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ASSESSMENT OF SIMPLE JOINT TIME/RISK PREFERENCE FUNCTIONS

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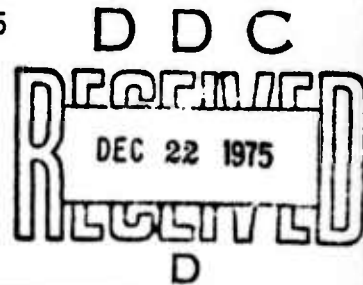
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DECISION ANALYSIS PROGRAM

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A new preference parameter is introduced. The parameter, called the coefficient of variation aversion, is a measure of how strongly an individual feels about undesirable variations in payoff vectors. It is shown that the coefficient of variation aversion exists and is strictly positive if the ordinal preference function has an additive representation and preferences satisfy a reasonable set of axioms.

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ASSESSMENT OF SIMPLE JOINT TIME/RISK PREFERENCE FUNCTIONS

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August, 1975

ABSTRACT

This article outlines a procedure for assessing a decision maker's cardinal utility function $U(x_1, \dots, x_1, \dots, x_N)$ where x_i is the payoff in the i^{th} period of an N-period future. The function $U(x)$ captures the decision maker's time preferences (his willingness to trade off payoffs between time periods) and his risk preferences (his attitude toward risk taking). The procedure outlined uses a straightforward but little known two-step method for assessing multiattribute utility. In the first step the decision maker is asked to reveal time preferences by choosing between sure payoff vectors. In the second step attitude toward uncertainty is measured by encoding risk aversion on an appropriate single-dimensional index. The two steps are combined mathematically to produce the utility function.

A new preference parameter is introduced. The parameter, called the coefficient of variation aversion, is a measure of how strongly an individual feels about undesirable variations in payoff vectors. It is shown that the coefficient of variation aversion exists and is strictly positive if the ordinal preference function has an additive representation and preferences satisfy a reasonable set of axioms.

The choice between available investment opportunities is an essential economic activity of virtually every individual, business, or government. The investment decision is complicated by two factors: time and uncertainty. Future costs and payoffs are often distributed over a long enough period that the timing of events is a significant consideration. Likewise, future costs and payoffs are uncertain. In general, when uncertainties are involved, logical choice between investment alternatives requires description of the decision maker's preferences for the timing of payoffs and his attitude toward risk taking.

The outcome of an investment can usually be described as a vector $(x_1, \dots, x_i, \dots, x_N)$ where x_i is the payoff in the i th period of an N period future. Payoffs might be such quantities as individual consumption expenditures, corporate dividend payments, or the benefits of a government supported research program. Under certainty, preferences for such outcomes can be described by an ordinal preference function $V(x)$. This real-valued function exists so long as the decision maker can perform a transitive ordering of the vectors, can make trade-offs between time periods, and always prefers more payoff to less in any time period [5]. The ordinal (time) preference function has the property that if \underline{x}^1 is strictly preferred to \underline{x}^2 then $V(\underline{x}^1) > V(\underline{x}^2)$. (In the two-period case the time preference function can be represented graphically by indifference curves.)

If an ordinal time preference function exists and if in addition the decision maker is rational in the von Neumann-Morgenstern sense, then a cardinal utility function $u(x)$ exists [8]. This function describes attitude toward risk taking as well as time preference. It has the property that

the preferred investment has the highest expected utility over all future time periods. This function can be used to evaluate payoff propositions that are uncertain as well as dynamic.

Given that a cardinal utility function exists, the practical problem is to assess its mathematical form. The function must be complex enough to realistically describe preferences but at the same time simple enough to be computable and be understood by mathematically unsophisticated decision makers.

One assessment approach is to prejudge, based on qualitative criteria, that indifference curves or utility have the form of a known analytical function with adjustable parameters. A few well chosen questions can then be used to estimate the parameter values. For example, in a two-period framework it might be assumed that an individual's curves are described by straight lines. The straight lines could then be specified uniquely by assessing only one parameter, the slope.

In the following development the parametric approach is extended to assess a cardinal utility function that describes both time and risk preference. The result is a method for constructing a cardinal utility function $u(x_1, \dots, x_1, \dots, x_N)$ by assessing $N + 2$ parameters. N parameters specify the relative interperiod weighting of payoffs (the preferred payoff pattern): One parameter, the coefficient of variation aversion, is a measure of how strongly an individual feels about undesirable variations in payoffs; the final parameter is the familiar coefficient of risk aversion, a measure of attitude toward risk taking [7]. A more complete description of preference can be obtained by assessing a coefficient of variation aversion for each time period. However, for most applications one coefficient probably provides an adequate detail.

A simple but fairly undiscovered two-stage process can be used to construct $u(\underline{x})$ [2,6]. The procedure consists of a deterministic time preference assessment followed by an assessment of the risk aversion coefficient on an appropriately chosen, real-valued index. This approach has the advantage that time preferences (the willingness to trade off sure payoffs between time periods) and risk preference (willingness to face uncertainty) are assessed independently.

Variation Aversion

We begin by assuming that the ordinal time preference function $V(\underline{x})$ exists and has an additive representation, i.e.,

$$V(\underline{x}) = \sum_{i=1}^N v_i(x_i) \quad (1)$$

where v_i is a function of x_i only. This is a restrictive assumption, although far less restrictive than the frequently made assumption that cardinal utility is additive. The behavioral implications of additivity will be discussed later.

A special case, identical v_i , will be used to explain the concept of variation aversion. The intuitive insights afforded by this special case are lost in a more general derivation. Equal v_i would be characteristic of a decision maker that preferred uniform payoffs.

It seems reasonable that an individual will be indifferent between a vector \underline{x} with varying payoffs ($x_i \neq x_j$ for some i and j) and some vector with uniform components $\underline{a} = (a, \dots, a, \dots, a)$. If a is

very small, \underline{x} will be preferred; if \underline{a} is very large \underline{a} will be preferred. For some intermediate value, \hat{a} , \underline{a} and \underline{x} will be equally desirable. \hat{a} will be called the uniform equivalent of \underline{x} . It satisfies the relation

$$V(\underline{x}) = \sum_{i=1}^N v(\hat{a}) = N v(\hat{a}) . \quad (2)$$

If the variation of the x_i about their average value \bar{x} is small then by a Taylor's series expansion

$$V(\underline{x}) \approx N v(\bar{x}) + \sum_{i=1}^N v'(\bar{x})(x_i - \bar{x}) + 1/2 \sum_{i=1}^N v''(\bar{x})(x_i - \bar{x})^2 \quad (3)$$

where v' and v'' are respectively the first and second derivative of v .

The term $v(\hat{a})$ can be approximated by a first-order expansion as

$$v(\hat{a}) \approx v(\bar{x}) + v'(\bar{x})(\hat{a} - \bar{x}) . \quad (4)$$

By combining (3) and (4) via Relation (2), the following result is obtained:

$$\hat{a} \approx \bar{x} - 1/2 v(\bar{x}) \sum_{i=1}^N (x_i - \bar{x})^2 / N \quad (5)$$

where

$$v(\bar{x}) = - v''(\bar{x}) / v'(\bar{x}) . \quad (6)$$

The function v will be called the coefficient of variation aversion. For a uniform preferred payoff pattern, the interpretation of the coefficient of variation aversion is clear from Relation (6). If v is

positive it is twice the decrease in the uniform equivalent due to a unit increase in the second moment of \underline{x} about its average value (for small second moments). Thus we postulate that v is a measure of how strongly an individual feels about deviations in payoffs away from some preferred pattern.

The function v has a form similar to the familiar coefficient of risk aversion defined by Pratt [7]. While there are strong parallels between variation aversion and risk aversion, the latter concept has no meaning in the deterministic time preference context under consideration.

Two important questions have been left unanswered in the above development. First, what are the behavioral implications of assuming that the ordinal preference function has an additive representation (Equation 1)? Second, what fundamental behavioral characteristics guarantee positive variation aversion? Both of these questions will now be answered.

In developing a mathematical description of an individual's preferences, it is customary to assume that the preferences can be represented by a set of three binary relations defined over all payoff vectors \underline{x} . The three binary relations are:

(i) Strict Preference P .

$\underline{x}_1 P \underline{x}_2$, if \underline{x}_1 is strictly preferred to \underline{x}_2 .

(ii) Indifference I .

$\underline{x}_1 I \underline{x}_2$, if \underline{x}_1 is indifferent to \underline{x}_2 .

(iii) Weak Preference R .

$\underline{x}_1 R \underline{x}_2$, if $\underline{x}_1 P \underline{x}_2$ or $\underline{x}_1 I \underline{x}_2$.

Assumptions related to the properties of preference relations are stated as axioms. A set of five axioms guarantee that the time preference function has an additive representation and that variation aversion is strictly positive. The first three axioms are quite standard and can be found in most developments of preference theory [5]. The fourth and fifth axioms are, perhaps, less casually accepted. The axioms and their explanations follow:

Axiom 1: Weak Ordering. The relation R is transitive and connected. R is transitive if $\underline{x}_1 R \underline{x}_2$ and $\underline{x}_2 R \underline{x}_3$ imply $\underline{x}_1 R \underline{x}_3$; it is connected if $\underline{x}_1 R \underline{x}_2$ and/or $\underline{x}_2 R \underline{x}_1$ for all \underline{x}_1 and \underline{x}_2 in a finite dimensional Euclidian space.

If the decision maker violates transitivity, for instance, if $\underline{x}_1 P \underline{x}_2$, $\underline{x}_2 P \underline{x}_3$, and $\underline{x}_3 P \underline{x}_1$, then he can be turned into a "money pump." That is, if he owns \underline{x}_1 he will be willing to pay some small amount to exchange \underline{x}_1 for \underline{x}_2 . Once he has \underline{x}_2 , he will pay something to exchange \underline{x}_2 for \underline{x}_3 ; likewise, with \underline{x}_3 and \underline{x}_1 . After three voluntary payments the decision maker holds \underline{x}_1 , the vector with which he began. Thus, violation of transitivity implies a conscious willingness to pay to accomplish nothing.

Axiom 2: Continuity. If $\underline{x}_1 R \underline{x}_2$ and $\underline{x}_2 R \underline{x}_3$, then there is a real number λ such that $0 \leq \lambda \leq 1$ and $[\lambda \underline{x}_1 + (1 - \lambda)\underline{x}_3] I \underline{x}_2$. This axiom captures the notion that individuals are willing to make trade offs. In this case payoff in one time period is being traded for payoff in another.

Axiom 3: Nonsatiety or Greed. If $x_i \geq y_i$ and strict inequality holds for at least one component of \underline{x} , then $\underline{x} P \underline{y}$. This axiom

requires simply that the individual always prefers more payoff to less in any time period.

Together the weak ordering, continuity, and nonsatiety axioms guarantee the existence of a continuous, real-valued deterministic time function $V(\underline{x})$ with the property that $V(\underline{x}_1) \geq V(\underline{x}_2)$ if and only if $\underline{x}_1 R \underline{x}_2$. The function $V(\underline{x})$ is unique only to a monotonically increasing transformation. For a proof of this fact and further discussion, see Luce and Suppes

Axiom 4: Decreasing Marginal Rates of Substitution. This axiom is satisfied if, for any two time periods i and j , the increase in x_j required to compensate for a loss Δx_i decreases as x_i increases, or, when V is differentiable

$$\left. \frac{\partial^2 x_j}{\partial x_i^2} \right|_{V(\underline{x}) = \text{constant}} > 0 .$$

This axiom states mathematically the belief that as total payoff in any time period i increases, the individual becomes less and less sensitive to small changes Δx_i .

Axiom 5: Deterministic Independence. All factors of the payoff vector \underline{x} are deterministically independent. Deterministic independence is defined by Debreu [3] as follows: Let I be any subset of $n = (1, \dots, N)$, and for every $i \in I$ let a_i represent a constant payoff in period i . If the preference ordering of \underline{x} conditional on $(x_i = a_i)_{i \in I}$ is invariant for all levels of $(a_i)_{i \in I}$ then the n factors of \underline{x} are said to be independent.

For $n = 2$, this means that if $\underline{x} = (a, x_2)$ is preferred to

$y = (a, y_2)$ for some a , then (a, x_2) is preferred to (a, y_2) for all a , and likewise for $\underline{x} = (x_1, a)$ and $\underline{y} = (y_1, a)$. Basically, independence implies that an individual can make consistent value judgments about payoffs in any subset of future years when the levels of the payoffs in all other years are held fixed. And further, these value judgments do not depend on the particular fixed levels.

Although there are doubtless special instances in which preferences violate the independence axiom, in general it seems to be a reasonable proposition. It certainly suffices as a first approximation to a more detailed preference description.

The independence axiom, together with the assumption that at least two components of \underline{x} affect choices, guarantees that an additive representative of the ordinal preference function exists (see Debreu [3], Definition 4 and Theorem 3; also Luce and Suppes [5]). The additive representation of the preference function is unique to a positive linear transformation. Notice also that an additive preference function implies deterministic independence.

A theorem can now be stated relating the preference axioms and the variation aversion coefficient. The theorem stipulates that positive variation aversion is guaranteed if and only if the five axioms are satisfied.

Theorem

Given a preference relation satisfying the axioms of

- (1) weak ordering (transitivity and connectedness),
- (2) continuity, and
- (3) nonsatiety,

then the relation satisfies the axioms of

- (4) decreasing marginal rates of substitution, and
- (5) deterministic independence

if, and only if,

- (a) $v''_{ij} = 0$, for all i and j , $i \neq j$, and
- (b) $-(v''_{ii}/v'_i) = v_i(c_i) > 0$, for all i ,

where

$$v'_i = \partial v / \partial c_i \quad \text{and} \quad v''_{ij} = \partial^2 v / \partial c_i \partial c_j .$$

(See the appendix for a proof of this theorem.)

The quantity $v_i(x_i)$ appearing in the theorem is called the coefficient of variation aversion with respect to x_i . We will continue to make the simplifying assumption that all v_i are identical.

Families of Time Preference Functions

The theorem of the last section guarantees that a family of preference functions is uniquely determined for any set of strictly positive coefficients $v_i(x_i)$. If $-(v''_{ii}/v'_i) = v_i(x_i)$, then by the rules of integration and the additivity property

$$v(\underline{x}) = \sum_{i=1}^N \int e^{-\int v_i(x_i) dx_i} dx_i . \quad (7)$$

We will investigate two particularly interesting families of preference functions. For the first family, $v(x_i)$ is a constant and for the second, $v(x_i)$ is inversely proportional to x_i .

If the coefficient of variation is a positive constant $v = v_0$, then by Equation (7)

$$V(\underline{x}) = - \sum_{i=1}^N a_i e^{-v_0 x_i} \quad (8)$$

where the a_i 's are also positive constants. Four sets of indifference curves are shown in Fig. 1. The curves correspond to $a_1 = a_2$, and $v_0 = .001, .005, .01, \text{ and } .05$. The graphs demonstrate that small values of v_0 correspond to flat, nearly linear preferences and that as v_0 gets larger the difference curves become more and more sharply bent along the line $x_1 = x_2$.

The exponential functional form has one salient behavioral implication. That is, if an individual with this preference function is indifferent between any two vectors \underline{x}_1 and \underline{x}_2 , then he will also be indifferent between $(\underline{x}_1 + \underline{\Delta})$ and $(\underline{x}_2 + \underline{\Delta})$ where $\underline{\Delta}$ is the vector (Δ, \dots, Δ) . This is true because, for the exponential function, $V(\underline{x}_1) = V(\underline{x}_2)$ implies that $V(\underline{x}_1 + \underline{\Delta}) = V(\underline{x}_2 + \underline{\Delta})$.

If the variation aversion is inversely proportional to payoff, i.e., $v_i = 1/(\sigma x_i)$, then by Equation (7)

$$V(\underline{x}) = \sum_{i=1}^N a_i x_i^{(1-(1/\sigma))} \quad \text{for } 1 < \sigma < \infty, \quad (9)$$

$$V(\underline{x}) = \sum_{i=1}^N a_i \ln x_i \quad \text{for } \sigma = 1, \text{ and} \quad (10)$$

$$V(\underline{x}) = - \sum_{i=1}^N a_i x_i^{(1-(1/\sigma))} \quad \text{for } 0 < \sigma < 1. \quad (11)$$

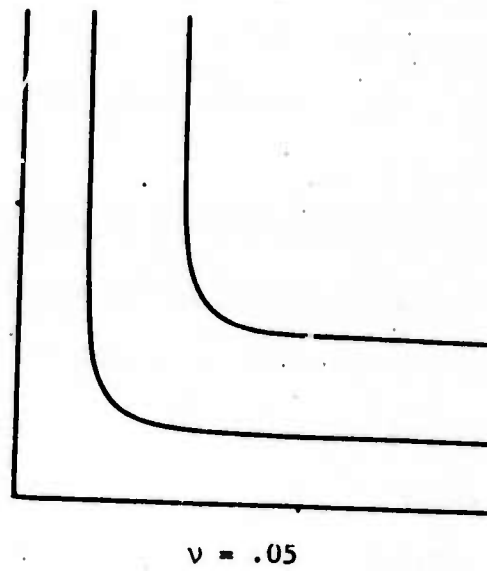
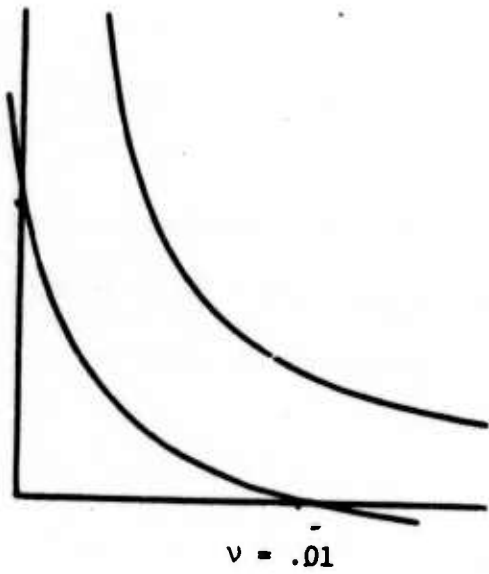
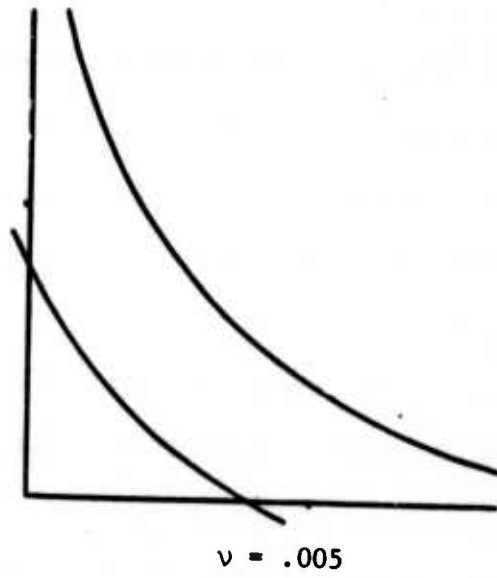
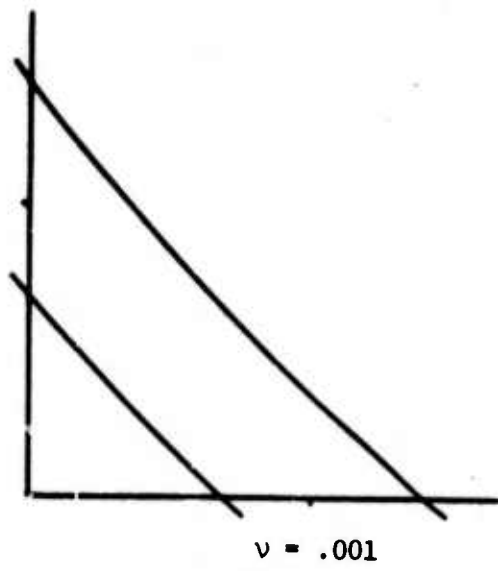


Figure 1. Indifference curves for constant variation aversion

All of the a_i 's are positive constants. The effect of changing σ is illustrated by the four sets of indifference curves in Fig. 2. The curves correspond to $a_1 = a_2$; and $\sigma = 20, 2.5, 1.43$, and 1 . A distinguishing feature of this family is that the curves get flatter (more linear) as both x_1 and x_2 increase. This "flattening" of the curves can be used to reflect a phenomenon that might be observed frequently in actual preferences. That is, as an individual has higher payoffs in all periods his overall welfare is less sensitive to small shifts between periods. It is also apparent in Fig. 2 that the rate at which the indifference curves flatten as wealth increases can be varied by adjusting σ .

The condition $v_i = 1/(\sigma x_i)$ implies that the preferences are invariant under scaling, i.e., for Equations (2.9), (2.10), and (2.11) $V(\underline{x}_1) = V(\underline{x}_2)$ implies that $V(b\underline{x}_1) = V(b\underline{x}_2)$, where b is a positive constant.

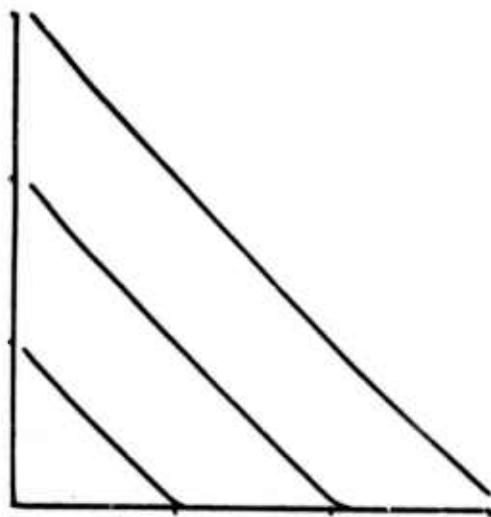
Use of a particular functional form imposes the associated behavioral characteristic on the decision maker's preferences. Which behavioral implications are acceptable depends of course on the decision maker and the situation.

Encoding

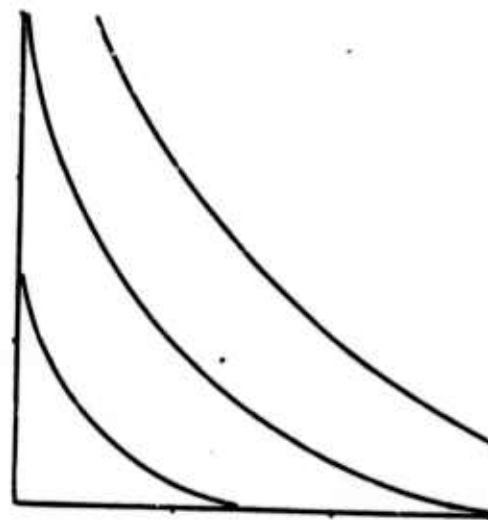
Enough information to specify a unique time preference function from either of the above families can be obtained by performing two encoding tasks. They are:

- (1) Assessment of preferred payoff pattern.

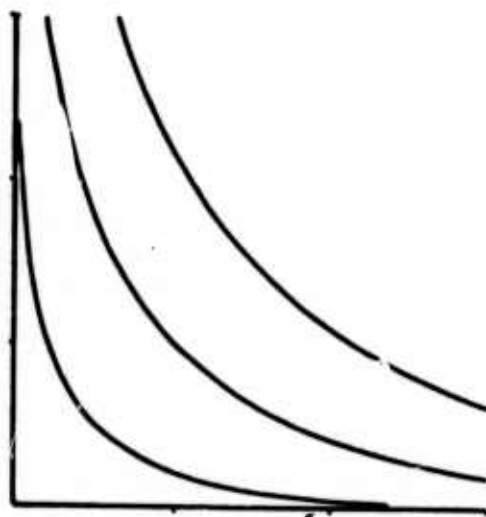
To establish this pattern the individual is asked to distribute a fixed total payoff over a given lifetime to reveal the payoff pattern he prefers to all others. This distribution \hat{x} can be called the preferred



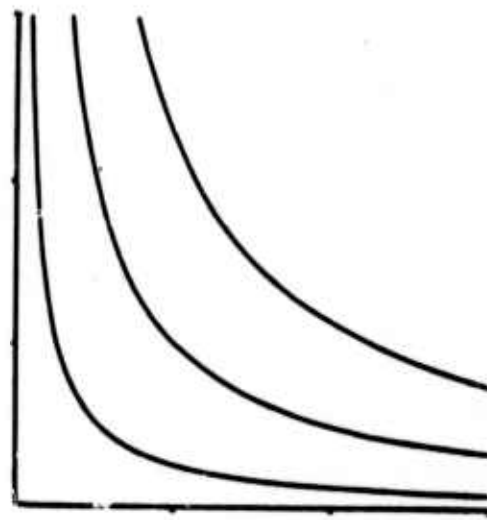
$\sigma = 20$



$\sigma = 2.5$



$\sigma = 1.43$



$\sigma = 1.$

Figure 2 Indifference curves for $v = 1/(\sigma x_1)$

payoff pattern conditional on the total payoff level X , where

$$X = \sum_{i=1}^N x_i . \quad (12)$$

This vector quantity will be denoted $(\hat{x}|X)$. The preferred payoff pattern can vary with total payoff. To illustrate, suppose x_i represents an individual's consumption expenditure in the i^{th} year of a five-year lifetime. If total income is large, say $X = 200$ thousand dollars, the individual might prefer an increasing expenditure pattern, e.g., $(\hat{x}|200) = (20,30,40,50,60)$. If total income is very low, say 10 thousand dollars, the simple desire to survive might prescribe a uniform preferred pattern, e.g., $(\hat{x}|10) = (2,2,2,2,2)$. In a particular encoding situation X can be chosen to be consistent with the range of \underline{x} over which $V(\underline{x})$ will be applied.

(2) Variation aversion assessment.

To assess variation aversion at a given payoff level, the individual is asked to reveal the level of Δ for which he is indifferent between the vector $(\hat{x}_1, \dots, \hat{x}_i, \hat{x}_{i+1}, \dots, \hat{x}_N | X)$ and $(\hat{x}_1, \dots, \hat{x}_i - \Delta/2, \hat{x}_{i+1} + \Delta, \dots, \hat{x}_N | X)$. The quantity Δ will be shown to be approximately equal to the decision maker's coefficient of variation aversion if X and i are properly selected.

In performing these encoding tasks the individual reveals two clearly identifiable aspects of his time preferences: first, the manner in which he would like to distribute payoffs over all future periods or, equivalently, the relative importance he places on payoffs in different periods; second, his attitude towards deviations from his preferred payoff pattern.

The preference function form appropriate for a particular decision-making situation is a matter of analytical judgment. Important factors to be considered include problem complexity and desired accuracy. If the exponential form of Equation (3) is appropriate, the encoding can proceed as follows. The total number of parameters to be encoded is $N + 1$ (N weighting factors a_i and the variation aversion coefficient v_0). The information needed to determine these values can be obtained by assessing one preferred payoff pattern $(\hat{x}|X)$ at some appropriate level of X , and assessing the variation tolerance coefficient at the same level.

All preferred payoff patterns $(\hat{x}|X)$ have the property

$$\left. \frac{dx_j}{dx_i} \right|_{\hat{x}} = -1. \quad (13)$$

For the exponential form,

$$\frac{dx_j}{dx_i} = - \frac{\partial V / \partial x_i}{\partial V / \partial x_j} = - \frac{a_i}{a_j} e^{(x_j - x_i)v_0}. \quad (14)$$

Combining (13) and (14) yields

$$\frac{a_i}{a_j} e^{(\hat{x}_j - \hat{x}_i)v_0} = 1 \text{ for all } i \text{ and } j. \quad (15)$$

Relation (15) yields $N - 1$ independent equations in $N + 1$ unknowns. Only two more independent equations are necessary in order to solve all the unknowns. Task 2 reveals that some vector $(\hat{x}_1, \dots, \hat{x}_i - \Delta/2, \hat{x}_{i+1} + \Delta, \dots, \hat{x}_N)$ is indifferent to $(\hat{x}_1, \dots, \hat{x}_1, \dots, \hat{x}_{i+1}, \dots, \hat{x}_N)$, or by Equation (8)

$$-a_i e^{-(\hat{x}_i - \Delta/2)v_0} - a_{i+1} e^{-(\hat{x}_i + \Delta)v_0} = -a_i e^{-\hat{x}_i v_0} - a_{i+1} e^{-\hat{x}_{i+1} v_0} \quad (16)$$

Equation (17) plus an arbitrary assumption that

$$\sum_{i=1}^N a_i = 1$$

yields the required $N + 1$ independent equations, which can be solved for v_0 and for all the a_i 's.

The process can be simplified somewhat if a_i and x_i are approximately equal to a_{i+1} and x_{i+1} , respectively, in Equation (16). The periods i and $i + 1$, can be chosen so that this approximation is valid. Equation (16) can then be written

$$e^{-(\hat{x}_i - \Delta/2)v_0} + e^{-(\hat{x}_i + \Delta)v_0} = 2e^{-\hat{x}_i v_0} \quad (17)$$

Factoring, Equation (17) becomes

$$e^{\Delta v_0 / 2} + e^{-\Delta v_0} = 2 \quad (18)$$

The exact solution of this relation is $v_0 = .96/\Delta$. Thus, if i is properly selected the coefficient of variation aversion is approximately equal to the reciprocal of the directly assessable value Δ .

This assessment procedure gives us some additional insight into the meaning of variation aversion. The reciprocal of variation aversion is approximately equal to the amount of the payoff from one period that an individual would be willing to defer to a later period if the rate of exchange were 100 percent ($\Delta/2$ dollars deferred yields Δ dollars).

In practice variation aversion may decrease as total payoff or wealth increases. If an individual expects small payoffs in all periods, Δ will probably be small, i.e., high variation aversion. As payoffs in all periods increase, Δ will probably tend to increase (decreasing variation aversion). Where this property is important it can be captured mathematically by using Equations (9) or (10) rather than the exponential.

Elsewhere it has been shown that variation aversion is an indirect indicator of a decision maker's sensitivity to delays in the resolution of future payoff uncertainties [1]. High variation aversion implies a willingness to pay to avoid resolution delays. With early resolution, saving or borrowing can be used to redistribute payoffs between time periods, thus avoiding undesirable variations.

So far, a procedure has been outlined for encoding a simple ordinal time preference function, $V(x)$. This function is ordinal but not necessarily cardinal. That is, the form of the function determines the shape of indifference surfaces, but the absolute magnitude of $V(x)$ has no significance for describing time preference.

The ordering implied by $V(x)$ is preserved by any monotonically increasing transformation. For the ordinal preference functions described by Relations (8) through (11) the uniform equivalent \hat{a} of any payoff vector is a monotonically increasing transformation of $V(x)$. For example, if preferences are described by Relation (10), then \hat{a} is determined by solving

$$\sum_{i=1}^N a_i \ln \hat{a} = \sum_{i=1}^N a_i \ln x_i \quad (19)$$

or

$$\hat{a} = e^{V(x)} . \quad (20)$$

Uniform equivalents are a cardinal measure and they can be used to combine time and risk preferences. We will discuss very briefly how this might be accomplished.

Risk Preference

If a decision maker satisfies the von Neumann-Morgenstern axiom for lotteries with outcomes measured in uniform equivalents, then a cardinal utility function $u(\hat{a})$ exists [8]. This function has the property that if $u(\hat{a}_1)$ is preferred to $u(\hat{a}_2)$ then $u(\hat{a}_1) > u(\hat{a}_2)$ and the utility of any lottery equals the expected utility of its prizes. $u(\hat{a})$ captures the decision maker's attitude toward uncertainty, i.e., his risk preference.

The coefficient of risk aversion $r(y)$ is defined by the relation $r(y) = -u''(y)/u'(y)$ where u'' and u' are respectively the second and first derivatives of u . If $r(y)$ is known, then $u(y)$ can be uniquely determined by integration.

As with time preference assessment, one approach for encoding risk preference is to assume a functional form for $r(\hat{a})$ based on qualitative criteria. A few choices between appropriate lotteries can then be used to estimate function parameters. For example, we might assume a decision maker is adverse to risk taking and risk attitude is independent of \hat{a} over an appropriate range. In this case, $r(\hat{a})$ can be approximated by a positive constant γ_0 . This in turn implies $u(y) = -A e^{-\gamma_0 \hat{a}} + B$ where A and B are constants, $A > 0$. The value of γ_0 most nearly describing a decision maker's preferences can be estimated by asking

him to choose between keeping the lottery in Figure 3 and giving it away. If \hat{a} is adjusted until the decision maker is indifferent, then a relation very similar to Equation (18) is obtained, i.e.,

$$1/2 e^{-\gamma_0 \hat{a}} + 1/2 e^{\gamma_0 (\hat{a}/2)} = 1 \quad (21)$$

or

$$\gamma_0 = 1/\hat{a} . \quad (22)$$

Joint Time/Risk Preferences

If $\hat{a} = T(V(x))$, where T is a monotonically increasing transformation, then by substitution

$$u(\hat{a}) = u(T(V(x))) = U(x) . \quad (23)$$

$U(x)$ is the desired cardinal utility for payoff vectors. It describes the decision maker's preferences for the timing of payoffs as well as his attitude toward risk taking.

Using the procedure described above, the functional form of $U(x)$ is uniquely determined by specifying two parameters: a coefficient of variation aversion and a coefficient of risk aversion (with respect to uniform equivalents). The functional forms of $U(x)$ corresponding to three pairs of v and r are summarized in Table 1. It is interesting that in each case $U(x)$ and $V(x)$ have the same form when v and r are equal which implies additive cardinal utility.

Uniform equivalents were used in the above development because of their simple interpretation and mathematical convenience. The construction procedure outlined is applicable whenever cardinal utility can be encoded as a function of a real-valued numeraire y and $y(x) = T(V(x))$

TABLE I
 Ordinal Preference Function and Cardinal Utility Function for
 Three Combinations of Variation and Risk Aversion

Coefficient of Variation Aversion	Coefficient of Risk Aversion with respect to CAE's	Ordinal Preference Function	Cardinal Utility Function
$v(x_1)$	$\gamma[\text{CAE}(\bar{x})]$	$V(\bar{x})$	$u(\bar{x})$
v_0	γ_0	$-\sum_{i=1}^N a_i e^{-v_0 x_i}$	$-[-V(\bar{x})] \gamma_0 / v_0$
$\frac{1}{\sigma x_i}$ ($1 < \sigma < \infty$)	$\frac{1}{\eta \text{CAE}(\bar{x})}$ ($1 < \eta < \infty$)	$\sum_{i=1}^N a_i x_i^{(\sigma-1) / \sigma}$	$V(\bar{x}) (\sigma(\eta-1)) / (\eta(\sigma-1))$
$\frac{1}{x_i}$	$\frac{1}{\text{CAE}(\bar{x})}$	$\sum_{i=1}^N a_i \ln x_i$	$V(\bar{x})$

$\sum_{i=1}^N a_i = 1$; $a_i > 0$ for all i .

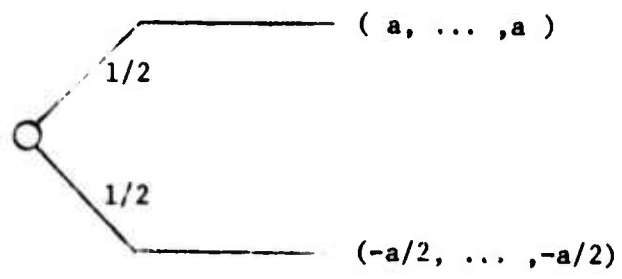


Figure 3 Lottery for encoding risk aversion

where T is a monotonically increasing transformation [5]. Such a transformation preserves the transitive ordering of the prizes x .

Summary

Variation aversion has been introduced as a measure of a decision maker's attitude toward undesirable variations in future payoff streams. This parameter has been used to generate simple additive functions that might prove useful in describing time preferences.

A procedure was outlined for constructing a cardinal utility function which captures both the decision maker's time preferences and his attitude toward risk taking. Simple, but flexible, utility functions can be specified uniquely by performing three encoding tasks. The tasks are:

- (1) Assessment of a preferred payoff pattern $(\hat{x}|X)$. This vector describes the way an individual desires to have payoffs distributed over all future time periods.
- (2) Assessment of a coefficient of variation aversion.
- (3) Assessment of a coefficient of risk aversion with respect to uniform payoffs.

The first two tasks are performed under the assumption of certainty. Uncertainty is introduced in the last step. This encoding procedure has the advantage that time and risk preference assessment are performed independently and then combined mathematically.

APPENDIX

Proof of Theorem

First, we will prove that (4) and (5) imply (a) and (b). By total differentiation,

$$dV(\underline{x}) = \sum_{k=1}^N \frac{\partial V(\underline{x})}{\partial x_k} dx_k . \quad (\text{A.1})$$

For $V(\underline{x})$ and all x_k constant, $k \neq i, j$, the above relation becomes

$$0 = \frac{\partial V}{\partial x_i} dx_i + \frac{\partial V}{\partial x_j} dx_j . \quad (\text{A.2})$$

Rearranging (A.2)

$$\frac{\partial x_j}{\partial x_i} = \frac{-(\partial V)/(\partial x_i)}{(\partial V)/(\partial x_j)} = -(V'_i)/(V'_j) . \quad (\text{A.3})$$

Differentiating (A.3) with respect to x_i and applying the decreasing marginal rates of substitution condition yields

$$\frac{\partial^2 x_j}{\partial x_i^2} = - \frac{V'_j V''_{ii} - V'_i V''_{ji}}{(V'_j)^2} > 0 . \quad (\text{A.4})$$

Rearranging the right side of (A.4) yields

$$\frac{V''_{ji}}{V'_j} - \frac{V''_{ii}}{V'_i} > 0 . \quad (\text{A.5})$$

By the independence axiom $V''_{ji} = 0$, and then (A.5) becomes

$$-\frac{V''_{ii}}{V'_i} = v_i(x_i) > 0, \quad (\text{A.6})$$

proving sufficiency.

To prove necessity we start with condition (a) and see that

$$V''_{ij} = 0 \quad (\text{A.7})$$

implies that

$$V'_i = f(x_i), \quad (\text{A.8})$$

which by integration yields

$$V(\underline{x}) = g(x_i) + h(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{N-1}), \quad (\text{A.9})$$

where $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ denote real-valued functions. Since (A.9) is true for all i ,

$$V(\underline{x}) = \sum_{i=0}^{N-1} v_i(x_i), \quad (\text{A.10})$$

which proves additivity and therefore deterministic independence. To prove nonincreasing marginal rates of substitution we rewrite (A.6) as

$$-\frac{\partial}{\partial x_i} \ln \frac{\partial V}{\partial x_i} = v_i(x_i). \quad (\text{A.11})$$

Integrating and rearranging (A.11) yields

$$\frac{\partial V}{\partial x_i} = e^{-\int v_i(x_i) dx_i}, \quad (\text{A.12})$$

and therefore

$$\frac{\partial x_j}{\partial x_i} = - \frac{V'_i}{V'_j} = - e^{-\int v_i(x_i) dx_i + \int v_j(x_j) dx_j} . \quad (\text{A.13})$$

Differentiating (A.13) with respect to x_i produces the result we are seeking, i.e.,

$$\frac{\partial^2 x_j}{\partial x_i^2} = v_i(x_i) e^{-\int v_i(x_i) dx_i + \int v_j(x_j) dx_j} . \quad (\text{A.14})$$

This expression is always greater than zero if $v_i(x_i) > 0$.

Q.E.D.

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