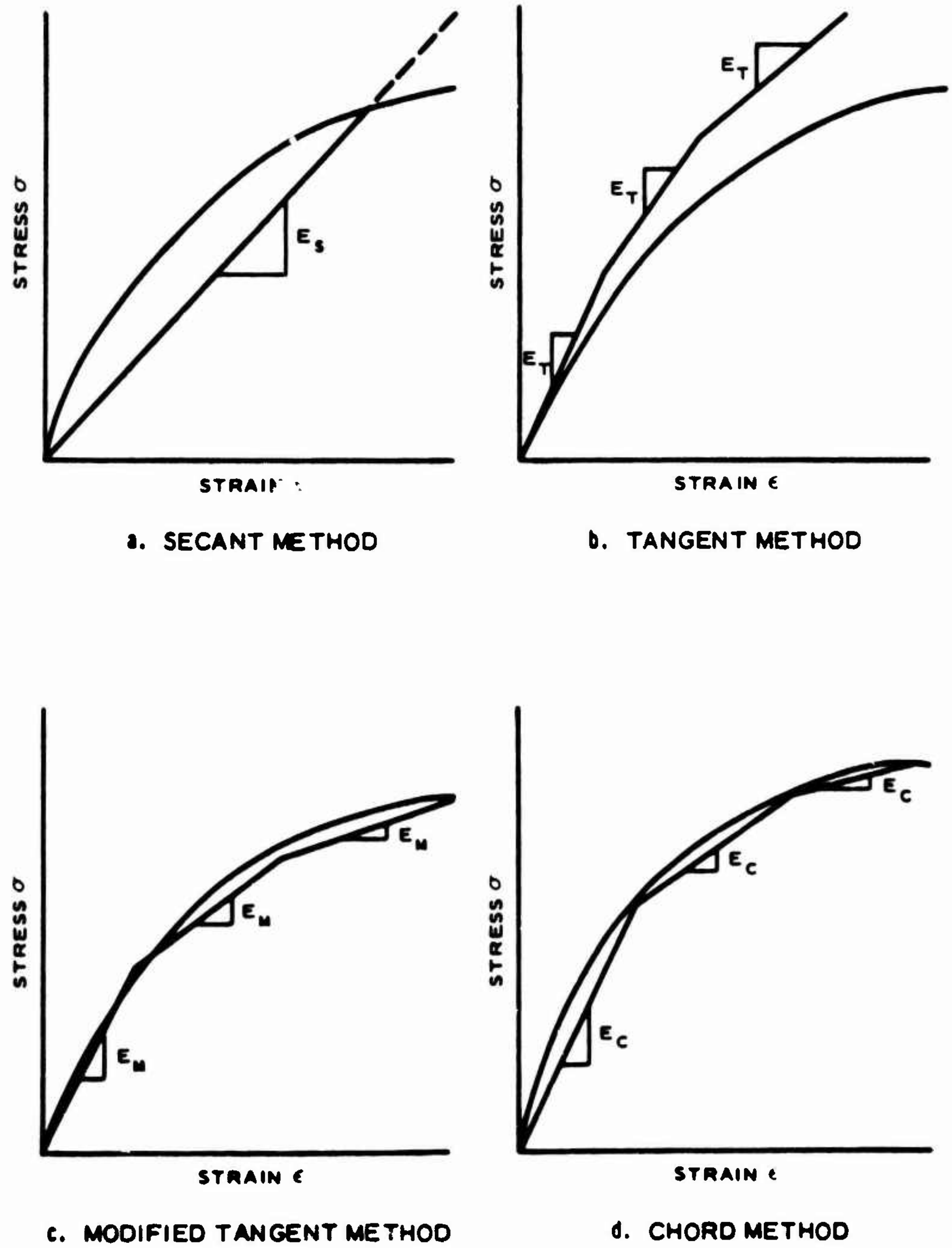


Figure 15. Solution strategies for nonlinear materials

As a general rule of thumb, the authors contend that all things being equal, the higher cost penalty should not deter engineers from using more accurate algorithms since the computer cost is usually miniscule compared to the overall project.

3.2 AN EXAMPLE OF THE EMPLOYMENT OF NONLINEAR MATERIAL MODELS USING THE FINITE ELEMENT PROCEDURE

Analysis of nonlinear materials generally consists of a sequence of "linearlike" analyses (referred to as steps) which are related to one another. At each step a linear response of the structure (i.e., displacement, stress, and strain) is computed. The material parameters used during a step for each element are determined from the response of the element computed in the previous step. The outcome of such an analysis is a sequence of linear responses. Taken as a whole, they represent a nonlinear response to the applied load (mathematically, this is known as a linear piecewise approximation of a nonlinear function).

The tangent method and the model illustrated in Figure 13 will be used to demonstrate the technique. In this example a single wheel is being supported by a concrete surface upon a single layer of soil. To simplify the discussion, only the response of a single soil element will be considered, marked S in the figure. The soil's nonlinear material properties are shown in Figure 16; for simplicity, only hydrostatic stress (i.e., $(\sigma_R + \sigma_Z + \sigma_\theta)/3$) versus volumetric strain (i.e., $\epsilon_V = -(\epsilon_R + \epsilon_Z + \epsilon_\theta)$) will be considered. Material unloading and preloading are omitted. Five steps will be used to approximate the nonlinear response.

For each step, the response of the soil element will be computed for one-fifth of the total load (i.e., 20 psi). At the beginning of the n^{th} step, the element's bulk modulus K_n is computed based on the strain in element S at the end of the previous step ϵ_n. Thus, for the first step, where $\epsilon_1 = 0$, K_1 is the tangent at the beginning of the material curve. Because of the linear approximation, an error is accumulated at the end of each step such that the material curve the element "sees" shadows the curve derived in the laboratory. The degree

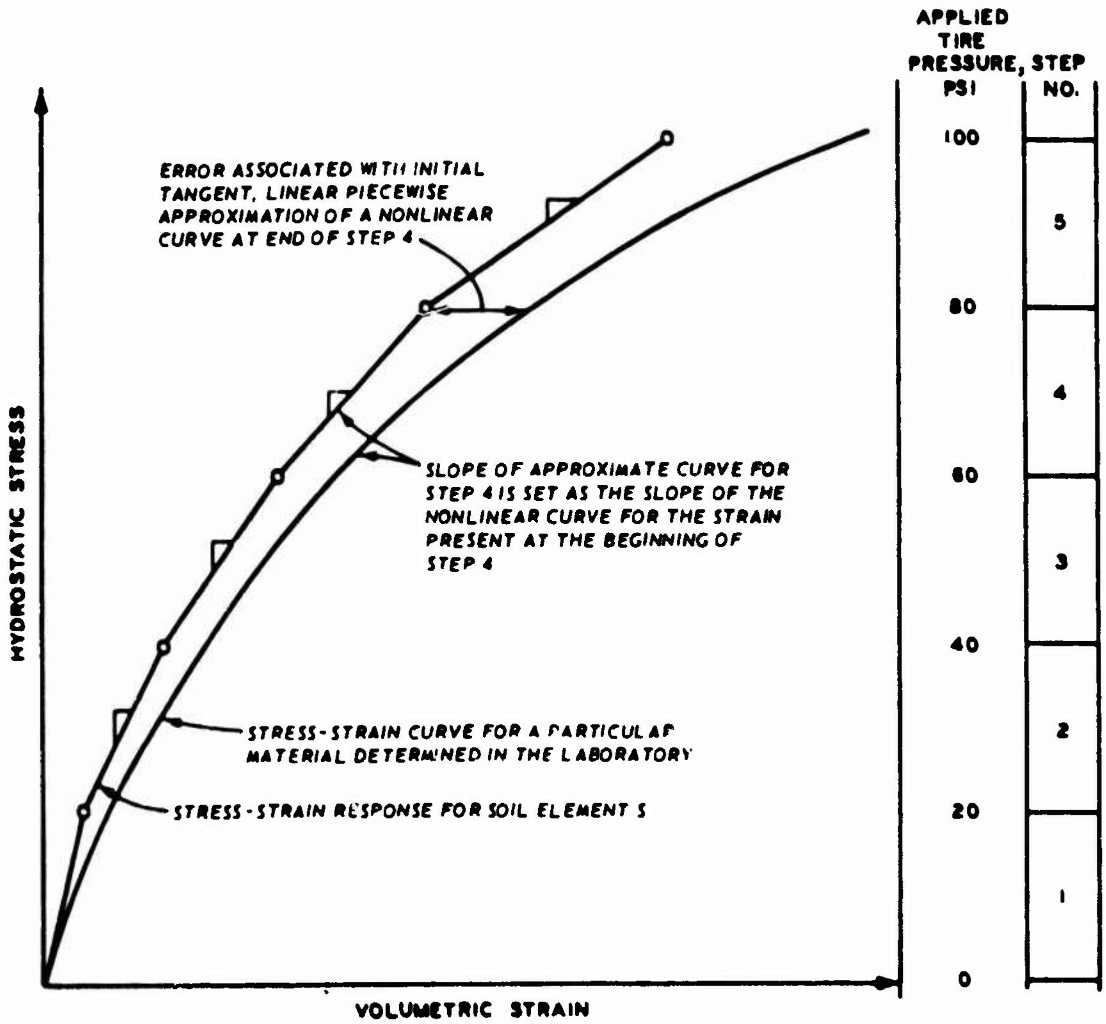


Figure 16. Stepwise approximation of nonlinear behavior

to which the shadow approximates the measured curve is for this example improved merely by increasing the number of steps used for nonlinear analysis. The responses computed during each step are cumulatively added to those of the previous steps, resulting at the end of the last step in the response of the pavement to the total applied load (for this example, 100 psi).

The number of steps used for the analysis is a function of the degree of nonlinearity present. Five steps have been found to give reasonable results while using more than 20 steps provides little added benefit. The procedures for this study were to use five steps for all the preliminary calculations needed to establish the various mesh parameters and to use 20 steps for the final runs. (This reduced the error associated with the linear piecewise approximation of the measured material curve to less than 10 percent.) For this study, only where concrete overlayed a "soft" soil resulting in extensive concrete cracking were there significant differences between the 5- and 20-step analyses.

3.3 CLASSIFICATION OF MATERIAL MODELS FOR PAVEMENTS

A convenient method for categorizing material laws is with the descriptors: time-dependent versus time-independent and nonlinear versus linear. Here, time-dependent implies that a real-time variable is included in the constitutive relation, and nonlinear implies that stress is not a linear function of strain when all other state variables are held constant. These descriptors are utilized in Figure 17 to denote a hierarchy of constitutive fo[illegible] The top box of this figure represents the most general form (time-dependent and nonlinear) of which examples are viscoplasticity and nonlinear creep laws. Much work has been done on these constitutive forms for metals[29] but very little for pavements. It is well known that pavements exhibit significant time-dependent behavior particularly for saturated cohesive soils and/or asphaltic surfaces. However, unless the real-time response history is of interest, the added complication of using time-dependent models is questionable. That is, from an engineering viewpoint the capacity of a pavement system can be adequately determined by considering all loads applied for long durations, thereby accounting for creep and relaxation effects. No doubt, future research may make these time-dependent nonlinear forms very attractive, particularly if failure criteria can be built into the models. However, until such time, these forms belong more to the realm of research than to today's design tools.

In the next level of Figure 17 the general constitutive form is subdivided into two categories: time-dependent and linear and time-independent and nonlinear. Examples of the first category include visco-plasticity and linear creep laws. Again, as previously discussed, the current benefits from including real time in the constitutive form are marginal. Worse still, the implied linear stress-strain assumption is simply not in accordance with observed behavior of any component of a pavement system. That is, all pavement components that exhibit a significant time-dependent response also exhibit a significant nonlinearity, e.g., saturated granular soils, cohesive soils, and asphaltic cements. Consequently, it does not appear that time-dependent and linear material models will ever play a significant role in pavement analysis with the

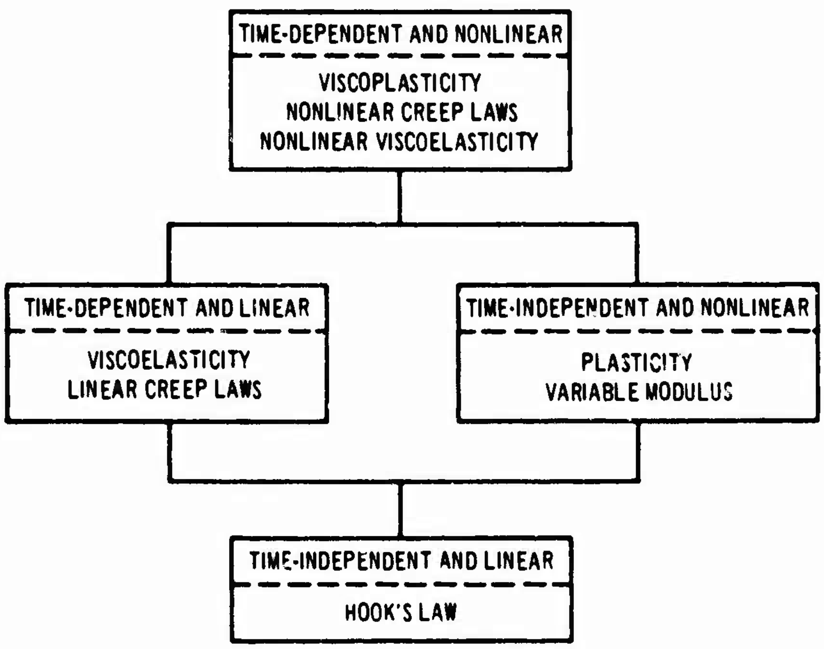


Figure 17. Classification of material models

possible exception of consolidation problems or pavements composed of sea-ice.[30]

The adjoining category, time-independent and nonlinear, is by far and away the most significant category; consequently this category will be discussed in greater depth.

It is convenient to further subdivide the time-independent and nonlinear classification into two groups: plasticity models and variable modulus models. The former grouping is based on the theory of plasticity which in general requires a yield criterion, a hardening rule, and a flow rule. A yield criterion defines the onset of plastic yielding and is usually assumed to be a function of the stress invariants. The hardening rule redefines the yield criterion after plastic deformation has occurred and is usually assumed to be a function of plastic work and stress level. Lastly, the flow rule relates increments of plastic strain to increments of stress after the yield criterion is satisfied. Examples of plastic models applied to pavements are the Drucker-Prager, Mohr-Coulomb, and capped models.[31]

From an academic viewpoint, the plasticity models are more glamorous than the variable modulus models (discussed next) because they generally satisfy rigorous theoretical requirements and are inherently capable of treating unloading and cyclic loading for fatigue considerations. On the negative side, plasticity models generally do not correlate well with triaxial data from soil specimens. Also, the parameters of the plasticity models are relatively difficult to determine.

Variable modulus models are based on the hypothesis that stress increments can be related to strain increments by an elastic constitutive matrix wherein the components of the constitutive matrix are dependent on the level of stress and/or strain (i.e., $\{\Delta\sigma\} = [C]\{\Delta\varepsilon\}$ where $\{\Delta\sigma\}$ and $\{\Delta\varepsilon\}$ are increments of stress and strain and $[C]$ is the elastic constitutive matrix whose components are dependent on the current total level of stress and strain). The variable modulus models represent materials of the hypoelastic classification; that is, the constitutive components are dependent upon initial conditions and the stress path. Consequently, the term nonlinear elastic is not appropriate for variable modulus models since nonlinear elastic implies path independence. The advantages of variable modulus models are their inherent ability to approximate experimental data closely and the relative ease of determining the parameters of the model. The disadvantages of this approach are that unloading is not inherently built into the model and must be treated in an ad hoc fashion. Also special programming features may have to be employed to avoid possible numerical and theoretical difficulties arising from energy violations (e.g., Poisson's ratio exceeding 0.5). Examples of variable modulus models are given in References 32-34. Further discussion of plasticity and variable modulus models as they relate to pavements is given in the next section.

The last category of material models shown in Figure 17 is the linear form. The linear form is a straightforward application of the generalized Hook's law. The obvious advantage of the linear form is its utter simplicity; it also permits the usage of classical solution techniques. Of course, the disadvantage of the linear form is the inability to model the nonlinear behavior of pavement systems, particularly the base course and subgrade components.

To summarize, it is felt that the time-independent and nonlinear category of Figure 17 embodies the most relevant models for pavement analysis. Time-dependent and nonlinear models generally impose a level of sophistication in excess of today's performance criteria; however, their future is very promising. Time-dependent and linear models do not now or in the future appear to be relevant in pavement analysis. Lastly, the simple linear forms have limited applications and may be entirely phased out in future developments.

In the next section, particular models in the time-independent and nonlinear category are discussed.

3.4 MATERIAL MODELS FOR PAVEMENTS

Wearing courses are the most significant component of a layered pavement system since they directly receive the load and distribute it to the layers below; proper material modeling is essential for meaningful results.* Both variable modulus and plasticity models have been extensively used for these materials with apparent success. In particular, the hybrid plasticity model developed by Kupfer et al.[35] is well suited for concrete. The yield surface of this model faithfully matches the data points of concrete specimens in all quadrants of the biaxial stress plane. Accordingly, the model can predict the onset of concrete cracking in tension zones as well as nonlinear responses in the compression zones. A finite element development of this model is given in Reference 36.

Base course layers are almost exclusively composed of various gradations of granular materials or stabilized soil while the subgrade layers may include granular, mixed, and cohesive soils. A model capable of treating any of these soils is a variable modulus model originated by Hardin.[32] The objective of the following discussion concerning the Hardin model is to demonstrate procedures that should be used to evaluate any soil model and to outline desirable model features.

The Hardin model presents a hyperbolic relationship for the secant shear modulus as follows:

$$G_s = \frac{G_{max}}{1 + \gamma_h}$$

where

G_s = secant shear modulus

G_{max} = maximum shear modulus, a function of spherical stress

γ_h = hyperbolic shear strain, a function of shear strain and spherical stress for a particular soil

The beauty of Hardin's approach is that not only does he present the form of the relationships for G_{max} and γ_h, but he also identifies the soil-dependent parameters and provides equations for evaluating them in terms of soil type (granular, mixed, or cohesive), void ratio, percent

* An error in material characterization for the top layer has a dramatic impact on the predicted response for the whole pavement. This is especially true for asphalt where the authors have observed others using values of E from 20,000 to a million psi, and for concrete where tensile cracking is extensive.

saturation, and plasticity index. In addition, Hardin's model accounts for the number of loading cycles and rate of loading, thereby allowing consideration for a range of aircraft loading.

Figure 18 demonstrates the accuracy of the Hardin shear model for Cook's bayou sand* wherein the observed shear modulus from four triaxial tests are compared with the Hardin prediction. It is observed that the agreement between the Hardin prediction and the experimental data is excellent. More significantly, it is important to note that the Hardin

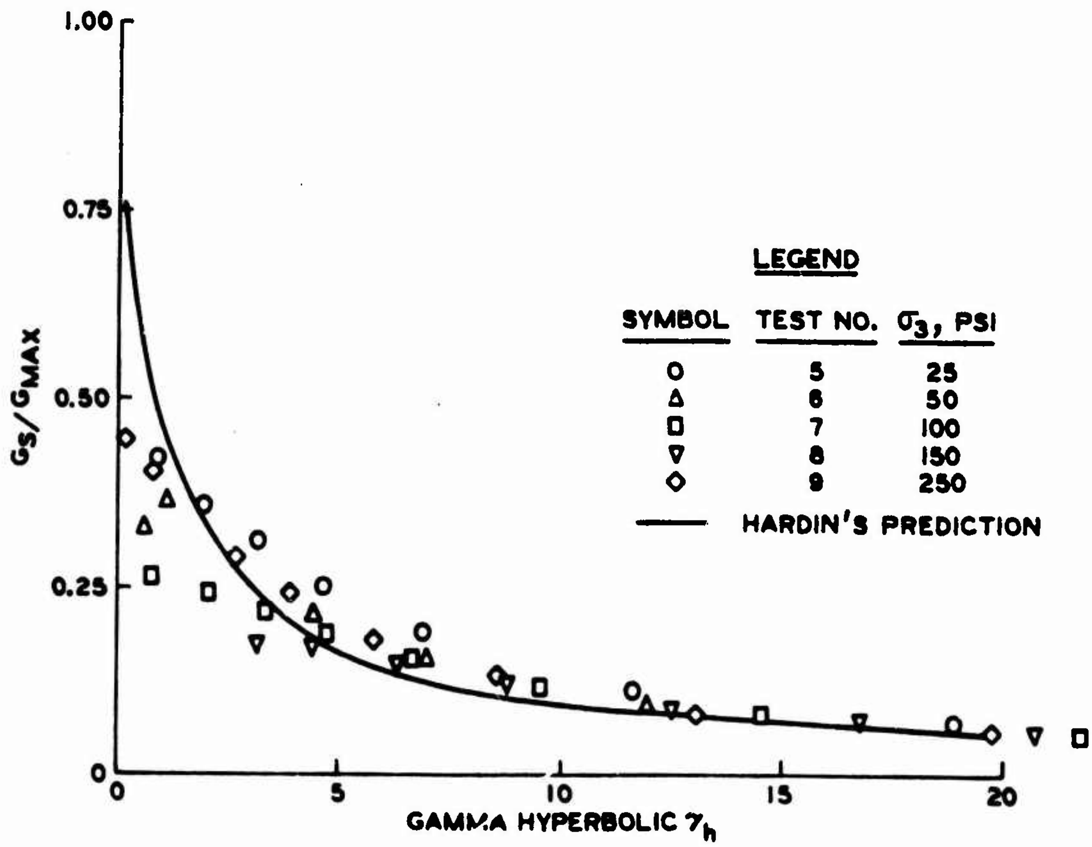


Figure 18. G_s/G_{max} versus γ_h

* J. T. Ballard, U. S. Army Engineer Waterways Experiment Station, personal communication with the Naval Civil Engineering Laboratory, 6 September 1973.

prediction was not obtained by curve fitting, but rather by a straightforward application of the model where the only soil information required was void ratio.

This result implies that it is possible to utilize the Hardin shear model in the absence of test data, a situation often encountered by pavement analysts.

By itself, Hardin's relationship for the shear modulus does not provide a complete variable modulus model. To complete the constitutive form, it is necessary to incorporate at least one additional elastic parameter (function) into the constitutive matrix. For example, Young's modulus, bulk modulus, or Poisson's ratio could be selected. Of these choices, Poisson's ratio is the most convenient based on the consideration of avoiding numerical difficulties and maintaining theoretical energy requirements. By developing a function for Poisson's ratio such that its range is within the limits $0.0 \leq \nu < 0.5$, the energy requirements are satisfied regardless of the current value of the shear modulus. On the other hand, if the bulk modulus or Young's modulus were used, their valid range is dependent on the current value of the shear modulus (i.e., bulk modulus must be greater than two-thirds of the shear modulus, and Young's modulus must be greater than two but less than three times the value of the shear modulus).

Many investigators assume Poisson's ratio is constant in soil models; however, this assumption is based more on convenience than on actual material behavior. The fact that Poisson's ratio does vary during loading is demonstrated in Figure 19. Here the observed Poisson's ratio of Cook's bayou sand* is plotted for four triaxial tests as a function of normalized shear strain. Clearly it is observed that Poisson's ratio is a function of both shear strain and confining pressure and varies from approximately 0.1 to 0.5. Accordingly, any serious attempt to model soil behavior must incorporate a relationship for Poisson's ratio. To this end, a general relationship for Poisson's ratio has been developed in a manner analogous to Hardin's shear modulus. The interested

* J. T. Ballard, op. cit.

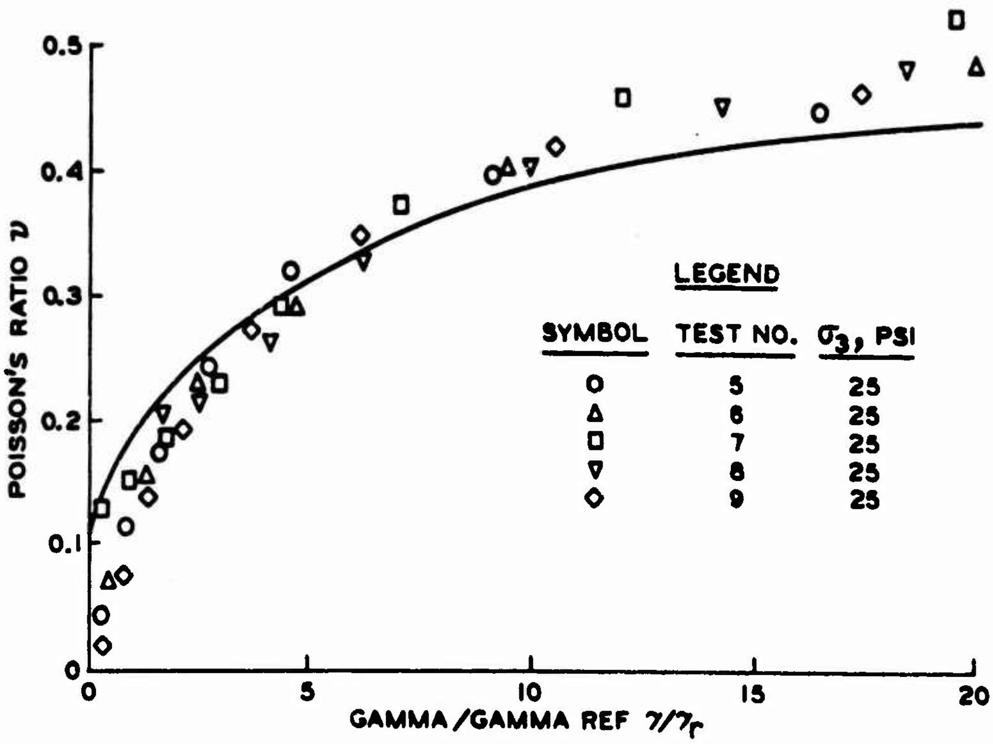


Figure 19. Poisson's ratio ν versus ratio of measured shear strain γ to a reference shear strain γ_r

reader may find a complete presentation of this development in Reference 37 along with a review of Hardin's and other relationships.

3.5 SUMMARY

The main points of this section are summarized below:

a. Finite element codes should be structured to operate on an incremental basis with allowance for iterating within the load step.

b. Material models should be isolated in subroutines to permit easy additions, deletions, and/or modifications.

c. Solution strategies for nonlinear equations should not be compromised to promote efficiency at the expense of accuracy.

d. Time-dependent and nonlinear material models are for the most part overly sophisticated with respect to the currently employed performance criteria.

e. Time-dependent and linear material models have little relevance in pavement analysis.

f. Time-independent and nonlinear material models are most relevant to the pavement problem. In particular, plasticity models are adequate for the wearing courses, and variable modulus models are adequate for the soil layers.

g. Hardin's shear modulus relationship is an excellent example of the generality and versatility that should be incorporated into soil models.

4 COMPARISON AND PRESENTATION OF ANALYTICAL AND MEASURED RESULTS

Previous sections have dealt in a general way with particular facets of prediction techniques. This section presents several applications of prediction techniques and by illustrating their application and results ties them to the foregoing general discussion. The techniques covered include Westergaard, elastic layer (Burmister problem), and linear and nonlinear finite element. Most of the examples presented have been taken from various previously published papers. Comparison of one technique versus another is shown wherever available.

4.1 WESTERGAARD AND ELASTIC LAYER ANALYSES

Figure 20 shows various pavement sections which are used to demonstrate the application of Westergaard and elastic layer analysis techniques. The first five are flexible sections while the second five are rigid. Elastic layer responses are shown in Table 1 for the sections of Figure 20 using four different elastic layer computer codes: ELAST (4), BISTRO (5), CRANLAY (7), and CHEVRON (6). Because these sections are assumed to be linear elastic and have no joints or edges, the layer elastic response is "exact" within the context of the theory of elasticity, assuming negligible computational errors.

The Westergaard equations shown in Section 2.2, which are those standardly used, are a first approximation for the solution of a plate on a fluid foundation (Figure 1). Westergaard in Reference 3 indicates that including more than the first term of the Kei and Ker function yields more accurate results. Responses for the single term (Equations 1 and 2) and a six-term approximation are shown in Table 1. Both of these approximations assume that the top layer contains a neutral axis; erroneous results occur when this is untrue. To eliminate this problem, the Westergaard idealization is solved using mathematics similar to that used for the layer elastic idealization.[4] In this case, the first layer is considered an axisymmetric solid rather than a plate. The seventh column of Table 1 contains these results.

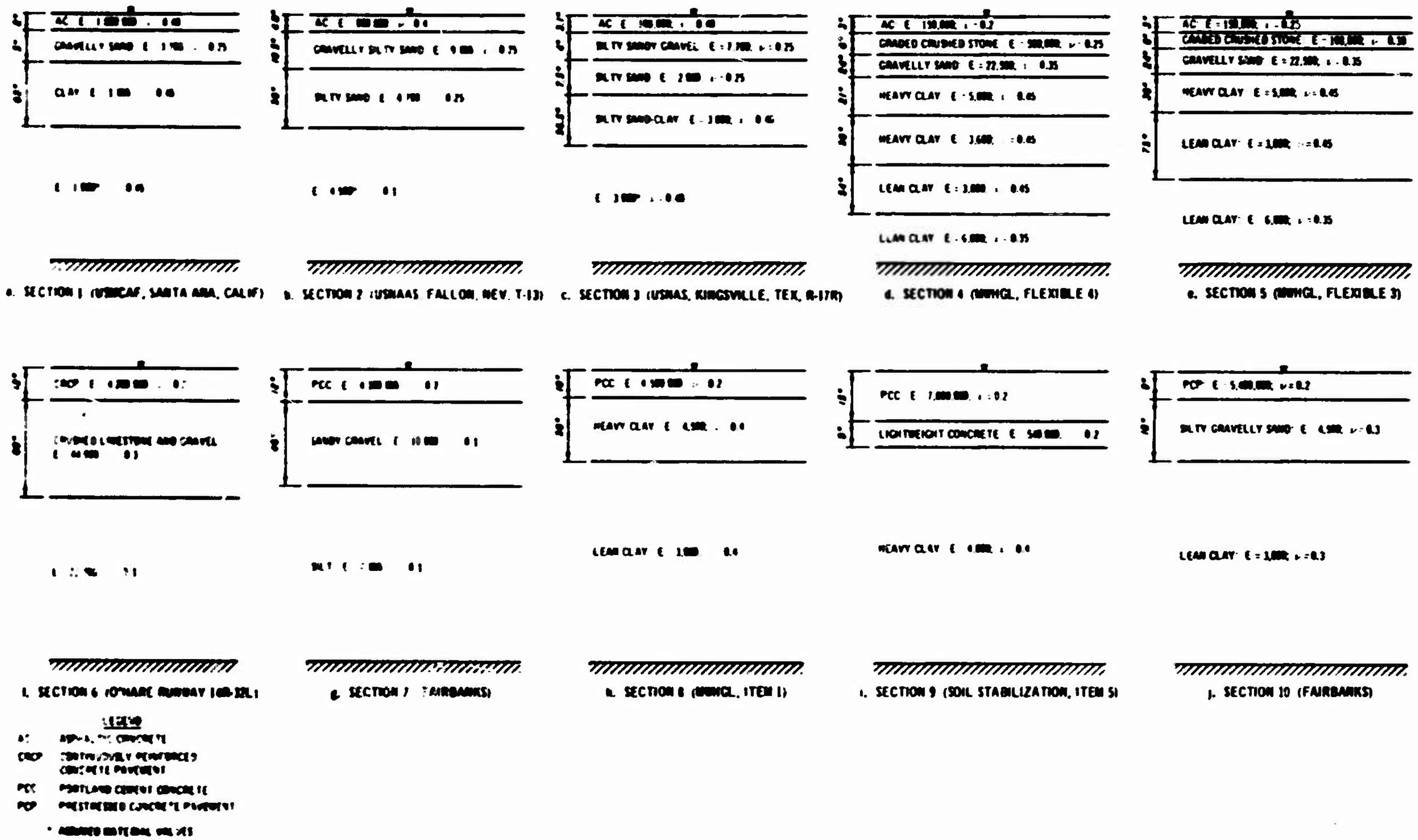


Figure 20. Selected pavement sections

Table 1

Predicted Linear Response of Selected Sections

Section No.	Linear Elastic Layer Response: ELAST δ/σ	Linear Elastic Layer Response: BISTRO $\delta/\sigma/\epsilon$*	Linear Elastic Layer Response: CRANLAY $\delta/\sigma/\epsilon$	Linear Elastic Layer Response: CHEVRON $\delta/\sigma/\epsilon$	Westergaard Response: First Approximation δ/σ	Westergaard Response: Six-Term δ/σ	Westergaard Response: Elastic Layer/Fluid δ/σ	Westergaard Response: Equivalent k_v	Measured k
1	0.605/4880	0.605/4880/0.014	0.603/4940/0.014	0.607/4870/0.014	0.492/5670	0.492/5960	0.372/5610	50	90
2	0.100/862	0.100/862/0.0045	0.100/867/0.0045	0.100/859/0.0045	0.079/1020	0.080/1110	0.063/1210	283	440
3	0.368/736	0.369/736/0.017	0.367/765/0.017	0.363/734/0.017	0.490/1510	0.487/1690	0.382/1870	114	400
4	0.087/-50	0.087/-50/0.0017	0.092/38/0.0020	0.091/-44/0.0017	0.190/910	0.188/1120	0.120/1100	502	Not available
5	0.086/-54	0.085/-54/0.0016	0.096/-45/0.0018	0.090/-48/0.0016	0.303/810	0.290/1590	0.320/1490	510	Not available
6	0.009/225	0.009/225/0.00015	0.009/224/0.00015	0.009/221/0.00015	0.004/230	0.004/230	0.005/210	1428	Not available
7	0.022/289	0.022/289/0.00023	0.022/289/0.00022	0.024/282/0.00023	0.007/270	0.007/270	0.009/243	339	Not available
8	0.041/472	0.041/471/0.00022	0.041/471/0.00022	0.046/478/0.00025	0.011/410	0.011/410	0.015/340	156	62
9	0.015/168	0.015/168/0.000021	0.015/167/0.000021	0.015/183/0.000022	0.003/180	0.003/170	0.003/160	627	420
10	0.049/555	0.049/555/0.00038	0.049/554/0.00039	0.056/546/0.00042	0.016/490	0.016/500	0.018/460	136	88

Note: All responses computed for load of 27,000 lb uniformly distributed over a 11.4-in.-diam circular area.
* δ is the peak deflection under load, σ is the maximum tensile stress at the bottom of the first layer, ϵ is maximum vertical compressive strain in the second layer.

The Westergaard and the elastic layer idealizations may be related through their common mechanistic basis using the following function (see Figures 1 and 2 for symbols).

$$k = F(h_2E_2\nu_2,\ h_3E_3\nu_3 \ldots E_N\nu_N)$$

The derivation of F is presented in detail in Reference 4, and the results of its usage are shown in Table 1 as k_w.

To enable direct comparison of elastic layer and Westergaard results, all of the Westergaard responses shown in the table are computed using k_w. Thus, at least for the sections of Figure 20, the predictions of the Westergaard idealization (whichever solution technique is used, i.e., columns 5, 6, or 7 of Table 1) are "spotty" when measured against the "exact" elastic layer solution (computed by any of the four codes shown).

Values for the Westergaard subgrade modulus k are shown where available in the last column of Table 1. Comparison of k_w (computed from layer properties) with k produces a measure of the accuracy with which the elastic layer material properties predict the results of a plate bearing test. In this manner k_w can be used to assess the overall performance of the elastic layer parameters. For the sections of Figure 20 where most of the E and ν values were based on rather sketchy data, it is expected that k and k_w will not agree very often.

The advantages of both these techniques, especially Westergaard, are their ease of application and relatively modest requirement for preparation of data or interpretation of results. If these techniques are carefully applied with knowledge of their limitations, useful design data can be generated. As an analysis procedure, however, these techniques are at a distinct disadvantage with respect to all but linear layered pavements. The crux of the matter is that while, for example, a nonlinear k can be used for a Westergaard analysis, the lack of capacity in these formulations to assess what these nonlinear approximations mean makes such results suspect. Possibly use of more refined methods, such as finite element, to prove the validity of such extreme extensions of relatively simple theories will make such approaches desirable.

4.2 LINEAR FINITE ELEMENT

Finite element techniques have two significant differences from the linear idealizations of Section 4.1: the capacity to vary material properties within the layer and the consideration of a soil island. A single- and multiwheel analysis for Section 4 of Figure 20 is presented to establish the relationship between finite element and elastic layer analyses and to study the peculiarities of the finite element procedure. This pavement section is Item 4 of the C-5A test track at Waterways Experiment Station (WES).

4.2.1 SINGLE WHEEL ON LAYERED PAVEMENT

Figure 21* shows the response predicted by an axisymmetric finite element code (WIL67) and an elastic layer code (BISTRO) for a single C-5A wheel loaded by 30,000 lb. Several features are illustrated by this figure:

a. The stress responses predicted by elastic layer and finite element codes are identical while the deflections are different by a constant amount. This is a typical result where the disparity is attributable to the soil island nature of the finite element procedure; that is, the values of vertical deflection are dependent on the integration of all the vertical strains along a vertical line. If the mesh is truncated before these strains are negligible, then a portion of displacement is lost, which leads to a constant error factor between the "exact" elastic layer and finite element idealizations. However, the stresses, strains, and deflection shapes, which are derivatives of the vertical displacement, are easily matched between the two methods for any reasonably designed mesh. To match the deflection requires a convergence study which is discussed in Section 4.2.3.

b. The linear pavement response is relatively insensitive to the distribution of stiffness in the lower layers. This is demonstrated by comparison of the responses for the seven actual layers to a smeared four-layer system. The smeared values are shown in Figure 21 as E_4' and ν_4' .

* This figure and the cost data were extracted from Reference 1.

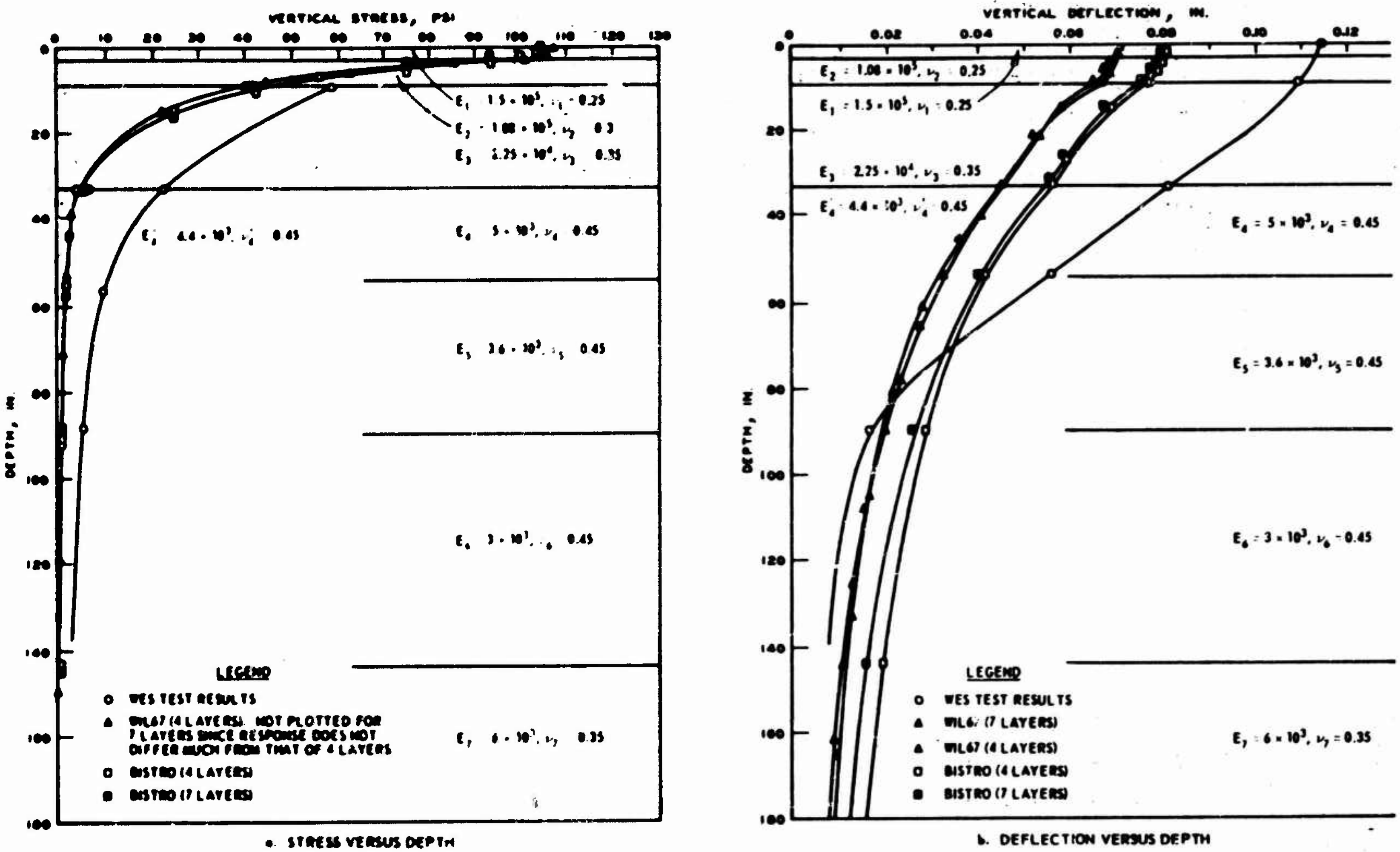


Figure 21. Response of Section 4 to a single-wheel load of 30,000 lb (from Reference 1)

c. The reversed curvature form of the measured displacements is typical of test results and is indicative of nonlinear behavior. The rather uniform curvature of the predicted displacements is typical of linear layered analysis, irrespective of the method applied.

d. Stiffness within a layer must be nonuniform to match the test results. The disparity between measured and predicted stress (Figure 21a) can be attributed to two possible failings of the model: first, that the material of layer 4 is too flexible, or second, that too much flexural stiffness is present in layer 3. Examination of these two possibilities yields the following: if the flexural stiffness E_3 is decreased, the magnitude and shape of the vertical deflection curve will become larger and less vertical in the third layer than that shown in Figure 21b. Second, if the fourth layer's stiffness E_4 is increased, the deflected shape across the fourth layer becomes steeper. Neither of these courses will produce an improvement in the predicted responses, and moreover, little change will occur in the stress field because of equilibrium considerations associated with layered systems. Thus, neither possibility offers the answer, and in fact, this disparity illustrates the fundamental flaw associated with elastic layer idealizations: that is, uniformity of stiffness within a layer. The only reasonable way that the measured responses for this section can be matched by analytical prediction is by resorting to a nonuniform layer stiffness, that is, (1) by an increase of stiffness in layer 3 directly under the load which causes an increase in vertical stress, but little increase in flexural stiffness; and (2) by a decrease in stiffness at the center of layers 4 and 5 under the load to provide the reverse curvature of the WES test results.

The computer time for running the BISTRO code is proportional to the number of locations at which responses are requested and the number of pavement layers (for Figure 21, 4 sec per location on a CDC 6600 for four layers and 6 sec for seven layers). For the WIL67 code the time is related to the mesh, which is selected relatively independent of the number of responses desired and pavement layers (40 sec on a CDC 6600 for a 480-element mesh). Thus, if more than 10 responses are to be computed, use of the finite element technique would be cheaper in this case. Because of the data generators associated with WIL67, the amount of data preparation is the same for both codes.

4.2.2 MULTIPLE-WHEEL GEAR ON LAYERED PAVEMENT

While single-wheel analyses have much educational value and are of practical significance at installations predominately used by fighter aircraft, most airfields are designed for the large multiwheeled aircraft such as the C-5A shown in Figure 22. To obtain multiwheel results requires use of superposition for elastic layer and axisymmetric idealizations, or results may be directly obtained using prismatic or three-dimensional solid idealizations. Figure 23 shows the predicted response for the 12-wheel assembly placed on the four-layer approximation of Section 4. Results similar to those of Figure 21 are obtained.

a. Figure 23a shows the predicted deflection basin along the gear centroid (i.e., along the X-axis). Two different mesh depths were taken, both fairly shallow, causing considerable disparity between finite element and elastic layer.

b. The data presented in Figure 23b and c show the maximum response recorded within a horizontal plane as a function of depth; that is, these curves are envelopes of peak response (denoted limiting curves in Figure 23). Also shown in these curves is the effect of increasing the stiffness of layer 4: no effect occurs in the stress; and while the displaced shape is the same, the curve is shifted to the left.

c. Figure 23d and e shows the horizontal and shear stress under the points shown in Figure 22. These values of stress are also unaffected by changing E_4 to 8000 psi.[1]

The time for running BISTRO for the 12 wheels, 4 layers, and 44 locations of response was 2896 sec on a CDC 6600, while for AFPAV, it was 926 sec for 350 elements and 31 Fourier terms. Because of the preprocessor associated with AFPAV, the amount of data preparation for both codes is about the same.

4.2.3 PARAMETER STUDIES

Several parameter studies have been conducted to assess the accuracy of the finite element predictions as well as to provide a better understanding of the effects of changes in pavement constituents. Unfortunately, this work is usually an unreported part of most studies. Some of the work reported concerning understanding and accuracy follows.

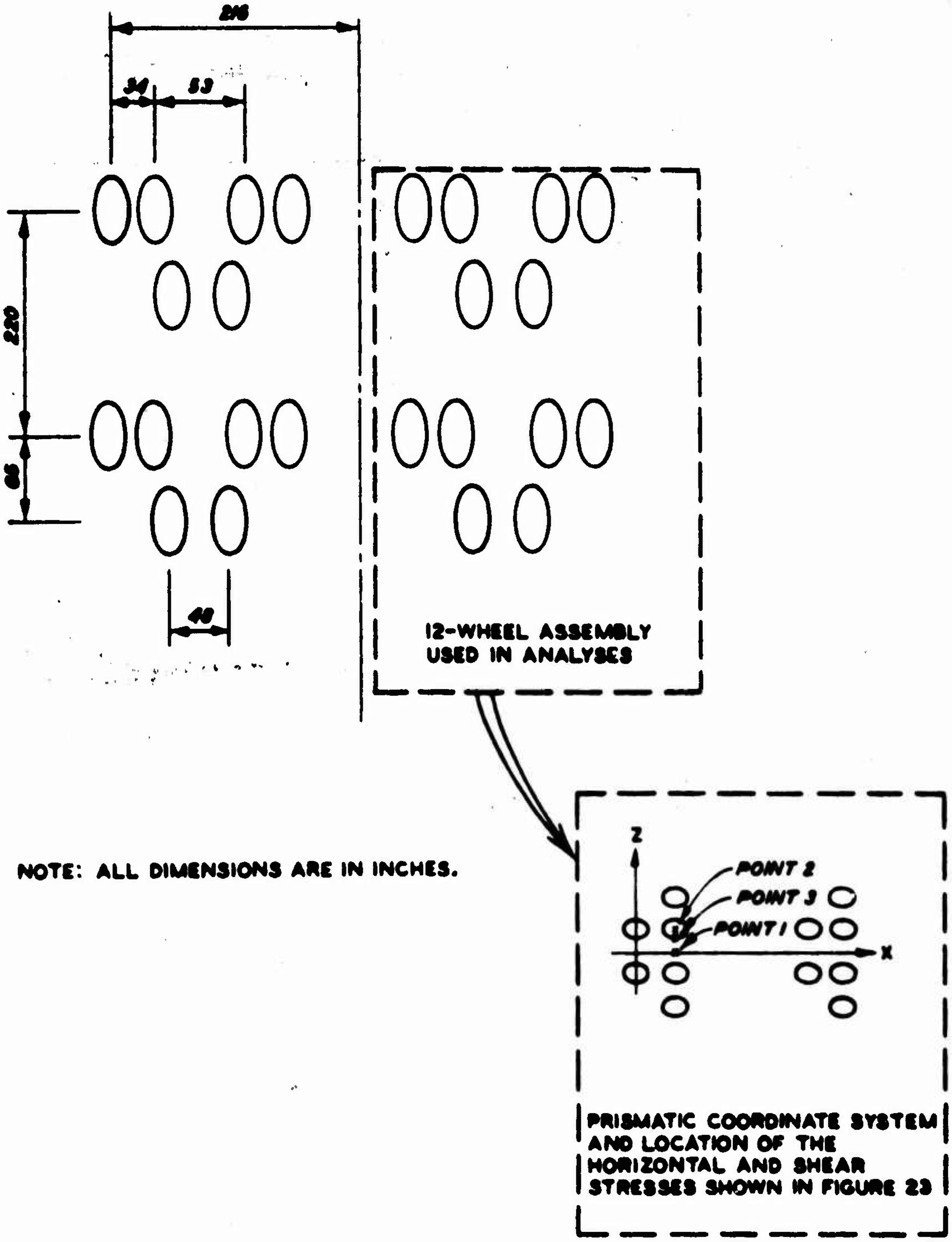


Figure 22. C-5A main landing gear configuration

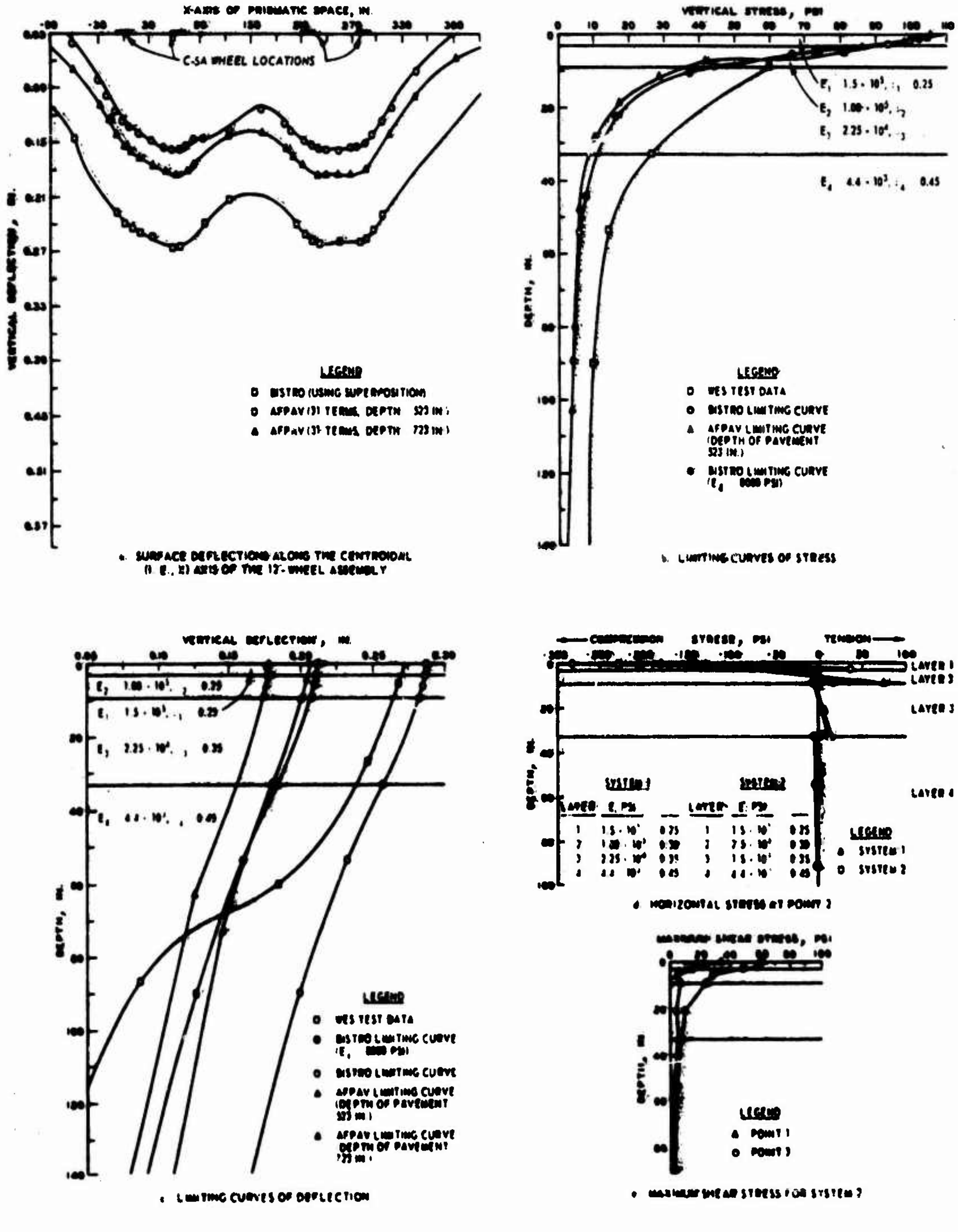
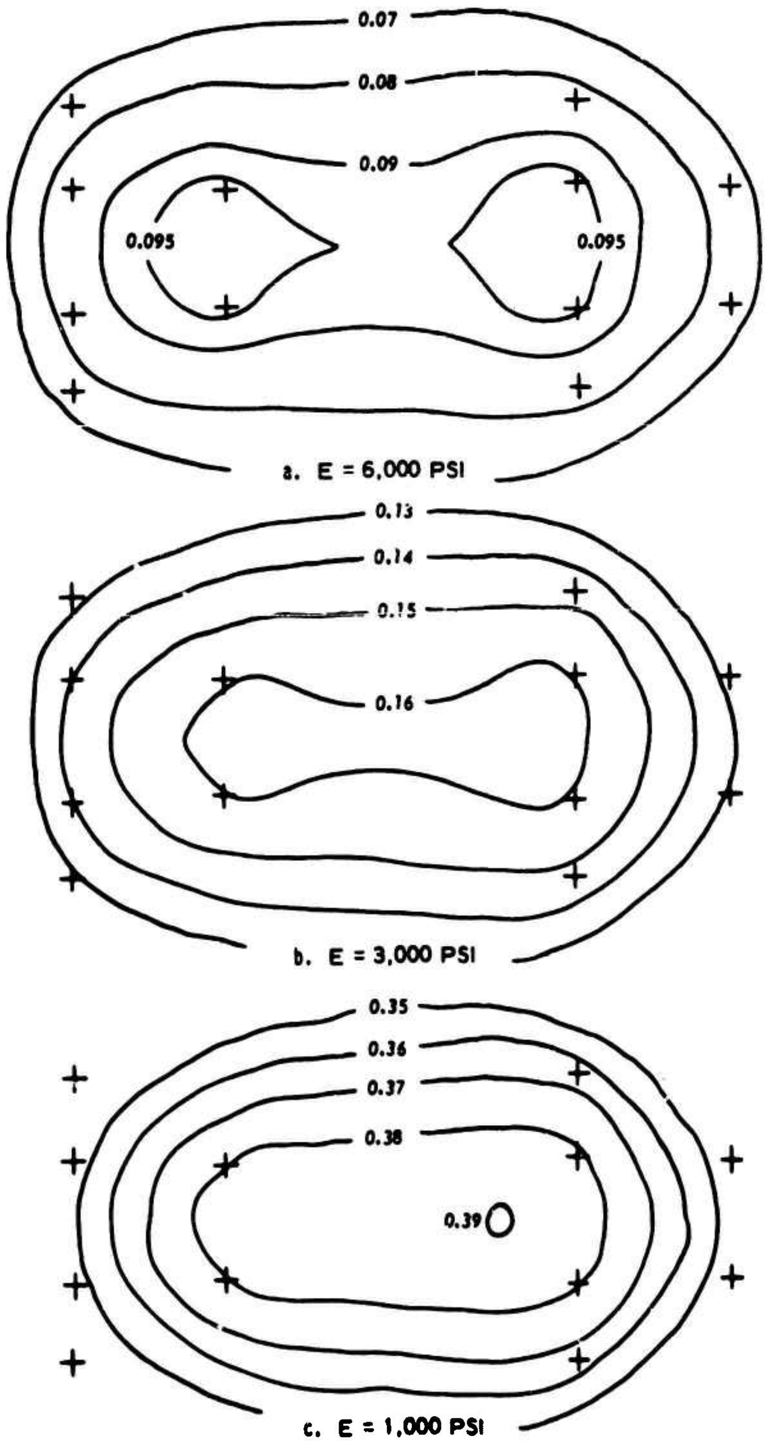


Figure 23. Response of Section 4 to a C-5A 12-wheel assembly

An extensive parameter study for both flexible and rigid linear layered systems is reported in Reference 19. This study is primarily concerned with the impact that changing the various pavement properties has on the predicted response. Two key points for both rigid and flexible pavements are derived. First, soil stresses (Figure 23b is an example) and second, the displacement gradients within the deflection basin are relatively unaffected by significant changes in base, subbase, and subgrade moduli. Figure 24 demonstrates that the displacement gradients at the surface (i.e., the spacing between contour lines) remain relatively unchanged for the rigid two-layer pavement systems shown.

To develop an accurate finite element prediction requires not only a good mesh but also an appropriate soil island. This leads to another form of parameter study in which the mesh and its extent and displacement boundary conditions are examined with respect to displacement and stress-strain convergence. A convenient method for establishing convergence is by comparison of finite element and elastic layer predictions. References 15 and 28 present such parameter studies with the basic conclusion that a sloping boundary of 30 deg off the vertical with a free edge or with restrained horizontal displacement produces the fastest convergence. Unpublished studies using the WINDAX Code[17] indicate that finite element displacements within 10 percent of those predicted by elastic layer techniques can be achieved by using 600 elements and soil island dimensions of 100 loading radii wide and 150 radii deep for typical rigid and flexible pavements.

Generally, stress-strain predictions are unaffected by the size and boundaries of the soil island for any reasonable mesh. Convergent displacements are considerably harder to obtain and are more dependent on the materials involved. However, the difficulty associated with displacement computation is not a serious liability for the finite element idealization. First, to predict a linear displacement has relatively little merit and as shown in Figures 21 and 23 has little resemblance to actual performance. Second, most useful displacement predictions are derived from nonlinear analyses which are much less affected by the soil island employed.



NOTE: + INDICATES THE WHEEL POSITIONS OF THE 12-WHEEL ASSEMBLY OF THE C-5A GEAR.

Figure 24. Effect of subgrade E-modulus on surface deflection in rigid pavement (from Reference 19)

4.3 NONLINEAR FINITE ELEMENT

Section 3 presents a variety of material laws and implementation algorithms which when coupled with the variety of prediction techniques yields many alternatives. However, in reviewing what is available and is being applied to pavements, there appear to be only four relatively widespread approaches:

a. Elastic layer analysis whereby the layer property is iteratively defined during a succession of linear analyses based on a typical layer strain value. Such a method is cheap and simple, and the predicted displacements are at least nonlinear functions of load. However, as was mentioned earlier, the stress distribution and the shape of the deflection basin will remain relatively unchanged as long as the basic premise of uniform layer properties is adhered to; this will make any precise usage of these responses suspect.

b. Axisymmetric finite element analysis using a one-step secant method (Figure 15a). There are various forms of this technique available based on the code reported in Reference 18. However, highly nonlinear material (e.g., that which cracks under load) cannot be modeled.

c. Axisymmetric finite element analysis using the tangent method. Using this technique, the response for the repaired crater shown in Figure 12 was computed; Reference 17 contains considerable information pertaining to the details of this study. The response of this system where the concrete was allowed and not allowed to crack is shown in Figure 25. In this instance, the need for a multiple-step loading, which allows a cracking capability, is shown to be important in ascertaining correlation between measured and predicted results.

d. Prismatic finite element analysis using a one-step secant method whereby the nonlinearity is by element in the X-Y plane and by the layer along the Z-axis (see Figure 7). This is only a first-order approximation to the full nonlinear analysis capability available using the prismatic idealization. Some results are reported in References 21 and 22; typical run times are 10-40 min on a CDC 6600.

The axisymmetric idealizations offer the best opportunity to study the nonlinear process. They are inexpensive to run (typically $3-30); results are relatively easy to interpret; and data preparation is straightforward. These codes provide an excellent environment in which to check the various input parameters and evaluate their adequacy. On the other hand, the prismatic idealization offers the most realistic procedure for modeling pavements loaded by multiple-wheel aircraft. Also, the prismatic idealization is easier to "black box" for usage by the typical engineer through the usage of pre- and postprocessing software (Section 5.1.2).

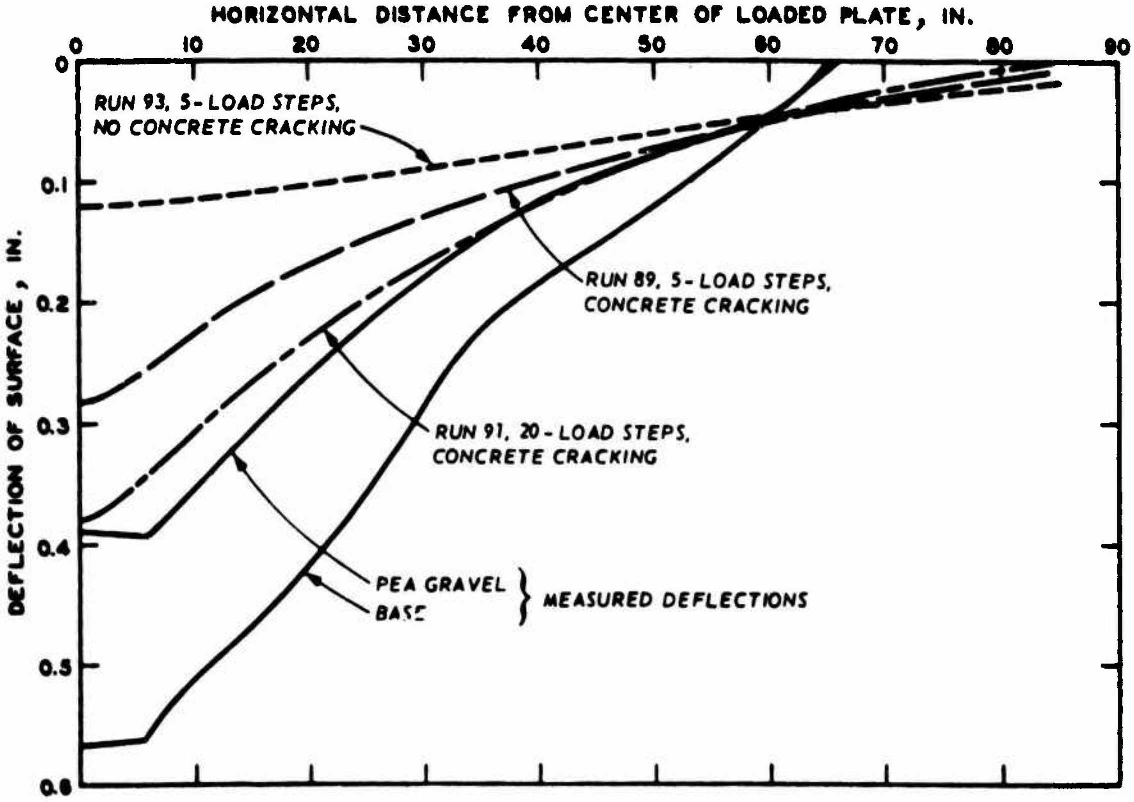


Figure 25. Deflection basins under 30-kip load for repaired crater shown in Figure 12

4.4 THREE-DIMENSIONAL ANALYSIS

The only relatively practical usage of three-dimensional analysis known to the authors is presented in Reference 23. The objective was to make a rational determination of the basic parameters (e.g., pile spacing and depth) of a soil-pile reinforced pavement. Because of the symmetry of the system and the lack of any other reasonable alternatives, three-dimensional analysis was used. Figures 8 and 26 illustrate some of the steps taken and the results obtained.

4.5 NONLINEAR KINEMATICS

Nonlinear kinematical considerations are required for such items as friction joints,[15] mechanically interacting multicomponent structures (e.g., the key joint shown in Figure 27[38]), or large deformation phenomena (e.g., membrane surfaces for helicopter landing zones shown in Figure 28[39]).

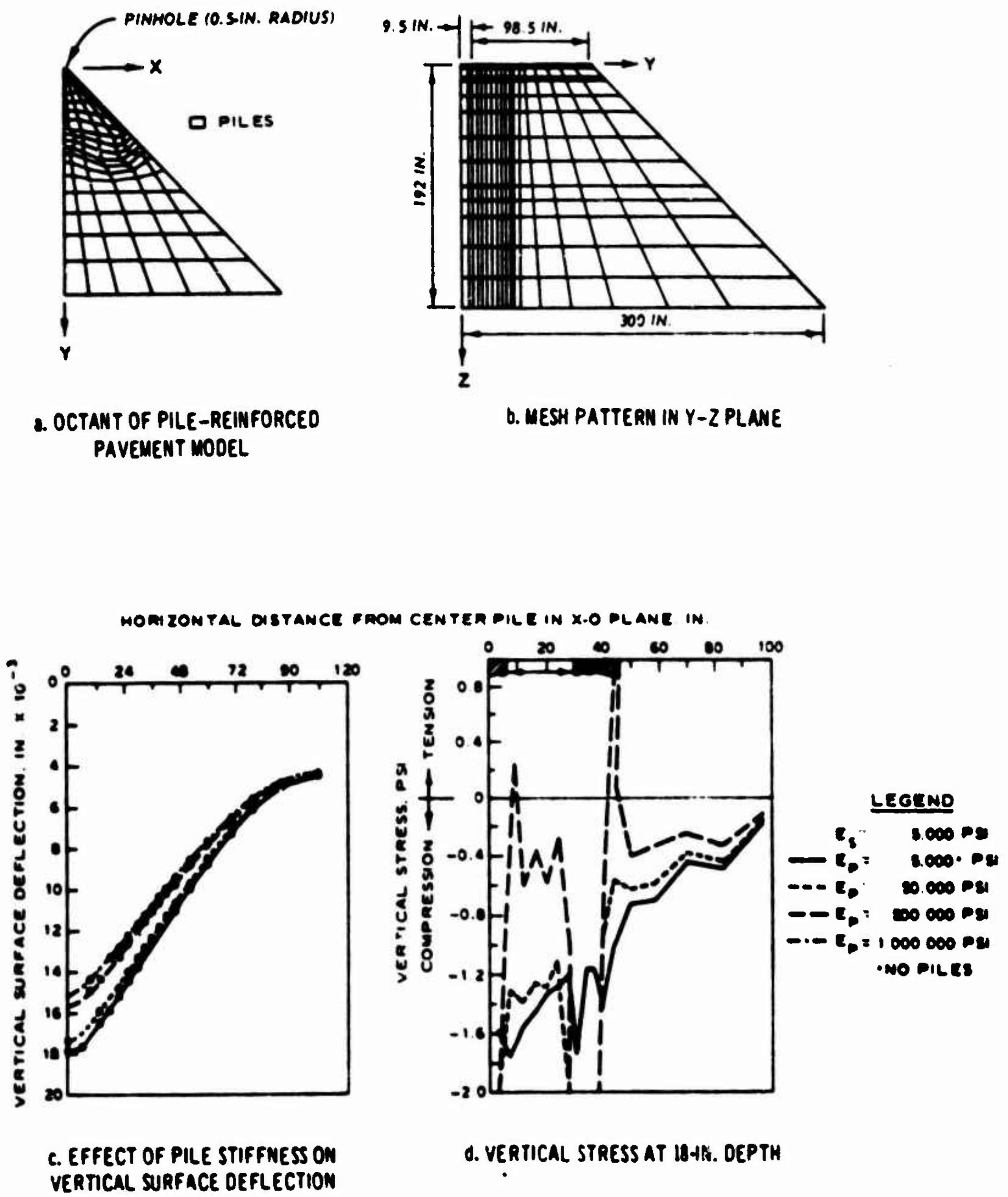
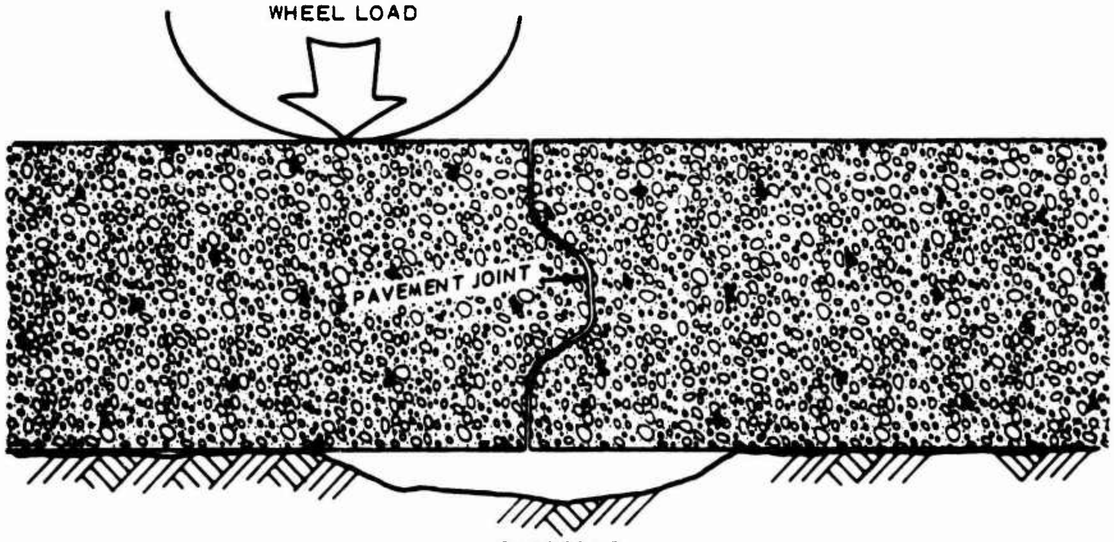
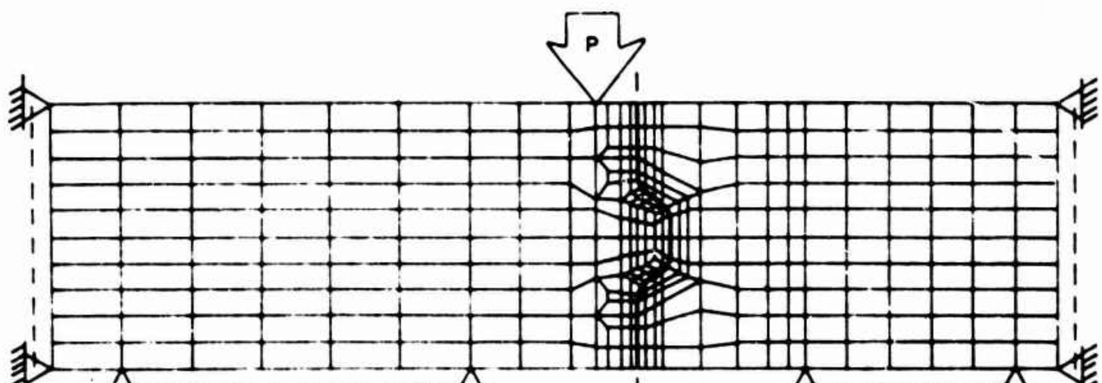


Figure 26. Three-dimensional analysis of a pile-reinforced pavement

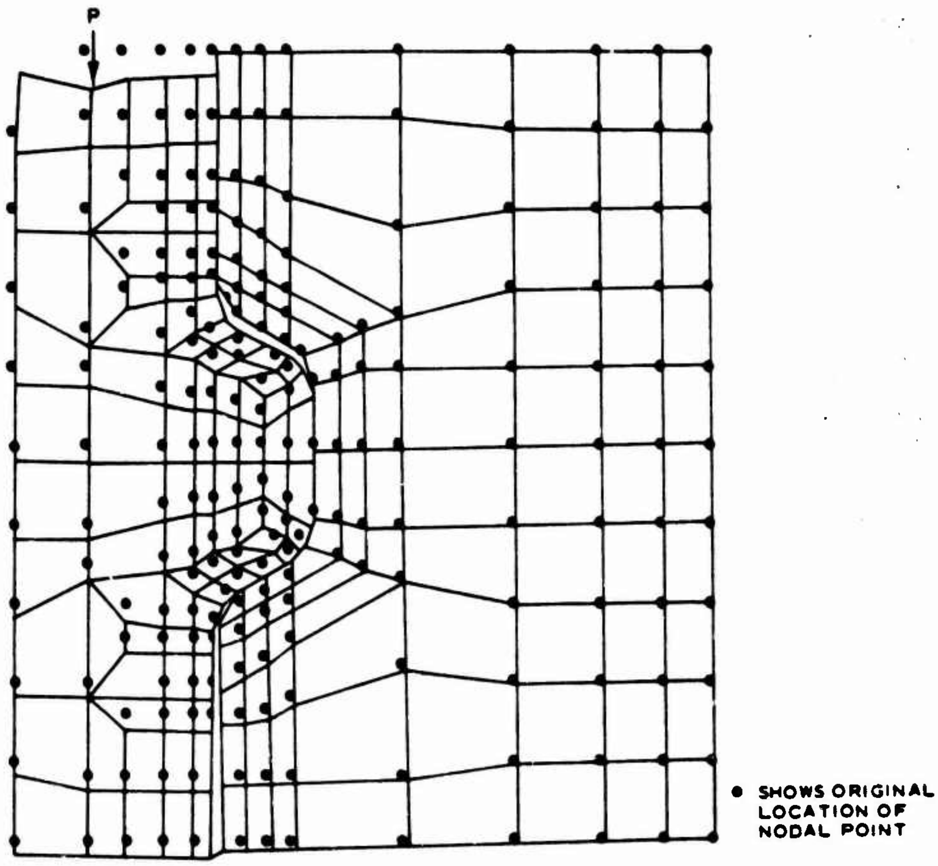


a. KEY JOINT MODEL

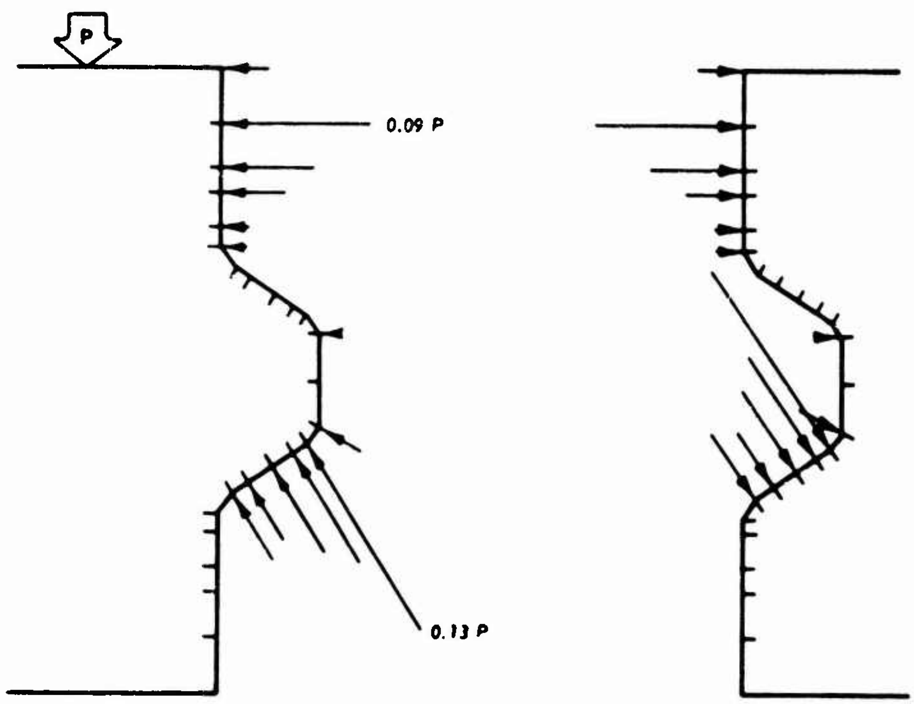


b. FINITE ELEMENT MESH

Figure 27. Analysis of a key joint (from Reference 38) (sheet 1 of 2)



c. DISPLACEMENT OF KEY JOINT UNDER LOAD



d. INTERFACE FORCES

Figure 27 (sheet 2 of 2)

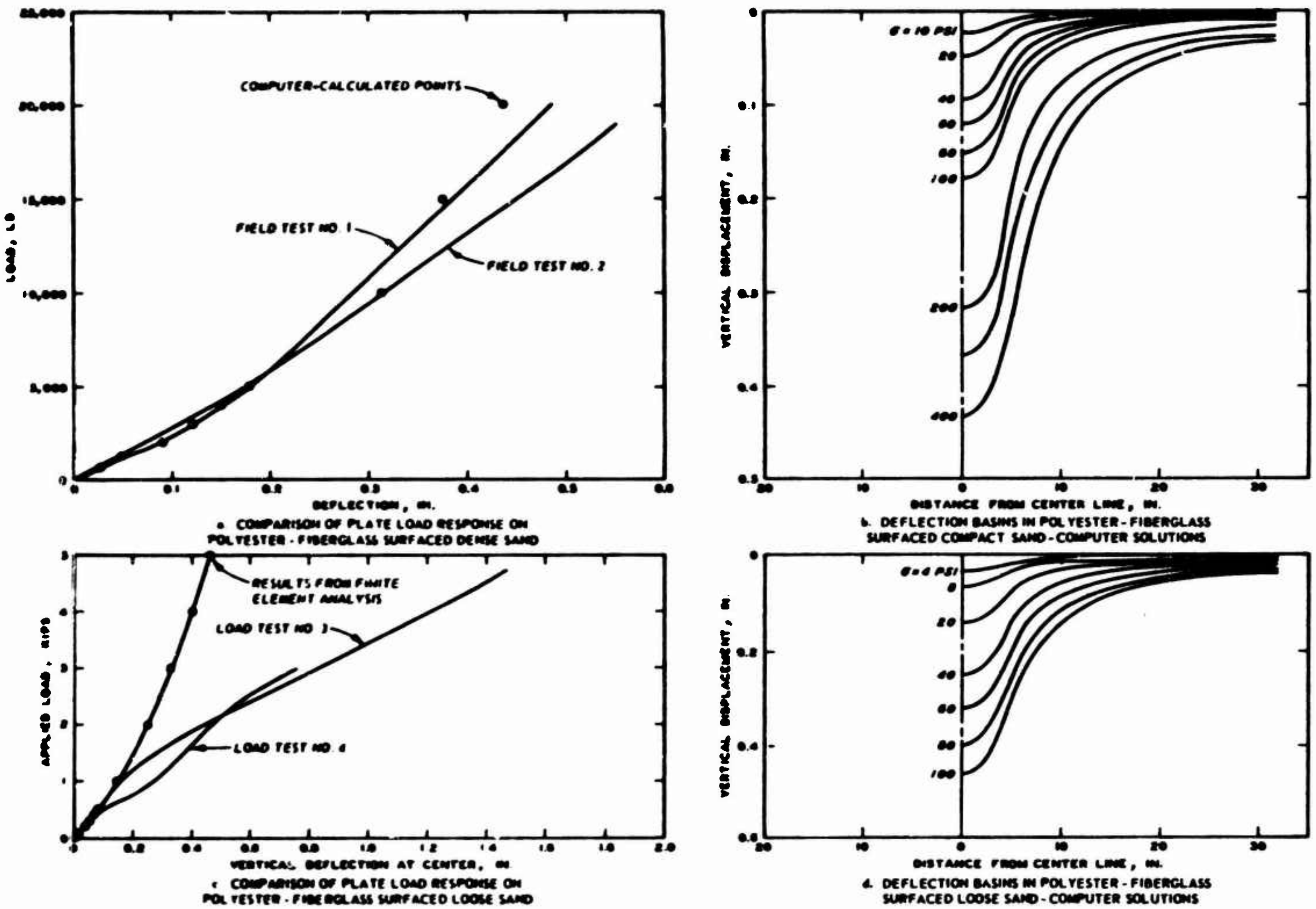


Figure 28. Response of membrane used as pavement surface (from Reference 39)

5 SELECTION OF A PREDICTION TECHNIQUE

Selection of a technique for response prediction requires a compromise between the limitations inherent in the chosen technique, the cost of utilizing the technique, and its availability and usability. Previous sections of this report have alluded to most of the limitations. The other factors are discussed in this section followed by some recommendations.

5.1 AVAILABILITY AND USABILITY

Application of most techniques requires a computer code which may not be readily available; for those that are, there is still the need to adapt the code to "your" computer and become familiar with its input and output. "Teaching an old dog new tricks" was never more applicable than to pavement engineers operating computer programs based on various prediction theories which must be coupled with performance and complicated laboratory and field tests.

The usability of an analytic technique is ultimately measured by its capability to aid in the design of pavements. Because most techniques are integrally tied to computer hardware and software, it is also important that special attention be focused on the man-machine interface While the use of computer-based analytic procedures is still embryonic (especially finite element), two areas have emerged which will play key roles in the future expansion of computer-aided design. These are performance models and pre- and postprocessing of analytical results.

5.1.1 PERFORMANCE MODELS

Although discussion of performance models is beyond the scope of this report, a brief comment on their relationship to analytic techniques follows. Computer-based analytical techniques predict the displacement, stress, and strain responses of a pavement structure composed of various idealized inputs for a specific, one-time loading. To have meaning in the design process, these responses must be related to pavement performance, ideally to allowable repetitions of loading prior to the functional failure of the pavement for a specific aircraft. Only performance criteria for flexible pavements have been related to load repetitions,[40,41] and these are based on relatively little data. Table 2 presents those criteria known to the authors.

An important "back door" approach to performance models is usage of repeated load tests to derive material parameters which reflect the worst possible condition of the material. These parameters, known as resilient moduli, are input into a prediction technique, and voila, instant performance criteria are built into the analysis. There are limitations to such a procedure: first, materials, usually in the wearing course, that exhibit any brittle or fatigue behavior (e.g., concrete) cannot be treated this way; and second, because soil materials lack homogeneous stiffness, which aggravates various problems such as asphalt rutting, something more than a resilient modulus is required. However, at this stage of the game, such quantities as resilient modulus provide useful results in an area so lacking in other data.

5.1.2 PRE- AND POSTPROCESSING SOFTWARE

Pre- and postprocessing software is an integral part of computer-based technology and to a certain extent is "the tail wagging the dog" in that the more complicated analysis techniques, especially finite element, are practically unusable without some computer preparation of input and computer generation of summary tables and graphical displays. Parenthetically, techniques which are not amendable to pre- and post-processing cannot compete. Software packages can also be developed which provide combinations of techniques. For example, the BISTRO, CHEVRON, WINDAX, and AFPAV programs can be combined to run using the same input and generating the same type of output where the analysis technique is selected based on the type of problem to be solved or the level of effort to be expended.

For preparation of finite element input, two basic approaches exist. First, various levels of automation within the finite element context are available (see References 15, 17, and 20 for examples). These primarily are schemes for generation of element and nodal data based on the prescription of a few key elements and nodes. This type

Table 2

Performance Criteria with Respect to Analytic Predictions

Pavement Structure	Base Subbase	Wearing Course	Subgrade	Pavement Systems
Flexible	Allowable stress unrelated to repetition (other than possible use of resilient modulus)	Maximum tensile strain related to repetition to failure	Vertical strain at top of subgrade related to allowable repetitions	Maximum surface deflection related to failure
Rigid		Horizontal tensile stress related to allowable stress that as yet is unrelated to fatigue damage	Allowable strain unrelated to repetition (except through use of resilient modulus)	None
Landing Mats or Membranes		Allowable stress for steel, aluminum, or membrane material can be determined as function of load repetitions		
Bomb Damage Repairs		None		

of automation probably requires an even greater familiarity with the finite element procedure than the relatively "brute force" methods associated with direct input into the programs. For the "initiated," this appears to provide the best situation.

The second approach involves completely shielding the user from the finite element process (see References 37 and 42 for examples). Input is specified in a context familiar to the design engineer. For example, specification of a set of uniform layer thicknesses is transformed by the preprocessor into a set of nodes and elements. This approach offers the best chance of incorporating the complex prediction techniques into the everyday design process. Paradoxically, it is the more sophisticated code, operated with fewer a priori assumptions, that is more easily "black boxed." Thus, making complicated procedures more widely available is somewhat linked to making these procedures even more complicated.

There are similar approaches relevant to postprocessing of finite element data plus a variety of available hardware (e.g., line printer, drum plotter, interactive and display cathode-ray tube). Some graphic examples are shown in Section 4 and References 15, 17, and 43.

5.2 COST EFFECTIVENESS

An important factor in application of most techniques is the trade-off between costs (of running the technique, obtaining the necessary input, and interpreting the output) and the benefits (derived from a more refined or different approach). The benefits derived to date, especially from the finite element procedures, are largely educational. While this has led to a better understanding of how pavements perform, it has not produced better pavements. To improve pavement designs with any of the more rigorous analysis schemes will take a more diligent and coordinated effort to establish the correlations between response predictions and pavement performance. Rudimentary forms of these correlations for flexible pavement exist and were presented in Section 5.1.1.

The cost of performing these analyses is more clearly defined. For example, assuming all software is in the production mode and is being run by experienced personnel, a man-week would cover a prismatic or axisymmetric analysis excluding any detailed materials investigation. Obtaining topnotch laboratory materials data suitable for sophisticated nonlinear material laws requires \$5,000-\$10,000 per material. These are the "bare bones" estimates and probably must be complemented with additional dollars for training inexperienced personnel, the necessity to write and check out new software, etc. The amount of these additional dollars can easily overshadow the "bare bones" figures; moreover, most design organizations are not in positions to bring together sufficient resources to make these additional dollars work effectively.

In sum, cost effectiveness is not easily defined. Obviously some prediction technique is necessary, but whether today's methods or the more analytic ones outlined in this report are used in future designs cannot be settled by appealing to cost-benefit arguments due to the lack of data.

5.3 RECOMMENDATIONS

Recommendations for selection of a prediction technique for typical pavement systems are presented separately for linear layered pavements and nonlinear and/or nonlayered pavements.

The dilemma of which technique to use is difficult to solve. Obviously, a jointed runway, a pile-supported pavement, or a pavement crossing over a bridge or culvert are nonlayered systems. In many cases where experimental data have been taken, pavement response is shown to be a nonlinear phenomenon (e.g., Figure 23). However, the state-of-the-art does suggest that extensive use of linear layered analysis procedures for design is compatible with acceptable practice and available criteria. Linear layered responses should also be computed even if only a prelude to nonlinear and/or nonlayered work. For the near future, it appears that nonlinear predictions will at least supplement linear ones. General acceptance of nonlinear procedures seems dependent on establishing better performance criteria, making available software (predominately the pre- and postprocessing variety) better suited for practicing engineers, and accepting the need for better determination of material properties.

5.3.1 LINEAR LAYERED PAVEMENTS

For a layered pavement system where linear responses are satisfactory, the elastic layer idealization is usually significantly better than any other technique. Exceptions would occur where a "hand" calculation is desirable--possibly using the Westergaard equations 1&2 - or when a complete picture of response is desired (then finite element would usually be more efficient). Availability of the software and hardware is a consideration for this method, but several programs[4-7] are relatively easy to obtain. These programs may be executed on a variety of hardware for less than $10 for a "typical" run. The only actual handicap of the elastic layer method is its dependence on the accuracy with which the layer properties are measured and represent the field conditions.

The most desirable alternative to elastic layer calculations are Westergaard and axisymmetric finite element idealizations. As indicated by Table 1, the Westergaard idealization produces results of variable quality although probably accurate enough for design calculations when a stiff layer tops relatively soft layers. Its major advantages are simplicity and usage of the parameter k . While k has some drawbacks associated with smearing all of the soil properties into a single parameter, it has several important advantages: first, it is relatively easily measured and is often available from airport records; second, it represents in situ conditions for at least the plate bearing test at the time taken; and third, it provides a check of elastic layer parameters (see Reference 4 or Section 4.1).

The axisymmetric finite element idealization, while competitive in cost and accuracy to elastic layer, lacks its basic simplicity of operation. Only when appropriate pre- and postprocessing software is available, is this method's use significantly more advantageous than elastic layer. Also, finite element usage generally requires added expertise (either in software operation or technical knowledge) over other methods.

5.3.2 NONLINEAR AND/OR NONLAYERED PAVEMENTS

It is recommended that only those nonlinear techniques which allow variable material properties within a layer be used. Employing this as a general criteria, the state-of-the-art suggests that most techniques will be based on either the axisymmetric (for a single-wheel loading) or prismatic (for a multiple-wheel loading) finite element idealizations while occasionally three-dimensional idealizations will be employed in peculiar circumstances (e.g., the nonlayered system shown in Figure 8) to predict primarily linear responses. The state-of-the-art also suggests that the material model for soil be of either the secant or tangent variety (Section 3.1), and the nonlinear solution algorithm be of the tangent type (Section 2.4.2) and be capable of approximating cracking and/or jointed materials.

6 FUTURE TRENDS AND NEEDS

The trend for pavement prediction techniques is toward more sophistication, accuracy, complication, and automation. This is generally true for all forms of structural analysis. Also, for the future, it appears that usage of the finite element method will increase significantly. Some trends which seem significant follow:

a. New material models for soils (e.g., Reference 44) which establish a framework adequate for the general situation. Also, modeling of other pavement materials will be accorded more importance. Particular attention will be paid to relating material models to performance criteria.

b. New analytic procedures, especially for the finite element method, will allow more confident and efficient modeling of nonlinear and/or nonlayered pavements. Use of the so-called "global-local" finite element formulation (References 45, 46) will help eliminate the uncertainties of the soil island approximation of semi-infinite domains. This method essentially combines the best features of the finite element method and those of classical elasticity. Finite elements are used in regions where loadings or high gradients of stress occur while elasticity functions--for example, those associated with elastic layer responses, as in equation 3--are used over the whole idealization. This allows the finite elements to be used only where they are needed and not wasted merely to approximate the semi-infinite nature of pavements. Another important procedure is substructuring, which provides an efficient means for dealing with large finite element models. This allows portions of the pavement system to be analyzed separately, ultimately being combined to produce a unified result. For example, to model joint intersections for a rigid pavement, the pavement could be idealized using three-dimensional elements coupled along joint interfaces, supported by a separate three-dimensional or prismatic idealization of the soil.

c. New solution algorithms for finite element codes (e.g., Reference 47) will be more efficient and will incorporate more advanced and accurate nonlinear solution strategies. This will lead not only to cost savings, but will facilitate designing pre- and postprocessing software that will allow "black boxing" of the complicated features associated with highly nonlinear methods.

Future needs can be viewed from "technical" and "political" standpoints. The technical needs are relatively easy to define;

however, their successful satisfaction is in part dependent on fulfilling the political needs. Some important technical needs are listed below:

a. The need for standardized, readily available, laboratory materials testing procedures and equipment for establishing accurate constitutive relations.

b. The need for comprehensive materials data available in the literature which is suitable for driving complex material laws.

c. The need for field testing procedures and equipment that allow better accumulation of data about in situ performance of materials.

d. The need for field data, especially measured strains, to verify code predictions and material models.

e. The need to improve the usability of nonlinear and/or nonlayered finite element techniques. The primary deterrent to widespread employment of these techniques is lack of pre- and postprocessing software (especially the type that completely shields the user from the complex or tedious portion of the simulation, see Section 5.1.2), and lack of accepted performance criteria (Section 5.1.1).

Politically, the need is to coordinate the pavement community in both research and field practices. Coordination is needed to deal with problems, such as software responsibility.

For example, if a Federal agency develops a design procedure based on a particular computer code, how is the code to be made available to others? Most techniques require large-size computers and significant expense per run and are of sufficient complexity to make them relatively susceptible to misuse. Thus, who is responsible for correctly installing the code on other computers and insuring its correct usage? Other groups (such as the American Society of Civil Engineers) are studying similar questions. Some major structural codes are widely distributed through commercial associations and academic or government agencies. However, the effective utilization and dissemination of the major resource that software has become is still to be achieved.

Also needed are more forums where the course of pavement research and practices may be discussed and evaluated.

What is aptly demonstrated by the listing of trends and needs is the increasing dependency that design of pavements has on a broadening range of disciplines. It is not adequate just to develop material models without regard to the availability of equipment for performing the associated laboratory testing, or whether the law can be incorporated into an appropriate nonlinear algorithm, etc. Those involved in development work should be aware of where their efforts fit in the total effort related to the prediction of pavement response.

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