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DERIVATION AND ANALYSIS OF THE COMPLETE NONLINEAR DYNAMICAL EQUATIONS OF THE MICOM STABILIZED MIRROR SYSTEM

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Redstone Arsenal, Alabama

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The complete nonlinear dynamical equations of the MICOM stabilized mirror system mounted on a vehicle are derived and analyzed for certain stabilized motion. Lagrange's formulation of D'Alembert's principle is used to derive the highly coupled equations of motion of the gimballed platform, balancer, mirror, and azimuth and elevation gyroscopes. Four special cases of the full dynamical equations are presented. They are the equations based on the assumptions of (a) small vehicular motion, (b) small mass unbalance, (c) small deviations from system looking forward, and (d) both small motions and small mass unbalance.

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Block 20, Abstract (continued)

A block diagram is given for the analog simulation of case (d). These equations are derived for the purposes of analyzing design changes in the basic structure and for designing stabilization and servo-control compensators for improved accuracy in tracking and stabilization. Although no new control system design is attempted, conclusions are drawn from the structure of the dynamical system equations.

CONTENTS

1. Introduction

This report is a portion of the research performed under the Guidance and Control Technology Program on Automatic Tracking Systems for Airborne Direct Fire and Integrated Fire Control. The dynamical systems equations of the MICOM stabilized mirror/gimbal tracking system are derived in this report. These equations are derived for the purposes of (1) analyzing design changes in the basic structure and (2) designing stabilization and servo control compensations for improved accuracy in tracking and stabilizing a target scene or in stabilizing a laser beam in a helicopter environment.

The system consists of an azimuth gimbal, G, mounted in a base structure, S, such as that of a helicopter pod (Figure 1). The platform, P, is gimbal mounted in G and rotates in elevation. Mounted on P are two rate integrating gyroscopes used for inertially rate stabilizing P via feedback to torque motors mounted on the gimbal axes of G and P. Figure 2 shows a schematic drawing of the elevation and azimuth gyros mounted on the platform. The mirror, M, is not mounted on the platform, but is gimbal mounted on G with axis of rotation parallel to that of P. The mirror is connected to the stable platform, P, by a wire belt pulley arrangement with a 2:1 drive ratio. The mirror is not mounted on the platform because a stable view of a fixed scene and laser beam pointing stability require the mirror to move one-half the angle of pitch of the gimbal, G. A 2:1 drive ratio consisting of a belt drive and an aided inertia drive can meet this requirement. The inertia drive, or what is misnomered the inertia balancer, B, is a gimbal mounted mass with axis parallel to those of P and M. The balancer, B, is constructed with the precise moment of inertia about its axis for "inertially balancing" the mirror. When the base, S, and the gimbal, G, pitch, the mirror should move in elevation through one-half the pitch angle so that the scene does not change when viewing the image of the scene reflected from the mirror. The "balancer" will accomplish this in the absence of other forces and is especially useful when the frequency of disturbance motion is greater than the bandwidth of the platform stabilization loop. Figure 1 shows the balancer, B, in contact with the belt drive and only operates correctly when there is no slip between the balancer and belt.

The equations derived herein describe the motion of the gimbal, platform, mirror, balancer, and gyros with the full nonlinear and cross-coupling terms and such imperfections as friction, deformation of the wire drive, and spring coupling due to electrical cables connected between elements of the system. The associated feedback and servo-control electronics, as they are presently designed, are given in transfer function form in the summary block diagram presented in Section 3.

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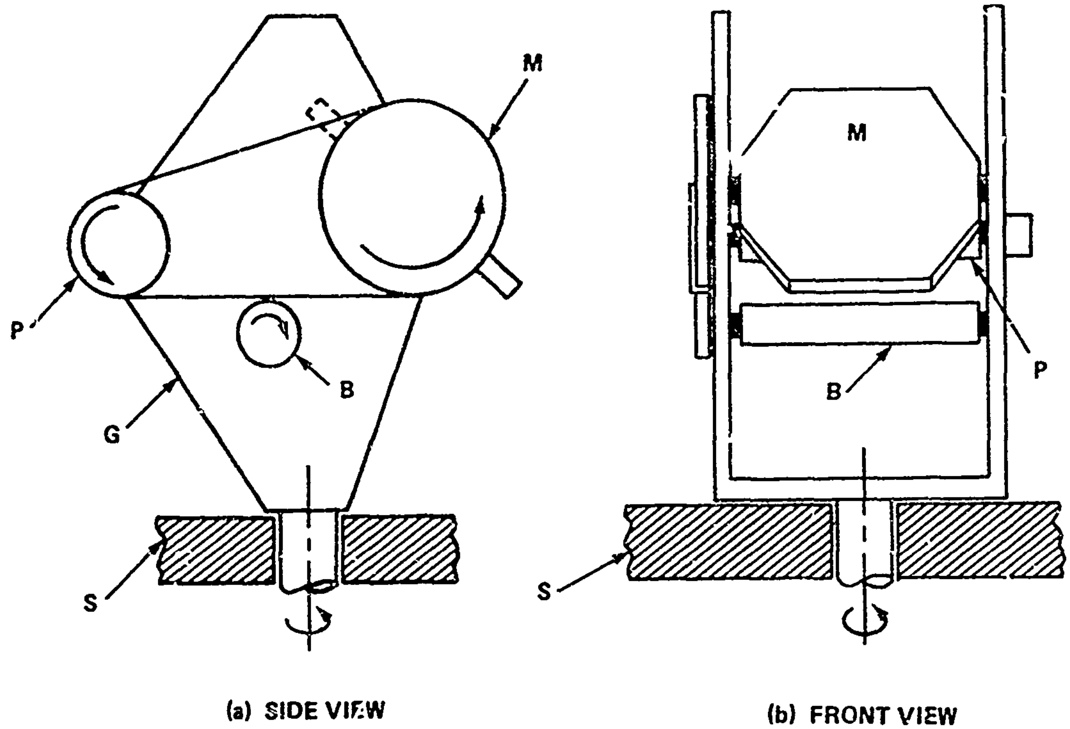


Figure 1. Schematic of rigid body system elements.

2. Equations of Motion for Rigid Body System Elements

a. Analytical Approach

In a system of rigid body elements there are for modeling purposes, three classifications of interactions of the elements. First, elements of significant inertia interact with each other and all of their motion variables (generalized coordinates) are coupled dynamically in the dynamical equations of motion. Second, if some of the element's inertias are much larger than those of other elements, the larger inertias are insignificantly affected by the motion of the smaller inertias so that the dynamical equations of the larger inertias do not include the effects of the smaller inertias, i.e., the generalized coordinates of the larger appear in the equations of the smaller, but not vice versa. For example, the smaller inertias of the gyroscopes have an insignificant effect on the platform's motion. However, platform motion significantly affects the motions of the gyroscopes. Thus, there are equations of motion for the platform as well as the gyroscopes but the coupling is from the platform to the gyroscopes and not vice versa except in the electrical feedback. Third, in a variation of the second classification is the case when the coupling is one way and the larger inertias have motions which can be measured and prescribed as a function of time rather than described by a system of dynamical equations. For example, the motion of a helicopter would not be significantly affected by the motion of the stabilized mirror/gimbal tracking system so that if we included the helicopter in the dynamical system (i.e., we determine its motion from

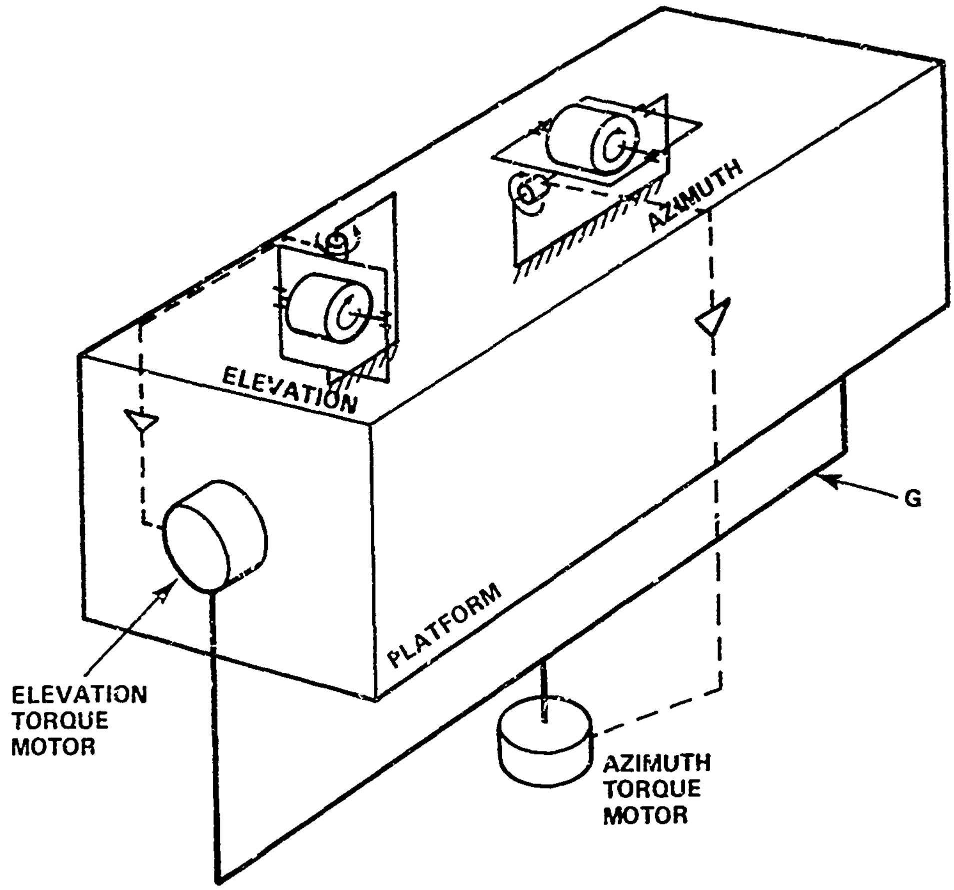


Figure 2. Rate integrating gyro stabilized platform.

its dynamical equations), the effects of the mirror system's motion would not be included. In this case, however, the helicopter is not dynamically modeled to determine its motion and, thus, its effect on the mirror system, but has its motion prescribed from measurement data to determine its motion's effects on the mirror. In a related program, linear and angular accelerometers are used to determine helicopter motion under typical flight profiles with dummy rigid masses simulating the stabilized mirror system attached.

In this development of the system's dynamical equations we use helicopter motion data as the prescribed motion of the base structure, S. The base motion can be completely described by three angles and three coordinates of a point of S as functions of time. Herein, we prescribe the azimuth, elevation, and roll angles $\phi_A(t)$, $\phi_E(t)$, and $\phi_R(t)$ and the position of a base point x(t), y(t), z(t). These are calculated from body-fixed accelerometers outputs. The point x, y, z of the base, S, can be any point of the rigid base structure.

However, for laboratory tests, we construct the test stand (on which the gimbal system is mounted) so that rotational motion is simulated separately from helicopter translational motion. Figure 3 is a schematic of the base test stand. This separation is accomplished by designing the test stand with rotational axes colinear with the system's gimbal axes. The test stand is mounted with either the elevation axis or the azimuth axis mounted in bearings in a fixed support for rotational motion. A large shaker provides the prescribed motion through a rigid linkage. For a linear motion test, the test stand is mounted directly onto the haker.

The equations of motion are derived in terms of the prescribed motion variables ϕ_A, ϕ_E, ϕ_R, x, y, z for this case in which rotational input motion is separated from translational input motion.

The method of analytical mechanics used in deriving the equations of motion of the system is Lagrange's formulation of D'Alembert's principle, i.e., Lagrange's Equations of First Kind, written as

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} - \frac{\partial K}{\partial \theta} = F_\theta \qquad (1)$$

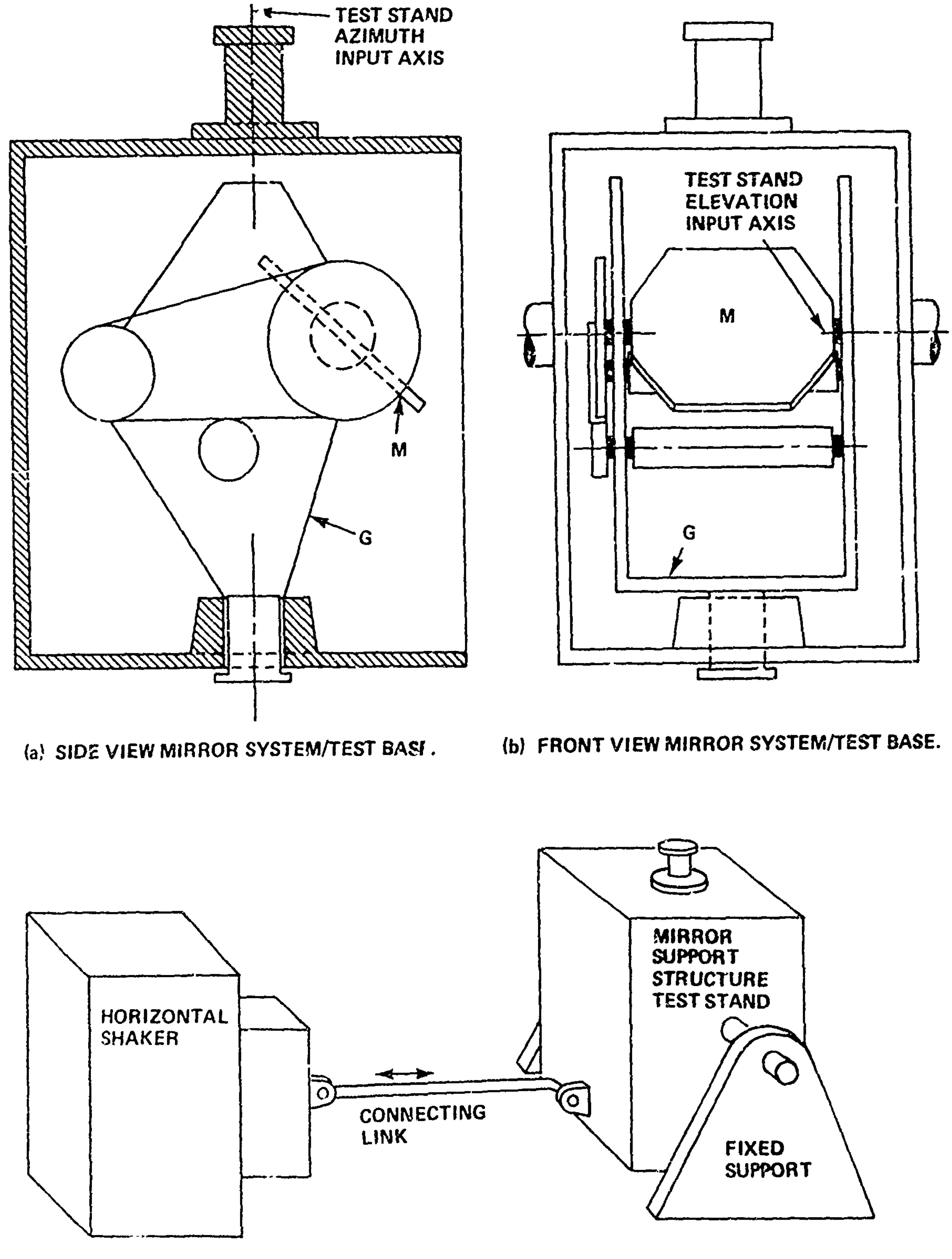
where K is the kinetic energy of the system, θ is a generalized coordinate, $\dot{\theta}$ is its time derivative, and F_θ is the generalized force of the system for the coordinate θ.

This method is used since it is conceptually straightforward and, most importantly, it does not require the inclusion of conservative forces of interaction between system elements and then their elimination. We are not interested in determining the forces of interaction since we are interested in only the system's motion under external disturbances and not in structural limit design.

The kinetic energy of the system is the sum of the kinetic energy of its parts. Thus,

$$K = K_M + K_B + K_P + K_G + K_A + K_E \qquad (2)$$

where K_A is the kinetic energy of the azimuth gyroscope and K_E is for the elevation gyroscope. The kinetic energy of any single rigid body, for example, B, can be written as a function of the mass of B, m_B, the angular velocity $\underline{\omega}^B$ of the body B with respect to an inertial frame, the velocity $\underline{V}^{B*}$ of the center of mass B* of B in an inertial frame, and the moments and products of inertia of B with respect to B* which are expressed compactly in the dyadic (or matrix) form $\underline{I}^{B/B*}$. Explicitly,



(a) SIDE VIEW MIRROR SYSTEM/TEST BASE.

(b) FRONT VIEW MIRROR SYSTEM/TEST BASE.

(c) TEST STAND LAYOUT FOR ELEVATION MOTION TEST

Figure 3. Schematic of base test stand.

K_B is

$$K_B = \frac{1}{2} m_B \underline{V}^{B*} \cdot \underline{V}^{B*} + \frac{1}{2} \underline{\omega}^B \cdot I^{B/B*} \cdot \underline{\omega}^B \tag{3}$$

where it is implicitly assumed that the axes through B* which define the moments and products of inertia are parallel to the right-hand set of unit vectors in terms of $\underline{\omega}^B$ is expressed; i.e., we express

$$\underline{\omega}^B = \omega_1^B \underline{b}_1 + \omega_2^B \underline{b}_2 + \omega_3^B \underline{b}_3 \tag{4}$$

where $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ is such a set of unit vectors. Such a set of unit vectors is sometimes called a basis. Likewise, we can express a vector such as $\underline{V}^{B*}$ in terms of components in this or some other basis. In this basis

$$\underline{V}^{B*} = V_1^B \underline{b}_1 + V_2^B \underline{b}_2 + V_3^B \underline{b}_3 \tag{5}$$

where $V_i^{B*} \underline{b}_i$ is simply the component of $\underline{V}^{B*}$ in the $\underline{b}_i$ direction.

The inertia dyadic is defined by

$$\underline{I}^{B/B*} = \sum_{i=1}^{3} \sum_{j=1}^{3} \underline{b}_i \, I_{ij}^{B/B*} \, \underline{b}_j$$

or in matrix rotation

$$\underline{I}^{B/B*} = (\underline{b}_1, \underline{b}_2, \underline{b}_3) \begin{bmatrix} B_1 & B_{12} & B_{13} \\ B_{12} & B_2 & B_{23} \\ B_{13} & B_{23} & B_3 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{6}$$

where, for a simplier rotation, we have

$$I_{11}^{B/B*} = B_1$$

$$I_{12}^{B/B*} = B_{12} \quad \text{etc.}$$

The kinetic energy of each rigid body in the system can be derived in this manner.

b. Kinematics

The kinematics of both the gimballed mirror, platform, and balancer system and the gyroscopes are developed in this section. Generalized coordinates are selected such that the dynamical equations of motion will be simple. The outer gimbal motion is described in terms of angles with respect to the base which has prescribed angular and translational motion. The other elements of the system have only angular motion with respect to the outer gimbal or an inner gimbal.

The angular velocities of each rigid body of the system and the velocities of the mass centers of each body are now derived. These are used in Section 2.c. in determining the generalized inertia forces.

(1) Gimballed Mirror, Platform, and Balancer. With the base, S, having prescribed motion, the only generalized coordinates required to specify the motion of the system, G, P, M, and B, are those of the system relative to the base. Thus, we take the generalized coordinates

θ_G: angle of rotation of G relative to S

θ_P: angle of rotation of P relative to G

θ_M: angle of rotation of M relative to G

θ_B: angle of rotation of B relative to G

so that if the base's motion is known, the motion of every point of the system is also known.

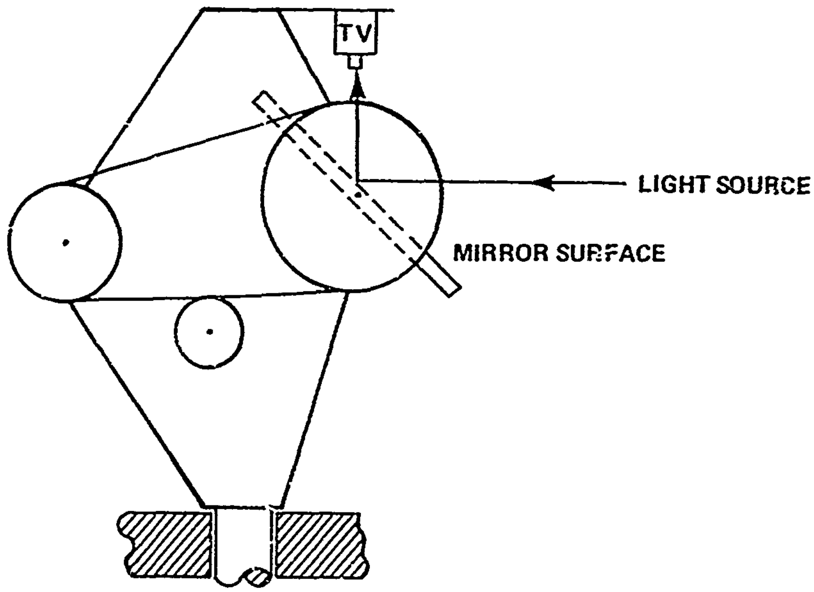
The system is designed so that if the base is level and a TV camera is mounted on the gimbal looking vertically down at the mirror, the TV will see a scene forward of the system. Thus, θ_M is chosen to be measured from a line at 45 degrees from the vertical so that under nominal operation the angles will be small. Figure 4 defines the generalized coordinates and illustrates the TV looking forward at a source of light.

The angular velocity of G in an inertial frame is

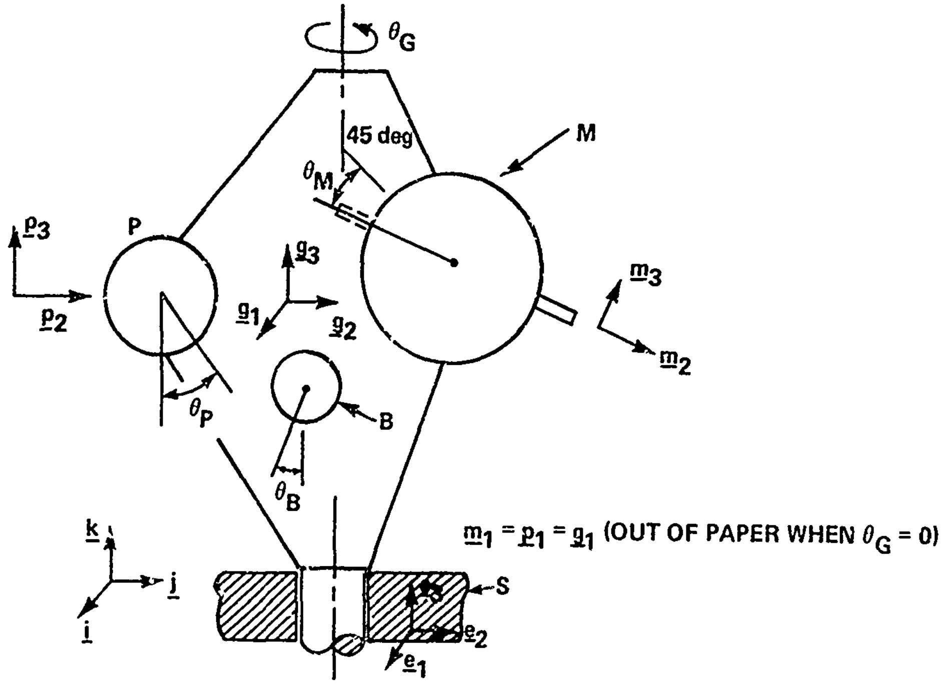
$$\underline{\omega}^G = \dot{\theta}_G \, \underline{g}_3 + \underline{\omega}^S \tag{7}$$

where $\underline{\omega}^S$ is the angular velocity of S in an inertial frame.

The angular velocity of S is now derived in terms of the base motion angles relative to an inertial frame, ϕ_R, ϕ_E, and ϕ_A, and their derivatives. The derivation of the equations of motion is simpler if



(a) TV CAMERA LOOKING FORWARD



(b) DEFINITION OF GENERALIZED COORDINATES AND UNIT VECTORS

Figure 4. TV viewing, generalized coordinates, and unit vectors.

$\underline{\omega}^S$ is expressed in components for $\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$. For convenience the same order and direction of rotation of the base is taken as the system relative to the base. Beginning a rotation in roll of angle ϕ_R about a vertical earth-fixed line (the earth suffices for an inertial frame in this derivation), let $\underline{i}$, $\underline{j}$, $\underline{k}$ be an earth-fixed basis with $\underline{k}$ vertical, let $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$ be fixed in S and coincide with $\underline{i}$, $\underline{j}$, $\underline{k}$ before the ϕ_R notation, and let $\underline{k}$ and $\underline{r}_3$ be directed vertically upward. Figure 5 illustrates the description of the rotation of S relative to the earth. The new orientation of S is completely determined by the orientation of $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$ and thus by ϕ_R. Next a rotation ϕ_A in azimuth is performed. Let $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ be a basis fixed in S during this rotation and let $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$ remain fixed relative to the inertial frame defined by $\underline{i}$, $\underline{j}$, $\underline{k}$; i.e., hold ϕ_R constant. After this second finite rotation, the new orientation of S is completely determined by the orientation of $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and thus by both ϕ_R and ϕ_A. Finally, an elevation rotation of ϕ_E is performed, and let $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ remain fixed; i.e., hold ϕ_R and ϕ_A constant, while $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ are fixed in S during this rotation. Now, the orientation of S is completely determined by the orientation of $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ and thus by ϕ_R, ϕ_A, and ϕ_E, and every rotation of S can be specified by specifying ϕ_R, ϕ_A, and ϕ_E. With these simple rotations describing the total rotation of S, it is possible to determine the angular velocity ${}^I\underline{\omega}^S \equiv \underline{\omega}^S$ of S in an inertial frame, I, from simple angular velocities. Observing from Figure 5 that

$$ {}^I\underline{\omega}^S = {}^A\underline{\omega}^S + {}^R\underline{\omega}^A + {}^I\underline{\omega}^R = \dot{\phi}_E\underline{e}_1 + \dot{\phi}_A\underline{a}_3 + \dot{\phi}_R\underline{r}_2 \tag{8} $$

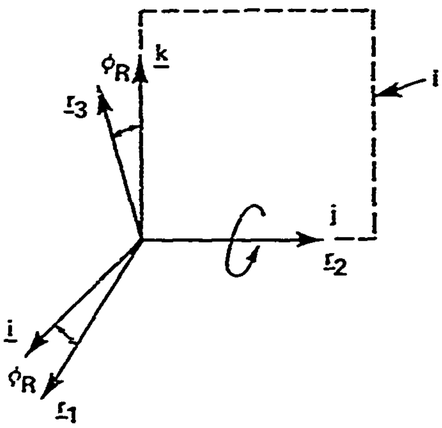
where, for example, ${}^A\underline{\omega}^S$ is the angular velocity of the S frame (whose orientation is determined by $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) relative to the A frame (determined by $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$). Equation (8) is an expression for $\underline{\omega}^S$ but is not convenient to use until it is expressed in a common basis. For this purpose, Figure 5 expresses

$$ \underline{a}_3 = s_E\underline{e}_2 + C_E\underline{e}_3 \tag{9} $$

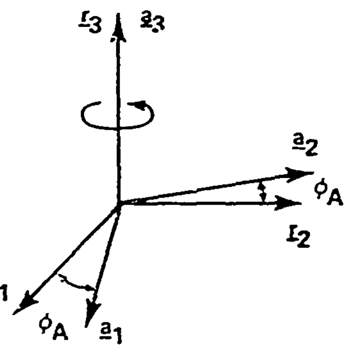
$$ \underline{r}_2 = s_A\underline{a}_1 + C_A\underline{a}_2 \tag{10} $$

$$ \underline{a}_1 = \underline{e}_1 \tag{11} $$

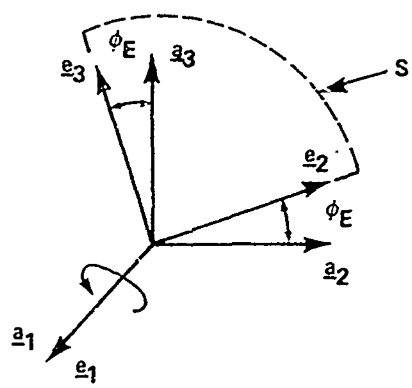
$$ \underline{a}_2 = C_E\underline{e}_2 - s_E\underline{e}_3 \tag{12} $$



(a) ROLL ROTATION OF BASE, S, AND ORIENTATION OF A BASE-FIXED BASIS $\underline{r}_1, \underline{r}_2, \underline{r}_3$



(b) AZIMUTH ROTATION OF BASE, S, AND ORIENTATION OF A BASE-FIXED BASIS $\underline{a}_1, \underline{a}_2, \underline{a}_3$



(c) ELEVATION ROTATION OF BASE, S, AND ORIENTATION OF THE FINAL BASE-FIXED BASIS $\underline{e}_1, \underline{e}_2, \underline{e}_3$

Figure 5. Unit vectors and angles for describing the orientation of the base, S, in an inertial frame, I.

where, for example, $s_E = \sin \phi_E$ and $C_A = \cos \phi_A$. Now substitute Equations (11) and (12) into Equation (10) and substitute this result with Equation (9) into Equation (8). This gives

$$\underline{\omega}^S = (\dot{\phi}_E + \dot{\phi}_R s_A)\underline{e}_1 + (\dot{\phi}_R C_A C_E + \dot{\phi}_A s_E)\underline{e}_2 + (\dot{\phi}_A C_E - \dot{\phi}_R s_E C_A)\underline{e}_3 \quad . \tag{13}$$

Since $\underline{e}_3 = \underline{g}_3$, $\underline{e}_1 = c_G\underline{g}_1 - s_G\underline{g}_2$, and $\underline{e}_2 = s_G\underline{g}_1 + C_G\underline{g}_2$, it is possible to substitute Equation (13) into Equation (7) to obtain $\underline{\omega}^G$ in the $\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$ basis as

$$\begin{aligned}\underline{\omega}^G &= \left(\omega_{S_1} C_G + \omega_{S_2} s_G\right)\underline{g}_1 + \left(\omega_{S_2} C_G - \omega_{S_1} s_G\right)\underline{g}_2 + \left(\dot{\theta}_G + \omega_{S_3}\right)\underline{g}_3 \\ &\equiv \omega_1^G \underline{g}_1 + \omega_2^G \underline{g}_2 + \omega_3^G \underline{g}_3\end{aligned} \tag{14}$$

where we define

$$\begin{aligned}\omega_{S_1} &= \dot{\phi}_E + \dot{\phi}_R s_A \\ \omega_{S_2} &= \dot{\phi}_R C_A C_E + \dot{\phi}_A s_E \\ \omega_{S_3} &= \dot{\phi}_A C_E - \dot{\phi}_R s_E C_A \quad .\end{aligned} \tag{15}$$

Since

$$\underline{\omega}^M \equiv {}^I\underline{\omega}^M = {}^G\underline{\omega}^M + {}^I\underline{\omega}^G = \dot{\theta}_M \underline{m}_1 + {}^I\underline{\omega}^G \tag{16}$$

$$\underline{\omega}^P \equiv {}^I\underline{\omega}^P = {}^G\underline{\omega}^P + {}^I\underline{\omega}^G = \dot{\theta}_P \underline{p}_1 + {}^I\underline{\omega}^G \tag{17}$$

$$\underline{\omega}^B \equiv {}^I\underline{\omega}^B = {}^G\underline{\omega}^B + {}^I\underline{\omega}^G = -\dot{\theta}_B \underline{g}_1 + {}^I\underline{\omega}^G \tag{18}$$

it is possible to determine $\underline{\omega}^M$ and $\underline{\omega}^P$ in their respective bases by expressing ${}^I\underline{\omega}^G$ in these bases. There is no need to express $\underline{\omega}^B$ in the $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ basis since the "balancer" B is symmetrically constructed about its axis of rotation and its inertia properties do not vary with respect to the $\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$ basis with rotations of B. Thus, since

$$\underline{g}_1 = \underline{m}_1 = \underline{p}_1$$

$$\underline{g}_2 = c_P\underline{p}_2 - s_P\underline{p}_3 = c_\alpha\underline{m}_2 + s_\alpha\underline{m}_3$$

$$\underline{g}_3 = s_P\underline{p}_2 + c_P\underline{p}_3 = s_\alpha\underline{m}_2 + c_\alpha\underline{m}_3 \tag{19}$$

where $\alpha = 45 \text{ deg} - \theta_M$, and Equation (16) can now be written in the $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$ basis and Equation (17) in the $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ basis. Substitution of Equations (19) into Equation (16) gives

$$\begin{aligned}\underline{\omega}^M &= \left(\dot{\theta}_M + \omega_1^G\right)\underline{m}_1 + \left(c_\alpha\omega_2^G - s_\alpha\omega_3^G\right)\underline{m}_2 + \left(s_\alpha\omega_2^G + c_\alpha\omega_3^G\right)\underline{m}_3 \\ &\equiv \omega_1^M\,\underline{m}_1 + \omega_2^M\,\underline{m}_2 + \omega_3^M\,\underline{m}_3 \quad .\end{aligned} \tag{20}$$

Similarly, Equations (17) and (18) become

$$\begin{aligned}\underline{\omega}^P &= \left(\dot{\theta}_P + \omega_1^G\right)\underline{p}_1 + \left(c_P\omega_2^G + s_P\omega_3^G\right)\underline{p}_2 + \left(c_P\omega_3^G - s_P\omega_2^G\right)\underline{p}_3 \\ &\equiv \omega_1^P\,\underline{p}_1 + \omega_2^P\,\underline{p}_2 + \omega_3^P\,\underline{p}_3\end{aligned} \tag{21}$$

$$\underline{\omega}^B = \left(-\dot{\theta}_B + \omega_1^G\right)\underline{g}_1 + \omega_2^G\,\underline{g}_2 + \omega_3^G\,\underline{g}_3 \equiv \omega_1^B\,\underline{g}_1 + \omega_2^B\,\underline{g}_2 + \omega_3^B\,\underline{g}_3 \quad . \tag{22}$$

In the beginning of Section 2.b., it was mentioned that the system is designed so that the mirror will have an orientation of 45 degrees with respect to the gimbal. With no torque motor power and no disturbances, the mirror-platform-wire drive is statistically balanced to seek a 45-degree orientation from vertical. This is accomplished by having the mass centers of M, say M*, and P, say P*, displaced from the axes of rotation of M and P. Let $\underline{m}$ be the position vector of M* from the center of rotation of M, say M_o. Likewise, let $\underline{p}$ be the position vector from P_o to P*. It has been observed that

$$\underline{m} = m\,\underline{m}_2 \tag{23}$$

$$\underline{p} = -P\,\underline{p}_2 \tag{24}$$

where m and P are the scalar distances.

Figure 6 gives the geometry of centers of rotation, centers of masses and vectors between these. These are now used to compute the velocities of mass centers. If P_1 and P_2 are two points of the same rigid body, B, then

$$\underline{V}^{P_2} = \underline{V}^{P_1} + \underline{\omega}^B \times \underline{p}^{P_2/P_1}$$

where $\underline{p}^{P_2/P_1}$ is the vector from P_1 to P_2. Thus, it is possible to write

$$\underline{V}^{M*} = \underline{V}^{M_o} + \underline{\omega}^M \times \underline{m} \quad (25)$$

where

$$\underline{V}^{M_o} = \underline{V}^{G_o} + \underline{\omega}^G \times \underline{m}_o$$

and

$$\underline{V}^{P*} = \underline{V}^{P_o} + \underline{\omega}^P \times \underline{p} \quad (26)$$

where

$$\underline{V}^{P_o} = \underline{V}^{G_o} + \underline{\omega}^G \times \underline{p}_o \; .$$

Previously, x, y, and z were allowed to be any point of S for prescribing its motion. Taking that point to be the point G_o which is common to S and G and fixed in both. Thus,

$$\underline{V}^{G_o} = V_1\, \underline{e}_1 + V_2\, \underline{e}_2 + V_3\, \underline{e}_3 = V_{G_1}\, \underline{g}_1 + V_{G_2}\, \underline{g}_2 + V_{G_3}\, \underline{g}_3 \quad (27)$$

wherein V_1, V_2, V_3 are computed from base-fixed accelerometer data. The vectors $\underline{m}_o$ and $\underline{p}_o$ can be expressed as

$$\underline{m}_o = m_2\, \underline{g}_2 + m_3\, \underline{g}_3$$

$$\underline{p}_o = P_2\, \underline{g}_2 + P_3\, \underline{g}_3$$

so that

$$\underline{\omega}^G \times \underline{m}_o = \left(m_3\omega_2^G - m_2\omega_3^G\right) \underline{g}_1 - m_3\omega_1^G\, \underline{g}_2 + m_2\omega_1^G\, \underline{g}_3 \quad (28)$$

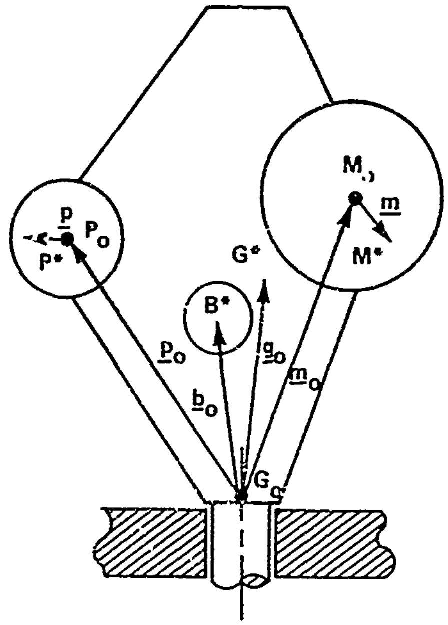


Figure 6. Centers of rotation, centers of masses, and their position vectors.

$$\underline{\omega}^G \times \underline{p}_o = \left(P_3\omega_2^G - P_2\omega_3^G\right)\underline{g}_1 - P_3\omega_1^G\,\underline{g}_2 + P_2\omega_1^G\,\underline{g}_3 \quad . \tag{29}$$

From Equations (20) and (23) the following is derived:

$$\underline{\omega}^M \times \underline{m} = -\bar{m}\left(\omega_3^M\,\underline{m}_1 - \omega_1^M\,\underline{m}_3\right) = -m\left[\omega_3^M\,\underline{g}_1 - \omega_1^M\,(s_\alpha\underline{g}_2 + c_\alpha\underline{g}_3)\right] \quad . \tag{30}$$

Likewise, Equations (21) and (24) give

$$\underline{\omega}^P \times \underline{p} = P\left(\omega_3^P\,\underline{p}_1 - \omega_1^P\,\underline{p}_3\right) \quad . \tag{31}$$

At this point V^{M*} and V^{P*} can be expressed in terms of the generalized coordinates, their rates, the prescribed motion ϕ_A, $\dot{\phi}_A$, ϕ_E, $\dot{\phi}_E$, ϕ_R, $\dot{\phi}_R$, V_1, V_2, V_3 and a common basis. The basis $\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$ is a convenient basis. On substituting Equations (27), (28), and (30) into Equation (25) the result is

$$\begin{aligned}\underline{V}^{M*} = {} & \left(V_{G_1} + m_3\omega_2^G - m_2\omega_3^G - m\omega_3^M\right)\underline{g}_1 + \left(V_{G_2} - m_3\omega_1^G + ms_\alpha\omega_1^M\right)\underline{g}_2 \\ & + \left(V_{G_3} + m_2\omega_1^G + mC_\alpha\omega_1^M\right)\underline{g}_3 \quad .\end{aligned} \tag{32}$$

Similarly, from Equations (26), (27), (29), and (31)

$$\underline{V}^{P*} = \left(V_{G_1} + P_3\omega_2^G - P_2\omega_3^G + P\omega_3^P\right) g_1$$

$$+ \left(V_{G_2} - P_3\omega_1^G + Ps_P\omega_1^P\right) g_2$$

$$+ \left(V_{G_3} + P_2\omega_1^G - P\ C_P\omega_1^P\right) g_3 \qquad (33)$$

is obtained. Likewise, for $\underline{V}^{G*}$ similar terms are obtained but with g_2 and g_3 instead of, say, P_2 and P_3 and with P = 0. Since B* coincides with B_o,

$$\underline{V}^{B*} = \underline{V}^{B_o} = \underline{V}^{G_o} + \underline{\omega}^G \quad \underline{b}_o$$

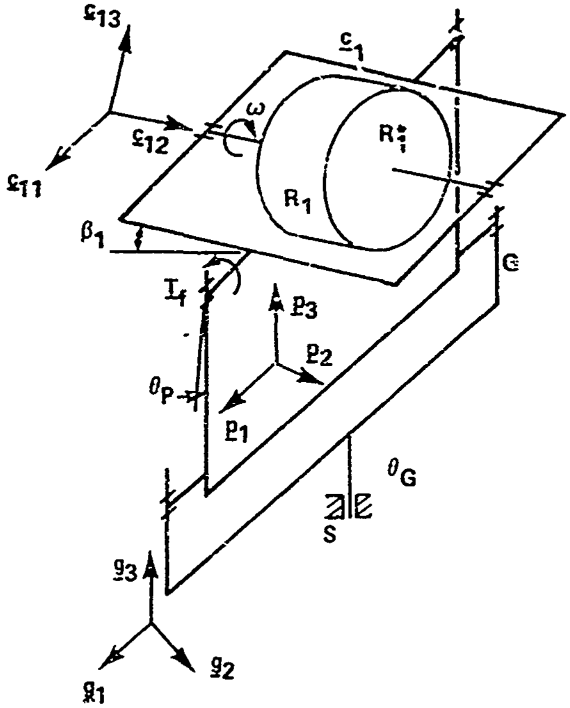
$$= \left(V_{G_1} + b_3\omega_2^G - b_2\omega_3^G\right) g_1$$

$$+ \left(V_{G_2} - b_3\omega_1^G\right) g_2 + \left(V_3 + b_2\omega_1^G\right) g_3 \quad . \qquad (34)$$

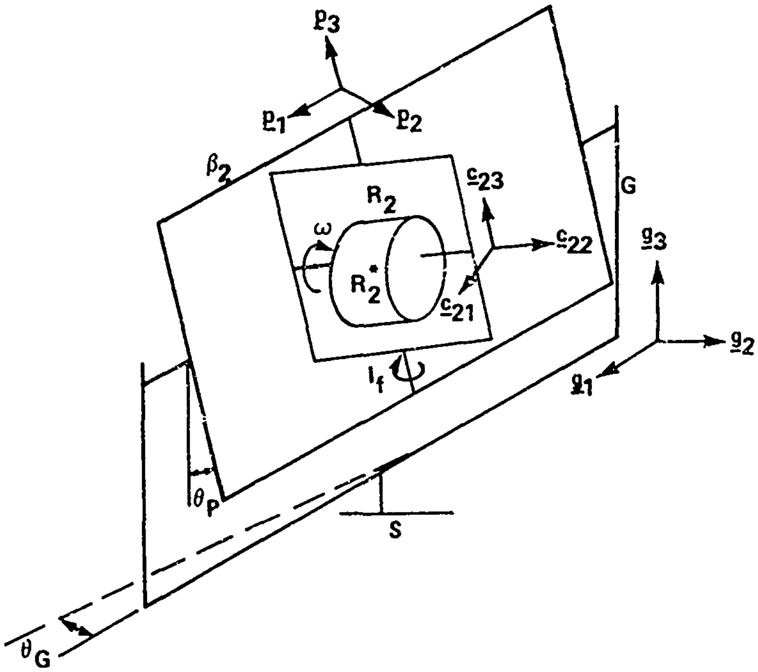
(2) Kinematics of Gyroscopes. The gyroscopes are mounted on the platform, P. One gyroscope has its sensitive axis parallel to the axis of the gimbal, G, to sense azimuth rates. The other gyroscope has its sensitive axis parallel to the axis of the platform, P, to sense elevation rates. A schematic of each gyroscope is shown in Figure 7. The gyroscopes are single degree-of-freedom rate integrating gyros with their gimbals mounted on the platform, P. The gyro's rotor R_i, i=1, 2, is motor driven at a constant rate ω with respect to the gyro gimbal C_i. The rotor is symmetric about its axis of rotation so that all lines through the mass center, R_i^*, and perpendicular to the rotation axis are principal axes for R_i^*. The center of mass of C_i, C_i^*, coincides with R_i^*. The generalized coordinates of the single degree-of-freedom gyros in an inertial frame are β_i, i=1, 2, and the coordinates of the platform, P.

The angular velocity of the rotor R_i is

$$\underline{\omega}^{R_i} = \underline{\omega}^{C_i} + \omega\ \underline{C}_{i2} \quad . \qquad (35)$$



(a) GENERALIZED COORDINATES AND VECTOR BASES FOR THE AZIMUTH GYRO.



(b) GENERALIZED COORDINATES AND VECTOR BASES FOR THE ELEVATION GYRO.

Figure 7. Schematics of azimuth and elevation gyroscopes mounted on platform.

The gyro gimbals have angular velocities

$$\underline{\omega}^{C_1} = \underline{\omega}^P - \dot{\beta}_1 \underline{C}_{11} \tag{36}$$

$$\underline{\omega}^{C_2} = \underline{\omega}^P + \dot{\beta}_2 \underline{C}_{23} \quad . \tag{37}$$

Previously, $\underline{\omega}^P$ was determined in the $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ basis. From Equation (21) the following is obtained in the $\underline{C}_{11}$, $\underline{C}_{12}$, $\underline{C}_{13}$ basis:

$$\underline{\omega}^P = \omega_1^P \underline{C}_{11} + \left(\omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}\right)\underline{C}_{12} + \left(\omega_2^P s_{\beta_1} + \omega_3^P c_{\beta_1}\right)\underline{C}_{13} \quad . \tag{38}$$

Likewise, in the $\underline{C}_{21}$, $\underline{C}_{22}$, $\underline{C}_{23}$ basis,

$$\underline{\omega}^P = \left(\omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}\right)\underline{C}_{21} + \left(\omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}\right)\underline{C}_{22} + \omega_3^P \underline{C}_{23}. \tag{39}$$

Finally, note that the velocities of the mass centers R_i^* and C_i^* do not depend on β_i and $\dot{\beta}_i$.

c. Inertia Forces

D'Alembert's principle states that a reference frame exists such that the system's active forces with the inertia forces in the frame are a zero force system. This frame is called an inertial frame. If certain forces of interaction within the rigid-body system are not to be determined, then the use of D'Alembert's principle is worthwhile since standard operations on zero force systems in statics allow one to equate moments about any point to zero. (We take moments about points on lines of action of forces that we wish to eliminate from the dynamical equations and thereby reduce the number of unimportant variables.)

Lagrange's formulation of D'Alembert's principle allows one to eliminate all unnecessary forces of interaction directly without using time-consuming algebraic elimination methods.

(1) Lagrange's Form of Inertia Forces for Gimballed Mirror System. In this section, we shall deal with only inertia forces which are derived only from properties of the system's elements and kinematics. The generalized inertia forces of Lagrange are given by

$$F_\theta^* = \frac{\partial K}{\partial \theta} - \frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} \tag{40}$$

where θ represents each of the generalized coordinates. To determine the generalized inertia forces in terms of the generalized coordinates, their derivatives, and the prescribed motion it is necessary to determine $\partial K/\partial\theta_M$, $\partial K/\partial\dot{\theta}_M$, $\partial K/\partial\theta_P$, etc. where K is given by Equations (2) and (3).

If the basis for each body of the system has unit vectors parallel to principal axes of inertia of the body for its mass center, then the products of inertia in Equation (6) are zero. The system is designed so that an axis of rotation of an element is precisely parallel to a principal axis of the body for its mass center. Thus, the kinetic energy of each body can be written as

$$K_B = \frac{1}{2}\left[m_B \sum_{i=1}^{3} (V_i^{B*})^2 + \sum_{i=1}^{3} B_i(\omega_i^B)^2\right] \tag{41}$$

since the products of inertia, B_{12}, B_{23}, etc., are zero. We now take the partial derivatives of the total kinetic energy [Equation (2)] with respect to the generalized coordinates. Since some of the velocity and angular velocity components are not functions of all of the generalized coordinates, we have

$$\partial K/\partial\theta_M = m_M \sum_{i=1}^{3} V_i^{M*} V_{i,\theta_M}^{M*} + \sum_{i=1}^{3} M_i\, \omega_i^M\, \omega_{i,\theta_M}^M \tag{42}$$

$$\partial K/\partial\theta_B = m_B \sum_{i=1}^{3} V_i^{B*} V_{i,\theta_B}^{B*} + \sum_{i=1}^{3} B_i\, \omega_i^B\, \omega_{i,\theta_B}^B \tag{43}$$

$$\partial K/\partial\theta_P = m_P \sum_{i=1}^{3} V_i^{P*} V_{i,\theta_P}^{P*} + \sum_{i=1}^{3} P_i\, \omega_i^P\, \omega_{i,\theta_P}^P$$

$$+ \partial K_E/\partial\theta_P + \partial K_A/\partial\theta_P \tag{44}$$

$$\partial K/\partial\theta_G = m_G \sum_{i=1}^{3} V_i^{G*} V_{i,\theta_G}^{G*} + \sum_{i=1}^{3} G_i\, \omega_i^G\, \omega_{i,\theta_G}^G$$

$$+ m_M \sum_{i=1}^{3} V_i^{M*} V_{i,\theta_G}^{M*} + \sum_{i=1}^{3} M_i\, \omega_i^M\, \omega_{i,\theta_G}^M$$

$$+ m_B \sum_{i=1}^{3} V_i^{B*} V_{i,\theta_G}^{B*} + \sum_{i=1}^{3} B_i \omega_i^B \omega_{i,\theta_G}^B$$

$$+ m_P \sum_{i=1}^{3} V_i^{P*} V_{i,\theta_G}^{P*} + \sum_{i=1}^{3} P_i \omega_i^P \omega_{i,\theta_G}^P$$

$$+ \partial K_E/\partial\theta_G + \partial K_A/\partial\theta_G \quad . \tag{45}$$

Terms in Equations (44) and (45), such as $\partial K_E/\partial\theta_P$ and $\partial K_A/\partial\theta_G$, are due to the elevation and azimuth gyroscopes. They are insignificant compared to the other terms and will not be included until the gyroscopes' inertia forces are derived in Section 2.c.(2). The task now is to express the terms in the summations as functions of the generalized coordinates. From Equation (32), the following is derived:

$$\left.\begin{aligned} V_{1,\theta_M}^{M*} &= m\left(C_\alpha\omega_2^G - s_\alpha\omega_3^G\right) = m\,\omega_2^M \\ V_{2,\theta_M}^{M*} &= -m\,C_\alpha\left(\dot{\theta}_M + \omega_1^G\right) = -m\,C_\alpha\,\omega_1^M \\ V_{3,\theta_M}^{M*} &= m\,s_\alpha\left(\dot{\theta}_M + \omega_1^G\right) = m\,s_\alpha\,\omega_1^M \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} V_{1,\theta_G}^{M*} &= V_2C_G - V_1s_G + m_3\omega_{2,\theta_G}^G - m\,s_\alpha\,\omega_{2,\theta_G}^G \\ &= V_{G_2} - (m_3 - ms_\alpha)\,\omega_1^G \\ V_{2,\theta_G}^{M*} &= -V_2s_G - V_1C_G - m_3\omega_{1,\theta_G}^G + ms_\alpha\,\omega_{1,\theta_G}^M \\ &= -V_{G_1} - (m_3 - ms_\alpha)\,\omega_2^G \\ V_{3,\theta_G}^{M*} &= (mC_\alpha + m_2)\,\omega_{1,\theta_G}^G = (m_2 + mC_\alpha)\,\omega_2^G \end{aligned}\right\} \tag{47}$$

The partial differentiation of components of Equation (14) with respect to θ_G results in

$$\left.\begin{aligned}
\omega^G_{1,\theta_G} &= \omega_{S_2}\, C_G - \omega_{S_1}\, s_G = \omega^G_2 \\
\omega^G_{2,\theta_G} &= -\omega_{S_2}\, s_G - \omega_{S_1}\, C_G = -\omega^G_1 \\
\omega^G_{3,\theta_G} &= 0
\end{aligned}\right\} \quad . \qquad (48)$$

Differentiation of Equation (20) gives

$$\left.\begin{aligned}
\omega^M_{1,\theta_M} &= 0 \\
\omega^M_{2,\theta_M} &= s_\alpha \omega^G_2 + C_\alpha \omega^G_3 = \omega^M_3 \\
\omega^M_{3,\theta_M} &= s_\alpha \omega^G_3 - C_\alpha \omega^G_2 = -\omega^M_2
\end{aligned}\right\} \quad . \qquad (49)$$

$$\left.\begin{aligned}
\omega^M_{1,\theta_G} &= \omega^G_{1,\theta_G} = \omega^G_2 \\
\omega^M_{2,\theta_G} &= C_\alpha \omega^G_{2,\theta_G} = -C_\alpha \omega^G_1 \\
\omega^M_{3,\theta_G} &= s_\alpha \omega^G_{2,\theta_G} = -s_\alpha \omega^G_1
\end{aligned}\right\} \quad . \qquad (50)$$

From Equation (21), the following is obtained:

$$\left.\begin{aligned}
\omega^P_{1,\theta_P} &= 0 \\
\omega^P_{2,\theta_P} &= C_P\, \omega^G_3 - s_P\, \omega^G_2 = \omega^P_3 \\
\omega^P_{3,\theta_P} &= -s_P\, \omega^G_3 - C_P\, \omega^G_2 = -\omega^P_2
\end{aligned}\right\} \quad . \qquad (51)$$

$$\left.\begin{aligned}
\omega^P_{1,\theta_G} &= \omega^G_{1,\theta_G} = \omega^G_2 \\
\omega^P_{2,\theta_G} &= c_P\ \omega^G_{2,\theta_G} = -c_P\omega^G_1 \\
\omega^P_{3,\theta_G} &= -s_P\ \omega^G_{2,\theta_G} = s_P\omega^G_1
\end{aligned}\right\} \quad . \tag{52}$$

Differentiation of Equation (22) gives

$$\omega^B_{1,\theta_B} = \omega^B_{2,\theta_B} = \omega^B_{3,\theta_B} = 0 \tag{53}$$

$$\left.\begin{aligned}
\omega^B_{1,\theta_G} &= \omega^G_{1,\theta_G} = \omega^G_2 \\
\omega^B_{2,\theta_G} &= \omega^G_{2,\theta_G} = -\omega^G_1 \\
\omega^B_{3,\theta_G} &= 0
\end{aligned}\right\} \quad . \tag{54}$$

The differentiation of the components of Equation (32) with respect to θ_P and θ_G results in

$$\left.\begin{aligned}
V^{P*}_{1,\theta_P} &= -P\left(s_P\omega^G_3 + c_P\omega^G_2\right) = -P\ \omega^P_2 \\
V^{P*}_{2,\theta_P} &= P\left(\dot{\theta}_P + \omega^G_1\right)c_P = P\ c_P\omega^P_1 \\
V^{P*}_{3,\theta_P} &= P\left(\dot{\theta}_P + \omega^G_1\right)s_P = P\ s_P\omega^P_1
\end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned}
V^{P*}_{1,\theta_G} &= V_2c_G - V_1s_G + (P_3 - Ps_P)\omega^G_{2,\theta_G} = V_{G_2} - (P_3 - Ps_P)\omega^G_1 \\
V^{P*}_{2,\theta_G} &= V_1c_G - V_2s_G - (P_3 - Ps_P)\omega^G_{1,\theta_G} = -V_{G_1} - (P_3 - Ps_P)\omega^G_2 \\
V^{P*}_{3,\theta_G} &= (P_2 - Pc_P)\omega^G_{1,\theta_G} = (P_2 - Pc_P)\omega^G_2
\end{aligned}\right\} \quad . \tag{56}$$

Similarly, Equation (39) gives

$$V^{B*}_{1,\theta_B} = V^{B*}_{2,\theta_B} = V^{B*}_{3,\theta_B} = 0 \tag{57}$$

$$\left.\begin{aligned} V^{B*}_{1,\theta_G} &= V_2C_G - V_1s_G + b_3\,\omega^G_{2,\theta_G} \\ &= V_{G_2} - b_3\omega^G_1 \\ V^{B*}_{2,\theta_G} &= -V_2s_G - V_1C_G - b_3\omega^G_{1,\theta_G} \\ &= -V_{G_1} - b_3\omega^G_2 \\ V^{B*}_{3,\theta_G} &= b_2\omega^G_{1,\theta_G} = b_2\omega^G_2 \end{aligned}\right\} \quad . \tag{58}$$

The substitution of Equations (32), (46), (20), and (49) into Equation (42) results in

$$\begin{aligned} \frac{\partial K}{\partial \theta_M} = m_M\, m\Bigg[&\left(V_{G_1} + m_3\omega^G_2 - m_2\omega^G_3 - m\omega^M_3\right)\omega^M_2 \\ &- \left(V_{G_2} - m_3\omega^G_1\right) C_\alpha\omega^M_1 + \left(V_3 + m_2\omega^G_1\right)s_\alpha\omega^M_1\Bigg] \\ &+ (M_2 - M_3)\ \omega^M_2\ \omega^M_3 \quad . \end{aligned} \tag{59}$$

From Equation (43) with Equations (22), (58), (34), and (57), the following is obtained:

$$\frac{\partial K}{\partial \theta_B} = 0 \quad . \tag{60}$$

From Equation (44) with Equations (21), (56), (33), and (55) and with $\partial K_E/\partial\theta_P = \partial K_A/\partial\theta_P = 0$, the following is obtained:

$$\frac{\partial K}{\partial\theta_P} = -m_P\ P\ \left(V_{G_1} + P_3\omega_2^G - P_2\omega_3^G + P\omega_3^P\right)\omega_2^P$$

$$- \left(V_{G_2} - P_3\omega_1^G\right)C_P\omega_1^P - \left(V_3 + P_2\omega_1^G\right)s_P\omega_1^P$$

$$+ (P_2 - P_3)\ \omega_2^P\ \omega_3^P \quad . \qquad (61)$$

With $\partial K_E/\partial\theta_G = \partial K_A/\partial\theta_G = 0$, the substitution of Equations (14), (48), (32), (47), (20), (50), (34), (58), (22), (54), (33), (56), (21), and (52) into Equation (45) results in

$$\frac{\partial K}{\partial\theta_G} = m_G\ g_2\left[\omega_2^G\left(V_3 + g_2\omega_1^G\right) + \omega_3^G\left(g_3\omega_1^G - V_{G_2}\right)\right] + (G_1 - G_2)\ \omega_1^G\ \omega_2^G$$

$$+ m_M\left\{(m_2 + mC_\alpha)\ \left[\omega_2^G\ V_3 + mC_\alpha\dot{\theta}_M + \omega_1^G(m_2 + mC_\alpha)\right.\right.$$

$$\left.\left.- \omega_3^G\left(V_{G_2} - \omega_1^G\ (m_3 - ms_\alpha)\right)\right] + \dot{\theta}_M ms_\alpha\left[V_{G_1} + (m_3 + ms_\alpha)\ \omega_2^G\right]\right\}$$

$$+ M_1\dot{\theta}_M\omega_2^G + (M_2 - M_3)\ C_\alpha s_\alpha\omega_3^G\omega_1^G + \left(M_1 - M_2C_\alpha^2 - M_3s_\alpha^2\right)\ \omega_1^G\ \omega_2^G$$

$$+ m_B\left[b_2\omega_2^G\ V_3 - b_2\omega_3^G\ V_{G_2} + b_3\omega_1^G\ \left(b_2\omega_3^G + b_3\omega_2^G\right)\right] - B_1\dot{\theta}_B\omega_2^G + (B_1 - B_2)\omega_1^G\omega_2^G$$

$$+ m_P\left\{(P_2 - PC_P)\ \left[\omega_2^G\left(V_3 - PC_P\dot{\theta}_P + \omega_1^G(P_2 - PC_P)\right)\right.\right.$$

$$\left.\left.- \omega_3^G\ \left(V_{G_2} - \omega_1^G\ (P_3 - Ps_P)\right)\right] - \dot{\theta}_P Ps_P\ \left[V_{G_1} + (P_3 - Ps_P)\omega_2^G\right]\right\}$$

$$+ P_1\dot{\theta}_P\omega_2^G + (P_3 - P_2)\ C_Ps_P\omega_3^G\omega_1^G + \left(P_1 - P_2C_P^2 - P_3s_P^2\right)\omega_1^G\omega_2^G \quad . \qquad (62)$$

It is necessary next to determine the partial derivatives of the system's kinetic energy with respect to the derivatives of the generalized coordinates. Since some of the velocity and angular velocity components are not functions of all of the generalized coordinate rates, we obtain only the following terms:

$$K,_{\dot{\theta}_M} = m_M \sum_{i=1}^{3} V_i^{M*}\ V_{i,\dot{\theta}_M}^{M*} + \sum_{i=1}^{3} M_i\ \omega_i^M\ \omega_{i,\dot{\theta}_m}^M \qquad (63)$$

$$K_{,\dot{\theta}_P} = m_P \sum_{i=1}^{3} V_i^{P*} V_{i,\dot{\theta}_P}^{P*} + \sum_{i=1}^{3} P_i \, \omega_i^P \, \omega_{i,\dot{\theta}_P}^P + K_{A,\dot{\theta}_P} + K_{E,\dot{\theta}_P} \tag{64}$$

$$K_{,\dot{\theta}_B} = \sum_{i=1}^{3} B_i \, \omega_i^B \, \omega_{i,\dot{\theta}_B}^B \tag{65}$$

$$\begin{aligned} K_{,\dot{\theta}_G} = {} & m_M \sum_{i=1}^{3} V_i^{M*} V_{i,\dot{\theta}_G}^{M*} + \sum_{i=1}^{3} M_i \, \omega_i^M \, \omega_{i,\dot{\theta}_G}^M \\ & + m_P \sum_{i=1}^{3} V_i^{P*} V_{i,\dot{\theta}_G}^{P*} + \sum_{i=1}^{3} P_i \, \omega_i^P \, \omega_{i,\dot{\theta}_G}^P \\ & + m_B \sum_{i=1}^{3} V_i^{B*} V_{i,\dot{\theta}_G}^{B*} + \sum_{i=1}^{3} B_i \, \omega_i^B \, \omega_{i,\dot{\theta}_G}^B \\ & + m_G \sum_{i=1}^{3} V_i^{G*} V_{i,\dot{\theta}_G}^{G*} + \sum_{i=1}^{3} G_i \, \omega_i^G \, \omega_{i,\dot{\theta}_G}^G \\ & + K_{A,\dot{\theta}_G} + K_{E,\dot{\theta}_G} \quad . \end{aligned} \tag{66}$$

(Again we neglect the gyroscopes' inertia terms such as $K_{A,\dot{\theta}_P}$ and $K_{E,\dot{\theta}_G}$.)

We now express Equations (63) through (66) in terms of the generalized coordinates and their derivatives. From Equation (14) we observe that

$$\left.\begin{aligned} \omega_{1,\dot{\theta}_G}^G = \omega_{2,\dot{\theta}_G}^G = 0 \\ \omega_{3,\dot{\theta}_G}^G = 1 \end{aligned}\right\} \quad . \tag{67}$$

From Equations (20) through (22) we obtain

$$\left.\begin{aligned} \omega_{1,\dot{\theta}_M}^M = 1 \\ \omega_{2,\dot{\theta}_M}^M = \omega_{3,\dot{\theta}_M}^M = 0 \end{aligned}\right\} \tag{68}$$

$$\left.\begin{aligned} \omega^{P}_{1,\dot{\theta}_P} &= 1 \\ \omega^{P}_{2,\dot{\theta}_P} &= \omega^{P}_{3,\dot{\theta}_P} = 0 \end{aligned}\right\} \tag{69}$$

$$\left.\begin{aligned} \omega^{B}_{1,\dot{\theta}_B} &= -1 \\ \omega^{B}_{2,\dot{\theta}_B} &= \omega^{B}_{3,\dot{\theta}_B} = 0 \end{aligned}\right\} \tag{70}$$

$$\left.\begin{aligned} \omega^{M}_{1,\dot{\theta}_G} &= 0 \\ \omega^{M}_{2,\dot{\theta}_G} &= -s_{\alpha} \\ \omega^{M}_{3,\dot{\theta}_G} &= C_{\alpha} \end{aligned}\right\} \tag{71}$$

$$\left.\begin{aligned} \omega^{P}_{1,\dot{\theta}_G} &= 0 \\ \omega^{P}_{2,\dot{\theta}_G} &= s_P \\ \omega^{P}_{3,\dot{\theta}_G} &= C_P \end{aligned}\right\} \tag{72}$$

$$\left.\begin{aligned} \omega^{B}_{1,\dot{\theta}_G} &= 0 \\ \omega^{B}_{2,\dot{\theta}_G} &= 0 \\ \omega^{B}_{3,\dot{\theta}_G} &= 1 \end{aligned}\right\} \tag{73}$$

Similarly, Equations (32) through (34) give

$$\left.\begin{aligned} V^{M*}_{1,\dot{\theta}_M} &= 0 \\ V^{M*}_{2,\dot{\theta}_M} &= ms_\alpha \\ V^{M*}_{3,\dot{\theta}_M} &= mC_\alpha \end{aligned}\right\} \tag{74}$$

$$\left.\begin{aligned} V^{P*}_{1,\dot{\theta}_P} &= 0 \\ V^{P*}_{2,\dot{\theta}_P} &= Ps_P \\ V^{P*}_{3,\dot{\theta}_P} &= -PC_P \end{aligned}\right\} \tag{75}$$

$$\left.\begin{aligned} V^{G*}_{1,\dot{\theta}_G} &= -g_2 \\ V^{G*}_{2,\dot{\theta}_G} &= 0 \\ V^{G*}_{3,\dot{\theta}_G} &= 0 \end{aligned}\right\} \tag{76}$$

$$\left.\begin{aligned} V^{M*}_{1,\dot{\theta}_G} &= -m_2 - mC_\alpha \\ V^{M*}_{2,\dot{\theta}_G} &= 0 \\ V^{M*}_{3,\dot{\theta}_G} &= 0 \end{aligned}\right\} \tag{77}$$

$$\left.\begin{aligned} V^{P*}_{1,\dot{\theta}_G} &= -P_2 + PC_p \\ V^{P*}_{2,\dot{\theta}_G} &= 0 \\ V^{P*}_{3,\dot{\theta}_G} &= 0 \end{aligned}\right\} \tag{78}$$

$$\left.\begin{aligned} V^{B*}_{1,\dot{\theta}_G} &= -b_2 \\ V^{B*}_{2,\dot{\theta}_G} &= 0 \\ V^{B*}_{3,\dot{\theta}_G} &= 0 \end{aligned}\right\} \tag{79}$$

The substitution of Equations (32), (74), (20), and (68) into Equation (63) results in

$$K,\dot{\theta}_M = m_M\, m\Big[C_\alpha V_3 + s_\alpha V_{G_2} + (m_2C_\alpha - m_3s_\alpha)\,\omega_1^G + m\,\omega_1^M\Big] + M_1\,\omega_1^M \quad . \tag{80}$$

From Equations (21), (33), (64), (69), and (75), we obtain

$$K,\dot{\theta}_P = m_P\, P\Big[s_P\, V_{G_2} - C_PV_3 - (P_3s_P + P_2C_P)\omega_1^G + P\,\omega_1^P\Big] + P_1\,\omega_1^P \quad . \tag{81}$$

From Equations (22), (34), (65), and (70), we obtain

$$K,\dot{\theta}_B = -B_1\,\omega_1^B \quad . \tag{82}$$

Substitution of Equations (14), (20), (21), (22), (32), (33), (34), (67), (71), (72), (73), (76), (77), (78), and (79) into Equation (66) results in

$$\begin{aligned} K,\dot{\theta}_G = {} & G_3\omega_3^G + m_Gg_2\left(g_2\omega_3^G - g_3\omega_2^G - V_{G_1}\right) + B_3\omega_3^G \\ & + \left(P_3C_P^2 + P_2s_P^2\right)\omega_3^G + (P_2 - P_3)s_PC_P\omega_2^G \\ & + \left(M_3C_\alpha^2 + M_2s_\alpha^2\right)\omega_3^G + (M_3 - M_2)s_\alpha C_\alpha\omega_2^G \\ & + V_{G_1}\,[m_P(PC_P - P_2) - m_Bb_2 - m_M(m_2 + mC_\alpha)] \\ & + m_M\,(m_2 + mC_\alpha)\left(m\omega_3^M + m_2\omega_3^G - m_3\omega_2^G\right) \\ & + m_P\,(P_2 - PC_P)\left(P\omega_3^P + P_2\omega_3^G - P_3\omega_2^G\right) \\ & + m_Bb_2\left(b_2\omega_3^G - b_3\omega_2^G\right) \end{aligned} \tag{83}$$

We now take the time derivatives of Equations (80) through (83). They are required in the generalized inertia forces given by Equation (40). These derivatives are

$$\frac{d}{dt} K,\dot{\theta}_M = \left(M_1 + m_M m^2\right)\dot{\omega}_1^M + m_M m\left[C_\alpha \dot{V}_3 + s_\alpha \dot{V}_{G_2} + (m_2 C_\alpha - m_3 s_\alpha)\dot{\omega}_1^G\right]$$

$$+ m_M m\left[s_\alpha V_3 - C_\alpha V_{G_2} + (m_2 s_\alpha + m_3 C_\alpha)\omega_1^G\right]\dot{\theta}_M \qquad (84)$$

$$\frac{d}{dt} K,\dot{\theta}_P = \left(P_1 + m_P P^2\right)\dot{\omega}_1^P - m_P P\left[C_P \dot{V}_3 - s_P \dot{V}_{G_2} + (P_2 C_P + P_3 s_P)\dot{\omega}_1^G\right]$$

$$+ m_P P\left[s_P V_3 + C_P V_{G_2} + (P_2 s_P - P_3 C_P)\omega_1^G\right]\dot{\theta}_P \qquad (85)$$

$$\frac{d}{dt} K,\dot{\theta}_B = -B_1 \dot{\omega}_1^B \qquad (86)$$

$$\frac{d}{dt} K,\dot{\theta}_G = \left(G_3 + B_3 + P_3 C_P^2 + P_2 s_P^2 + M_3 C_\alpha^2 + M_2 s_\alpha^2\right)\dot{\omega}_3^G$$

$$+ 2\,(P_2 - P_3)\, s_P C_P \omega_3^G \dot{\theta}_P + 2\,(M_3 - M_2)\, s_\alpha C_\alpha \omega_3^G \dot{\theta}_M$$

$$+ m_G g_2\left(g_2 \dot{\omega}_3^G - g_3 \dot{\omega}_2^G + \dot{V}_{G_1}\right)$$

$$+ (P_2 - P_3)\, s_P C_P \dot{\omega}_2^G + (P_2 - P_3)\left(C_P^2 - s_P^2\right)\omega_2^G \dot{\theta}_P$$

$$+ (M_3 - M_2)\, s_\alpha C_\alpha \dot{\omega}_2^G + (M_2 - M_3)\left(C_\alpha^2 - s_\alpha^2\right)\omega_2^G \dot{\theta}_M$$

$$- \dot{V}_{G_1}\,[m_P\,(P_2 - PC_P) + m_B b_2 + m_M\,(m_2 + mC_\alpha)]$$

$$- V_{G_1}\left(m_P\, P\, s_P\, \dot{\theta}_P + m_M\, m\, s_\alpha\, \dot{\theta}_M\right)$$

$$+ m_M\,(m_2 + mC_\alpha)\left(m\,\dot{\omega}_3^M + m_2 \dot{\omega}_3^G - m_3 \dot{\omega}_2^G\right)$$

$$+ m_M\, m\, s_\alpha\, \dot{\theta}_M\left(m\omega_3^M + m_2 \omega_3^G - m_3 \omega_2^G\right)$$

$$+ m_P (P_2 - PC_P) \left(P\dot{\omega}_3^P + P_2\dot{\omega}_3^G - P_3\dot{\omega}_2^G\right)$$

$$+ m_P \, P \, s_P \, \dot{\theta}_P \left(P\omega_3^P + P_2\omega_3^G - P_3\omega_2^G\right)$$

$$+ m_B b_2 \left(b_2\dot{\omega}_3^G - b_3\dot{\omega}_2^G\right) \quad . \tag{87}$$

The generalized inertia forces can now be written in equation form.

The combination of Equations (59) and (84) as in Equation (40) results in

$$F^*_{\theta_M} = \left(M_2 - M_3 - m_M m^2\right) \omega_2^M \, \omega_3^M + m_M m \left[\left(V_{G_1} + m_3\omega_2^G - m_2\omega_3^G\right) \omega_2^M\right.$$

$$\left. - \left(V_{G_2} - m_3\omega_1^G\right) C_\alpha\omega_1^M + \left(V_3 + m_2\omega_1^G\right) s_\alpha\omega_1^M\right]$$

$$- \left(M_1 + m_M m^2\right) \dot{\omega}_1^M - m_M m \left[C_\alpha\dot{V}_3 + s_\alpha\dot{V}_{G_2}\right.$$

$$\left. + (m_2C_\alpha - m_3s_\alpha) \, \dot{\omega}_1^G\right] - m_M m \left[s_\alpha V_3 - C_\alpha V_{G_2}\right.$$

$$\left. + (m_2s_\alpha + m_3C_\alpha) \, \omega_1^G\right] \dot{\theta}_M \quad . \tag{88}$$

Equations (61) and (85) are substituted into Equation (40). The result is

$$F^*_{\theta_P} = \left(P_2 - P_3 - m_P P^2\right) \omega_2^P \, \omega_3^P - m_P \, P\left[\left(V_{G_1} + P_3\omega_2^G - P_2\omega_3^G\right) \omega_2^P\right.$$

$$\left. - \left(V_{G_2} - P_3\omega_1^G\right) C_P\omega_1^P - \left(V_3 + P_2\omega_1^G\right) s_P\omega_1^P\right]$$

$$- \left(P_1 + m_P P^2\right) \dot{\omega}_1^P + m_P \, P\left[C_P\dot{V}_3 - s_P\dot{V}_{G_2} + (P_2C_P + P_3s_P) \, \dot{\omega}_1^G\right]$$

$$- m_P \, P \left[s_P V_3 + C_P V_{G_2} + (P_2s_P - P_3C_P) \, \omega_1^G\right] \dot{\theta}_P \quad . \tag{89}$$

The generalized inertia force for θ_B obtained from Equations (40), (60), and (86) is

$$F^*_{\theta_B} = B_1 \dot{\omega}_1^B \quad . \tag{90}$$

Similarly, from Equations (40), (62), and (87) we obtain

$$\begin{aligned}
F^*_{\theta_G} = \Big[& G_1 - G_2 + m_G g_2^2\, m_M (m_2 + mC_\alpha)^2 + M_1 - M_2 C_\alpha^2 - M_3 s_\alpha^2 \\
& + m_B b_3^2 + B_1 - B_2 + m_P (P_2 - PC_P)^2 + P_1 - P_2 C_P^2 - P_3 s_P^2 \Big] \omega_1^G\, \omega_2^G \\
& + [m_M (m_2 + mC_\alpha)\ (m_3 - ms_\alpha) + (M_2 - M_3)\ C_\alpha s_\alpha \\
& + m_B b_2 b_3 + m_P (P_2 - PC_P)\ (P_3 - Ps_P) + m_G g_2 g_3 \\
& + (P_3 - P_2)\ C_P s_P]\ \omega_3^G\, \omega_1^G \\
& + [m_M\, m(C_\alpha m_2 + s_\alpha m_3 + m) + M_1]\ \omega_2^G\, \dot{\theta}_M - B_1 \omega_2^G\, \dot{\theta}_B \\
& + [m_P\, P(P - C_P P_2 - s_P P_3) + P_1]\ \omega_2^G\, \dot{\theta}_P \\
& + V_3 \omega_2^G\ [m_G g_2 + m_M (m_2 + mC_\alpha) + m_B b_2 \\
& + m_P (P_2 - PC_P)] - V_{G_2} \omega_3^G\ [m_M (m_2 + mC_\alpha) + m_B b_2 + m_G g_2 \\
& + m_P (P_2 - PC_P)] + V_{G_1} \left(m_M\, m\, s_\alpha\, \dot{\theta}_M - m_P\, P\, s_P\, \dot{\theta}_P \right) \\
& - \Big[G_3 + B_3 + P_3 C_P^2 + P_2 s_P^2 + M_3 C_\alpha^2 + M_2 s_\alpha^2 \\
& + m_G g_2^2 + m_M (m_2 + mC_\alpha)\ m_2 + m_P (P_2 - PC_P)\ P_2 \\
& + m_B b_2^2 \Big] \dot{\omega}_3^G + [m_G g_2 g_3 + (P_3 - P_2)\ s_P C_P \\
& + (M_2 - M_3)\ s_\alpha C_\alpha + m_M\ (M_2 + mC_\alpha)\ m_3 \\
& + m_P (P_2 - PC_P)\ P_3 + m_B b_2 b_3]\ \dot{\omega}_2^G
\end{aligned}$$

$$
\begin{aligned}
&- \dot{V}_{G_1} \left[m_G g_2 - m_P(P_2 - PC_P) + m_B b_2 + m_M(m_2 + mC_\alpha)\right] \\
&- m_M(m_2 + mC_\alpha)\, m\, \dot{\omega}_3^M - m_P(P_2 - PC_P)\, P\, \dot{\omega}_3^P \\
&+ \Big[2(P_3 - P_2)\, s_P C_P \dot{\theta}_P + 2(M_2 - M_3)\, s_\alpha C_\alpha \dot{\theta}_M \\
&- m_M\, m\, s_\alpha\, m_2\, \dot{\theta}_M - m_P\, P\, s_P\, P_2\, \dot{\theta}_P\Big]\, \omega_3^G \\
&+ \Big[(P_3 - P_2)\left(C_P^2 - s_P^2\right)\dot{\theta}_P + (M_3 - M_2)\left(C_\alpha^2 - s_\alpha^2\right)\dot{\theta}_M \\
&+ m_M\, m\, s_\alpha\, m_3\, \dot{\theta}_M + m_P\, P\, s_P\, P_3\, \dot{\theta}_P\Big]\, \omega_2^G \\
&+ m_P\, P\, s_P\left(V_{G_1} - P\omega_3^P\right)\dot{\theta}_P + m_M\, m\, s_\alpha\left(V_{G_1} - m\omega_3^M\right)\dot{\theta}_M \quad . \qquad (91)
\end{aligned}
$$

(2) Lagrange's Form of Inertia Forces for Gyroscopes. Here, as in the last section for the gimballed mirror system, we derive the generalized inertia forces of Lagrange for the gyroscopes. For any system of rigid bodies, the Lagrange generalized forces are expressed as in Equation (40). The kinetic energy of each rigid body element of the gyro can be expressed as in Equation (41).

Since the velocities of the mass centers of R_i and C_i do not depend on β_i and $\dot{\beta}_i$, there is no need to include the translational velocity terms in the kinetic energy. Thus, we simply write the kinetic energies of the systems as

$$K_A = K_{R_1} + K_{C_1} \quad \text{(azimuth gyro)} \qquad (92)$$

$$K_E = K_{R_2} + K_{C_2} \quad \text{(elevation gyro)} \qquad (93)$$

where

$$K_{R_i} = \frac{1}{2}\left(R_{i_1}\, {\omega_1^{R_i}}^2 + R_{i_2}\, {\omega_2^{R_i}}^2 + R_{i_3}\, {\omega^{R_i}}^2\right)$$

$$K_{C_i} = \frac{1}{2}\left(C_{i_1}\, {\omega_1^{C_i}}^2 + C_{i_2}\, {\omega_2^{C_i}}^2 + C_{i_3}\, {\omega_3^{C_i}}^2\right)$$

and R_{ij}, C_{ij}, i=1,2, j=1,2,3, are principal moments of inertia of R_i and C_i for their mass centers.

We now determine the generalized inertia forces as expressed by Equation (40) for the gyroscopes. The gyroscope's Equation (40) is written as

$$F^*_{\beta_1} = \frac{\partial K_A}{\partial \beta_1} - \frac{d}{dt}\frac{\partial K_A}{\partial \dot{\beta}_1} \tag{94}$$

$$F^*_{\beta_2} = \frac{\partial K_E}{\partial \beta_2} - \frac{d}{dt}\frac{\partial K_E}{\partial \dot{\beta}_E} \quad . \tag{95}$$

The derivatives in Equations (94) and (95) are to be expressed in terms of the generalized coordinates, their rates, and the prescribed motion of the system's base mount. From Equations (92) and (93) we have

$$\frac{\partial K_A}{\partial \beta_1} = \sum_{i=1}^{3} \left(R_{1i}\, {}^{R_1}\omega_i\, {}^{R_1}\omega_{i,\beta_1} + C_{1i}\, {}^{C_1}\omega_i\, {}^{C_1}\omega_{i,\beta_1} \right) \tag{96}$$

$$\frac{\partial K_E}{\partial \beta_2} = \sum_{i=1}^{3} \left(R_{2i}\, {}^{R_2}\omega_i\, {}^{R_2}\omega_{i,\beta_2} + C_{2i}\, {}^{C_2}\omega_i\, {}^{C_2}\omega_{i,\beta_2} \right) \tag{97}$$

$$\frac{\partial K_A}{\partial \beta_1} = \sum_{i=1}^{3} \left(R_{1i}\, {}^{R_1}\omega_i\, {}^{R_1}\omega_{i,\dot{\beta}_i} + C_{1i}\, {}^{C_1}\omega_i\, {}^{C_1}\omega_{i,\dot{\beta}_1} \right) \tag{98}$$

$$\frac{\partial K_E}{\partial \beta_2} = \sum_{i=1}^{3} \left(R_{2i}\, {}^{R_2}\omega_i\, {}^{R_2}\omega_{i,\dot{\beta}_2} + C_{2i}\, {}^{C_2}\omega_i\, {}^{C_2}\omega_{i,\dot{\beta}_2} \right) \tag{99}$$

wherein from Equations (35) through (39) we find

$${}^{R_1}\omega_{1,\beta_1} = {}^{C_1}\omega_{1,\beta_1} = 0$$

$${}^{R_1}\omega_{2,\beta_1} = {}^{C_1}\omega_{2,\beta_1} = -\left(\omega^P_2\, s_{\beta_1} + \omega^P_3 c_{\beta_1} \right)$$

$${}^{R_1}\omega_{3,\beta_1} = {}^{C_1}\omega_{3,\beta_1} = \omega^P_2 c_{\beta_1} - \omega^P_3 s_{\beta_1}$$

$$\omega_{1,\beta_2}^{R_2} = \omega_{1,\beta_2}^{C_2} = \omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}$$

$$\omega_{2,\beta_2}^{R_2} = \omega_{2,\beta_2}^{C_2} = -\left(\omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}\right)$$

$$\omega_{3,\beta_2}^{R_2} = \omega_{3,\beta_2}^{C_2} = 0$$

$$\omega_{1,\dot\beta_1}^{R_1} = \omega_{1,\dot\beta_1}^{C_1} = -1$$

$$\omega_{2,\dot\beta_1}^{R_1} = \omega_{2,\dot\beta_1}^{C_1} = 0$$

$$\omega_{3,\dot\beta_1}^{R_1} = \omega_{3,\dot\beta_1}^{C_1} = 0$$

$$\omega_{1,\dot\beta_2}^{R_2} = \omega_{1,\dot\beta_2}^{C_2} = 0$$

$$\omega_{2,\dot\beta_2}^{R_2} = \omega_{2,\dot\beta_2}^{C_2} = 0$$

$$\omega_{3,\dot\beta_2}^{R_2} = \omega_{3,\dot\beta_2}^{C_2} = 1 \quad .$$

Thus, we can write Equations (96) through (99) as

$$\frac{\partial K_A}{\partial \beta_1} = -\left(R_{12}\omega_2^{R_1} + C_{12}\omega_2^{C_1}\right)\left(\omega_2^P s_{\beta_1} + \omega_3^P c_{\beta_1}\right)$$
$$+\left(R_{13}\omega_3^{R_1} + C_{13}\omega_3^{C_1}\right)\left(\omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}\right) \tag{100}$$

$$\frac{\partial K_E}{\partial \beta_2} = \left(R_{21}\omega_1^{R_2} + C_{21}\omega_1^{C_2}\right)\left(\omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}\right)$$

$$- \left(R_{22}\omega_2^{R_2} + C_{22}\omega_2^{C_2}\right)\left(\omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}\right) \tag{101}$$

$$\frac{\partial K_A}{\partial \dot{\beta}_1} = - \left(R_{11}\omega_1^{R_1} + C_{11}\omega_1^{C_1}\right) \tag{102}$$

$$\frac{\partial K_E}{\partial \dot{\beta}_2} = \left(R_{23}\omega_3^{R_2} + C_{23}\omega_3^{C_2}\right) \quad . \tag{103}$$

The time derivatives of Equations (102) and (103) are

$$\frac{d}{dt} K_{A,\dot{\beta}_1} = -(R_{11} + C_{11})\dot{\omega}_1^{C_1} \tag{104}$$

$$\frac{d}{dt} K_{E,\dot{\beta}_2} = (R_{23} + C_{23})\dot{\omega}_3^{C_2} \quad . \tag{105}$$

The substitution of Equations (100), (101), (104), and (105) into Equations (94) and (95) results in

$$F^*_{\beta_1} = - \left(R_{12}\omega_2^{R_1} + C_{12}\omega_2^{C_1}\right)\left(\omega_2^P s_{\beta_1} + \omega_3^P c_{\beta_1}\right)$$

$$+ \left(R_{13}\omega_3^{R_1} + C_{13}\omega_3^{C_1}\right)\left(\omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}\right)$$

$$+ (R_{11} + C_{11})\ \dot{\omega}_1^{C_1} \tag{106}$$

$$F^*_{\beta_2} = \left(R_{21}\omega_1^{R_2} + C_{21}\omega_1^{C_2}\right)\left(\omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}\right)$$

$$- \left(R_{22}\omega_2^{R_2} + C_{22}\omega_2^{C_2}\right)\left(\omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}\right)$$

$$- (R_{23} + C_{23})\ \dot{\omega}_3^{C_2} \tag{107}$$

where from Equations (35), (36), and (37)

$$\omega_2^{R_1} = \left(\omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}\right) - \omega$$

$$\omega_2^{C_1} = \omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}$$

$$\omega_3^{R_1} = \omega_3^{C_1} = \omega_3^P c_{\beta_1} + \omega_2^P s_{\beta_1}$$

$$\dot{\omega}_1^{C_1} = \dot{\omega}_1^P - \ddot{\beta}_1$$

$$\omega_1^{R_2} = \omega_1^{C_2} = \omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}$$

$$\omega_2^{R_2} = \left(\omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}\right) - \omega$$

$$\omega_2^{C_2} = \omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}$$

$$\dot{\omega}_3^{C_2} = \dot{\omega}_3^P + \ddot{\beta}_2$$

so that Equations (106) and (107) are finally written as

$$\begin{aligned} F^*_{\beta_1} = & -(R_{12} + C_{12})\left(\omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}\right)\left(\omega_2^P s_{\beta_1} + \omega_3^P c_{\beta_1}\right) \\ & + R_{12}\ \omega\left(\omega_3^P c_{\beta_1} + \omega_2^P s_{\beta_1}\right) \\ & + (R_{13} + C_{13})\left(\omega_3^P c_{\beta_1} + \omega_2^P s_{\beta_1}\right)\left(\omega_2^P c_{\beta_1} - \omega_3^P s_{\beta_1}\right) \\ & + (R_{11} + C_{11})\left(\dot{\omega}_1^P - \ddot{\beta}_1\right) \end{aligned} \tag{108}$$

$$\begin{aligned} F^*_{\beta_2} = & (R_{21} + C_{21})\left(\omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}\right)\left(\omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}\right) \\ & - (R_{22} + C_{22})\left(\omega_2^P c_{\beta_2} - \omega_1^P s_{\beta_2}\right)\left(\omega_1^P c_{\beta_2} + \omega_2^P s_{\beta_2}\right) \\ & + R_{22}\ \omega\left(\omega_1^P c_{\beta_2} - \omega_2^P s_{\beta_2}\right) \\ & - (R_{23} + C_{23})\left(\dot{\omega}_3^P + \ddot{\beta}_2\right) \quad . \end{aligned} \tag{109}$$

d. Active Forces

Active forces acting on the system of rigid bodies are those due to friction, gravity, deformable wire drive (massless) connections, spring tension due to electrical conductor connections between moving parts and torque motors.

The wire drive is assumed to deform according to linear elasticity. The wire drive is attached to the platform and mirror wheels in tension. That is, there is a force of F_o in a wire segment when there is no movement of P, M, and B and $\theta_P = \theta_M = \theta_B = 0$. The wire can be stretched or its tension can be reduced by relative movement of the wheels of P, M, and B. Figure 8 shows the configuration of the wheels and wire drive with critical dimensions. Free-body diagrams of the wheels showing the wire drive forces and the relative rotations of the wheels are also depicted. Consider first the wire of length S_1 between the mirror and platform wheels. The wire has a cross-sectional area, A, and a modulus of elasticity, E. The increment in force in this wire due to the relative rotation of P and M is

$$\Delta F_1 = (EA/S_1)(r_P\theta_P - r_M\theta_M). \tag{110}$$

Similarly, the increment in force in the wire between P and B is

$$\Delta F_P = (EA/S_P)(r_B\theta_B - r_P\theta_P) \tag{111}$$

and the increment in force in the wire between M and B is

$$\Delta F_M = (EA/S_M)(r_M\theta_M - r_B\theta_B) \quad . \tag{112}$$

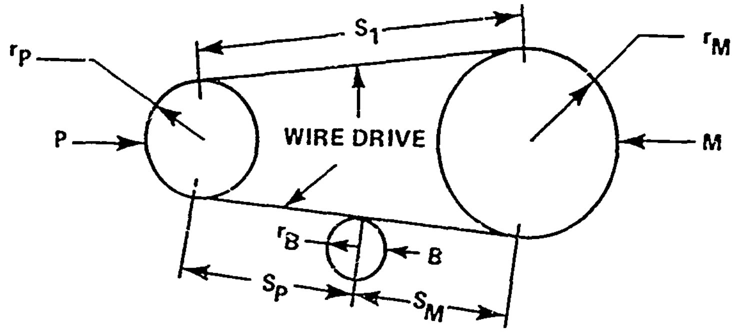
The moment of the forces due to the wire acting on M obtained from Equations (110) and (112) is

$$\begin{aligned} \underline{T}_{MW} &= r_M(F_o + \Delta F_1 - F_o - \Delta F_M)\ \underline{g}_1 \\ &= r_M\,[K_1(r_P\theta_P - r_M\theta_M) + K_M(r_B\theta_B - r_M\theta_M)]\ \underline{g}_1 \end{aligned} \tag{113}$$

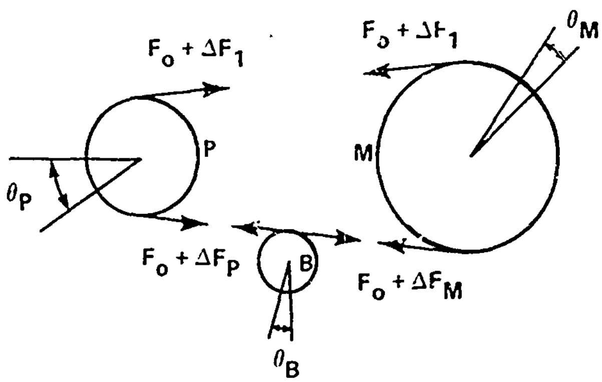
where

$K_1 = EA/S_1$

$K_M = EA/S_M$.



(a) CONFIGURATION OF THE WHEELS AND WIRE DRIVE WITH DIMENSIONS.



(b) WIRE DRIVE FORCES ON WHEELS FOR RELATIVE ROTATIONS.

Figure 8. Wheels and wire drive configuration, dimensions, and forces.

Equations (110) and (111) are used to obtain the moment of the forces due to the wire acting on P and give

$$\underline{T}_{PW} = r_P(F_o + \Delta F_P - F_o - \Delta F_1)\ \underline{g}_1$$

$$= r_P[K_P(r_B\theta_B - r_P\theta_P) + K_1(r_M\theta_M - r_P\theta_P)]\ \underline{g}_1 \qquad (114)$$

where

$K_P = EA/S_P$. Likewise, from Equations (111) and (112) we obtain for B under the assumption of no slip between B and the wire

$$\underline{T}_{BW} = r_B(F_o + \Delta F_P - F_o - \Delta F_M)\ \underline{g}_1$$

$$= r_B[K_P(r_B\theta_B - r_P\theta_P) + K_M(r_B\theta_B - r_M\theta_M)]\ \underline{g}_1 \quad . \qquad (115)$$

Coulomb friction torque on the mirror is written as

$$\underline{T}_{MC} = -C_M \operatorname{sgn} (\dot{\theta}_M) \, \underline{g}_1 \quad . \tag{116}$$

For the platform we have

$$\underline{T}_{PC} = -C_P \operatorname{sgn} (\dot{\theta}_P) \, \underline{g}_1 \tag{117}$$

and for the balancer

$$\underline{T}_{BC} = C_B \operatorname{sgn} (\dot{\theta}_B) \, \underline{g}_1 \quad . \tag{118}$$

Similarly, we denote viscous friction torques on M, P, and B by

$$\underline{T}_{MV} = -N_M \, \dot{\theta}_M \, \underline{g}_1 \tag{119}$$

$$\underline{T}_{PV} = -N_P \, \dot{\theta}_P \, \underline{g}_1 \tag{120}$$

$$\underline{T}_{BV} = N_B \, \dot{\theta}_B \, \underline{g}_1 \quad . \tag{121}$$

For the gimbal, G, the Coulomb effects are expressed as

$$\begin{aligned} \underline{T}_{GC} = & -C_G \operatorname{sgn} (\dot{\theta}_G) \, \underline{g}_3 \\ & + [C_M \operatorname{sgn} (\dot{\theta}_M) + C_P \operatorname{sgn} (\dot{\theta}_P) - C_B \operatorname{sgn} (\dot{\theta}_B)] \, \underline{g}_1 \end{aligned} \tag{122}$$

and those due to viscous effects are

$$\begin{aligned} \underline{T}_{GV} = & -N_G \operatorname{sgn} (\dot{\theta}_G) \, \underline{g}_3 \\ & + (N_M \, \dot{\theta}_M + N_P \, \dot{\theta}_P - N_B \, \dot{\theta}_B) \, \underline{g}_1 \quad . \end{aligned} \tag{123}$$

Gravity force torques are not zero since there are mass unbalances. The mirror's gravity torque in the $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$ basis is

$$\begin{aligned} \underline{T}_{MG} = m_M \, m \, g \Big\{ & [s_\alpha (s_R s_A C_E + C_R s_E) C_G + s_\alpha (s_R C_A) s_G \\ & + C_\alpha \, (C_R C_E) - C_\alpha \, (s_R s_A s_E)] \, \underline{m}_1 \\ & + [(s_R C_A) C_G - (s_R s_A C_E + C_R s_E) s_G] \Big\} \, \underline{m}_3 \quad . \end{aligned} \tag{124}$$

Likewise, the platform's gravity torque in terms of $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ is

$$\underline{T}_{PG} = m_P\, P\, g \Big\{ [C_P(s_R s_A s_E) - C_P(C_R C_E)$$

$$+ s_P(s_R s_A C_E + C_R s_E)C_G + s_P(s_R C_A)s_G]\, \underline{p}_1$$

$$+ [(s_R s_A C_E + C_R s_E)s_G - (s_R C_A)C_G] \Big\}\, \underline{p}_3 \qquad (125)$$

and for the gimbal, G, we have

$$\underline{T}_{GG} = m_G\, g \Big\{ [g_2(C_R C_E - s_R s_A s_E) - g_3(C_G(s_R s_A C_E + C_R s_E)$$

$$+ (s_R C_A)s_G)]\underline{g}_1 + g_3\, [s_G(s_R s_A C_E + C_R s_E) - (s_R C_A)C_G]\, \underline{g}_2$$

$$+ g_2\, [(s_R C_A)C_G - (s_R s_A C_E + C_R s_E)s_G]\, \underline{g}_3 \Big\} \quad . \qquad (126)$$

The electrical conductors which connect between the gimbal, G, and the platform, P, exent a spring torque on the platform and gimbal. These torques are expressed as

$$\underline{T}_{PS} = -K_S\, \theta_P\, \underline{g}_1 \qquad (127)$$

for P and

$$\underline{T}_{GS} = K_S\, \theta_P\, \underline{g}_1 \quad . \qquad (128)$$

(1) Generalized Active Forces for the Gimballed Mirror System. The generalized active forces for each element of the system is determined by

$$F_\theta \Big)_B = \underline{\omega}^{B}_{,\dot{\theta}} \cdot \underline{T}_B \qquad (129)$$

where B denotes one of the rigid body elements and $\underline{T}_B$ is the torque of the moment about the axis of rotation of all forces acting on B. From Equation (14) we find

$$\underline{\omega}^{G}_{,\dot{\theta}_M} = \underline{\omega}^{G}_{,\dot{\theta}_B} = \underline{\omega}^{G}_{,\dot{\theta}_P} = \underline{0} \qquad (130)$$

$$\underline{\omega}^{G}_{,\dot{\theta}_G} = \underline{g}_3 \quad . \qquad (131)$$

Equation (20) is observed to give

$$\underline{\omega}^{M}_{,\dot{\theta}_P} = \underline{\omega}^{M}_{,\dot{\theta}_B} = \underline{0} \tag{132}$$

$$\underline{\omega}^{M}_{,\dot{\theta}_M} = \underline{m}_1 \tag{133}$$

$$\underline{\omega}^{M}_{,\theta_G} = -s_\alpha \underline{m}_2 + C_\alpha \underline{m}_3 = \underline{g}_3 \quad . \tag{134}$$

Similarly, we obtain from Equation (21)

$$\underline{\omega}^{P}_{,\dot{\theta}_M} = \underline{\omega}^{P}_{,\dot{\theta}_B} = \underline{0} \tag{135}$$

$$\underline{\omega}^{P}_{,\dot{\theta}_P} = \underline{p}_1 \tag{136}$$

$$\underline{\omega}^{P}_{,\dot{\theta}_G} = s_P \underline{p}_2 + C_P \underline{p}_3 = \underline{g}_3 \tag{137}$$

and from Equation (22)

$$\underline{\omega}^{B}_{,\dot{\theta}_M} = \underline{\omega}^{B}_{,\dot{\theta}_P} = \underline{0} \tag{138}$$

$$\underline{\omega}^{B}_{,\dot{\theta}_B} = -\underline{g}_1 \tag{139}$$

$$\underline{\omega}^{B}_{,\dot{\theta}_G} = \underline{g}_3 \quad . \tag{140}$$

Now, from Equations (122), (123), (126), and (130) through (140), we obtain

$$\left.F_{\theta_M}\right)_G = \left.F_{\theta_P}\right)_G = \left.F_{\theta_B}\right)_G = 0 \tag{141}$$

$$\left.F_{\theta_G}\right)_G = -C_G \operatorname{sgn}\left(\dot{\theta}_G\right) - N_G \operatorname{sgn}\left(\dot{\theta}_G\right)$$

$$+ m_G g\, g_2 \left[(s_R C_A) C_G - (s_R s_A C_E + C_R s_E) s_G\right] \quad . \tag{142}$$

The contributions to the generalized forces by the forces acting on M are

$$F_{\theta_B}\Big)_M = F_{\theta_P}\Big)_M = 0 \tag{143}$$

$$F_{\theta_G}\Big)_M = m_M mg\, C_\alpha[(s_R C_A)C_G - (s_R s_A C_E + C_R s_E)s_G] \tag{144}$$

$$\begin{aligned} F_{\theta_M}\Big)_M = {} & r_M[K_1(r_P\theta_P - r_M\theta_M) + K_M(r_B\theta_B - r_M\theta_M)] \\ & - C_M \operatorname{sgn}\left(\dot{\theta}_M\right) - N_M\dot{\theta}_M \\ & + m_M mg\, [s_\alpha(s_R s_A C_E + C_R s_E)C_G \\ & + s_\alpha(s_R C_A)s_G + C_\alpha(C_R C_E) - C_\alpha(s_R s_A s_E)] \end{aligned} \tag{145}$$

as seen from Equations (113), (116), (119), (124), (132), (133), and (134). Similarly, from Equations (114), (117), (120), (125), (135), (136), and (137), we find

$$F_{\theta_B}\Big)_P = F_{\theta_M}\Big)_P = 0 \tag{146}$$

$$\begin{aligned} F_{\theta_P}\Big)_P = {} & r_P[K_P(r_B\theta_B - r_P\theta_P) + K_1(r_M\theta_M - r_P\theta_P)] \\ & - C_P \operatorname{sgn}\left(\dot{\theta}_P\right) - N_P\dot{\theta}_P \\ & + m_P Pg\, [C_P(s_R s_A s_E) - C_P(C_R C_E) \\ & + s_P\, (s_R s_A C_E + C_R s_E)C_G + s_P(s_R C_A)s_G] \end{aligned} \tag{147}$$

$$F_{\theta_G}\Big)_P = m_P Pg C_P[(s_R s_A C_E + C_R s_E)s_G - (s_R C_A)C_G] \quad . \tag{148}$$

The contributions to the generalized forces by the forces acting on B are

$$F_{\theta_M}\Big)_B = F_{\theta_P}\Big)_B = F_{\theta_G}\Big)_B = 0 \tag{149}$$

$$F_{\theta_B}\Big)_B = -r_B[K_P(r_B\theta_B - r_P\theta_P) + K_M(r_B\theta_B - r_M\theta_M)]$$
$$- C_B \operatorname{sgn}\left(\dot{\theta}_B\right) - N_B\dot{\theta}_B \quad . \tag{150}$$

(2) Generalized Active Forces for the Gyroscopes. The only forces acting on the gyroscope elements are those due to friction, gravity, and the servo drive, gimbal precession, torque motor. The friction forces are described as viscous friction. The torques of their moments for the azimuth and elevation gyros are

$$\underline{T}_{AV} = N_A\,\dot{\beta}_1\,\underline{C}_{11} \tag{151}$$

$$\underline{T}_{EV} = -N_E\,\dot{\beta}_2\,\underline{C}_{23} \quad . \tag{152}$$

The gravity forces do not affect the system's motions since the torques about the mass centers are zero. The gimbal precession torque motors of the gyroscopes are active only when the platform is servo driven in a slewing or tracking mode. When the precession torque motors are active, they provide torques proportional to an external rate signal generated by, for example, a manual tracking stick or contrast TV tracker. These torques are not functions of the generalized coordinates and can be added in later.

To determine the generalized active forces for the gyroscope systems, we refer to Equation (129):

$$F_{\beta_1}\Big)_{C_1} = {}^{C_1}\underline{\omega}_{,\beta_1} \cdot \underline{T}_{AV} = -\underline{C}_{11} \cdot N_A\dot{\beta}_1\underline{C}_{11} = -N_A\dot{\beta}_1 \tag{153}$$

$$F_{\beta_2}\Big)_{C_1} = F_{\beta_1}\Big)_{C_2} = F_{\beta_2}\Big)_{R_1} = F_{\beta_1}\Big)_{R_2} = F_{\beta_2}\Big)_{R_2}$$
$$= F_{\beta_1}\Big)_{R_1} = 0 \tag{154}$$

$$F_{\beta_2}\Big)_{C_2} = {}^{C_2}\underline{\omega}_{,\dot{\beta}_2} \cdot \underline{T}_{EV} = \underline{C}_{23} \cdot -N_E\dot{\beta}_2\underline{C}_{23} = -N_E\dot{\beta}_2 \tag{155}$$

wherein $\underline{\omega}^{C_1}$, $\dot{\beta}_1$ etc. are determined from Equations (235), (236), and (237) and $\underline{T}_{AV}$, $\underline{T}_{EV}$ are given in Equations (151) and (152).

e. Complete Nonlinear Equations of Motion

The equations of motion of the gimballed mirror system and the gyroscope systems are now obtained by simply setting to zero the sum of the generalized active forces for each generalized coordinate and the corresponding generalized inertia forces; i.e., we form

$$F_\theta + F_\theta^* = F_\theta + \frac{\partial K}{\partial \theta} - \frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} = 0 \qquad (156)$$

for θ replaced with each of θ_M, θ_P, θ_B, θ_G, β_1, and β_2. From Equations (88), (141), (145), (146), and (149) and the fact that

$$F_{\theta_M} = F_{\theta_M}\Big)_M + F_{\theta_M}\Big)_P + F_{\theta_M}\Big)_B + F_{\theta_M}\Big)_G$$

we obtain for θ_M

$$\begin{aligned}
& r_M[K_1(r_P\theta_P - r_M\theta_M) + K_M(r_B\theta_B - r_M\theta_M)] \\
& - C_M \operatorname{sgn}\left(\dot{\theta}_M\right) - N_M\dot{\theta}_M + \mu_M mg[s_\alpha(s_R s_A C_E + C_R s_E)C_G \\
& + s_\alpha(s_R C_A)s_G + C_\alpha(C_R C_E) - C_\alpha(s_R s_A s_E)] \\
& - \left(M_1 + m_M m^2\right)\dot{\omega}_1^M + \left(M_2 - M_3 - m_M m^2\right)\omega_2^M\omega_3^M \\
& + m_M m\left[\left(V_{G_1} + m_3\omega_2^G - m_2\omega_3^G\right)\omega_2^M - \left(V_{G_2} - m_3\omega_1^G\right)C_\alpha\omega_1^M\right. \\
& \left. + \left(V_3 + m_2\omega_1^G\right)s_\alpha\omega_1^M\right] - m_M m\left[C_\alpha\dot{V}_3 + s_\alpha\dot{V}_{G_2}\right. \\
& \left. + (m_2C_\alpha - m_3s_\alpha)\dot{\omega}_1^G\right] - m_M m\left[s_\alpha V_3 - C_\alpha V_{G_2}\right. \\
& \left. + (m_2s_\alpha + m_3C_\alpha)\omega_1^G\right]\dot{\theta}_M = 0 \quad .
\end{aligned} \qquad (157)$$

Similarly, from Equations (89), (141), (143), (147), and (149) we have for θ_P

$$
\begin{aligned}
& r_P[K_P(r_B\theta_B - r_P\theta_P) + K_1(r_M\theta_M - r_P\theta_P)] - C_P \operatorname{sgn}\left(\dot{\theta}_P\right) \\
& - N_P\dot{\theta}_P + m_P\ P\ g[C_P(s_R s_A s_E) - C_P(C_R C_E) \\
& + s_P(s_R s_A C_E + C_R s_E)C_G + s_P(s_R C_A)s_G] \\
& - \left(P_1 + m_P\ P^2\right)\dot{\omega}_1^P + \left(P_2 - P_3 - m_P\ P^2\right)\omega_2^P\omega_3^P \\
& - m_P\ P\left[\left(V_{G_1} + P_3\omega_2^G - P_2\omega_3^G\right)\omega_2^P - \left(V_{G_2} - P_3\omega_1^G\right)C_P\omega_1^P\right. \\
& \left. - \left(V_3 + P_2\omega_1^G\right)s_P\omega_1^P\right] + m_P\ P\left[C_P\dot{V}_3 - s_P\dot{V}_{G_2}\right. \\
& \left. + (P_2\ C_P + P_3\ s_P)\dot{\omega}_1^G\right] - m_P\ P\left[s_P V_3 + C_P V_{G_2}\right. \\
& \left. + (P_2\ s_P - P_3\ C_P)\omega_1^G\right]\dot{\theta}_P = 0 \quad .
\end{aligned}
\tag{158}
$$

Likewise, from Equations (90), (141), (143), (146), (150), and (156), we find for θ_B

$$
\begin{aligned}
& - r_B[K_P(r_B\theta_B - r_P\theta_P) + K_M(r_B\theta_B - r_M\theta_M)] \\
& - C_B \operatorname{sgn}\left(\dot{\theta}_B\right) - N_B\dot{\theta}_B + B_1\ \dot{\omega}_1^B = 0 \quad .
\end{aligned}
\tag{159}
$$

For θ_G we determine from Equations (91), (142), (144), (148), (149), and (156) that

$$
\begin{aligned}
& - C_G \operatorname{sgn}\left(\dot{\theta}_G\right) - N_G\dot{\theta}_G + m_G\ g\ g_2\ [(s_R C_A)C_G - (s_R s_A C_E + C_R s_E)\ s_G] \\
& + m_M\ m\ g\ C_\alpha[(s_R C_A)C_G - (s_R s_A C_E + C_R s_E)\ s_G] \\
& + m_P\ P\ g\ C_P[(s_R s_A C_E + C_R s_E)\ s_G - (s_R C_A)\ C_G] \\
& - \left[G_3 + B_3 + P_3\ C_P^2 + P_2\ s_P^2 + M_3\ C_\alpha^2 + M_2\ s_\alpha^2\right.
\end{aligned}
$$

$$+ m_G g_2^2 + m_M(m_2 + mC_\alpha) m_2 + m_P(P_2 - P C_P) P_2$$

$$+ m_B b_2^2\Big] \dot{\omega}_3^G + \Big[G_1 - G_2 + m_G g_2^2 + m_M(m_2 + mC_\alpha)^2$$

$$+ M_1 - M_2 C_\alpha^2 - M_3 s_\alpha^2 + m_B b_3^2 + B_1 - B_2 + m_P (P_2 - P C_P)^2$$

$$+ P_1 - P_2 C_P^2 - P_3 s_P^2\Big] \omega_1^G \omega_2^G + [\text{other terms in Equation (91)}]. \quad (160)$$

The equation for β_1 is obtained from Equations (106), (153), and (154) and is

$$- N_A\dot{\beta}_1 + (R_{11} + C_{11})\dot{\omega}_1^{C_1} - \left(R_{12}\omega_2^{R_1} + C_{12}\omega_2^{C_1}\right)\left(\omega_2^P s_{\beta_1} + \omega_3^P C_{\beta_1}\right)$$

$$+ \left(R_{13}\omega_3^{R_1} + C_{13}\omega_3^{C_1}\right)\left(\omega_2^P C_{\beta_1} - \omega_3^P s_{\beta_1}\right) = 0. \quad (161)$$

From Equations (107), (154), and (155) we obtain for β_2 that

$$- N_E\dot{\beta}_2 - (R_{23} + C_{23}) \dot{\omega}_3^{C_2} + \left(R_{21}\omega_1^{R_2} + C_{21}\omega_1^{C_2}\right)\left(\omega_2^P C_{\beta_2} - \omega_1^P s_{\beta_2}\right)$$

$$- \left(R_{22}\omega_2^{R_2} + C_{22}\omega_2^{C_2}\right)\left(\omega_1^P C_{\beta_2} + \omega_2^P s_{\beta_2}\right) = 0 \quad . \quad (162)$$

3. Analysis of Equations of Motion for Special Cases

The equations of motion given in Equations (157) through (162) are nonlinear and quite complicated with various orders of cross-coupling terms. Besides the usual single rigid body cross-coupling terms, the equations contain significant cross-coupling terms due to mass unbalance, base motion, and band drive deformation. We shall now put the equations of motion into a standard form of a system of first-order differential equations suitable for numerical solution by a forward integration scheme such as a fourth-order Runge-Kutta.

In the equations of motion, the variables V_1, V_2, V_3, Φ_A, Φ_E, Φ_R, $\dot{\Phi}_A$, $\dot{\Phi}_E$, and $\dot{\Phi}_R$ and their derivatives are known functions of time from the dynamics of the base or from measurement data. We must specify initial conditions in solving the system of differential equations; i.e., we must specify that at time t = 0, the values θ_G, $\dot{\theta}_G$, θ_M, $\dot{\theta}_M$, θ_P, $\dot{\theta}_P$, θ_B, $\dot{\theta}_B$, β_1, $\dot{\beta}_1$, β_2, and $\dot{\beta}_2$. Then from Equation (14) we can calculate the values of $\omega_1^G(0)$, $\omega_2^G(0)$, $\omega_3^G(0)$, and from Equations (20), (21), and

(22) we can calculate $\omega_1^M(0)$, $\omega_2^M(0)$, $\omega_3^M(0)$, $\omega_1^P(0)$, $\omega_2^P(0)$, $\omega_3^P(0)$, $\omega_1^B(0)$. Following these calculations, we determine from Equations (36), (37), (38), and (39) the initial values $\omega_1^{C_1}(0)$, $\omega_2^{C_1}(0)$, $\omega_3^{C_1}(0)$, $\omega_1^{C_2}(0)$, $\omega_2^{C_2}(0)$, $\omega_3^{C_2}(0)$ and from Equation (35) the initial values $\omega_2^{R_1}(0)$, $\omega_3^{R_1}(0)$, $\omega_1^{R_2}(0)$, and $\omega_2^{R_2}(0)$. With these initial conditions we solve the gyro Equations (161) and (162) and the following set of 33 first-order differential and algebraic equations:

$$V_{G_1} = c_G V_1 + s_G V_2 \tag{163}$$

$$V_{G_2} = c_G V_2 - s_G V_1 \tag{164}$$

$$\omega_{S_1} = \dot{\phi}_E + \dot{\phi}_R s_A \tag{165}$$

$$\omega_{S_2} = \dot{\phi}_R c_A c_E + \dot{\phi}_A s_E \tag{166}$$

$$\omega_{S_3} = \dot{\phi}_A c_E - \dot{\phi}_R s_E c_A \tag{167}$$

$$\omega_1^G = \omega_{S_1} c_G + \omega_{S_2} s_G \tag{168}$$

$$\dot{\omega}_1^G = \dot{\theta}_G \left(\omega_{S_2} c_G - \omega_{S_1} s_G\right) + \dot{\omega}_{S_1} c_G + \dot{\omega}_{S_2} s_G \tag{169}$$

$$\omega_2^G = \omega_{S_2} c_G - \omega_{S_1} s_G \tag{170}$$

$$\dot{\omega}_2^G = -\dot{\theta}_G \left(\omega_{S_1} c_G + \omega_{S_2} s_G\right) + \dot{\omega}_{S_2} c_G - \dot{\omega}_{S_1} s_G \tag{171}$$

$$\omega_2^P = c_P \omega_2^G + s_P \omega_3^G \tag{172}$$

$$\omega_3^P = c_P \omega_3^G - s_P \omega_2^G \tag{173}$$

$$\omega_2^{C_i} = c_{\beta_1} \omega_2^P - s_{\beta_1} \omega_3^P \tag{174}$$

$$\omega_3^{C_1} = s_{\beta_1}\omega_2^P + c_{\beta_1}\omega_3^P \tag{175}$$

$$\omega_2^{R_1} = \omega_2^{C_1} - \omega \tag{176}$$

$$\omega_3^{R_1} = \omega_3^{C_1} \tag{177}$$

$$\omega_2^M = c_\alpha\omega_2^G - s_\alpha\omega_3^G \tag{178}$$

$$\omega_3^M = s_\alpha\omega_2^G + c_\alpha\omega_3^G \tag{179}$$

$$\omega_1^{C_2} = c_{\beta_2}\omega_1^P + s_{\beta_2}\omega_2^P \tag{180}$$

$$\omega_2^{C_2} = c_{\beta_2}\omega_2^P - s_{\beta_2}\omega_1^P \tag{181}$$

$$\omega_1^{R_2} = \omega_1^{C_2} \tag{182}$$

$$\omega_2^{R_2} = \omega_2^{C_2} - \omega \tag{183}$$

$$\dot{\theta}_M = \omega_1^M - \omega_1^G \tag{184}$$

$$\dot{\theta}_G = \omega_3^G - \omega_{S_3} \tag{185}$$

$$\dot{\theta}_P = \omega_1^P - \omega_1^G \tag{186}$$

$$\dot{\theta}_B = \omega_1^G - \omega_1^B \tag{187}$$

$$\dot{\beta}_1 = \omega_1^P - \omega_1^{C_1} \tag{188}$$

$$\dot{\beta}_2 = \omega_3^{C_2} - \omega_3^P \tag{189}$$

$$\dot{\omega}_1^M = \left(M_1 + m_M m^2\right)^{-1} \left\{\left(M_2 - M_3 - m_M m^2\right) \omega_2^M \omega_3^M\right.$$

$$- C_M \operatorname{sgn}\left(\dot{\theta}_M\right) - N_M \dot{\theta}_M$$

$$+ r_M[K_1(r_P\theta_P - r_M\theta_M) + K_M(r_B\theta_B - r_M\theta_M)]$$

$$+ m_M\, m\, g\, [s_\alpha(s_R s_A C_E + C_R s_E)C_G + s_\alpha(s_P C_A)s_G$$

$$+ C_\alpha(C_R C_E) - C_\alpha(s_R s_A s_E)]$$

$$+ m_M\, m\left[\left(V_{G_1} + m_3\omega_2^G - m_2\omega_3^G\right)\omega_2^M\right.$$

$$\left. - \left(V_{G_2} - m_3\omega_1^G\right) C_\alpha\, \omega_1^M + \left(V_3 + m_2\omega_1^G\right) s_\alpha\, \omega_1^M\right]$$

$$- m_M\, m\left[C_\alpha \dot{V}_3 + s_\alpha \dot{V}_{G_2} + (m_2 C_\alpha - m_3 s_\alpha)\, \dot{\omega}_1^G\right]$$

$$\left. - m_M\, m\left[s_\alpha V_3 - C_\alpha V_{G_2} + (m_2 s_\alpha + m_3 C_\alpha)\, \omega_1^G\right] \dot{\theta}_M\right\} \qquad (190)$$

$$\dot{\omega}_1^P = \left(P_1 + m_P P^2\right)^{-1} \left\{\left(P_2 - P_3 - m_P P^2\right) \omega_2^P \omega_3^P\right.$$

$$- C_P \operatorname{sgn}\left(\dot{\theta}_P\right) - N_P\dot{\theta}_P + r_P[K_P(r_B\theta_B - r_P\theta_P)$$

$$+ K_1(r_M\theta_M - r_P\theta_P)] + m_P\, P\, g[C_P(s_R s_A s_E)$$

$$- C_P(C_R C_E) + s_P(s_R s_A C_E + C_R s_E)C_G$$

$$+ s_P(s_R C_A)s_G] - m_P\, P\left[\left(V_{G_1} + P_3\omega_2^G - P_2\omega_3^G\right)\omega_2^P\right.$$

$$\left. - \left(V_{G_2} - P_3\omega_1^G\right) C_P\omega_1^P - \left(V_3 + P_2\omega_1^G\right) s_P\omega_1^P\right]$$

$$+ m_P\, P\left[C_P\dot{V}_3 - s_P\dot{V}_{G_2} + (P_2 C_P + P_3 s_P)\, \dot{\omega}_1^G\right]$$

$$\left. - m_P\, P\left[s_P V_3 + C_P V_{G_2} + (P_2 s_P - P_3 C_P)\, \omega_1^G\right] \dot{\theta}_P\right\} \qquad (191)$$

$$\dot{\omega}_1^B = B_1^{-1} \left\{ C_B \operatorname{sgn} \left(\dot{\theta}_B \right) + N_B \dot{\theta}_B \right.$$

$$\left. + r_B [K_P (r_B \theta_B - r_P \theta_P) + K_M (r_B \theta_B - r_M \theta_M)] \right\} \tag{192}$$

$$\dot{\omega}_3^G = \left[G_3 + B_3 + P_3 C_P^2 + P_2 s_P^2 + M_3 C_\alpha^2 + M_2 s_\alpha^2 + m_G g_2^2 \right.$$

$$+ m_M (m_2 + m C_\alpha) m_2 + m_P (P_2 - P\, C_P) P_2$$

$$\left. + m_B b_2^2 \right]^{-1} \left\{ \left[G_1 - G_2 + m_G g_2^2 + m_M (m_2 + m C_\alpha)^2 \right. \right.$$

$$+ M_1 - M_2 C_\alpha^2 - M_3 s_\alpha^2 + m_B b_3^2 + B_1 - B_2$$

$$\left. + m_P (P_2 - P\, C_P)^2 + P_1 - P_2 C_P^2 - P_3 s_P^2 \right] \omega_1^G \omega_2^G$$

$$- C_G \operatorname{sgn} \left(\dot{\theta}_G \right) - N_G \dot{\theta}_G$$

$$+ m_G\, g\, g_2 [s_R C_A C_G - s_G (s_R s_A C_E + C_R s_E)]$$

$$+ m_M\, g\, m\, C_\alpha [s_R C_A C_G - s_G (s_R s_A C_E + C_R s_E)]$$

$$- m_P\, g\, P\, C_P [s_R C_A C_G - s_G (s_R s_A C_E + C_R s_E)]$$

$$\left. + [\text{other similar terms from Equation (91)}] \right\} \tag{193}$$

$$\dot{\omega}_1^{C_1} = (R_{11} + C_{11})^{-1} \left[\left(R_{12} \omega_2^{R_1} + C_{12} \omega_2^{C_1} \right) \left(\omega_2^P s_{\beta_1} + \omega_3^P C_{\beta_1} \right) \right.$$

$$\left. + \left(\omega_3^P s_{\beta_1} - \omega_2^P C_{\beta_1} \right) \left(R_{13} \omega_3^{R_1} + C_{13} \omega_3^{C_1} \right) + N_A \dot{\beta}_1 \right] \tag{194}$$

$$\dot{\omega}_3^{C_2} = (R_{23} + C_{23})^{-1} \left[\left(R_{21} \omega_1^{R_2} + C_{21} \omega_1^{C_2} \right) \left(\omega_2^P C_{\beta_2} - \omega_1^P s_{\beta_2} \right) \right.$$

$$\left. - \left(R_{22} \omega_2^{R_2} + C_{22} \omega_2^{C_2} \right) \left(\omega_1^P C_{\beta_2} + \omega_2^P s_{\beta_2} \right) - N_E \dot{\beta}_2 \right] \tag{195}$$

These equations become quite simple for the three special cases of:

a) Small vehicle motion: $\phi_A \approx \phi_E \approx \phi_R \approx V_1 \approx V_2 \approx V_3 \approx 0$ with their derivatives

b) Small mass unbalance: $m \approx P \approx g_2 \approx 0$

c) Small deviations from system looking forward:

$$\theta_M \approx \dot{\theta}_M \approx \theta_G \approx \dot{\theta}_G \approx \theta_B \approx \dot{\theta}_B \approx \theta_P \approx \dot{\theta}_P \approx \phi_A \approx \phi_E \approx \phi_R$$

$$\approx \beta_1 \approx \dot{\beta}_1 \approx \beta_2 \approx \dot{\beta}_2 \approx 0$$

d) Small motions and mass unbalances.

In Case a) the equations become:

$$V_{G_1} = V_{G_2} = V_3 = 0$$

$$\omega_{S_1} = \omega_{S_2} = \omega_{S_3} = 0$$

$$\omega_1^G = \omega_2^G = 0$$

$$\dot{\omega}_1^G = \dot{\omega}_2^G = 0$$

$$\omega_2^M = -s_\alpha \omega_3^G$$

$$\omega_3^M = C_\alpha \omega_3^G$$

$$\omega_2^P = s_P \omega_3^G$$

$$\omega_3^P = C_P \omega_3^G$$

$$\omega_2^{C_1} = C_{\beta_1} \omega_2^P - s_{\beta_1} \omega_3^P$$

$$\omega_3^{C_1} = s_{\beta_1} \omega_2^P + C_{\beta_1} \omega_3^P$$

$$\omega_2^{R_1} = \omega_2^{C_1} - \omega$$

$$\omega_3^{R_1} = \omega_3^{C_1}$$

$$\omega_1^{C_2} = c_{\beta_2}\omega_1^P + s_{\beta_2}\omega_2^P$$

$$\omega_2^{C_2} = c_{\beta_2}\omega_2^P - s_{\beta_2}\omega_1^P$$

$$\omega_1^{R_2} = \omega_1^{C_2}$$

$$\omega_2^{R_2} = \omega_2^{C_2} - \omega$$

$$\dot{\theta}_M = \omega_1^M$$

$$\dot{\theta}_G = \omega_3^G - \omega_{S_3}$$

$$\dot{\theta}_P = \omega_1^P$$

$$\dot{\theta}_B = -\omega_1^B$$

$$\dot{\beta}_1 = \omega_1^P - \omega_1^{C_1}$$

$$\dot{\beta}_2 = \omega_3^{C_2} - \omega_3^P$$

$$\dot{\omega}_1^M = \left(M_1 + m_M m^2\right)^{-1} \Big\{\left(M_2 - M_3 - m_M m^2\right) \omega_2^M\, \omega_3^M$$

$$- C_M \operatorname{sgn} \dot{\theta}_M - N_M \dot{\theta}_M$$

$$+ r_M[K_1(r_P\theta_P - r_M\theta_M) + K_M(r_B\theta_B - r_M\theta_M)]$$

$$+ m_M m\, g\, C_\alpha - m_M\, m\, m_2\, \omega_3^G\, \omega_2^M \Big\}$$

$$\dot{\omega}_1^P = \left(P_1 + m_P P^2\right)^{-1} \left\{\left(P_2 - P_3 - m_P P^2\right) \omega_2^P \, \omega_3^P\right.$$

$$- C_P \operatorname{sgn}\left(\dot{\theta}_P\right) - N_P \dot{\theta}_P$$

$$+ r_P[K_P(r_B\theta_B - r_P\theta_P) + K_1(r_M\theta_M - r_P\theta_P)]$$

$$\left. - m_P \, P \, g \, C_P + m_P \, P \, P_2 \, \omega_3^G \, \omega_2^P\right\}$$

$$\dot{\omega}_1^B = B_1^{-1} \left\{C_B \operatorname{sgn}\left(\dot{\theta}_B\right) + N_B \dot{\theta}_B\right.$$

$$\left. + r_B[K_P(r_B\theta_B - r_P\theta_P) + K_M(r_B\theta_B - r_M\theta_M)]\right\}$$

$$\dot{\omega}_3^G = \left\{G_3 + B_3 + P_3 C_P^2 + P_2 s_P^2 + M_3 C_\alpha^2 + M_2 s_\alpha^2 + m_G g_2^2\right.$$

$$+ m_M(m_2 + mC_\alpha) \, m_2 + m_P(P_2 - P \, C_P) \, P_2$$

$$\left. + m_B b_2^2\right\}^{-1} \left\{-C_G \operatorname{sgn}\left(\dot{\theta}_G\right) - N_G \dot{\theta}_G\right.$$

$$- m_M(m_2 + mC_\alpha) m \, \dot{\omega}_3^M - m_P(P_2 - P \, C_P) P \, \dot{\omega}_3^P$$

$$+ \; 2(P_3 - P_2) s_P C_P \dot{\theta}_P + 2(M_2 - M_3) s_\alpha C_\alpha \dot{\theta}_M + (P_3 - P_2)\left(C_p^2 - s_p^2\right) \dot{\theta}_P$$

$$- m_M \, m \, m_2 \, s_\alpha \dot{\theta}_M - m_P \, P \, P_2 \, s_P \dot{\theta}_P \; \omega_3^G + (M_3 - M_2)\left(C_\alpha^2 - s_\alpha^2\right) \dot{\theta}_M$$

$$- m_P \, P^2 \, s_P \, \omega_3^P \, \dot{\theta}_P - m_M \, m^2 \, s_\alpha \, \omega_3^M \, \dot{\theta}_M$$

Equations (194) and (195) remain the same in this case.

In Case b), Equations (163) through (189), (192), (194), and (195) remain the same. The other equations become

$$\dot{\omega}_1^M = M_1^{-1} \{(M_2 - M_3) \, \omega_2^M \, \omega_3^M - C_M \operatorname{sgn} \dot{\theta}_M$$

$$- N_M \dot{\theta}_M + r_M[K_1(r_P\theta_P - r_M\theta_M)$$

$$+ K_M(r_B\theta_B - r_M\theta_M)]\}$$

$$
\begin{aligned}
\dot{\omega}_1^P = P_1^{-1} \Big\{ & (P_2 - P_3)\, \omega_2^P \omega_3^P - C_P \operatorname{sgn} \dot{\theta}_P \\
& - N_P \dot{\theta}_P + r_P [K_P (r_B \theta_B - r_P \theta_P) \\
& + K_1 (r_M \theta_M - r_P \theta_P)] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\omega}_3^G = \Big[& G_3 + B_3 + P_3 c_P^2 + P_2 s_P^2 + M_3 c_\alpha^2 + M_2 s_\alpha^2 \\
& + m_M m_2^2 + m_P P_2^2 + m_B b_2^2 \Big]^{-1} \Big[G_1 - G_2 \\
& + M_1 - M_2 c_\alpha^2 - M_3 s_\alpha^2 + m_B b_3^2 \\
& + B_1 - B_2 + m_P P_2^2 + P_1 - P_2 c_P^2 - P_3 s_P^2 \Big] \omega_1^G \omega_2^G \\
& - C_G \operatorname{sgn} \dot{\theta}_G - N_G \dot{\theta}_G \\
& + [m_M m_2 m_3 + (M_2 - M_3)\, s_\alpha c_\alpha + m_B b_2 b_3 \\
& + m_P P_2 P_3 + (P_3 - P_2) c_P s_P]\, \omega_3^G \omega_1^G + M_1 \omega_2^G \theta_M \\
& - B_1 \omega_2^G \dot{\theta}_B + V_3 \omega_2^G (m_M m_2 + m_B b_2 + m_P P_2) + P_1 \omega_2^G \theta_P \\
& - V_{G_2} \omega_3^G (m_M m_2 + m_B b_2 + m_P P_2) \\
& + [(P_3 - P_2) s_P c_P + (M_2 - M_3) s_\alpha c_\alpha + m_M m_2 m_3 \\
& + m_P P_2 P_3 + m_B b_2 b_3]\, \dot{\omega}_2^G + (m_P P_2 - m_B b_2 - m_M m_2)\, \dot{V}_{G_1} \\
& + 2[(P_3 - P_2) s_P c_P \dot{\theta}_P + (M_2 - M_3) s_\alpha c_\alpha \dot{\theta}_M]\, \omega_3^G \\
& + [(P_3 - P_2)\left(c_P^2 - s_P^2\right)\dot{\theta}_P + (M_3 - M_2)\,(c_\alpha^2 - s_\alpha^2)\, \dot{\theta}_M]\, \omega_2^G
\end{aligned}
$$

In Case c), we obtain the linearized differential equations which have only linear terms in the dependent variables except for essential nonlinearities like Coulomb friction. Also added are torque terms which would result from a torque motor driving the platform (T_{m_p}), gimbal (T_{m_G}), azimuth gyro (T_{m_A}), and elevation gyro (T_{m_E}). In this case we obtain

$$V_{G_1} = V_1$$

$$V_{G_2} = V_2$$

$$\omega_{S_1} = \dot{\phi}_E$$

$$\omega_{S_2} = \dot{\phi}_R$$

$$\omega_{S_3} = \dot{\phi}_A$$

$$\omega_1^G = \omega_{S_1}$$

$$\dot{\omega}_1^G = \dot{\omega}_{S_1}$$

$$\omega_2^G = \omega_{S_2}$$

$$\dot{\omega}_2^G = \dot{\omega}_{S_2}$$

$$\omega_2^P = \omega_2^G$$

$$\omega_3^P = \omega_3^G$$

$$\omega_2^M = C_\alpha \omega_2^G - S_\alpha \omega_3^G$$

$$\omega_3^M = S_\alpha \omega_2^G + C_\alpha \omega_3^G$$

$$\omega_2^{C_1} = \omega_2^P$$

$$\omega_3^{C_1} = \omega_3^P$$

$$\omega_2^{R_1} = \omega_2^{C_1} - \omega$$

$$\omega_3^{R_1} = \omega_3^{C_1}$$

$$\omega_1^{C_2} = \omega_1^{P}$$

$$\omega_2^{C_2} = \omega_2^{P}$$

$$\omega_1^{R_2} = \omega_1^{C_2}$$

$$\omega_2^{R_2} = \omega_2^{C_2} - \omega$$

$$\dot{\theta}_M = \omega_1^M - \omega_1^G$$

$$\dot{\theta}_G = \omega_3^G - \omega_{S_3}$$

$$\dot{\theta}_P = \omega_1^P - \omega_1^G$$

$$\dot{\theta}_B = \omega_1^G - \omega_1^B$$

$$\dot{\beta}_1 = \omega_1^P - \omega_1^{C_1}$$

$$\dot{\beta}_2 = \omega_3^{C_2} - \omega_3^P$$

$$\dot{\omega}_1^M = \left(M_1 + m_M m^2\right)^{-1} \Big\{-C_M \operatorname{sgn} \dot{\theta}_M - N_M \dot{\theta}_M + r_M[K_1(r_P\theta_P - r_M\theta_M) + K_M(r_B\theta_B - r_M\theta_M)] + m_M\, m\, g\, C_\alpha - m_M\, m\left(C_\alpha \dot{V}_3 + (m_2 C_\alpha - m_3 s_\alpha)\, \dot{\omega}_1^G\right)\Big\}$$

$$\dot{\omega}_1^P = \left(P_1 + m_P P^2\right)^{-1} \Big\{-C_P \operatorname{sgn} \dot{\theta}_P - N_P \dot{\theta}_P + r_P[K_P(r_B\theta_B - r_P\theta_P) + K_1(r_M\theta_M - r_P\theta_P)] - m_P\, g\, P + m_P\, P\left(\dot{V}_3 + P_2 \dot{\omega}_1^G\right) + T_{m_P}\Big\}$$

$$\dot{\omega}_1^B = B_1^{-1}\Big\{C_B \operatorname{sgn} \dot{\theta}_B + N_B\dot{\theta}_B$$

$$+ r_B[K_P(r_B\theta_B - r_P\theta_P) + K_M(r_B\theta_B - r_M\theta_M)]\Big\}$$

$$\dot{\omega}_3^G = G_{33}^{-1}\Big\{-C_G \operatorname{sgn} \dot{\theta}_G - N_G\dot{\theta}_G + \Big[m_G g_2 g_3$$

$$+ (M_2 - M_3)C_\alpha^2 + m_M m_3(m_2 + mC_\alpha)$$

$$+ m_P P_3(P_2 - P) + m_B b_2 b_3\Big]\dot{\omega}_2^G - [m_G g_2$$

$$- m_P(P_2 - P) + m_B b_2 + m_M(m_2 + mC_\alpha)]\dot{V}_{G_1}$$

$$- m_M(m_2 + mC_\alpha)m\,\dot{\omega}_3^M - m_P(P_2 - P)P\,\dot{\omega}_3^P + T_{m_G}\Big\}$$

$$\dot{\omega}_1^{C_1} = (R_{11} + C_{11})^{-1}\left[N_A\dot{\beta}_1 + \omega_3^P\left(R_{12}\omega_2^{R_1} + C_{12}\omega_2^{C_1}\right) - T_{m_A}\right]$$

$$\dot{\omega}_3^{C_2} = -(R_{23} + C_{23})^{-1}\left[N_E\dot{\beta}_2 + \omega_1^P\left(R_{22}\omega_2^{R_2} + C_{22}\omega_2^{C_2}\right) - T_{m_E}\right] .$$

If we assume that the mass unbalance of M and P are negligible and all motions are small, we have Case d). In this case the equations of motion are:

$$M_1\left(\ddot{\theta}_M + \ddot{\phi}_E\right) + N_M\dot{\theta}_M + C_M \operatorname{sgn} \dot{\theta}_M + r_M^2(K_1 + K_M)\theta_M$$

$$- r_M r_P K_1\theta_P - r_M r_B K_M\theta_B = 0 \qquad (196)$$

$$P_1\left(\ddot{\theta}_P + \ddot{\phi}_E\right) + N_P\dot{\theta}_P + C_P \operatorname{sgn} \dot{\theta}_P + r_P^2(K_1 + K_P)\theta_P$$

$$- r_P r_M K_1\theta_M - r_P r_B K_P\theta_B = T_{m_P} \qquad (197)$$

$$B_1\left(\ddot{\theta}_B - \ddot{\phi}_E\right) + N_B\dot{\theta}_B + C_B \operatorname{sgn} \dot{\theta}_B + r_B^2(K_P + K_M)\theta_B$$

$$- r_B r_P K_P\theta_P - r_B r_M K_M\theta_M = 0 \qquad (198)$$

$$G_{33}\left(\ddot{\theta}_G + \ddot{\phi}_A\right) + N_G\dot{\theta}_G + C_G \operatorname{sgn} \dot{\theta}_G - \ddot{\phi}_R\Big[m_G g_2 g_3$$

$$+ (M_2 - M_3)C^2 + m_M m_2 m_3 + m_P P_2 P_3 + m_B b_2 b_3\Big]$$

$$+ \dot{V}_1[m_G g_2 - m_P P_2 + m_B b_2 + m_M m_2] = T_{m_G} \qquad (199)$$

$$(R_{11} + C_{11})\left(\ddot{\beta}_1 - \ddot{\theta}_P - \ddot{\phi}_E\right) + N_A\dot{\beta}_1 - \omega R_{12}\left(\dot{\theta}_G + \dot{\phi}_A\right) = T_{m_A}(\chi_1) \qquad (200)$$

$$(R_{23} + C_{23})\left(\ddot{\beta}_2 + \ddot{\theta}_G + \ddot{\phi}_A\right) + N_E\dot{\beta}_2 - \omega R_{22}\left(\dot{\theta}_P + \dot{\phi}_E\right) = T_{m_E}(\chi_2) \quad . \qquad (201)$$

These linearized equations for a perfectly balanced system are normally used in the design of control compensators for system. Equations (196) through (201) are put in state variable form in a block diagram in Section 4.

4. State Variable Block Diagram

A block diagram of Equations (196) through (201) for Case d), suitable for programming an analog computer, is presented in Figure 9. The output of each integrator is a state variable. Figure 9 is shown for a closed-loop system; i.e., it not only shows the gimbal dynamics and the gyro sensor's dynamics but includes blocks representing torque generators (motors) for driving the gimbals and blocks for compensators to give the desired closed-loop response. The details of the torque generator blocks and compensator blocks are not shown since they have been adequately defined in References 1 and 2, or they are to be designed. It should be noticed in the block diagram that the gyro blocks could be simplified by reducing their orders, i.e., the number of integrators. This is equivalent to pole-zero cancellation and can cause a reduction in system analysis information.

5. Conclusions

The full nonlinear equations of motion can be used to analyze the system for both large and small base motion inputs and for large platform motions such as occur in tracking or acquisition. However, only the inner loop or stabilization loop has been considered in the derivation.

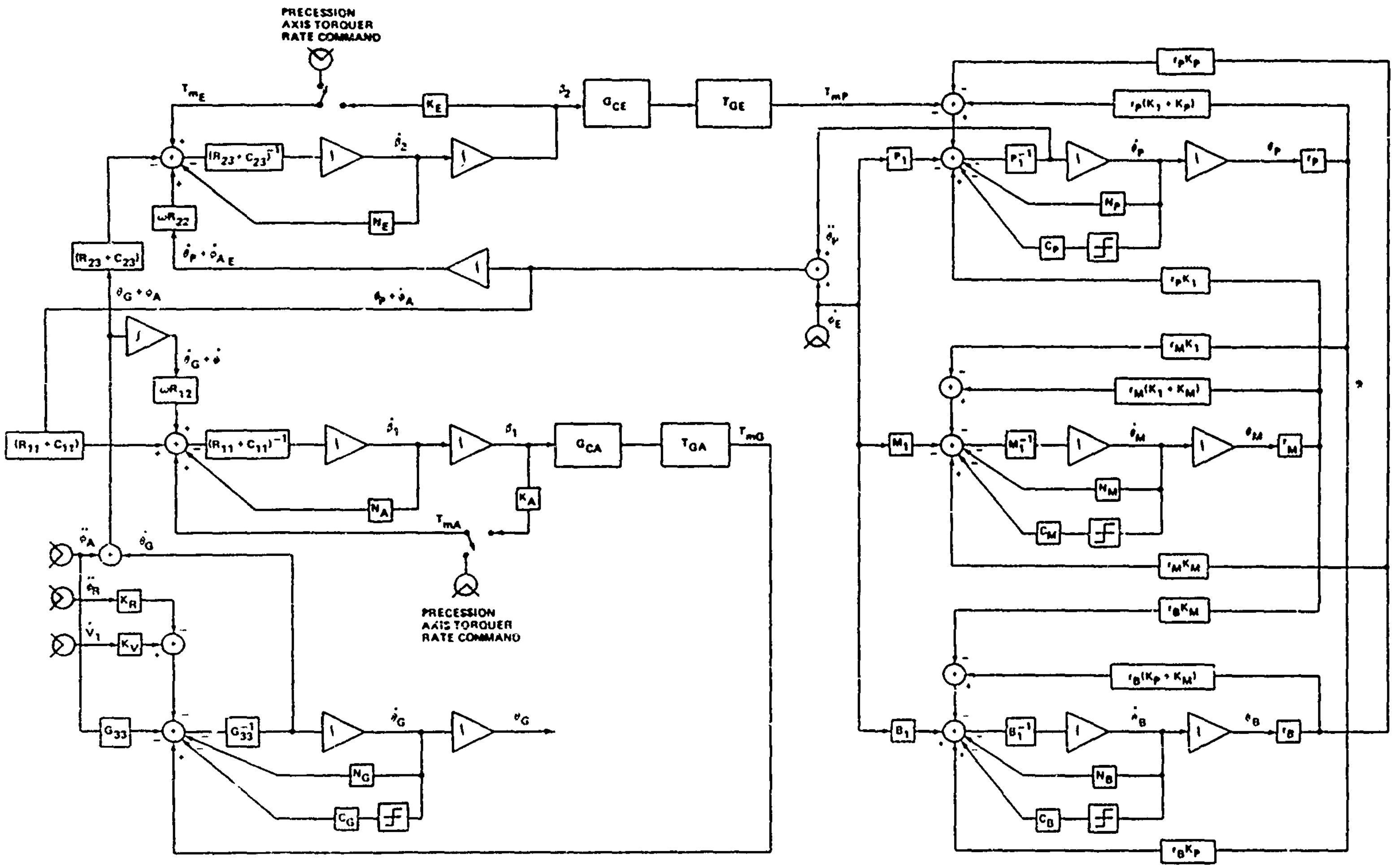


Figure 9. State variable block diagram.

Linearized equations are valid for large angles of pointing from the forward-pointing orientation if angular rates are small and one linearizes about the new, large angles.

All forces acting on the system have been expressed in terms of the states of the system by carefully considering such affects as prestress in the wire drive, precise geometry and stretch of the wires, Coulomb and viscous friction, mass unbalance torques due to gravity and accelerations, and torques produced by the electrical wiring. From observations of the movement of the system in the laboratory it was found that Coulomb friction and electrical wiring (flex lead) torques were greater than those due to mass unbalance.

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SYMBOLS

S, G, M, P, B	Characterizing the Base, Gimbal, Mirror, Platform, and Balancer
$\phi_A(t)$, $\phi_E(t)$, $\phi_R(t)$	Independent azimuth, elevation, and roll angles
θ_G	Angle of rotation of the gimbal, G, with respect to the Base, S
θ_P	Angle of rotation of the platform, P, with respect to the gimbal, G
θ_M	Angle of rotation of the mirror, M, with respect to the gimbal, G
θ_B	Angle of rotation of the balancer, B, with respect to the gimbal, G
$\underline{\omega}^S$, $\underline{\omega}^G$, $\underline{\omega}^M$, $\underline{\omega}^P$, $\underline{\omega}^B$	Inertial angular velocities of S, G, M, P, and B
$\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$	Base vectors fixed in S
$\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$	Base vectors fixed in the platform, P
$\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$	Base vectors fixed in the mirror, M
$\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$	Base vectors fixed in the gimbal, G
$\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$	Base vectors fixed in the balancer, B
G_0	Point in S and G for which motion is prescribed about
P_0, M_0, B_0	Centers of rotation of P, M, and B
$\underline{p}_0$, $\underline{m}_0$, $\underline{b}_0$	Distance vector from G_0 to P_0, M_0, and B_0
G^*, P^*, M^*, B^*	Centers of mass of G, P, M, and B
$\underline{g}_0$, $\underline{p}$, $\underline{m}$, $\underline{b}_0$	Position vector of center of mass from center of rotation
V^{M*}, V^{P*}, V^{B*}	Velocity of centers of mass of M, P, B in the inertial frame
$\frac{\partial K}{\partial \theta_M}$, $\frac{\partial K}{\partial \theta_B}$, $\frac{\partial K}{\partial \theta_P}$, $\frac{\partial K}{\partial \theta_G}$	Partial derivatives of the total kinetic energy with respect to the generalized coordinates θ_M, θ_B, θ_P, θ_G
$K,\dot{\theta}_M$; $K,\dot{\theta}_B$; $K,\dot{\theta}_P$; $K,\dot{\theta}_G$	Partial derivatives of the total system kinetic energy with respect to the derivatives of the generalized coordinates

$F^{*}_{\theta_M}$, $F^{*}_{\theta_P}$, $F^{*}_{\theta_B}$, $F^{*}_{\theta_G}$	Generalized inertia forces for M, P, B, and G
$F^{*}_{B_1}$, $F^{*}_{B_2}$	Generalized inertia forces for the gyroscopes
$\underline{T}_{MW}$, $\underline{T}_{PW}$, $\underline{T}_{BW}$	Moments of the forces due to the wire band drives acting on M, P, and B
$\underline{T}_{MC}$, $\underline{T}_{PC}$, $\underline{T}_{BC}$, $\underline{T}_{GC}$	Coulomb friction torque acting on M, P, B, and G
$\underline{T}_{MV}$, $\underline{T}_{PV}$, $\underline{T}_{BV}$, $\underline{T}_{GV}$	Viscous friction torques acting on M, P, B, and G
$\underline{T}_{MG}$, $\underline{T}_{PG}$, $\underline{T}_{GG}$	Torques acting on M, P, and G due to mass unbalance
$\underline{T}_{PS}$, $\underline{T}_{GS}$	Spring torque forces acting on P and G due to electrical conductors