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TARGET TRACKING: UNIFORMLY DIRECTED MOTION FROM A NORMALLY DISTRIBUTED POSITION

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Following observations of a target position, a set of probabilities, or a position density function, is obtained. If no further information is available, the corresponding probabilities, or density function, after a time interval of arbitrary duration, are obtained from the initial probabilities and an assumed distribution of target course and speed.

Point by point computation of the final position probabilities is always possible, but such a procedure is inefficient where all of the distributions may be specified by

20 means of a few parameters. In this case, a preferred approach is to derive the form of the final position density and compute the values of the parameters necessary for a complete specification from the values of the parameters of the initial position density and the assumed distribution of course and speed.

In this research contribution, the initial position density is bivariate normal. It is assumed that the distribution of target speed may be approximated by a discrete distribution and that the distribution of target course is independent of speed and uniform on the interval 0 to 2π. It is shown that the final position density is expressible as an infinite series of Modified Bessel Functions and that a formal similarity with a well-documented density could be utilized in the computation of its values.

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1. INTRODUCTION

A central problem in target tracking is the determination of position probabilities after an arbitrary time interval for an assumed distribution of target course and speed. A general treatment of this problem which requires only that all contributing distributions are discrete is given in reference 1. One consequence of the distribution-free nature of this treatment, however, is that the procedures which result are excessive in terms of machine time or storage requirements where standard distributions are involved. In such cases, probabilities can be generated by algorithms based on standard mathematical forms and the storage and manipulation of large numerical arrays is thus circumvented.

If mathematical expressions for both the initial position distribution and that of the motion are available, the final position distribution is expressible in terms of the parameters of these two distributions. Thus, if the mathematical form of the final distribution can be established, the computation of the required probabilities consists of the following steps.

1. The computation of the parameters of the final position distribution from those of the constituent distributions of initial position and motion.
2. The computation of the final position probabilities from the fully-specified distribution function so obtained.

For the standard bivariate distributions under consideration, the number of parameters necessary for their specification is small, so the extent of the computation should compare very favorably with that necessary to implement the more general procedures derived in the referenced memorandum. If it turns out that the stated computations can be based on simple, closed-form expressions, this is self-evident, but unfortunately, such relationships cannot be guaranteed. We argue, however, that the lack of a closed-form expression does not necessarily preclude a satisfactory computational procedure, and that the inherent economy provided by constituent distributions with standard mathematical forms is not necessarily eliminated by a transcendental outcome.

The specific problem to be addressed is related to current efforts to improve tracking procedures in the fleet. For the purpose of this research contribution, the problem may be formulated as follows.

Following observations of a target of interest, the probability density function of its position (x, y) is closely approximated by the bivariate normal density with mean (μ_x, μ_y) variances σ_x^2 and σ_y^2, and correlation coefficient ρ. The direction of target motion, φ, is uniformly distributed and independent of target speed. That is, if V is the effective speed,

$$p_{V\varphi}(V, \varphi) = p_V(V)p_\varphi(\varphi)$$

$$= \frac{1}{2\pi}\, p_V(V) \qquad \text{for } 0 \le \varphi < 2\pi \quad .$$

Further, the probability density of the effective speed over a time interval t may be approximated by means of a discrete density function, that is,

$$p_V(V) = \sum_{i=1}^{n} p_i \, \delta(V - V_i)$$

where $V_1, V_2, \ldots, V_n$ are known constants. Required are the probability density functions of the cartesian and polar coordinates, (u, v) and (R, θ) respectively, of the position at the end of the time interval. The maximum target speed is such that a flat earth approximation is acceptable.

It should be noted that the derivation of the mathematical forms of these final position densities is but an intermediate result. The computation of the position probabilities with accuracies adequate for particular applications by means of the facilities available may require additional effort other than at the programming level. While the principal emphasis of this research contribution is on obtaining relationships applicable to the problem stated above, the advantages of a solution in terms of functions for which computational procedures already exist have not been overlooked. A solution of this nature based on the formal solution derived in the main text is presented in appendix B.

2. GENERAL FORMULATION

In this section the general formulation that will be used is outlined. The succeeding section is devoted to the specific problem just described.

Let (x, y) be the initial position of the target of interest. The joint probability density function of this position, $p_{xy}(x, y)$, is assumed known.

Let $p_{V\varphi}(V, \varphi)$ be the joint density of target speed V and direction φ, also assumed known. This joint density, which describes the motion, may be conveniently replaced by:

$$p_{r\varphi}(r, \varphi) = \frac{1}{t}\, p_{V\varphi}\left(\frac{r}{t}, \varphi\right)$$

where $r = Vt$ is the range traversed in the time interval t.

Finally, let (u, v) and (R, θ) be the cartesian and polar coordinates of the target position at the end of this time interval, and let $p_{uv}(u, v)$ and $p_{R\theta}(R, \theta)$ be the corresponding position densities.

To formulate the relationship of these densities to $p_{xy}(x, y)$ and $p_{r\varphi}(r, \varphi)$, we proceed as follows.

The target coordinates (u, v) after time $t > 0$, are related to the initial coordinates (x, y) by:

$$\begin{aligned} u &= x + r \cos \varphi \\ v &= y + r \sin \varphi \quad . \end{aligned} \tag{2.1}$$

Taking x and y as fixed, the Jacobian of the transformation is:

$$r = \{(u-x)^2 + (v-y)^2\}^{\frac{1}{2}}$$

and the conditional density $p_{uv}(u, v \mid x, y)$ is given by:

$$p_{uv}(u, v \mid x, y) = \{(u-x)^2 + (v-y)^2\}^{-\frac{1}{2}} p_{r\varphi}\left[\{(u-x)^2 + (y-v)^2\}^{\frac{1}{2}}, \arctan \frac{v-y}{u-x}\right] \quad .$$

The unconditional density $p_{uv}(u, v)$ is given by:

$$p_{uv}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{uv}(u, v \mid x, y)\, p_{xy}(x, y)\, dx\, dy \quad .$$

Replacing $p_{uv}(u,v|x,y)$ with the expression derived above and using the transformation (2.1), with u and v considered as fixed, to change the variables of integration from x and y to r and φ, we obtain:

$$p_{uv}(u,v) = \int_o^{2\pi}\int_o^{\infty} p_{r\varphi}(r,\varphi)\, p_{xy}(u-r\cos\varphi,\ v-r\sin\varphi)\,dr\,d\varphi \ . \tag{2.2}$$

If direction and speed are independent:

$$p_{r\varphi}(r,\varphi) = p_r(r)\, p_\varphi(\varphi)$$

and

$$p_{uv}(u,v) = \int_o^{2\pi} p_\varphi(\varphi) \int_o^{\infty} p_r(r) p_{xy}(u-r\cos\varphi,\ v-r\sin\varphi)\,dr\,d\varphi \ . \tag{2.3}$$

If, in addition, $p_\varphi(\varphi)$ is uniform,

$$p_{uv}(u,v) = \frac{1}{2\pi}\int_o^{2\pi}\int_o^{\infty} p_r(r) p_{xy}(u-r\cos\varphi,\ v-r\sin\varphi)\,dr\,d\varphi \ . \tag{2.4}$$

Finally, if course and speed are independent and the former is uniformly distributed and, in addition, the distribution of speed is discrete, that is,

$$p_V(V) = \sum_{i=1}^{n} p_i\, \delta(V - V_i) \quad ,$$

it is readily shown that:

$$p_r(r) = \sum_{i=1}^{n} p_i\, \delta(r-r_i)\ , \qquad\qquad r_i = V_i t \ .$$

When these conditions all hold,

$$p_{uv}(u,v) = \frac{1}{2\pi}\int_o^{2\pi}\int_o^{\infty} \sum_{i=1}^{n} p_i\, \delta(r-r_i)\, p_{xy}(u-r\cos\varphi,\ v-r\sin\varphi)\,dr\,d\varphi$$

$$= \sum_{i=1}^{n} p_i P_i(u,v) \quad , \tag{2.5}$$

where

$$P_i(u,v) = \frac{1}{2\pi} \int_0^{2\pi} p_{xy}(u - r_i \cos\varphi,\ v - r_i \sin\varphi)\, d\varphi \quad .$$

Clearly $P_i(u,v)$ is the final position density for the special case of the problem under consideration in which

$$p_V(V) = \delta(V - V_i) \quad ,$$

that is, the speed is assigned the constant value V_i with certainty. It follows that if $P_i(u,v)$ can be determined for $i=1,2,\ldots,n$, the relationship (2.5) provides the final position density, $p_{uv}(u,v)$, for all cases in which an adequate description of the effective speed is provided by a discrete density function.

Where the polar form of the final position coordinates is preferred, the required density $p_{R\theta}(R,\theta)$ is obtained from $p_{uv}(u,v)$ by means of the transformation

$$R = (u^2 + v^2)^{\frac{1}{2}}$$

$$\theta = \arctan v/u \quad .$$

Thus,

$$\begin{aligned} p_{R\theta}(R,\theta) &= R p_{uv}(R\cos\theta,\ R\sin\theta) \\ &= \sum_{i=1}^{n} R\, p_i P_i(R\cos\theta,\ R\sin\theta) \quad . \end{aligned} \tag{2.6}$$

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For this case, the density $p_{xy}(x, y)$ is bivariate normal with parameters μ_x, μ_y, σ_x, σ_y and ρ, and a solution to the problem is obtained by evaluating the integral on the right of the expression

$$P_i(u,v) = \frac{1}{2\pi} \int_o^{2\pi} p_{xy}(u-r_i \cos\varphi,\ v-r_i \sin\varphi)\, d\varphi \tag{3.1}$$

and applying (2.5).

It is clear that this task would have been simplified if the original coordinate system had been translated and rotated to provide a new system in which the coordinate variables were uncorrelated. This would, of course, result in a final position density with respect to the transformed system, and a further transformation to effect the restoration of the original coordinate system would then be necessary. The approach we shall take to simplify the form of the integral is essentially equivalent to this.

We introduce the transformation:

$$\begin{aligned} u' &= (u-\mu_x)\cos\beta + (v-\mu_y)\sin\beta \\ v' &= -(u-\mu_x)\sin\beta + (v-\mu_y)\cos\beta \end{aligned} \tag{3.2}$$

when β satisfies the equation

$$\tan 2\beta = \frac{2\rho\sigma_x\sigma_y}{(\sigma_x+\sigma_y)(\sigma_x-\sigma_y)} \quad . \tag{3.3}$$

The inverse transformation is:

$$\begin{aligned} u &= \mu_x + \mu' \cos\beta - v' \sin\beta \\ v &= \mu_y + \mu' \sin\beta + v' \cos\beta \quad . \end{aligned} \tag{3.4}$$

The density function of the transformed variables, $P_i(u', v')$, is therefore related to that of the untransformed variables, $P_i(u,v)$, by:

$$P_i(u', v') = P_i \{(\mu_x + u' \cos\beta - v' \sin\beta), (\mu_y + u' \sin\beta + v' \cos\beta)\}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} p_{xy} \{(\mu_x + u' \cos\beta - v' \sin\beta - r_i \cos\varphi) ,$$

$$(\mu_y + u' \sin\beta + v' \cos\beta - r_i \sin\varphi)\} d\varphi \quad .$$

Introducing the identities:

$$\begin{aligned} \cos(\varphi - \beta) &= \cos\varphi \cos\beta + \sin\varphi \sin\beta \\ \sin(\varphi - \beta) &= \cos\varphi \sin\beta + \sin\varphi \cos\beta \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned} \cos\varphi &= \cos(\varphi - \beta) \cos\beta - \sin(\varphi - \beta) \sin\beta \\ \sin\varphi &= \cos(\varphi - \beta) \sin\beta + \sin(\varphi - \beta) \cos\beta \end{aligned} \tag{3.5b}$$

and using the latter pair to replace $\cos\varphi$ and $\sin\varphi$ in the integrand of the above expression, we obtain:

$$P_i(u', v') = \frac{1}{2\pi} \int_0^{2\pi} p_{xy} \{A_i(u', v', \varphi), \; B_i(u', v', \varphi)\} d\varphi$$

where

$$A_i(u', v', \varphi) = \mu_x + \{u' - r_i \cos(\varphi - \beta)\} \cos\beta - \{v' - r_i \sin(\varphi - \beta)\} \sin\beta$$

$$B_i(u', v', \varphi) = \mu_y + \{u' - r_i \cos(\varphi - \beta)\} \sin\beta + \{v' - r_i \sin(\varphi - \beta)\} \cos\beta \quad .$$

With β chosen in accordance with (3.3), this reduces to:

$$P_i(u', v') = (2\pi)^{-2} (S_u S_v)^{-\frac{1}{2}} \int_0^{2\pi} \exp\left[- \frac{\{u' - r_i \cos(\varphi - \beta)\}^2}{2S_u} - \frac{\{v' - r_i \sin(\varphi - \beta)\}^2}{2S_v} \right] d\varphi$$

$$= (2\pi)^{-2} (S_u S_v)^{-\frac{1}{2}} \int_{-\beta}^{2\pi - \beta} \exp\left[- \frac{1}{2} \left\{ \frac{(u' - r_i \cos\varphi)^2}{S_u} + \frac{(v' - r_i \sin\varphi)^2}{S_v} \right\} \right] d\varphi \tag{3.6}$$

where S_u and S_v are given in terms of the parameters of the initial position density by:

$$\frac{1}{S_u}, \frac{1}{S_v} = \frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2} \mp \left\{\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}^2 - \frac{4(1-\rho^2)}{\sigma_x^2\sigma_y^2}\right\}^{\frac{1}{2}}\right]. \tag{3.7}$$

The integral appearing on the right of this relationship cannot be expressed in closed form, but may be reduced to an infinite series of Modified Bessel Functions. The method used to achieve this reduction is given in reference 2 and is outlined in the appendix. It gives:

$$P_i(u',v') = (2\pi)^{-1}(S_uS_v)^{-\frac{1}{2}} \exp(-q_i') \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(F_i) I_{2k}(w_i') \cos(2k\gamma_i') \tag{3.8}$$

where

$\epsilon_k = 1 \quad \text{for } k = 0$

$\quad = 2 \quad \text{for } k > 0$

and the parameters q_i', F_i, w_i', and γ_i' are as defined in appendix A.

This gives the final position density in the transformed coordinate system. By transforming back to the variables u and v, the coordinate system with reference to which the problem was originally stated is restored. The required transformation is given by (3.4), namely:

$u = \mu_x + u' \cos\beta - v' \sin\beta$

$v = \mu_y + u' \sin\beta + v' \cos b$

and

$$\begin{aligned} P_i(u,v) &= P_i\left[\{(u-\mu_x)\cos\beta + (v-\mu_y)\sin\beta\}, \{-(u-\mu_x)\sin\beta + (v-\mu_y)\cos\beta\}\right] \\ &= (2\pi)^{-1}(S_uS_v)^{-\frac{1}{2}} \exp(-q_i) \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(F_i) I_{2k}(w_i) \cos(2k\gamma_i) \end{aligned} \tag{3.9}$$

where $q_i(u,v)$, $w_i(u,v)$ and $\gamma_i(u,v)$, the coordinate-dependent parameters of this density function are obtained from the corresponding expressions for $q'_i(u',v')$,

$w'_i(u',v')$ and $\gamma'_i(u', v')$, given in the appendix, by making the replacements required by the transformation, namely:

$$u' = (u-\mu_x)\cos\beta + (v-\mu_y)\sin\beta$$

$$v' = -(u-\mu_x)\sin\beta + (v-\mu_y)\cos\beta \quad .$$

This permits the computation of $P_i(u,v)$ for $i=1,2,\ldots,n$ and hence, from (2.5), the computation of the required final position density $p_{uv}(u,v)$.

If the final position density is required in polar form, an additional transformation is necessary. For any given values of R and θ, $P_i(R\cos\theta, R\sin\theta)$ is given by (3.9) with parameters $q_i(R\cos\theta, r\sin\theta)$, $w_i(R\cos\theta, R\sin\theta)$ and $\gamma_i(R\cos\theta, R\sin\theta)$. The required final position density, $p_{R\theta}(R,\theta)$, is then obtained from (2.6).

4. CONCLUSIONS

It follows from equation (2.5) and the discussion following its derivation that a solution to the problem stated in the Introduction is dependent on a solution being obtained for the special case of this problem where the effective speed can be assigned with certainty.

The problem of obtaining a solution for the special case is apparent from (3.1). If adequate computational facilities are available, a direct numerical evaluation of the integral expression in which $p_{xy}(x, y)$ is a bivariate normal density of the most general type, is, at least, feasible. In the absence of such facilities, a reduction to an equivalent, but more tractable, form is mandatory. Consistent with this reasoning, an equivalent form has been determined and, in a formal sense, equation (3.9) and the discussion of its parameters, constitute the principal results of this research contribution.

This concluding section would not be complete, however, without a brief examination of these results in the context of our opening remarks. These remarks list two steps as being necessary for the computation of position probabilities consequent to the mathematical form of the density being available.

No special problem is evident in implementing the first of these steps. Instances will occur where the transformed coordinate system is adequate or preferable for the treatment of the problem at hand. When this is the case, the parameters of the final density are directly provided by the simple algebraic formulas listed in appendix A. If results are to be referred to the original coordinate system or if the range and bearing of the final position are required, only the three coordinate-dependent parameters are affected and the modifications involved are readily implemented.

If the form of the expression obtained for the final density were entirely novel, some discussion of its properties would be required before the second of the steps listed could be carried out in practice. In view, however, of the striking resemblance of our principal result to the mathematical form of the Nakagami Density Function (reference 3) which has extensive application in the theory of tropospheric scatter, referral to the literature on this topic appears a preferred alternative. There can be little doubt that programs that will provide much of the information that is required, with only minor modification, already exist. It is also apparent that tabulations of the Nakagami Density Function may be used to determine the final position probabilities subject to a minor preliminary computation to determine the appropriate point of entry.

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APPENDIX A

INTEGRAL REDUCTION

From (3.6) and (3.7),

$$P_i(u',v') = (2\pi)^{-2} (S_u S_v)^{-\frac{1}{2}}$$

$$\int_{-\beta}^{2\pi-\beta} \exp\left[-\tfrac{1}{2}\left\{\frac{(u'-r_i\cos\varphi)^2}{S_u} + \frac{(v'-r_i\sin\varphi)^2}{S_v}\right\}\right] d\varphi$$

where

$$\frac{1}{S_u}, \frac{1}{S_v} = \frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2} \mp \left\{\left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)^2 - \frac{4(1-\rho^2)}{\sigma_x^2\sigma_y^2}\right\}^{\frac{1}{2}}\right] .$$

Let

$$F_i = \frac{r_i^2}{4}\left(\frac{1}{S_u} - \frac{1}{S_v}\right)$$

$$q_i'(u';v') = \frac{u'^2}{2S_u} + \frac{v'^2}{2S_v} + r_i^2\left(\frac{1}{4S_u} + \frac{1}{4S_v}\right)$$

$$w_i'(u',v') = r_i\left(\frac{u'^2}{S_u^2} + \frac{v'^2}{S_v^2}\right)^{\frac{1}{2}}$$

$$\nu_i'((u',v') = \arctan\frac{v'/S_v}{u'/S_u} \quad ,$$

then

$$P_i(u',v') = (2\pi)^{-2}(S_u S_v)^{-\frac{1}{2}} \exp(-q_i') \int_{-\beta}^{2\pi-\beta} \exp\{-F_i \cos 2\varphi + w_i' \cos(\varphi - \nu_i')\} d\varphi .$$

From the theory of Bessel Functions,

$$I_k(x) = \frac{1}{2\pi}\int_C^{2\pi+C} \exp(-x\cos\varphi + ik\varphi)\, d\varphi$$

where C is any real constant, and

$$\exp(-x\cos\varphi) = \sum_{k=-\infty}^{\infty} (-1)^k I_k(x) \exp(ik\varphi) .$$

Therefore,

$$P_i(u',v') = (2\pi)^{-2}(S_uS_v)^{-\frac{1}{2}}\exp(-q'_i)$$

$$\int_{-\beta}^{2\pi-\beta}\sum_{k=-\infty}^{\infty}(-1)^k I_k(F_i)\exp\{2ik\varphi + w'_i\cos(\varphi-\gamma'_i)\}\,d\varphi$$

$$= (2\pi)^{-2}(S_uS_v)^{-\frac{1}{2}}\exp(-q'_i)\sum_{k=-\infty}^{\infty}(-1)^k I_k(F_i)$$

$$\int_{-\beta}^{2\pi-\beta}\exp\{2ik\varphi + w'_i\cos(\varphi-\gamma'_i)\}\,d\varphi$$

$$= (2\pi)^{-1}(S_uS_v)^{-\frac{1}{2}}\exp(-q'_i)\sum_{k=-\infty}^{\infty}(-1)^k I_k(F_i)$$

$$\exp(2ik\gamma'_i)\frac{1}{2\pi}\int_{-\beta-\gamma'_i}^{2\pi-\beta-\gamma'_i}\exp(2ik\varphi + w'_i\cos\varphi)\,d\varphi$$

$$= (2\pi)^{-1}(S_uS_v)^{-\frac{1}{2}}\exp(-q'_i)\sum_{k=-\infty}^{\infty}(-1)^k I_k(F_i)I_{2K}(-w'_i)\exp(2ik\gamma'_i)$$

$$= (2\pi)^{-1}(S_uS_v)^{-\frac{1}{2}}\exp(-q'_i)\sum_{k=-\infty}^{\infty}(-1)^k I_k(F_i)I_{2k}(w'_i)\exp(2ik\gamma'_i) \quad ,$$

since, for k even, $I_k(-x) = I_k(x)$. Combining terms with indices k and $-k$,

$$P_i(u',v') = (2\pi)^{-1}(S_uS_v)^{-\frac{1}{2}}\exp(-q'_i)\sum_{k=0}^{\infty}(-1)^k \epsilon_k I_k(F_i)I_{2k}(w'_i)\cos(2k\gamma'_i)$$

where

$$\epsilon_k = 1 \quad \text{for } k = 0$$
$$= 2 \quad \text{for } k > 0 \quad .$$

This is the required result and the relationship (3.8) is thus established.

APPENDIX B

SOLUTION VIA THE NAKAGAMI DENSITY FUNCTION

The density function referred to as the Nakagami Density Function in the concluding section may be introduced in the following way. Let x and y be normally distributed with arbitrary parameters μ'_x, μ'_y, σ'_x, σ'_y, and ρ', and let

$$\chi = \sqrt{x^2 + y^2}$$

$$\varphi = \arctan y/x \quad ;$$

then

$$p_\chi(\chi) = \int_0^{2\pi} p_{\chi\varphi}(\chi, \varphi)\, d\varphi = \chi \int_0^{2\pi} p_{xy}(\chi \cos\varphi, \chi \sin\varphi)\, d\varphi \quad .$$

This relationship may be used to define the Nakagami Density Function. Although the integral expression in this form is unsuitable for computation, we may nevertheless write:

$$p_\chi(\chi \,|\, \mu'_x, \mu'_y, \sigma'_x, \sigma'_y, \rho') = \chi \int_0^{2\pi} p_{xy}(\chi\cos\varphi, \chi\sin\varphi \,|\, \mu'_x, \mu'_y, \sigma'_x, \sigma'_y, \rho')\, d\varphi \ .$$

We make use of this result by substituting variables defined in section 2 and in the opening sentence of section 3 as follows:

$$p_\chi(r_i \,|\, u-\mu_x, v-\mu_y, \sigma_x, \sigma_y, \rho) = r_i \int_0^{2\pi} p_{xy}(r_i\cos\varphi, r_i\sin\varphi \,|\, u-\mu_x, v-\mu_y, \sigma_x, \sigma_y, \rho)\, d\varphi$$

$$= r_i \int_0^{2\pi} p_{xy}(u - r_i\cos\varphi, v - r_i\sin\varphi \,|\, \mu_x, \mu_y, \sigma_x \sigma_y, \rho)\, d\varphi$$

$$= 2\pi r_i P_i(u, v \,|\, \mu_x, \mu_y, \sigma_x, \sigma_y, \rho)\, d\varphi$$

by equation (3.1). Then, from (2.5),

$$p_{uv}(u, v \,|\, \mu_x, \mu_y, \sigma_x, \sigma_y, \rho) = \sum_{i=1}^{n} p_i P_i(u, v \,|\, \mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$$

$$= \frac{1}{2\pi} \sum_{i=1}^{n} \frac{p_i}{r_i}\, p_\chi(r_i \,|\, u-\mu_x, v-\mu_y, \sigma_x, \sigma_y, \rho) \quad .$$

This expression provides a solution to the problem stated in the Introduction, where

$$p_V(V) = \sum_{i=1}^{n} p_i(V - V_i)$$

$$r_i = V_i t,$$

and $\mu_x, \mu_y, \sigma_x, \sigma_y$ and ρ are the parameters of the initial position density.