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OPTICAL BEAM PROPAGATION IN TURBULENT MEDIA

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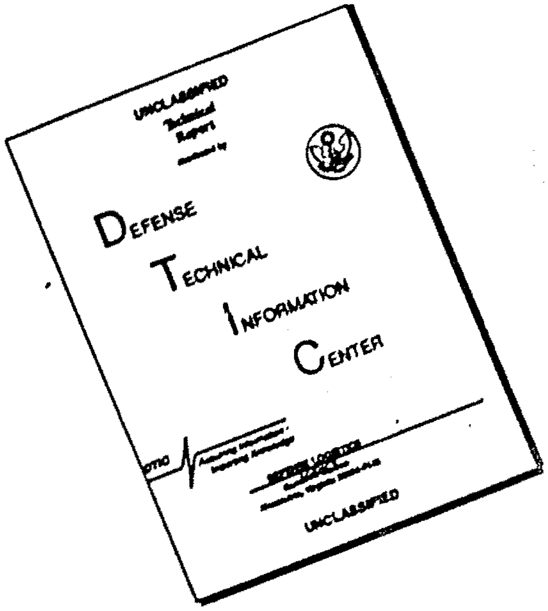

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We have discussed and extended the most recent developments on the propagation of microwave and optical beams in turbulent media, such as the clear atmosphere. Among the phenomena considered are beam spreading, beam wander, loss of coherence, scintillations, angle-of-arrival variations, and short pulse effects. Also included is a discussion of methods of compensation of the effect of turbulence on communications and imaging systems.

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1. INTRODUCTION

For many years astronomers have been concerned with the effect of fluctuations in the index of refraction of the earth's atmosphere on light ray propagation, since this is the origin of the twinkling of stars and image jitter in a telescope. These same effects limit the useful atmospheric path for laser communications and radar systems. That is, suppose we consider a light beam traversing a medium with random fluctuations in its index of refraction. Because of scatter of the light beam by the random fluctuations there will be a spreading of the beam, beyond that normally caused by diffraction, with a corresponding decrease in the beam intensity.* In addition, there will be scintillations of the received intensity, a decrease in the spatial and temporal coherence and even, in some cases, a distinct wander of the beam from position to position. These and other similar effects can seriously degrade the performance of a laser communication or radar system. For example, it is readily shown that the possible signal-to-noise ratio of an optical heterodyne received is limited by atmospheric turbulence effects.

The study of the interaction of light beams with random media is important in other applications besides astronomy and optical communications. Another

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(Received for publication 12 August 1975.)

*The quantity we call the "intensity" is referred to as the "irradiance" in texts on optics.

significant application is in the field of remote sensing, where lasers are used to probe the properties of turbulent plasmas, hydrodynamic flows, and the earth's atmosphere. This requires a thorough understanding of the relationship of the scattered laser radiation to the physical parameters of the medium being probed.

It is therefore important to have a theory for predicting the nature of the propagation of a light beam in a random medium, and in the decade from 1960 to 1970 a great deal of progress was made on this problem.[1-4] In particular, useful expressions were obtained for the beam intensity, scintillations, coherence, etc.; these were shown to agree quite well with measured data in the limiting case of short propagation paths. However, when these same theories were applied to longer propagation paths it was found that the theoretical predictions no longer correlated well with measured data. This spurred attempts to develop improved theories, and the last several years have been the development of some more general theoretical models which are applicable to long propagation paths.

In this report, we will review the properties of wave propagation in random media and then summarize the many important results which have been obtained since 1970. Our philosophy of presentation will be to present a minimum of detail describing how the results were obtained; rather we shall concentrate on expressing the mose useful results in their simplest possible form so that they can be readily used in systems applications without the necessity of first employing a digital computer.

2. THEORETICAL BACKGROUND

Here, we will first outline the model assumed for the index of refraction fluctuations. We will next review the Rytov method which is applicable to short propagation paths, then review the "Markov Approximation" which is applicable for paths of arbitrary length, and finally will present the linear system representation of the propagation channel.

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1. Tatarskii, V. (1971) The Effect of the Turbulent Atmosphere on Wave Propagation, U.S. Dept. of Commerce, Springfield, Virginia.
2. Lawrence, R., and Strohbehn, J. (1970) A survey of clear-air propagation effects relevent to optical communications, Proc IEEE 58:1523-1545.
3. Barabanenkov, Y., Kravtsov, Y., Rytov, S., and Tatarskii, V. (1971) Status of the theory of propagation of waves in randomly inhomogeneous media, Soviet Physics Usp. 13:551-580.
4. Strohbehn, J. (1968) Line of sight wave propagation through the turbulent atmosphere, Proc. IEEE 56:1301-1318.

2.1 Index-of-Refraction Model

Random fluctuations of the temperature in the earth's atmosphere lead to corresponding fluctuations in the index of refraction; these fluctuations are functions of the position $\underline{r}$ and time t, so that the index of refraction n can be written as

$$n(\underline{r}, t) = 1 + n_1(\underline{r}, t), \tag{1}$$

where n_1 is the fluctuation in the index of refraction. For clear-air atmospheric turbulence it is generally reasonable to assume that n_1 is small and that its temporal dependence is mainly due to atmospheric winds, so that* $n_1(\underline{r}, t) \simeq n_1[\underline{r} - \underline{V}(\underline{r})t]$ where $\underline{V}(\underline{r})$ is the local wind velocity. This last assumption is known as "Taylor's frozen-flow hypothesis," and appears to hold in most practical situations.

Let us now ignore, for the moment, the effect of atmospheric winds and concentrate on the spatial variations of n_1. In this report we shall only need to calculate integrals of n_1 of the form $\int f(x)\, n_1(x)\, dx$, and these are generally gaussian random variables. Therefore, we shall only be interested in the first two moments of n_1. We have already commented that the first moment $\langle n_1 \rangle = 0$ where $\langle\ \rangle$ denotes an ensemble average. We must now specify the covariance $\langle n_1(\underline{r})\, n_1(\underline{r}') \rangle$. We shall assume that the random medium is locally stationary; that is, if we define $\underline{\tilde{R}} = (\underline{r} + \underline{r}')/2$ and $\underline{\tilde{\rho}} = \underline{r} - \underline{r}'$ then the moment $\langle n_1(\underline{r})\, n_1(\underline{r}') \rangle$ varies much more rapidly with $\underline{\tilde{\rho}}$ than it does with $\underline{\tilde{R}}$. For this case, the covariance of n_1 may be written as

$$\langle n_1(\underline{r})\, n_1(\underline{r}') \rangle = \iiint_{-\infty}^{\infty} d^3\kappa\, \Phi_n(\underline{\tilde{R}}, \underline{\kappa})\, e^{i\underline{\kappa} \cdot \underline{\tilde{\rho}}} . \tag{2}$$

The function $\Phi(R, \kappa)$ is known as the wavenumber spectrum of the index of refraction fluctuations, and for well-developed turbulence in the earth's atmosphere is given approximately by

$$\Phi_n(\underline{\tilde{R}}, \underline{\kappa}) = \frac{0.033\, C_n^2(\underline{\tilde{R}}) \exp\left[-\left(\frac{\kappa \ell_o}{2\pi}\right)^2\right]}{\left[\kappa^2 + L_o^{-2}\right]^{11/6}} . \tag{3}$$

---

*Strictly speaking, we should state that the correlation of n_1 is a function of $\underline{r} - \underline{V}t$, because n_1 is a random quantity.

In Eq. (3) the quantity C_n^2 is known as the index-of-refraction structure constant, and is a measure of the magnitude of the fluctuations in the index of refraction. The quantity L_o is a measure of the largest distances over which fluctuations in the index of refraction are correlated, whereas ℓ_o is a measure of the smallest correlation distances. It is interesting to note that the index-of-refraction fluctuations are sometimes referred to as "turbulent eddies," and the correlation distances L_o and ℓ_o are usually referred to as the outer and inner scale sizes of the turbulent* eddies, respectively. In atmospheric turbulence, L_o may range anywhere from 1 to 100 meters, and ℓ_o is usually on the order of 0.001 meter.

The spectrum given in Eq. (3) is applicable to situations other than the earth's atmosphere. It can also be used to describe[5-10] fluctuations in the index of refraction of a turbulent plasma such as those generated on hypersonic spacecraft, and in fusion experiments.

2.2 The Rytov Method

Now that we have specified the nature of the index of refraction fluctuations, we can begin our discussion of their effect on the propagation of a light beam or other electromagnetic signal. The earliest[8-10] attempts to study propagation in a random medium employed the geometric optics approximation; however, this was shown to be of very limited utility since results obtained by geometric optics are valid only for propagation paths of order $k\ell_o^2$, where k is the signal wavenumber. In the late 1950's, Tatarski[9] developed a new technique, based on the Rytov approximation, which had a much greater range of validity than the geometric optics method. In this section will will outline the development of this method, and then discuss the limitations on its application to long-path propagation.

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*Throughout the remainder of this report, we will use the words "turbulent" and "random" interchangeably. Furthermore, some authors use the terminology "macroscale" in place of "outer scale" and "microscale" in place of "inner scale."

5. Houbolt, J. (1973) Atmospheric turbulence, AIAA Journal 11:421-437.

6. Fox, J., and Rungaldier, H. (1972) Electron density fluctuation measurements in projectile wakes, AIAA Journal 10:790-795.

7. Fox, J., and Rungaldier, H. (1973) Turbulent scattering spectra of electron density fluctuations in Mach 16 projectile wakes, AIAA Journal 11:1059-1064.

8. Bergmann, P. (1946) Propagation of radiation in a medium with random inhomogeneities, Phys. Rev. 70:486-492.

9. Tatarskii, V. (1967) Wave Propagation in a Turbulent Medium, Dover Publications, Inc., New York.

10. Chernov, L. (1967) Wave Propagation in a Random Medium, Dover Publications, Inc., New York.

The electric field $\underline{E}$ of a narrow-band* beam propagating in a random medium is governed by the Maxwell wave equation

$$\nabla^2 \underline{E} + k^2 (1 + n_1)^2 \underline{E} - \nabla(\nabla \cdot \underline{E}) = 0 \quad . \tag{4}$$

It can be shown[11, 12] that the last term in Eq. (4), which represents depolarization effects, is negligible for optical waves in the atmosphere. Consequently, it is acceptable to approximate Eq. (4) by

$$\nabla^2 E + k^2 (1 + n_1)^2 E = 0 \quad . \tag{5}$$

We next write

$$E = \exp(\psi) = \exp(\chi + iS) \tag{6}$$

and substitute Eq. (6) into (5) to get

$$\nabla^2 \psi + (\nabla\psi)^2 + k^2 (1 + n_1)^2 = 0 \quad . \tag{7}$$

In the Rytov method ψ is next written** as $\psi = \psi_o + \psi_1$, where ψ_o satisfies the vacuum equation

$$\nabla^2 \psi_o + (\nabla\psi_o)^2 + k^2 = 0. \tag{8}$$

If we substitute $\psi = \psi_o + \psi_1$ into Eq. (7), use Eq. (8) and then neglect $|\nabla\psi_1|$ in comparison with $|\nabla\psi_o|$ and n_1^2 in comparison with $2n_1$, it is found that ψ_1 satisfies

$$\nabla^2 \psi_1 + 2\nabla\psi_o \cdot \nabla\psi_1 + 2k^2 n_1 \simeq 0 \, , \tag{9}$$

*By narrow band we mean that the spread of the signal in frequency space is sufficiently small that $\partial^2 E/\partial t^2$ can be replaced by $\omega^2 (1 + n_1)^2 E$, where ω is the signal frequency. In Section 8, we will consider the case of wide-band signals.

**ψ_o is the nonfluctuating portion of ψ, whereas ψ_1 is the random component.

11. Strohbehn, J., and Clifford, S. (1967) Polarization and angle-of-arrival fluctuations for a plane wave propagated through a turbulent medium, IEEE Trans. Ant. and Prop. AP-15:416-421.

12. Collett, E., and Alferness, R. (1972) Depolarization of a laser beam in a turbulent medium, J. Opt. Soc. Am. 62:529-533.

which has a solution $\psi_1 = \chi_1 + i S_1$ given by

$$\psi_1(\underline{r}) = \frac{k^2}{2\pi E_o(\underline{r})} \iiint d^3r' \, n_1(\underline{r}') \, E_o(\underline{r}') \, \frac{e^{ik|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|} \, , \tag{10}$$

where $E_o = \exp(\psi_o)$. From Eq. (10) it is possible to calculate the moments of the log-amplitude χ_1 and the phase S_1; this will not be done here, but will rather be presented later. However, it is quite clear from Eq. (10) that the moments of ψ_1 can be expressed in terms of the moments of the index of refraction fluctuation n_1.

The assumption that $|\nabla\psi_1| << |\nabla\psi_o|$ leads to an important restriction on the range of validity of the Rytov method. When the Rytov method was first developed it appeared to give quite good agreement with all the available experimental data, which had been taken over propagation paths of less than 1 km in the atmosphere. However, in the late 1960's when experiments were performed using horizontal propagation paths much greater than 1 km, it was found[13] that the experimental data deviated significantly from predictions made using the Rytov method. In particular, it was found that if the propagation path x is such that the parameter $\sigma_1^2 = 1.23\, k^{7/6} C_n^2\, x^{11/16}$ is greater than 0.3, the Rytov approximation is invalid. It was soon recognized that, because the Rytov approximation is equivalent to the scatter of the incident wave by a series of random phase screens,[14] it did not adequately account for multiple scatter of the electromagnetic wave by the turbulent eddies. This spurred attempts to develop new theories which properly include multiple scatter, and led to an exploration of diagrammatic techniques,[3, 15-20]

13. Gracheva, M., Gurvich, A., and Kallistrova, M. (1970) Dispersion of strong atmospheric fluctuations in the intensity of laser radiation, Radiophysics and Quantum Electronics, 13:40-42.

14. Lee, R., and Harp, J. (1969) Weak scattering in random media, with applications to remote probing, Proc. IEEE 57:375-406.

15. Brown, W. (1972) Moment equations for waves propagated in random media, J. Opt. Soc. Amer. 62:45-54.

16. Brown, W. (1972) Fourth moment of a wave propagating in a random medium, J. Opt. Soc. Amer. 62:966-971.

17. deWolf, D. (1973) Strong irradiance fluctuations in turbulent air: plane waves, J. Opt. Soc. Amer. 63:171-179.

18. deWolf, D. (1973) Strong irradiance fluctuations in turbulent air II: spherical waves, J. Opt. Soc. Amer. 63:1249-1253.

19. deWolf, D. (1974) Strong irradiance fluctuations in turbulent air III: diffraction cutoff, J. Opt. Soc. Amer. 63:360-365.

20. Sancer, M., and Varvatsis, A. (1970) Saturation calculation for light propagation in the turbulent atmosphere, J. Opt. Soc. Amer. 60:654-659.

transport methods,[21, 22] coherence theory,[23-25] and a host of other approaches[26, 27] which we will discuss further in Section 5. Up until now, however, the Markov approximation[1, 28-30] technique appears to be the most successful in overcoming the limitations of the Rytov method and the easiest to understand. This approximation is discussed below.

2.3 The Markov Approximation

We concluded in the last section that the Rytov method is inadequate for predicting the properties of a light beam propagating over long paths through clear-air atmospheric turbulence. We will not outline a method which gives acceptable results over long paths.

We consider a light beam which is propagating along the x-axis in a medium with random index of refraction fluctuations, and write the electric field as

$$E = u(x, y, z)\, e^{ikx} \quad . \tag{11}$$

If Eq. (11) is substituted into (5) and the term $\partial^2 u/\partial x^2$ is neglected, we obtain (approximating $2n_1 + n_1^2$ by $2n_1$)

$$2ik \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2k^2 n_1 u = 0. \tag{12}$$

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21. Dolin, L. (1964) Propagation of a narrow light beam in a random medium, Radiofizika (Russian) 7:380-391.
22. Fante, R. (1973) Propagation of electromagnetic waves through a turbulent plasma using transport theory, IEEE Trans. Ant. and Prop. AP-21:750-755.
23. Beran, M. (1970) Propagation of a finite beam in a random medium, J. Opt. Soc. Amer. 60:518-521.
24. Beran, M., and Ho, T. (1969) Propagation of the fourth-order coherence function in a random medium, J. Opt. Soc. Amer. 59:1134-1138.
25. Molyneux, J. (1971) Propagation of N^{th} order coherence functions in a random medium, J. Opt. Soc. Amer. 61:369-377.
26. Lutomirski, R., and Yura, H. (1971) Propagation of a finite optical beam in an inhomogeneous medium, Appl. Optics 10:1652-1658.
27. Furutsu, K. (1972) Statistical theory of wave propagation in a random medium and the irradiance distribution function, J. Opt. Soc. Amer. 62:240-254.
28. Tatarskii, V. (1969) Light propagation in a medium with random refractive index inhomogeneities in the Markov approximation, Soviet Phys. JETP 29:1133-1138.
29. Klyatskin, V. (1970) Applicability of the approximation of a Markov random process in problems related to the propagation of light in a medium with random inhomogeneities, Soviet Phys. JETP 30:520-523.
30. Klyatskin, V., and Tatarskii, V. (1970) The parabolic equation approximation for propagation of waves in a medium with random inhomogeneities, Soviet Physics JETP 31:335-339.

Equation (12) is known as the parabolic approximation, and is applicable to the propagation of light beams with narrow angular spread. Later in this section we will give the quantitative conditions under which $\partial^2 u/\partial x^2$ can be neglected. We can be neglected. We can solve Eq. (12) for the moments $\langle u(x, \underline{\rho}_1)\rangle$, $\langle u(x, \underline{\rho}_1) u^*(x, \underline{\rho}_2)\rangle$ etc. where $\underline{\rho} \equiv (y, z)$, by a method known as the "Markov approximation."[1] In this method it is first assumed that the index of refraction fluctuation n_1 is delta-function correlated in the direction of propagation, so that the turbulent eddies look like flat discs oriented normally to the propagation path. That is, it is assumed that

$$\langle n_1(x, \underline{\rho})\, n_1(x', \underline{\rho}')\rangle = \delta(x - x')\, A(\underline{\rho} - \underline{\rho}') \, , \tag{13}$$

where

$$A(x, \underline{\rho}) = 8\pi \iint_{-\infty}^{\infty} dk_y\, dk_z\, \Phi_n(x, k_x = 0, k_y, k_z)\, e^{i\underline{k}\cdot\underline{\rho}} \, . \tag{14}$$

Equation (12) can then be solved for the moments of the field by using (13) along with the Novikov-Furutsu formula,[31] which states that if n_1 is a gaussian random variable and $\phi[n_1]$ is an arbitrary function of n_1, then

$$\langle n_1(\underline{r})\, \phi[n_1]\rangle = \iiint_{-\infty}^{\infty} d^3r' \,\langle n_1(\underline{r})\, n_1(\underline{r}')\rangle \left\langle \frac{\delta\phi[n_1]}{\delta n_1(\underline{r}')} \right\rangle \, , \tag{15}$$

where $\delta\phi/\delta n_1$ is a variational derivative.*

If Eq. (12) is ensemble averaged directly, and Eqs. (13) and (15)—with the functional ϕ set equal to u—are used to evaluate $\langle u n_1\rangle$, one finds (reference 1, Section 65-72)

$$2ik \frac{\partial\langle u\rangle}{\partial x} + \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\langle u\rangle + \frac{ik^3}{4} A(x, 0)\langle u\rangle = 0. \tag{16}$$

---

*The use of a delta-function index-of-refraction correlation is a mathematical artifice. In a recent review of research in the Soviet Union, Prokhorov et al[32] have provided a more physical means of deriving Eqs. (16), (21), and (24).

31. (a) Novikov, E. (1965) Functionals and the method of random forces in the theory of turbulence, Soviet Physics JETP 20:1290-1294.
(b) Furutsu, K. (1963) On the statistical theory of electromagnetic waves in a fluctuating medium, J. Res NBS 67D:303-310.

32. Prokhorov, A., Bunkin, F., Gochelashvily, K., and Shishov, V. (1975) Laser irradiance propagation in turbulent media, Proc. IEEE 63:790-811.

This equation is readily solved for $<u>$. The result is

$$\langle u(x,\underline{\rho})\rangle = \left(\frac{k}{2\pi i x}\right) \iint_{-\infty}^{\infty} d^2\rho' \, u_o(\underline{\rho}') \exp\left[i\frac{k(\underline{\rho}-\underline{\rho}')^2}{2x} - \frac{k^2}{8}\int_0^x A(x',0)dx'\right] \tag{17}$$

where $u_o(\underline{\rho}')$ is the electric field in the plane of the transmitter; we have chosen this to be the $x = 0$ plane. For the spectrum in Eq. (3 the function A(0) appearing in Eq. (17) is readily evaluated to give

$$\frac{k^2}{8}\int_0^x A(x',0)\,dx' = 0.391\, k^2 L_o^{5/3} \int_0^x C_n^2(x')\,dx' \quad . \tag{18}$$

If we use Eq. (18) in (17) and then consider the limiting case of a plane wave, it is readily shown that

$$\langle u(x)\rangle = \exp\left\{-0.391\, k^2 L_o^{5/3} \int_0^x C_n^2(x')\,dx'\right\} \quad . \tag{17a}$$

For an optical wave propagating in the lower atmosphere, $\langle u\rangle \simeq 0$.* For example, for a signal of wavelength μm propagating over a path of 1 km in homogeneous turbulence with $C_n^2 = 10^{-14}\,m^{-2/3}$ and $L_o = 100$ m, we find that $|\langle u\rangle| = \exp(-3.4 \times 10^5)$. It can also be shown that $|\langle uu\rangle| \simeq 0$, but as we shall see later in this section, this is not true for $\langle uu^*\rangle$.

It can be demonstrated[1] that the conditions required for the validity of both the parabolic approximation in Eq. (12) and the "Markov Approximation" leading to Eq. (16) are that

$$k\,\ell_o >> 1 \tag{19a}$$

and

$$k\,C_n^2\,L_o^{5/3} << 1. \tag{19b}$$

The condition in Eq. (19a) requires that the scattering pattern of even the smallest turbulent eddies must be primarily in the forward direction, while that requirement expressed by Eq. (19b) is that there be very little attenuation (due to scatter) of

*The vanishing of the field is a consequence of the randomization of the phase.

the signal over one wavelength. This condition can alternately be written as $\alpha_e \lambda << 1$, where α_e is the extinction coefficient. In addition to the conditions given in Eqs. (19a) and (19b), a number of sufficient conditions for application of the Markov approximation have been obtained.[1] Among these are that $C_n^2 \ell_o^{-1/3} x << 1$ and $k C_n^2 L_o^{5/6} \ell_o^{-1/6} x << 1$. However, it is not presently clear whether these are also necessary conditions.

If we substitute numbers typical of C_n^2, L_o and ℓ_o in the earth's atmosphere, we find that the condition in Eq. (19b) is really not very restrictive and that the Markov approximation is valid over horizontal propagation paths of many hundreds or thousands of kilometers in the earth's atmosphere; this is quite an improvement over the Rytov approximation, discussed in the last section, which was valid only over distances less than about one kilometer.

It is also possible to derive the equations satisfied by the higher order moments of the field; the equation satisfied by second moment

$$\Gamma_2 (x, \underline{\rho_1}, \underline{\rho_2}) = \langle u(x, \underline{\rho_1})\, u^* (x, \underline{\rho_2}) \rangle$$

can be obtained by multiplying Eq. (12) by u^* and then using Eq. (15) with $\phi[n_1] = u(x, \underline{\rho_1}) u^*(x, \underline{\rho_2})$ to evaluate the moment $\langle n_1 u u^* \rangle$. The result is

$$\left(2ik \frac{\partial}{\partial x} + T_1 - T_2 \right) \Gamma_2 + \frac{ik^3}{2} \left[A(x, 0) - A(x, \underline{\rho_1} - \underline{\rho_2}) \right] \Gamma_2 = 0 , \tag{20}$$

where

$$T_j = \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2} .$$

Equation (20) can be solved in a straight forward fashion using Fourier transform methods, and the result is

$$\Gamma_2 (x, \underline{\rho_1}, \underline{\rho_2}) = \left(\frac{k}{2\pi x}\right)^2 \iint_{-\infty}^{\infty} d^2\rho_1' \iint_{-\infty}^{\infty} d^2\rho_2' \; u_o(\underline{\rho_1'})\, u_o^*(\underline{\rho_2'})$$
$$\times \exp\left\{ i \frac{k}{2x}\left[(\underline{\rho_1} - \underline{\rho_1'})^2 - (\underline{\rho_2} - \underline{\rho_2'})^2 \right] - \frac{\pi k^2}{4} \int_0^x dx' \, H\left[x'; \frac{x'}{x}(\underline{\rho_1} - \underline{\rho_2}) + (\underline{\rho_1'} - \underline{\rho_2'})\left(1 - \frac{x'}{x}\right) \right] \right\} \tag{21}$$

where

$$H(x, \underline{\xi}) = 8 \iint_{-\infty}^{\infty} (1 - \cos \underline{\kappa} \cdot \underline{\xi})\, \Phi_n(x, \kappa_x = 0, \kappa_y, \kappa_z)\, d\kappa_y\, d\kappa_z \; . \tag{22}$$

For the spectrum Φ_n given in Eq. (3), it is straightforward to show that

$$H(x, \underline{\xi}) \simeq 1.88\, C_n^2(x) \left|\underline{\xi}\right|^{5/3} \left[1 - 0.805\left(\frac{|\underline{\xi}|}{L_o}\right)^{1/3}\right] \tag{23}$$

provided* $\ell_o << |\xi| < L_o$. Equation (21) gives the general spatial coherence function of the beam; the intensity distribution $I(x, \underline{\rho}_1) \equiv u(x, \underline{\rho}_1)\, u^*(x, \underline{\rho}_1)$ can be obtained from Eq. (21) simply by setting $\underline{\rho}_2$ equal to $\underline{\rho}_1$.

It is also possible to derive the equation satisfied by the fourth moment of the field

$$\Gamma_4(x, \underline{\rho}_1, \underline{\rho}_2, \underline{\rho}_3, \underline{\rho}_4) \equiv \langle u(x, \underline{\rho}_1)\, u^*(x, \underline{\rho}_2)\, u(x, \underline{\rho}_3)\, u^*(x, \underline{\rho}_4) \rangle \; .$$

The procedure is the same as was followed in deriving Eq. (18) and the result is

$$\begin{aligned} &\left[\frac{\partial}{\partial x} - \frac{i}{2k}(T_1 + T_3 - T_2 - T_4)\right] \Gamma_4 + \frac{k^2}{4}\Big[2A(x, 0) - A(x, \underline{\rho}_1 - \underline{\rho}_2) \\ &- A(x, \underline{\rho}_3 - \underline{\rho}_4) - A(x, \underline{\rho}_1 - \underline{\rho}_4) - A(x, \underline{\rho}_3 - \underline{\rho}_2) + A(x, \underline{\rho}_3 - \underline{\rho}_1) + A(x, \underline{\rho}_1 - \underline{\rho}_2)\Big] \Gamma_4 = 0 \end{aligned} \tag{24}$$

Unlike (20), Eq. (24) cannot be solved in closed form. This is unfortunate since a knowledge of Γ_4 is necessary in order to calculate intensity scintillations, beam wander, aperture averaging, and a number of other measurable effects. However, a number of approximate solutions, valid for different ranges of σ_1^2, have recently been developed and will be discussed in detail in Section 5.

We could also go on to present the equations satisfied by yet higher order field moments such as Γ_6, Γ_8 etc; however, these are generally of little practical importance and do not warrant detailed discussion. Rather, most of the remainder of this report will be devoted to the study of the properties of the solutions for Γ_2 and Γ_4, and their application to develop expressions for the beam spread, coherence, etc. Before we begin the detailed discussion of the solutions, however, it is useful to discuss an alternate representation of the random propagation

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*For $\underline{\xi} \le \ell_o$, we have $H(x, \underline{\xi}) = 1.88\, C_n^2(x) [(\xi^2 + \ell_o^2)^{5/6} - \ell_o^{5/3}]$.

channel as a linear system. This representation is being increasingly used by optical designers and should be included in our preliminary discussion.

2.1 Linear Systems Representation

It is possible to demonstrate[26,33] that the solution of Eq. (12) can be written as

$$u(x, \underline{\rho}_1) = \frac{k}{2\pi i x} \iint_{-\infty}^{\infty} d^2 \rho_1' \; u_o(\underline{\rho}_1') \exp\left[\frac{ik(\underline{\rho}_1 - \underline{\rho}_1')^2}{2x} + \psi(\underline{\rho}_1, \underline{\rho}_1')\right] \tag{25}$$

where $u_o(\underline{\rho}_1')$ is the field distribution in the $x = 0$ plane and $\psi(\underline{\rho}_1, \underline{\rho}_1')$ is the random part of the complex phase of a spherical wave propagating in the turbulent medium from the point $(0, \underline{\rho}_1')$ to the point $(x, \underline{\rho}_1)$. Equation (25) is merely an extended version of the Huygens-Fresnel formula which has been shown[34] to be the solution to Eq. (12) in the limit when $n_1 = 0$.

The form of the solution in Eq. (25) is quite appealing for optical systems applications, since it can be rewritten as

$$u(\underline{\rho}_1) = \iint_{-\infty}^{\infty} d^2 \rho_1' \; u_o(\underline{\rho}_1') \, h(\underline{\rho}_1, \underline{\rho}_1') \quad , \tag{26}$$

where $h(\underline{\rho}_1, \underline{\rho}_1') = (k/(2\pi i x)) \exp[\, ik(\underline{\rho}_1 - \underline{\rho}_1')^2/2x + \psi(\underline{\rho}_1, \underline{\rho}_1')\,]$ can be interpreted as the (spatial) impulse response of the system. It should be pointed out that $h(\underline{\rho}_1, \underline{\rho}_1')$ can be shown[35] to be reciprocal for propagation in atmospheric turbulence. That is, suppose $\underline{\rho}_1$ lies in a receiving aperture Σ and $\underline{\rho}_1'$ lies in some transmitting aperture Σ'. Now suppose the roles of Σ and Σ' are reversed so that Σ is the transmitter and Σ' is the receiver. Then at a point $\underline{\rho}_1'$ in the receiving aperture Σ', we can write

$$u(\underline{\rho}_1') = \iint_{-\infty}^{\infty} d^2 \rho_1 \; \tilde{u}_o(\underline{\rho}_1) \, h(\underline{\rho}_1, \underline{\rho}_1') \quad , \tag{27}$$

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33. Feizulin, Z., and Kravtsov, Y. (1967) Broadening of a laser beam in a turbulent medium, Radiophysics and Quantum Electronics 10:33-35.

34. Born, M., and Wolf, E. (1965) Principles of Optics, Pergamon Press, Oxford, England.

35. Shapiro, J. (1971) Reciprocity of the turbulent atmosphere, J. Opt. Soc. Amer. 61:492-495.

where $\tilde{u}_0(\underline{\rho}_1)$ is the aperture distribution in the transmitting aperture Σ. This result is important in a number of practical systems applications, as we shall see later.

We next consider the moments of Eq. (26). We have

$$\langle u(x,\underline{\rho}_1)\rangle = \iint_{-\infty}^{\infty} d^2\rho_1' \, u_0(\underline{\rho}_1') \, \langle h(\underline{\rho}_1,\underline{\rho}_1')\rangle \quad . \tag{28}$$

If we compare Eq. (28) with Eq. (17), it is readily seen that

$$\langle h(\underline{\rho}_1,\underline{\rho}_1')\rangle = \frac{k}{2\pi i x} \exp\left[\frac{ik(\underline{\rho}_1-\underline{\rho}_1')^2}{2x} - \frac{k^2}{8}\int_0^x A(x',0)\,dx'\right] . \tag{29}$$

For the second moment, we have

$$\langle u(x,\underline{\rho}_1)\,u^*(x,\underline{\rho}_2)\rangle = \iint_{-\infty}^{\infty} d^2\rho_1' \iint_{-\infty}^{\infty} d^2\rho_2' \, u_0(\underline{\rho}_1')\,u_0(\underline{\rho}_2')\,\langle h(\underline{\rho}_1,\underline{\rho}_1')\,h^*(\underline{\rho}_2,\underline{\rho}_2')\rangle \; . \tag{30}$$

Upon comparing Eqs. (30) and (21), we can readily identify

$$\langle h(\underline{\rho}_1,\underline{\rho}_1')\,h^*(\underline{\rho}_2,\underline{\rho}_2')\rangle = \frac{k^2}{2\pi x}\exp\left\{i\frac{k}{2x}\left[(\underline{\rho}_1-\underline{\rho}_1')^2 - (\underline{\rho}_2-\underline{\rho}_2')^2\right.\right.$$

$$\left.\left. - \frac{\pi k^2}{4}\int_0^x dx'\,H\left[x', \frac{x'}{x}(\underline{\rho}_1-\underline{\rho}_2) + (\underline{\rho}_1'-\underline{\rho}_2')(1-\frac{x'}{x})\right]\right\} \quad . \tag{31}$$

Of course, the representation in Eqs. (26), (28) and (30) does not contain any new information, beyond what we have derived in Section 2.3. However, this systems notation has now become so widely used that we felt it should be presented separately.

We are now in a position to begin our discussion of the various measurable properties of a light beam in a random medium.

3. BEAM SPREAD AND AVERAGE INTENSITY

In the absence of turbulence a laser beam exiting from an aperture of diameter D would, in the far field, have an angular spread θ_0 of order λ/D, where λ is the signal wavelength. When turbulence is present, the situation becomes much more complex because the beam is scattered by the moving turbulent eddies. This gives rise to an angular beam spread which may be much greater than θ_o; in addition, other effects such as beam wander or even breakup of the beam into an ensemble of individual beams may occur. In this section we will discuss these effects, along with the corresponding degradation of the beam intensity (or irradiance).

3.1 Short and Long Term Beam Spread

When discussing the radius of a beam propagating in a turbulent medium, it is necessary to distinguish between its short and long term spread. In general, when a laser beam interacts with the turbulent eddies it is found that those eddies which are large compared with the diameter of the beam tend to deflect the beam, whereas those eddies which are small compared with the beam diameter tend to broaden the beam, but do not deflect it significantly. Consequently, if we had a photographic plate at a distance x into the random medium and took a very short exposure picture, we would observe a broadened laser spot (due to the small eddies) of radius ρ_s which is deflected (due to the large eddies) by a distance ρ_c, as indicated pictorially in Figure 1. Now, because the turbulent eddies are flowing across the beam, the beam will be continually deflected in different directions in time intervals or order $\Delta t = D/\underline{V}$, where $\underline{V}$ is the transverse flow velocity of the turbulent eddies. The time history of the beam wander is shown in Figure 2. Because the spot dances from position to position in times of order Δt, it is clear from Figure 2 that if we took a picture — with an exposure time much longer than Δt — of the received spot we would see a broadened spot with a mean square radius $\langle \rho_L^2 \rangle$ given by

$$\langle \rho_L^2 \rangle = \langle \rho_s^2 \rangle + \langle \rho_c^2 \rangle . \qquad (32)$$

where ρ_s is the short-term beam spread and ρ_L is the long-term spread.

Unfortunately, the model given above is not the whole story, and only holds in the limit when the turbulence is relatively weak. When the turbulence is strong the beam no longer wanders significantly, but rather breaks up into multiple beams. In this case a short exposure picture of the received spot would consist not of a single spot, but of a multiplicity of spots at random locations on the receiving aperture. The long-exposure picture, however, would be a blurred version of the short exposure, but with approximately the same total diameter.

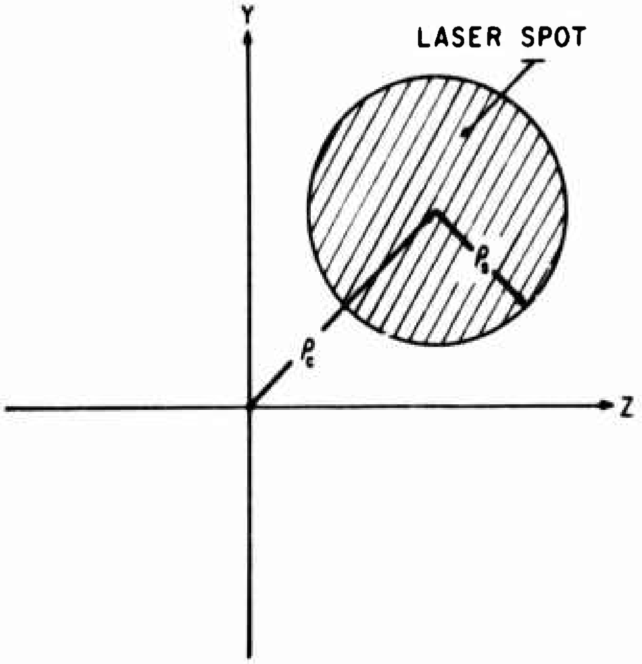

Figure 1. Short-exposure-time Broadening and Deflection of a Laser Spot on a Receiver on a Turbulent Medium

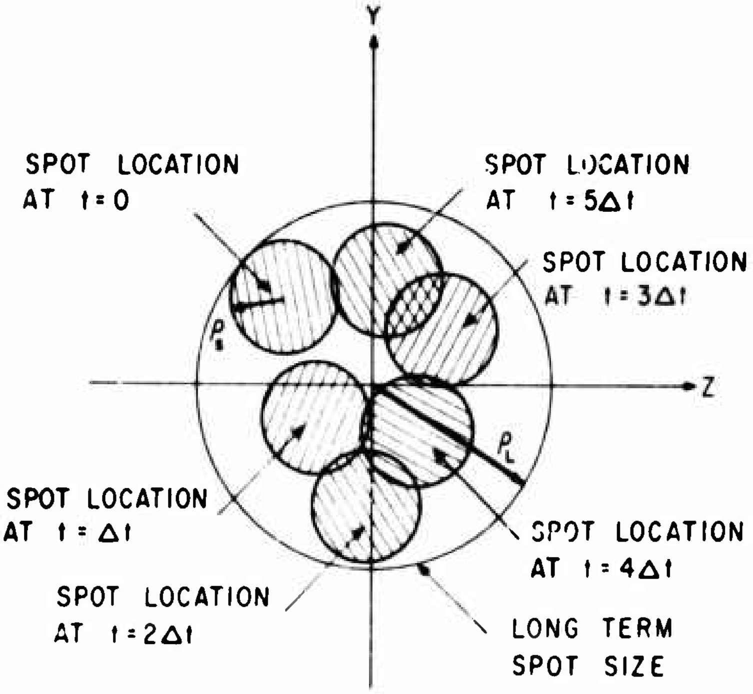

Figure 2. Time History of the Wander of a Laser Beam on a Receiver in a Turbulent Medium

We now desire to pursue the previous discussion further in a more quantitative fashion. It is conventional to define the beam centroid $\underline{\rho}_c$ by

$$\underline{\rho}_c = \frac{\iint_{-\infty}^{\infty} d^2\rho \, \underline{\rho} \, I(x, \underline{\rho})}{\iint_{-\infty}^{\infty} d^2\rho \, \langle I(x, \underline{\rho}) \rangle} \tag{33}$$

where $I(x, \underline{\rho})$ is the intensity of the beam. The mean square beam deflection can then be written as

$$\left\langle \rho_c^2 \right\rangle = \frac{\iint_{-\infty}^{\infty} d^2\rho_1 \iint_{-\infty}^{\infty} d^2\rho_2 \, (\underline{\rho}_1 \cdot \underline{\rho}_2) \, \Gamma_4(x, \underline{\rho}_1, \underline{\rho}_1, \underline{\rho}_2, \underline{\rho}_2)}{\left[\iint_{-\infty}^{\infty} d^2\rho_1 \, \Gamma_2(x, \underline{\rho}_1, \underline{\rho}_1) \right]^2} \tag{34}$$

where Γ_2 and Γ_4 are the solutions of Eqs. (20) and (24) respectively. Also, the mean-square long term beam radius $\left\langle \rho_L^2 \right\rangle$ is usually defined by

$$\left\langle \rho_L^2 \right\rangle = \frac{\iint_{-\infty}^{\infty} d^2\rho \, \rho^2 \, \Gamma_2(x, \underline{\rho}, \underline{\rho})}{\iint_{-\infty}^{\infty} d^2\rho \, \Gamma_2(x, \underline{\rho}, \underline{\rho})} . \tag{35}$$

Therefore, if Γ_2 and Γ_4 were completely known we could readily calculate $\left\langle \rho_c^2 \right\rangle$ and $\left\langle \rho_L^2 \right\rangle$. However, as was pointed out in Section 2.3, a rigorous solution is available for Γ_2 but we have only limited approximate solutions for Γ_4. Furthermore, the definitions in Eqs. (34) and (35) for $\left\langle \rho_c^2 \right\rangle$ and $\left\langle \rho_L^2 \right\rangle$ are clearly not unique, and there are a number of other definitions possible. We have therefore chosen to calculate $\left\langle \rho_L^2 \right\rangle$, $\left\langle \rho_s^2 \right\rangle$ and $\left\langle \rho_c^2 \right\rangle$ in a different fashion. In particular we will define $\left\langle \rho_L^2 \right\rangle$ and $\left\langle \rho_s^2 \right\rangle$ as the radii at which the long and short-term averaged intensity distributions are reduced by a factor of e^{-1} from their maximum values. The long-term-averaged beam radius can be deduced directly from Eq. (21), and the short-term beam radius can be obtained by modifying Eq. (21) in such a way that the contributions from turbulent eddies, which are larger than the beam diameter, are excluded because they cause only beam tilt. We will first present the results for $\langle \rho_L^2 \rangle$, and then discuss the calculation of $\langle \rho_s^2 \rangle$ and $\langle \rho_c^2 \rangle$.

In order to simplify the calculation of $\langle \rho_L^2 \rangle$, we shall assume that the initial field distribution has the gaussian form

$$u_o(\underline{\rho}) = \exp\left\{ -\frac{2\rho^2}{D^2} - \frac{ik\rho^2}{2F} \right\} . \tag{36}$$

This corresponds to a beam with an initial diameter D and a radius of curvature -F. For this distribution it is possible to show, by using the solution for Γ_2 presented in Eq. (21) that[36, 37]

$$\langle \rho_L^2 \rangle \simeq \frac{4x^2}{k^2 D^2} + \frac{D^2}{4}\left(1 - \frac{x}{F}\right)^2 + \frac{4x^2}{k^2 \rho_o^2} , \tag{37}$$

where

$$\rho_o = \left[1.46\, k^2 x \int_0^1 d\xi\, (1 - \xi)^{5/3}\, C_n^2\, (\xi x) \right]^{-3/5} . \tag{38}$$

Equation (37) is an excellent approximation for $\langle \rho_L^2 \rangle$ when $x \ll (k^2 C_n^2 \ell_o^{5/3})^{-1}$. For $x \ll (k^2 C_n^2 \ell_o^{5/3})^{-1}$, we have

$$\langle \rho_L^2 \rangle \simeq \frac{4x^2}{k^2 D^2} + \frac{D^2}{4}\left(1 - \frac{x}{F}\right)^2 + \frac{6.6\, x^3 \int_0^1 (1-\xi)^2 C_n^2 (\xi x)\, d\xi}{\ell_o^{1/3}} . \tag{37a}$$

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*Results equivalent to Eqs. (37) and (39) have also been obtained by the method of small perturbations[38] and the method of smooth perturbations.[39]

36. Yura, H. (1973) Short term average optical-beam spread in a turbulent medium, J. Opt. Soc. Amer. 63:567-572.

37. Fante, R. (1974) Mutual coherence function and frequency spectrum of a laser beam propagating through atmospheric turbulence, J. Opt. Soc. Amer. 64:592-598.

38. Poirier, J., and Korff, D. (1972) Beam spreading in a turbulent medium, J. Opt. Soc. Amer. 62:893-898.

39. Bunkin, F., and Gochelashvili, K. (1970) Spreading of a light beam in a turbulent medium, Radiophysics and Quantum Electronics 13:811-821.

The first two terms in Eqs. (37) and (37a) represent the beam spreading in vacuum; the last term represents the additional spread due to the scattering of the beam by the turbulent eddies.

The validity of Eq. (37) has been studied in some detail. Measurements in the Soviet Union[40] appear to confirm the validity of the result; however, other experiments[41] in this country do not. That is, Eq. (37) predicts a dependence of $\langle \rho_L^2 \rangle$ on C_n of $C_n^{12/5}$, but Dowling and Livingston[42] found a dependence of $C_n^{6/5}$. However, as pointed out by Wesley and Derzko[42] this discrepancy is because Dowling and Livingston used exposure times which exceeded L_0/V.

In order to present the approximate results for $\langle \rho_s^2 \rangle$ and $\langle \rho_c^2 \rangle$, we shall have to consider a number of separate cases:

(i) If $\rho_o << D < L_0$ and $x \leq k\, L^2$, where ρ_o is given in Eq. (33) and L is the smaller of the beam diameter D and the coherence length ρ_o, then the beam centroid and short-term beam spread are given approximately by[36, 43-45]

$$\langle \rho_s^2 \rangle \simeq \frac{4x^2}{k^2 D^2} + \frac{D^2}{4}\left(1 - \frac{x}{F}\right)^2 + \frac{4x^2}{k^2 \rho_o^2}\left[1 - 0.62\left(\frac{\rho_o}{D}\right)^{1/3}\right]^{6/5}, \quad (39)$$

$$\langle \rho_c^2 \rangle \simeq \frac{2.97\, x^2}{k^2 \rho_o^{5/3} D^{1/3}} . \quad (40)$$

In writing Eqs. (39) and (40), we have also assumed that $D > \ell_0$. The validity of Eq. (40) has been verified experimentally,[40, 41, 46, 47] although the measured

40. Kallistrova, M., and Pokasov, V. (1971) Defocusing and fluctuations of the displacement of a focused laser beam in the atmosphere, Radiophysics and Quantum Electronics 14:940-945.

41. Dowling, J., and Livingston, P. (1973) Behavior of focused beams in atmospheric turbulence: measurements and comments on the theory, J. Opt. Soc. Amer. 63:846-858.

42. Wesley, M., and Derzko, Z. (1975) Atmospheric turbulence parameters from visual resolution, Appl. Optics 14:847-853.

43. Fried, D. (1966) Optical resolution through a randomly inhomogeneous medium for very long and very short exposures, J. Opt. Soc. Amer. 56:1372-1379.

44. Andreev, G., and Gel'fer, E. (1971) Angular random walks of the center of gravity of the cross section of a diverging light beam, Radiophysics and Quantum Electronics 14:1145-1147.

45. Kon, A. (1970) Focusing of light in a turbulent medium, Radiophysics and Quantum Electronics 13:43-50.

46. Gel'fer, E., Kravtsov, V., and Finkel'shtein, S. (1972) Measurement of the light intensity on the axis at the center of gravity of a focused light beam, Radiophysics and Quantum Electronics 15:696-699.

47. Chiba, T. (1971) Spot dancing of the laser beam propagated through the turbulent atmosphere, Appl. Optics 10:2456-2461.

coefficient appears to be slightly smaller than the theoretical value of 2.97*. In addition, the wander of the center of gravity of the beam appears to satisfy a gaussian distribution;[41] based on this, several authors[49, 50] have developed theories for the statistics of the laser beam fade induced by beam wander.

(ii) If $\rho_o \sim D$ and $x \lesssim kL^2$, there are no simple expressions for $\langle \rho_c^2 \rangle$ and $\langle \rho_s^2 \rangle$. In this case, $\langle \rho_s^2 \rangle$ can be obtained from the numerical results in Figure 3. In that figure

$$\beta^2 = \left(\frac{kD^2}{4x}\right)^2 \left(1 - \frac{x}{F}\right)^2 \tag{41a}$$

and

$$\mu = \left[\frac{\langle \rho_s^2 \rangle}{\langle \rho_L^2 \rangle}\right]^{1/2} . \tag{41b}$$

Once $\langle \rho_s^2 \rangle$ has been obtained from Figure 3, $\langle \rho_c^2 \rangle$ follows immediately from Eq. (32).

(iii) If $\rho_o >> D$ and $x \lesssim kL^2$, there is very little beam wander, and the short and long term beam spreads are approximately equal, and are given by Eqs. (37) or (37a), depending on whether x is less than or greater than $k^2 C_n^2 \ell_o^{5/3}$.

(iv) If $x >> kL^2$, we expect the beam will be broken up into multiple patches, with negligible (compared with the total beam spread) wander of the beam centroid. Some calculations for this case have been made by Klystskin and Kon.[51] For $x >> kD^2$ and $D/\rho_o >> 1$, it is found that in homogeneous turbulence**

$$\langle \rho_c^2 \rangle \simeq C_n^{8/5} k^{-1/15} x^{37/15} . \tag{42}$$

*The temporal correlation function $\langle \rho_c(t) \rho_c(t + \tau) \rangle$ has also been obtained;[48] as expected it is found that the correlation time for beam wander is approximately D/V.

**By "homogeneous" turbulence, we will mean that C_n^2 is independent of position.

48. Gel'fer, E., Kon, A., and Cheremukhin, A. (1973) Correlation of the shift of the center of gravity of a focused light beam in a turbulent medium, Radiophysics and Quantum Electronics 16:182-187.

49. Fried, D. (1973) Statistics of laser beam fade induced by pointing jitter, Appl. Optics, 12:422-423.

50. Titterton, P. (1973) Power reduction and fluctuations caused by narrow laser beam motion in the far field, Appl. Optics 12:423-425.

51. Klystskin, V., and Kon, A. (1972) On the displacement of spatially-bounded light beams in a turbulent medium in the Markovian-random-process approximation, Radiophysics and Quantum Electronics 15:1056-1061.

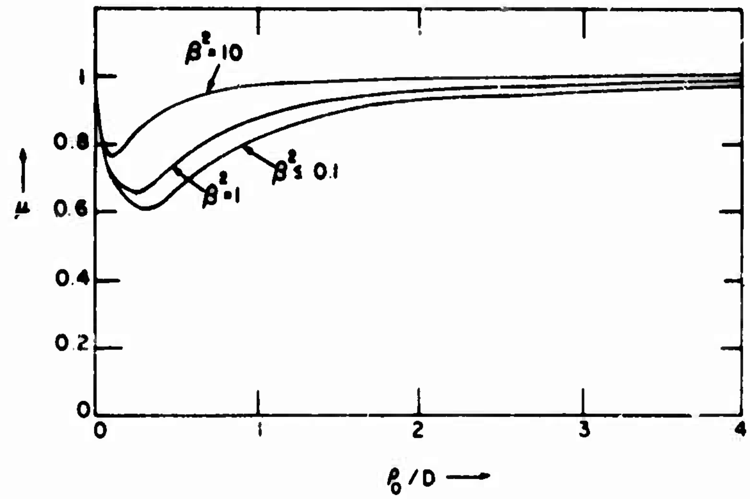

Figure 3 Ratio of the Short to Long-term-averaged Beam Spread

If we use Eqs. (37) and (42) to form the ratio $\langle\rho_c^2\rangle / \langle\rho_L^2\rangle$, we find that

$$\frac{\langle\rho_c^2\rangle}{\langle\rho_L^2\rangle} \sim \left(\frac{k\rho_o^2}{x}\right)^{1/3} .$$

Because $x >> k\rho_o^2$, it is clear that $\langle\rho_c^2\rangle << \langle\rho_L^2\rangle$, so that any motion of the beam centroid is negligible in comparison with the beam spread.

In the previous case where the beam is broken up into multiple beams, a knowledge of $\langle\rho_c^2\rangle$ and $\langle\rho_L^2\rangle$ does not tell us how many beams will be formed, or correspondingly how many bright patches will be formed on a receiver at some distance x into the random medium. All we can say is that the mean square radius of the region where the bright patches will be formed is $\langle\rho_L^2\rangle$.

3.2 Average Intensity

The formal expressions for the long-term averaged beam intensity is given by Eq. (21) with $\underline{\rho}_1 = \underline{\rho}_2 = \underline{\rho}$. In particular, for an arbitrary initial field distribution $u_o(\underline{\rho})$ the long-term-averaged intensity is

$$\left\langle I(x, \underline{\rho})\right\rangle = \left(\frac{k}{2\pi x}\right)^2 \iint_{-\infty}^{\infty} d^2\rho_1' \iint_{-\infty}^{\infty} d^2\rho_2' \, u_o(\underline{\rho}_1') \, u_o^*(\underline{\rho}_2')$$

$$\times \exp\left\{ i \frac{k}{2x}\left[(\underline{\rho} - \underline{\rho}_1')^2 - (\underline{\rho} - \underline{\rho}_2')^2\right] - \frac{\pi k^2}{4} \int_0^x dx' \, H\left[x', (\underline{\rho}_1' - \underline{\rho}_2')\left(1 - \frac{x'}{x}\right)\right]\right\}. \quad (43)$$

In our discussions, we will not explicitly evaluate Eq. (43). Rather, we will assume that the initial field $u_o(\underline{\rho})$ is given by Eq. (36) and then use the results of Eqs. (37) and (37a), along with energy conservation to calculate $\langle I(x, 0)\rangle$. That is, since the turbulent eddies are much larger than the signal wavelength, nearly all of the energy is scattered in the forward direction. Consequently $\langle I(x, 0)\rangle \langle \rho^2(x)\rangle$ is approximately a constant. If we denote the intensity at x = 0 by I_o, we then have for the axial intensity

$$I_o \frac{D^2}{4} \simeq \langle I(x, 0)\rangle \langle \rho^2(x)\rangle . \quad (44)$$

We next substitute Eq. (37) into (44) to obtain for $x << (k^2 C_n^2 \ell_o^{5/3})^{-1}$

$$\langle I(x, 0)\rangle \simeq \frac{\left(\frac{D^2}{4}\right) I_o}{\frac{4x^2}{k^2 D^2} + \frac{D^2}{4}\left(1 - \frac{x}{F}\right)^2 + \frac{4x^2}{k^2 \rho_o^2}} . \quad (45)$$

For $x >> (k^2 C_n^2 \ell_o^{5/3})^{-1}$ we get, upon using Eq. (37a) in (44)

$$\langle I(x, 0)\rangle \simeq \frac{\left(\frac{D^2}{4}\right) I_o}{\frac{4x^2}{k^2 D^2} + \frac{D^2}{4}\left(1 - \frac{x}{F}^2\right) + \frac{6.6 x^3}{\ell_o^{1/3}} \int_0^1 (1-\xi)^2 d\xi \, C_n^2(\xi x)} . \quad (46)$$

The approximate results in Eqs. (45) and (46) agree quite well with more rigorous theoretical results[1, 37] obtained by using Eq. (43), and also with experimental data.[52, 53] In almost all practical situations, we may use the

52. Starobinets, E. (1972) The average illumination and intensity fluctuations at the focus of a light beam in a turbulent atmosphere, Radiophysics and Quantum Electronics 15:738-742.

53. Mironov, V. and Khmelevstov, S. (1972) Broadening of a laser beam propagating in a turbulent atmosphere along inclined routes, Radiophysics and Quantum Electronics 15:567-571.

result of Eq. (45) for the long-term beam intensity, since in atmospheric turbulence $(k^2 C_n^2 \ell_o^{5/3})^{-1}$ is of order of 100 km.

We now observe that if the flux $P_o = \pi D^2 I_o/4$ through the transmitting aperture is held fixed, Eq. (45) predicts that, in turbulence, there is a limiting value of the intensity at the focus (x = F) of the beam, no matter how large the initial diameter D of the beam is made. This maximum intensity is

$$\langle I \rangle_{MAX} = \frac{0.051\, P_o}{F^{16/5}\, k^{2/5} \left[\int_0^1 d\xi\, (1-\xi)^{5/3}\, C_n^2\, (\xi F) \right]^{6/5}} \tag{47}$$

Kallistrova and Kon[54] have made detailed measurements of the focused beam intensity in turbulence; they have found that there is a limit to the intensity of a focused beam, and that Eq. (47) is a good approximation to that limit.

As we mentioned above, Eqs. (45) and (46) give the long time averaged axial beam intensity in the plane x = x in a random medium. It is also possible to use energy conservation to estimate the short term-averaged intensity. If Eq. (39) is used in (44), we have (for $\rho_o \ll D < L_o$ and $x \leq k L^2$ where L is the smaller and ρ_o and D) for the short term averaged intensity on the instantaneous beam axis

$$\left\langle I_s(x) \right\rangle \simeq \frac{\frac{D^2}{4} I_o}{\frac{4x^2}{k^2 D^2} + \frac{D^2}{4}\left(1 - \frac{x}{F}\right)^2 + \frac{4x^2}{k^2 \rho_o^2}\left[1 - 0.62\left(\frac{\rho_o}{D}\right)^{1/3}\right]^{6/5}}. \tag{48}$$

Equation (48) gives the spot intensity we would measure if we took a short exposure photograph of the beam spot in the plane x. We note, upon comparing Eqs. (48) and (45), that the short term spot will be brighter than the long term averaged spot.

4. SPATIAL COHERENCE OF THE ELECTRIC FIELD

The function $\Gamma_2(x, \rho_1, \rho_2) = \left\langle u(x, \rho_1)\, u^*(x, \rho_2) \right\rangle$ is a measure of the long-term spatial coherence of the electric field, in a plane transverse to the direction

54. Kallistrova, M., and Kon, A. (1972) On the effect of the size of optical systems on the definition of light beams in a turbulent atmosphere, Radiophysics and Quantum Electronics 15:545-549.

of propagation of the beam.* This function is important for interferometry experiments in radio astronomy, and because it determined the signal-to-noise ratio in an optical heterodyne receiver.[54] Exact expressions for Γ_2 in the limiting cases of plane and spherical waves have been available[57,58] for some time; however, the general solution for Γ_2 for a finite beam has only more recently been derived[1,37,59] and numerically evaluated.[37] In particular, the general solution for the coherence relative to the average center of the beam is given by Eq. (21) with $\underline{\rho}_1$ set equal to zero. We will not present the general solution for Γ_2 here, but will rather present an approximate evaluation of Eq. (21) for the case when the initial electric field of the beam is given by Eq. (36). We then find that the long-term coherence function $M(x, \underline{\rho})$ which measures coherence relative to the undisplaced center of the beam, is[59]

$$M(x, \underline{\rho}) = \frac{\Gamma_2(x, 0, \underline{\rho})}{\Gamma_2(x, 0, 0)} \simeq \exp\left[-\left(\frac{\rho}{\rho_b}\right)^{5/3}\right] , \tag{49}$$

where

$$\rho_b = \rho_p \left[\frac{\left(1 - \frac{x}{F}\right)^2 + \frac{4x^2}{k^2 D^4}\left\{1 + \frac{1}{3}\left(\frac{D}{\rho_p}\right)^2\right\}}{1 - \frac{13}{3}\left(\frac{x}{F}\right) + \frac{11}{3}\left(\frac{x}{F}\right)^2 + \frac{4x^2}{3k^2 D^4}\left\{1 + \frac{1}{4}\left(\frac{D}{\rho_p}\right)^2\right\}} \right] , \tag{50}$$

ρ_p is the plane-wave coherence length given by

$$\rho_p = \left[1.46\, k^2\, x \int_0^1 d\xi\, C_n^2\, (\xi x) \right]^{-3/5} , \tag{51}$$

---

*In some cases, it may also be desirable to know the longitudinal coherence length of the beam. Bremmer[55] has evaluated the longitudinal coherence length and found it to be of order $(k^2 C_n^2 L_o^{5/3})^{-1}$. That is, $\langle u(x,\underline{\rho})\, u^*(x',\underline{\rho}) \rangle$ is correlated over distances $|\underline{x} - \underline{x}'|$ of order $(k^2 C_n^2 L_o^{5/3})^{-1}$.

55. Bremmer, H. (1973) General remarks concerning theories dealing with scattering and diffraction in random media, Radio Science 8:511-534.

56. Fried, D., (1967) Optical heterodyne detection of an atmospherically distorted signal wavefront, Proc IEEE 55:57-67.

57. Hufnagel, R., and Stanley, N. (1964) Modulation transfer function associated with image transmission through turbulent media, J. Opt. Soc. Am. 54:52-61.

58. Kon, A., and Feizulin, V. (1970) Fluctuations in the parameters of spherical waves propagating in a turbulent atmosphere, Radiophysics and Quantum Electronics 13:51-53.

59. Yura, H. (1972) Mutual coherence function of a finite cross section optical beam propagating in a turbulent medium, Appl. Optics 11:1399-1406.

and it is assumed that $\ell_o << \rho_p < L_o$. The simplified expression in Eq. (49) can be shown[37] to be a good approximation to the exact results obtained from Eq. (21). It can also be demonstrated that the same trends, as predicted by Eq. (49), are observed experimentally. For example, consider the plane wave limit in which

$$M(x, \rho) = \exp\left\{-\left(\frac{\rho}{\rho_p}\right)^{5/3}\right\} . \tag{52}$$

Gilmartin and Holtz[60a] and Dainty and Scanlon[60b] have found experimentally that ρ_p varies as $k^{-6/5}$, as predicted by Eq. (51). Further verification is found in recent Soviet experiments[61] which indicate that ρ_p varies as $k^{-6/5}\, x^{-3/5}\, C_n^{-6/5}$ in complete agreement with Eqs. (51) and (52).

It is important to emphasize that ρ_b is the long-term averaged coherence length of the beam. This means that if we performed an interferometry experiment over times much longer than $\Delta t = D/\underline{V}$, we would measure a coherence length ρ_b. However, if we made measurements, relative to the instantaneous center of the wandering beam, over times much shorter than Δt we would not measure a coherence length equal to ρ_b, but rather some short-term length ρ_{bs}, which is greater than ρ_b. The short term coherence length has not been studied in any rigorous fashion; however, a rough approximation[36] is readily obtained, and it is found that the short-term beam coherence length ρ_{bs} is also given by Eq. (50), except with ρ_b replaced by ρ_{ps}, which for $\rho_p/D < 1$ is given by

$$\rho_{ps} \simeq \rho_p \left[1 + 0.37\left(\frac{\rho_p}{D}\right)^{1/3}\right] . \tag{53}$$

Before leaving our discussion of the spatial coherence of the field, it is useful for us to point out that in most optical systems applications the coherence function of the beam is not explicitly computed; rather, it is generally sufficient to know the coherence function for a purely spherical wave. For example, let us consider a direct-detection optical communications system. Suppose we have a transmitting

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60. (a) Gilmartin, T., and Holtz, J. (1974) Focused beam and atmospheric coherence measurements at 10.6 and 0.63 micrometers, Appl. Optics 13:1906-1912.
    (b) Dainty, J., and Scanlon, R. (1975) Measurements of the atmospheric transfer function at Mauna Kea, Hawaii, Monthly Not. Royal Astr. Soc. 170:519-532.

61. Gurvich, A., Time, N., Turevtseva, L., and Turchin, V. (1974) Reproduction of temperature pulsation spectra from optical measurements in the atmosphere Izv. Akad. Sci. USSR, Fiz. Atmos. Okeana (Russian) 10:484-492.

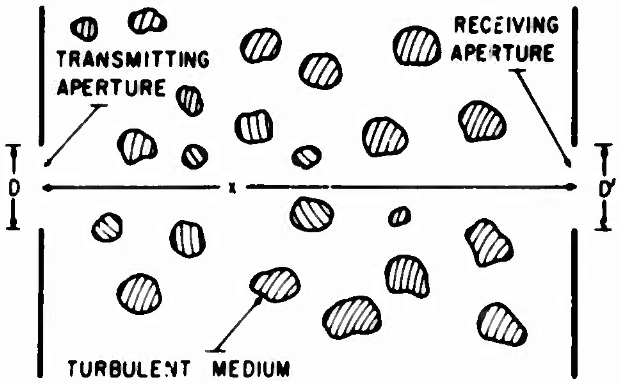

Figure 4. Typical Direct-detection Receiver System Operating Through a Turbulent Medium

aperture of diameter D in the x = 0 plane and a receiving aperture of diameter D' in the plane x, as shown in Figure 4. Then using Eq. (30) we can easily show that the received power P_R is

$$P_R = \iint_{D'} d^2\rho \iint_{D} d^2\rho_1' \iint_{D} d^2\rho_2' \, u_o(\rho_1')\, u_o(\rho_2') \left\langle h(\rho, \rho_1')\, h^*(\rho, \rho_2') \right\rangle \tag{54}$$

where

$$\left\langle h(\rho, \rho_1') h^*(\rho, \rho_2') \right\rangle = \left(\frac{k}{2\pi x}\right)^2 \exp\left\{\frac{ik}{2x}\left[(\rho - \rho_1')^2 - (\rho - \rho_2')^2\right]\right\} \Gamma_s(0, \rho_1' - \rho_2') \tag{55}$$

and $\Gamma_s(\rho_1 - \rho_2, \rho_1' - \rho_2')$ is the two-source spherical-wave coherence function defined as

$$\Gamma_s(\rho_1 - \rho_2, \rho_1' - \rho_2') = \exp\left\{-1.46\, k^2 \int_0^x dx'\, C_n^2(x') \left| \frac{x'}{x}(\rho_1 - \rho_2) + (\rho_1' - \rho_2')\left(1 - \frac{x'}{x}\right) \right|^{5/3}\right\} \tag{56}$$

In obtaining Eq. (56), we have used Eq. (23) in (31) assuming D and D' were both small compared with L_o, so that the term $0.805\, \xi^{1/3}/L_o^{1/3}$ can be ignored in Eq. (23). From Eqs. (54) - (56), we see that in evaluating the performance of the system we generally do not need to explicitly calculate the beam coherence function.

3. INTENSITY SCINTILLATIONS

If we measured the intensity of a laser beam in the plane x in a turbulent medium, we would find that the measured value of I would fluctuate with time about its average value $\langle I \rangle$. It is desirable to be able to predict the magnitude of the intensity scintillations, since this is an important consideration in the design of any receiver system. At present, extensive experimental data on the intensity scintillations is available[13, 62-68] but the theory is complete only for the limiting case of weak turbulence.* That is, as we pointed out in Section 2, extensive calculations[1, 2, 69] of the intensity fluctuations have been made using the Rytov approximation, but these are valid only for $\sigma_1^2 = 1.23\, k^{7/6} C_n^2 x^{11/6} \lesssim 0.3$. This point is clearly demonstrated in Figure 5 where we compare theoretical predictions of the normalized variance of the intensity fluctuations

$$\sigma_I^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2}$$

made using the Rytov method[2] with measured data. It is evident that the Rytov method fails for $\sigma_1^2 \gtrsim 0.3$; for this case a number of theories have been proposed to calculate the intensity fluctuations, but none is completely satisfactory, as we shall see later.

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*We shall call the turbulence "weak" if the parameter $\sigma_1^2 = 1.23\, k^{7/6} C_n^2 x^{11/6} << 1$; for $\sigma_1^2 > 1$, we call the turbulence "strong."

62. Dunphy, J., and Kerr, J. (1973) Scintillation measurements for large integrated-path turbulence, J. Opt. Soc. Am. 63:981-986.

63. Gracheva, M., Gurvich, A., Kashkarov, S., and Pokasov, V. (1973) Similarity Correlations and their Experimental Verification in the case of Strong Intensity Fluctuations of Laser Radiation (Russian). English Translation No. LRG-73-T-28 available from Aerospace Corp., Library Services, P.O. Box 92957, Los Angeles, California.

64. S'edin, V., Khmelevstov, S., and Tsvyk, R. (1972) Intensity fluctuations in a focused light beam that has passed through a stratum of turbulent atmosphere, Radiophysics and Quantum Electronics 15:612-613.

65. Dietz, P., and Wright, N. (1969) Saturation of scintillation magnitude in near-earth optical propagation, J. Opt. Soc. Amer. 59:527-535.

66. Kerr, J. (1972) Experiments on turbulence characteristics and multiwavelength scintillation phenomena, J. Opt. Soc. Amer. 62:1040-1051.

67. (a) S'edin, V., Khmelevstov, S., and Nevol'sin, M. (1970) Intensity fluctuations in a pulsed laser beam propagating up to 9.8 km in the atmosphere, Radiophysics and Quantum Electronics, 13:32-35.
(b) Khmelevstov, S. (1973) Propagation of laser radiation in a turbulent atmosphere, Appl. Optics 12:2421-2432.

68. Young, A. (1970) Saturation of scintillation, J. Opt. Soc. Amer. 60:1495-1501.

69. Ishimaru, A. (1969) Fluctuations of a beam propagating through a locally homogeneous medium, Radio Science 4:295-305.

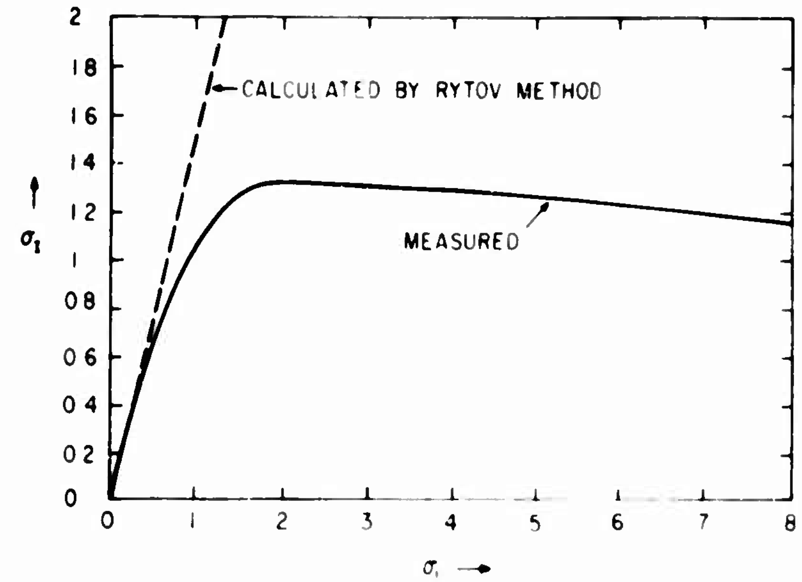

Figure 5. Average of the Measured Values of the Variance of the Intensity Fluctuations of a Plane Wave Propagating Through a Turbulent Medium

In the remainder of this section, we will first present the theoretical results for the variance and covariance of the intensity fluctuations in weak turbulence. We will then present those results presently available for strong turbulence, and finally we will discuss the temporal frequency spectrum of the scintillations.

5.1 Scintillations in Weak Turbulence

For the case of weak turbulence, it is conventional to calculate the variance and covariance of the logarithm of the amplitude fluctuations rather than of the intensity fluctuations. This presents no great difficulty because if the log-amplitude χ_1 is normally distributed, it can be shown (see Appendix A) that

$$B_\chi(x, \rho_1, \rho_2) \equiv \langle \chi_1 (x, \rho_1) \chi_1 (x, \rho_2) \rangle \tag{57}$$

is related to

$$b_I(x, \rho_1, \rho_2) = \frac{\langle I(x, \rho_1)\, I(x, \rho_2)\rangle - \langle I(x, \rho_1)\rangle \langle I(x, \rho_2)\rangle}{\langle I(x, \rho_1)\rangle \langle I(x, \rho_2)\rangle} \tag{58}$$

via

$$B_\chi(x, \underline{\rho}_1, \underline{\rho}_2) = \frac{1}{4} \ln [1 + b_I(x, \underline{\rho}_1, \underline{\rho}_2)] . \qquad (59)$$

We shall therefore present the theoretical results for B_χ ; the results for b_I follow from Eq. (59).

The ensemble average in Eq. (57) can be readily evaluated by using the Rytov method; the quantity $\chi_1(x, \underline{\rho}_1)$ can be calculated by taking the real part of Eq. (10). This result is then multiplied by $\chi_1(x, \underline{\rho}_2)$ and ensemble averaged, with Eq. (2) used to evaluate $\langle n_1(x, \underline{\rho}_1)\, n_1(x, \underline{\rho}_2)\rangle$. The phase fluctuations can be calculated by taking the imaginary part of Eq. (10) to obtain the phase $S_1(x, \underline{\rho}_1)$ and then obtaining

$$B_S(x, \underline{\rho}_1, \underline{\rho}_2) \equiv \langle S_1(x, \underline{\rho}_1)\, S_1(x, \underline{\rho}_2)\rangle . \qquad (60)$$

The results for both the log-amplitude and phase fluctuations for a beam with the inifial field distribution in Eq. (36), are[69]

$$B_{\substack{\chi \\ s}}(x, \underline{\rho}_1, \underline{\rho}_2) = \pi^2 \int_0^x d\eta \int_0^\infty \kappa\, d\kappa\, \Phi_n(\eta, \kappa) \left[\left\{ J_0(\kappa P) + J_0(\kappa P^*) \right\} |H|^2 \pm J_0(\kappa Q) H^2 \pm J_0(\kappa Q^*) (H^*)^2 \right] , \qquad (61)$$

where $J_0(\ldots)$ is the zero-order Bessel function, $P^2 = (\gamma_1 \hat{\underline{\rho}} - i 2 \gamma_2 \underline{\rho}_+)^2$, $Q^2 = \gamma^2 \hat{\rho}^2$, $\hat{\underline{\rho}} = \underline{\rho}_1 - \underline{\rho}_2$, $\rho_+ = (\underline{\rho}_1 + \underline{\rho}_2)/2$, $\gamma = \gamma_1 + i\gamma_2$,

$$H^2 = -k^2 \exp \left\{ \frac{- i\gamma (x - \eta) \kappa^2}{k} \right\} ,$$

$$\gamma_1 = \frac{1 - \alpha_2 x + \left[(\alpha_1^2 + \alpha_2^2) x - \alpha_2 \right] \eta}{(1 - \alpha_2 x)^2 + (\alpha_1 x)^2} ,$$

$$\gamma_2 = \frac{\alpha_1 (x - \eta)}{(1 - \alpha_2 x)^2 + (\alpha_1 x)^2} ,$$

$$\alpha_1 = \frac{2\lambda}{\pi D^2} , \qquad \alpha_2 = \frac{1}{F} ,$$

and

$$\Phi_n(\eta, \kappa) = \Phi_n\left(\eta,\ \kappa_x = 0,\ \kappa = \sqrt{\kappa_y^2 + \kappa_z^2}\right) .$$

The expression given in Eq. (61) for B_χ is valid only for $\sigma_1^2 \lesssim 0.3$, but that for B_s is a good approximation for both large and small values of σ_1^2. Calculations of B_χ, made using Eq. (61), have been compared in detail with experimental data and the agreement is quite favorable,[67] provided $\sigma_1^2 < 0.3$. This point is quite evident from Figure 5; there calculations of

$$\sigma_I^2 = \left\langle I^2 \right\rangle - \left\langle I \right\rangle^2 = \exp\left\{4 B_\chi (x, 0, 0)\right\} - 1$$

for a unit-amplitude plane wave are compared with experimental data.

In the limiting case of a plane wave ($F \to \infty$, $D \to \infty$) Eq. (61) reduces to

$$B_{\substack{\chi \\ s}} (x, \underline{\rho}) = 4\pi^2 k^2 \int_0^x d\eta \int_0^\infty \kappa d\kappa\, J_0(\kappa \hat{\rho})\, \Phi_n(\eta, \kappa) \begin{Bmatrix} \sin^2\left[\dfrac{\kappa^2 (x-\eta)}{2k}\right] \\ \cos^2\left[\dfrac{\kappa^2 (x-\eta)}{2k}\right] \end{Bmatrix} . \qquad (62)$$

A good approximation to Eq. (62) for B_χ can be obtained for the case when the turbulence is homogeneous (that is, C_n^2 independent of x). In that case, for $\lambda = k/2\pi$ = signal wavelength

$$B_\chi (x, \rho) \simeq \frac{1}{4}\, b_I (x, \rho) \simeq \frac{\sigma_1^2}{4} \left[1 - 10.9 \left(\frac{\hat{\rho}^2}{\lambda x}\right)^{5/6} - 10.7 \left(\frac{\hat{\rho}^2}{\lambda x}\right)\right] , \qquad (63)$$

provided $\ell_0 \ll \hat{\rho} \ll (\lambda x)^{1/2}$. We note from Eq. (63) that in weak turbulence, the characteristic transverse length for correlation of the intensity fluctuations is $(\lambda x)^{1/2}$; we will see later that in strong turbulence there is an entirely different characteristic correlation length for the intensity fluctuations.

5.2 Scintillations in Strong Turbulence

A great deal of effort[15-20, 24, 25, 27, 30, 32, 70-74] has been devoted in attempting to explain the reasons for the failure* of the Rytov method when the turbulence is strong and to develop new theories which are adequate when $\sigma_1 > 1$. Most of this effort has been an attempt[70-72, 75-84] to obtain solutions to Eq. (24), since the intensity fluctuations are directly related to Γ_4, through

$$\langle I(x, \underline{\rho}_1)\, I(x, \underline{\rho}_2) \rangle - \langle I(x, \underline{\rho}_1) \rangle \langle I(x, \underline{\rho}_2) \rangle$$

$$= \Gamma_4 (x, \underline{\rho}_1, \underline{\rho}_1, \underline{\rho}_2, \underline{\rho}_2) - \Gamma_2 (x, \underline{\rho}_1, \underline{\rho}_1)\, \Gamma_2 (x, \underline{\rho}_2, \underline{\rho}_2) .$$

*Physically, the reason why the Rytov method fails is because it does not properly account for the fact that the incident wave becomes more and more incoherent as it propagates into the medium. Thus the turbulent eddies are not scattering a coherent wave, but rather a partially coherent wave.

70. Clifford, S., Ochs, G., and Lawrence, R. (1974) Saturation of optical scintillations by strong turbulence, J. Opt. Soc. Amer. 64:148-154.

71. Yura, H. (1974) Physical model for strong optical-amplitude fluctuations in a turbulent medium, J. Opt. Soc. Amer. 64:59-67.

72. deWolf, D. (1974) Waves in turbulent air: a phenomenological model, Proc. IEEE 62:1523-1529.

73. Gochelashvili, K. (1971) Saturation of the fluctuations of focused radiation in a turbulent medium, Radiophysics and Quantum Electronics 14:470-473.

74. Dagkesamanskaya, I., and Shishov, V. (1970) Strong intensity fluctuations during wave propagation in statistically homogeneous and isotropic media, Radiophysics and Quantum Electronics 13:9-12.

75. Shishov, V. (1972) Strong fluctuations of the intensity of a plane wave propagating in a random refractive medium, Soviet Phys. JEPT 34:744-748.

76. Shishov, V. (1972) Strong fluctuations of the intensity of a spherical wave propagating in a randomly refractive medium, Radiophysics and Quantum Electronics 15:689-695.

77. Fante, R. (1975) Some new results on propagation of electromagnetic waves in strongly turbulent media, IEEE Trans. Ant. and Prop. AP-23:382-385.

78. Fante, R. (1975) Electric field spectrum and intensity covariance of a wave in a random medium, Radio Science 10:77-85.

79. Gochelashvili, K., and Shishov, V. (1974) Saturated intensity fluctuations of laser radiation in a turbulent medium Zh. Eksp. Teor. Fiz. (Russian) 66:1237-1247.

80. Fante, R. (1975) Irradiance scintillations: comparison of theory with experiment, J. Opt. Soc. Amer. 65:548-550.

81. Gochelashvili, K., and Shishov, V. (1971) Laser beam scintillation beyond a turbulent layer, Optica Acta 18:313-320.

82. Fante, R. (1975) Some results for the variance of the irradiance of a finite beam in a random medium, Opt. Soc. Amer. 65:608-610.

83. Gochelashvili, K., and Shishov, V. (1972) Focused irradiance fluctuations beyond a layer of turbulent atmosphere, Optica Acta 19:327-332.

84. Banakh, V., Krekov, G., Mironov, V., Khmelevtsov, S., and Tsvik, S. (1974) Focused-laser-beam scintillations in thr turbulent atmosphere, J. Opt. Soc. Amer. 64:516-518.

Although a great deal of effort has been expended, approximate asymptotic solutions (for $\sigma_1^2 >> 1$) have been obtained only in the limit of plane and spherical waves; there is, at present, no acceptable analytical solution for the intensity fluctuations of an arbitrary beam in strong turbulence, although some progress has been made in obtaining solutions to Eq. (24) by purely numerical methods. Dagkesamanskaya and Shishov[74] have numerically evaluated Eq. (24) in the plane wave limit, for the special case when the spectrum $\Phi_n(\kappa)$ is gaussian; this is of course unrealistic for clear-air atmospheric turbulence. Brown[16] has obtained numerical plane wave solutions to Eq. (24) in the two-dimensional limit; his solutions show the same trends as measured values for $b_I(x, \rho)$, but the two-dimensional approximation is unrealistic and consequently a direct comparison of his results with measured data is impossible. Banakh et al[84] have obtained a computer solution to Eq. (24) for the intensity scintillations on the axis of a finite beam in strong turbulence; despite several unjustified assumptions in their analysis, their results do agree reasonably well with experimental data.

We will now present the available analytical results in a little more detail. In the plane wave limit, it has been shown by several different techniques[32, 78, 79] that for $\sigma_1^2 >> 1$ an approximate solution to Eq. (24) for the normalized intensity scintillations $b_I(x, \rho)$ in homogeneous turbulence is

$$b_I(x, \hat{\rho}) \simeq \exp\left[-2\left(\frac{\hat{\rho}}{\rho_p}\right)^{5/3}\right] + \frac{1}{(\sigma_1^2)^{2/5}}\left\{f\left[\frac{\hat{\rho}}{(\lambda x)^{1/2}\sigma_1^{6/5}}\right] + g\left[\frac{\hat{\rho}\,\sigma_1^{6/11}}{(\lambda x)^{1/2}}\right]\right\}, \quad (64)$$

where b_I is defined in Eq. (58), $\hat{\rho} = |\underline{\rho}_1 - \underline{\rho}_2|$, ρ_p is the plane wave coherence length, defined in Eq. (51), $\sigma_1^2 = 1.23\, k^{7/6} C_n^2\, x^{11/6}$ and the functions $f(S)$ and $g(S)$ are shown in Figure 6. We can obtain the variance of the intensity fluctuations by setting $\hat{\rho} = 0$ in Eq. (64); the result is

$$\sigma_I^2 = \frac{\langle I^2\rangle - \langle I\rangle^2}{\langle I\rangle^2} = 1 + \frac{0.99}{(\sigma_1^2)^{2/5}} \quad . \quad (65)$$

This expression is in good agreement[80] with measurements; the decay of σ_I^2 to unity as $(\sigma_1^2)^{-2/5}$ has been predicted by a number of authors.[32,70,71,78,79,81,83] In Figure 7, we present a comparison of the analytical predictions, made using Eq. (64), and recent Soviet measured data[63] for the covariance of the intensity fluctuations. The agreement is quite good; further comparisons are given elsewhere.[80] In Figure 7, the abscissa R is defined as $R = \hat{\rho}/(\lambda x)^{1/2}$. The sharp decay in $b_I(R)$ near $R = 0$ is governed by the first term in Eq. (64), whereas the long tail is governed by the second term; the last term is important in the transition region between the first two terms.

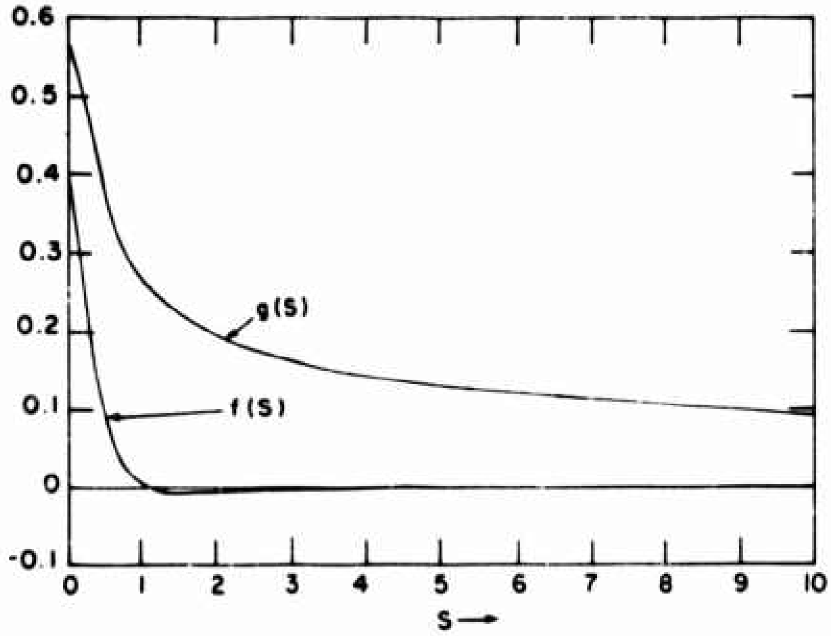

Figure 6. Plots of the Functions f(S) and g(S) Which Appear in Eq. (64)

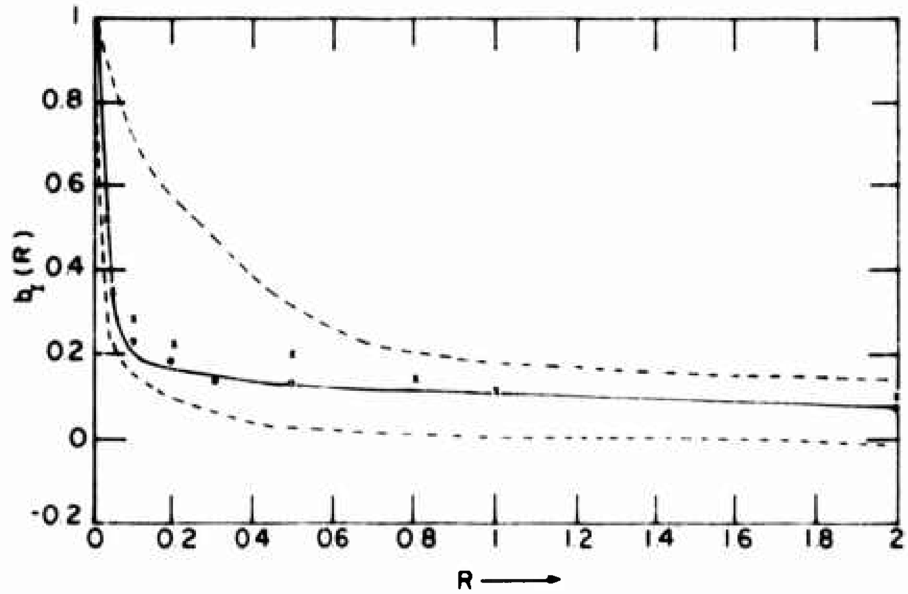

Figure 7. Comparison of Analytical and Measured Values of the Intensity Covariance when $\sigma_1^2 = 27$. The solid curve is calculated from Eq. (64), and the dashed curves are the envelopes of the band over which data points were found experimentally. The averages of the data points for collimated and divergent beams are indicated by x and 0. In this figure, $R = \rho/\lambda x)^{1/2}$

It is important to note that the nature of the intensity covariance function, b_I, is quite different in strong turbulence than in weak turbulence; this is evident from Figure 8. We observe that in weak turbulence the intensity fluctuations are correlated over transverse distances ρ of order $(\lambda x)^{1/2}$. However, in strong turbulence we see from Eq. (64) and Figure 8 that the correlation is over transverse separations*

$$\rho \sim \rho_p = \frac{0.36\,(\lambda x)^{1/2}}{(\sigma_1^2)^{3/5}} . \tag{66}$$

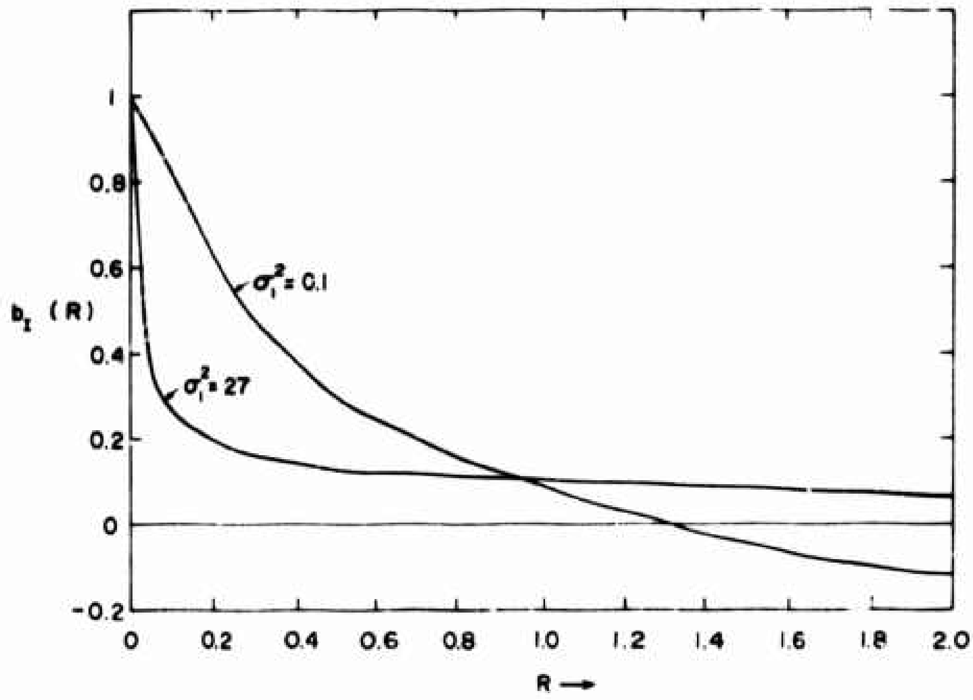

Figure 8. Calculations of the Intensity Covariance for $\sigma_1^2 >> 1$ and $\sigma_1^2 << 1$. In this figure $R = \rho/(\lambda x)^{1/2}$

*A recent Soviet paper[85] has questioned whether the transverse correlation distance is given by ρ_p since a correlation distance of order $(\lambda x)^{1/2}(\sigma_1^2)^{-3/11}$ seems to give a better fit with some of their data (of course, Eq. (64) also contains this scale length). Further experiments appear necessary to examine this point in more detail.

85. (a) Gurvich, A., and Tatarskii, V. (1975) Coherence and intensity fluctuations of light in the turbulent atmosphere, Radio Science 10:3-14.
(b) Gurvich, A., and Tatarskii, V. (1973) The frequency spectra of strong fluctuations of laser radiation in a turbulent atmosphere, Radiophysics and Quantum Electronics 16:701-704.

Since $\sigma_1^2 >> 1$ in strong turbulence, it is clear from Eq. (66) that the correlation distance is much shorter in strong turbulence than in weak turbulence. The result in Eq. (66) has also been predicted on physical grounds by Clifford et al[70] and Yura[71] who generalized the physical model, based on geometric optics, proposed by Tatarsk [1] ection 47. That is, Tatarskii has modeled the turbulent eddies by focusing lenses and has shown that the intensity scintillations are produced by the smallest lenses which are capable of focusing the radiation at the receiver. In weak turbulence, these lenses (eddies) were shown to be those with diameters of order $(\lambda x)^{1/2}$. The aforementioned papers by Clifford and Yura have generalized Tatarskii's model to include the effect of diffraction and the loss of transverse spatial coherence of the inciden w as it propagates into the turbulent medium. They have demonstrated that in stro turbulence, the eddies most effective in producing intensity scintillations are no es with diameters of order of the coherence length ρ_p (for plane wave case) and not $(\lambda x)^{1/2}$. Predictions of the intensity scintillations and covariance made with the aforementioned model are in qualitative agreement with measured data, and with the result in Eq. (64).

Although Eqs. (64) and (65) are strictly valid only for plane waves, it is expected that they will also yield a fair approximation for finite beams, except possibly in the focal plane. In fact it has been shown[82] that for $\sigma_1^2 >> 1$ and

$$\sigma_1^2 >> 0.5\left[\frac{4x}{kD^2} + \left(\frac{kD^2}{4x}\right)\left(1 - \frac{x}{F}\right)^2\right]^{5/6} \tag{67}$$

the properties of the intensity scintillations of the beam are independent of its initial structure. Therefore, for sufficiently strong turbulence the normalized scintillations of a plane wave are the same as those of a finite beam; this effect has also been observed experimentally.[13, 67]

We should also point out that some approximate results for the focussed beam case have been given by Prokhorov et al.[32]

5.3 Aperture Averaging

The intensity fluctuations σ_1^2 shown in Figure 5 and discussed in Sections 5.1 and 5.2 are really those which would be measured by a receiving aperture with an infinitesimally small diameter. In practice, the receiving aperture has a finite diameter and the intensity fluctuations measured will not be σ_1^2 but rather an average of the fluctuations over the whole aperture. In order to discuss this point quantitatively, let us consider the fluctuation δP_R in the power received by circular aperture of diameter D' when a signal of intensity I is incident on it.

This is

$$\delta P_R = P_R - \langle P_R \rangle = \iint_{D'} d^2\rho_1 \, [I - \langle I \rangle] \, .$$

Therefore, the mean square fluctuation in the received power is

$$\langle \delta P_R^{\,2} \rangle = \iint_{D'} d^2\rho_1 \iint_{D'} d^2\rho_2 \, [\langle I(\underline{\rho}_1) I(\underline{\rho}_2) \rangle - \langle I(\underline{\rho}_1) \rangle \langle I(\underline{\rho}_2) \rangle]$$

$$= \iint_{D'} d^2\rho_1 \iint_{D'} d^2\rho_2 \, B_I(x, \underline{\rho}_1, \underline{\rho}_2) \tag{68}$$

where $B_I(x, \underline{\rho}_1, \underline{\rho}_2) = \langle I(\underline{\rho}_1) \rangle \langle I(\underline{\rho}_2) \rangle \, b_I(x, \underline{\rho}_1, \underline{\rho}_2)$, and b_I is defined in Eq. (58). Let us now define $G(D')$ as the ratio of the received power fluctuations in an aperture of diameter D' to those measured by a point aperture. By using Eq. (68), it is straightforward to show[1] that for a plane wave

$$G(D') = \frac{16}{\pi (D')^2} \int_0^{D'} \frac{b_I(x, \hat{\rho})}{b_I(x, 0)} \left\{ \cos^{-1}\left(\frac{\hat{\rho}}{D'}\right) - \frac{\hat{\rho}}{D'} \left[1 - \left(\frac{\hat{\rho}}{D'}\right)^2 \right]^{1/2} \right\} \hat{\rho} \, d\hat{\rho} \, . \tag{69}$$

If we use Eqs. (62) and (64) in (69), we can readily calculate $G(D')$ for both strong and weak turbulence; the result is shown in Figure 9. We observe that as the diameter of the receiving aperture is increased, the magnitude of the fluctuations in the received power decreases;* this effect is known as aperture averaging, and has been observed experimentally.[86, 87] From Figure 9 we can note that in weak turbulence ($\sigma_1^2 \ll 1$), the fluctuations in the received power are significantly reduced whenever the aperture diameter D' exceeds $(\lambda x)^{1/2}$. In strong turbulence ($\sigma_1^2 \gg 1$), however, there is a significant reduction in the fluctuations whenever the receiving aperture diameter exceeds $\rho_p = 0.36 \, (\lambda x)^{1/2} (\sigma_1^2)^{-3/5}$, which is much smaller than $(\lambda x)^{1/2}$ because $\sigma_1^2 \gg 1$.

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*The reciprocal phenomenon of transmitter-aperture averaging is responsible for the well known result that planets twinkle less than stars, because of their larger disc size.

86. Homstad, G., Strohbehn, J., Berger, R., and Heneghan, J. (1974) Aperture-Averaging effects for weak scintillations, J. Opt. Soc. Amer. 64:162-165.

87. Kerr, J., and Dunphy, J. (1973) Experimental effects of finite transmitter apertures on scintillations, J. Opt. Soc. Amer. 63:1-7.

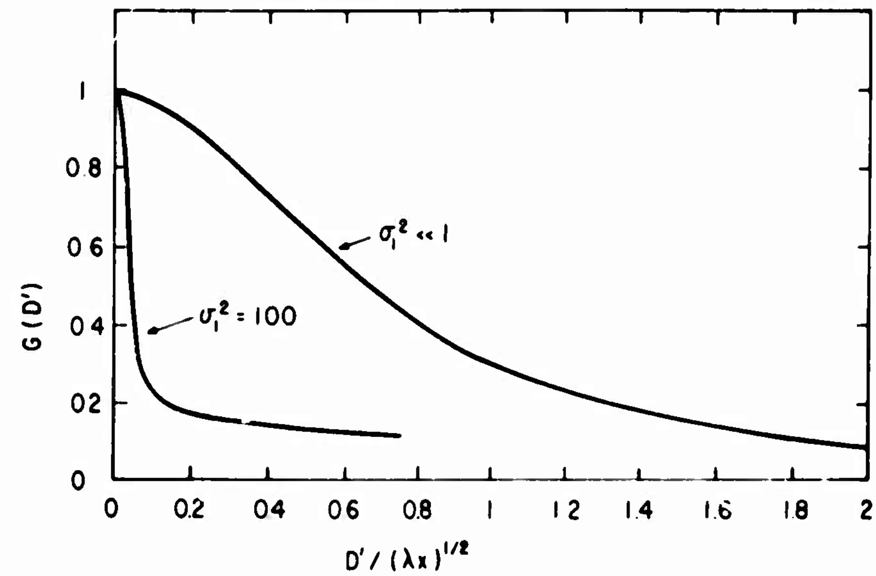

Figure 9. Aperture Averaging in Strong and Weak Turbulence. The function G(D') is defined in Eq. (69) and is the ratio of the power fluctuations measured by a receiver of diameter D' to those which would be measured by a point aperture

5.4 Frequency Spectrum of the Intensity Scintillations

In some applications it is desirable to know the frequency spectrum of the intensity fluctuations of a light beam which would be measured by a receiver in a turbulent medium. This spectrum $W_I(\omega)$ is given by

$$W_I(x, \rho_1, \omega) = \int_0^\infty d\tau \cos \omega\tau \left[\langle I(x, \rho_1, t)\, I(x, \rho_1, t+\tau)\rangle - \langle I(x, \rho_1, t)\rangle \langle I(x, \rho_1, t+\tau)\rangle\right], \quad (70)$$

where ω is the radian frequency and $I(x, \rho_1, t)$ is the instantaneous intensity at the position (x, ρ_1) at the time t. In writing Eq. (70), it has been implicitly assumed that the turbulence is a (temporally) stationary random process; for frozen flow this is a good assumption. For strong turbulence the expression in Eq. (70) has been evaluated in the plane wave limit by using Eq. (64) and equivalent results;[77, 88] the result is lengthy and will not be presented here. For weak

88. Yura, H. (1974) Temporal-frequency spectrum of an optical wave propagating under saturation conditions, J. Opt. Soc. Amer. 64:357-360

turbulence, it is customary to study[89-95] the frequency spectrum $W_\chi(\omega)$ of the log-amplitude fluctuations. This is related to the results obtained in Section 5.1 via

$$W_\chi(x, \rho_1, \omega) = \int_0^\infty d\tau \cos \omega\tau \, B_\chi(x, \rho_1, \rho_1, \tau) .$$

The time lagged version of B_χ is readily obtained[89] from Eq. (61) by replacing P^2 by $P^2 = (\gamma_1 \rho_1 - i\, 2\gamma_2 \rho_+ + \underline{V}\tau)^2$ and Q^2 by $Q^2 = (\gamma \rho + \underline{V}\tau)^2$ where $\underline{V}$ is the transverse flow velocity of the turbulent eddies across the beam. Approximate analytical expressions for $W_\chi(\omega)$ have been obtained in a number of limiting cases, but because they are extremely lengthy they will not be written out here. Tatarskii[1] has evaluated W_χ in the plane wave limit, and Clifford[90] and Reinhardt and Collins[91] have studied the case of a spherical wave. Approximate expressions for W_χ for a finite beam have been obtained by Time.[92]

Experimental measurements of W_χ and W_I have been made for both weak and strong turbulence.[62, 63, 85, 96, 97] For the case of propagation in the clear atmosphere, it is found that for propagation paths such that the turbulence parameter $\sigma_1^2 << 1$ the width of the frequency spectrum $W_\chi(\omega)$ is on the order $V/(\lambda x)^{1/2}$, which is typically about 10 to 100 Hz; for paths such that $\sigma_1^2 >> 1$ the width of the frequency spectrum is of order V/ρ_p, which is typically about 100 to 1000 Hz.

89. Ishimaru, A. (1969) Fluctuations of a focused beam wave for atmospheric turbulence probing, Proc. IEEE 57:407-414.

90. Clifford, S. (1971) Temporal-frequency spectra for a spherical wave propagating through atmospheric turbulence, J. Opt. Soc. Amer. 61:1285-1292.

91. Reinhardt, G., and Collins, S. (1972) Outer-scale effects in turbulence-degraded light beam spectra, J. Opt. Soc. Amer. 62:1526-1528.

92. Time, N. (1971) The spectrum of the amplitude fluctuations in a bounded light beam, Radiophysics and Quantum Electronics 14:936-939.

93. Strohbehn, J. (1974) Covariance functions and spectra for waves propagating in a turbulent medium to or from moving vehicles, IEEE Trans. Ant. and Prop. AP-22:303-311.

94. Livingston, P., Dietz, P., and Alcaraz, A. (1970) Light propagation through a turbulent atmosphere: measurement of the optical filter function, J. Opt. Soc. Amer. 60:925-935.

95. Lawrence, R., Ochs, G., and Clifford, S. (1972) Use of scintillations to measure average wind across a light beam, Appl. Optics 11:239-243.

96. Mandics, P., Lee, R., and Waterman, A. (1973) Spectra of short-term fluctuations of line-of-sight signals: electromagnetic and acoustic, Radio Science 8:185-201.

97. Gurvich, A., and Pokasov, V. (1973) Frequency spectra of strong laser radiation fluctuations in the turbulent atmosphere Izv. Vyssh. Vcheb. Zaved., Radiofiz. (Russian) 16:913-917.

6. PROBABILITY DISTRIBUTION OF THE INTENSITY

In Section 3, we studied the first moment $\langle I \rangle$ of the received intensity, and in Section 5 we considered the second moment $\langle I^2 \rangle$. In most applications, knowledge of these two moments is sufficient; however, in some applications, such as the calculation of the probability of error in a communications link, it is desirable to know the probability distribution satisfied by the received intensity. For the case when $\sigma_I^2 << 1$, it has been found[63, 98] that the probability distribution of the intensity is very nearly log-normal. That is, for a unit-amplitude plane wave the probability density p(I) satisfies

$$p(I) = \frac{1}{(2\pi)^{1/2}\sigma I} \exp\left\{ - \left(\ln I + \frac{\sigma^2}{2}\right)^2 (2\sigma^2)^{-1} \right\} , \tag{71}$$

where $\sigma^2 = 4B_\chi(x, 0) = \ln[1 + \sigma_I^2]$ can be obtained from Eqs. (59) and (62). The above result is physically reasonable since it implies that $\chi = (1/2) \ln(I)$ is normally distributed. By returning to Eq. (10), we see* that $\chi = \mathrm{Re}(\psi_1)$ is essentially the sum of a large number of independent forward scatterings; therefore, by virtue of the central limit theorem[99] χ_1 is a normally distributed random variable.

For $\sigma_I^2 > 0.3$, it is no longer true that p(I) is log-normal. Experimental measurements have been made[63, 98] for $0 < \sigma_I^2 < 100$, and for $\sigma_I^2 < 0.3$ the distribution in Eq. (71) is reasonably accurate; also, for $25 < \sigma_I^2 < 100$ Eq. (71) appears to be a reasonable approximation. However, for $1 \leq \sigma_I^2 \leq 25$ the measured probability distribution appears to deviate significantly from the result in Eq. (71). This is especially true for $1 \leq \sigma_I^2 \leq 4$. In no case studied[63] did the results indicate that the probability distribution is Rayleigh, as predicted by deWolf.[100] However, it is expected from physical considerations[72, 81] that for $\sigma_I^2 \rightarrow \infty$, the probability distribution p(I) should approach

$$p(I) = \exp(-I) . \tag{72}$$

*In reaching this conclusion, we have decomposed the integral in Eq. (10) into a sum over independent regions of order of an eddy size. By a similar argument, we may also conclude that the fluctuations in the phase S are also normally distributed.

98. Gurvich, A., Kallistrova, M., and Time, N. (1968) Fluctuation in the parameters of a light wave from a laser during propagation in the atmosphere, Radiophysics and Quantum Electronics 11:771-776.

99. Beckman, P. (1967) Probability in Communication Engineering, Harcourt, Brace and World, Inc. New York.

100. deWolf, D. (1968) Saturation of irradiance fluctuations due to turbulent atmosphere, J. Opt. Soc. Amer. 58:461-466.

In deWolf's[72] physical model, it is argued that there are two principal components which contribute to the field received at the point (x, 0): one is the component forward scattered by the large eddies on the propagation axis, and the other is the component which arrives at (x, 0) after multiple scattering by the off-axis eddies. This latter component is small in weak turbulence. By employing physical arguments, it can be shown[99] that the amplitude of the former component satisfies a log-normal probability distribution, whereas the amplitude of the latter component obeys a Rayleigh distribution. The general form of the probability distribution of the sum of these components in the aforementioned physical model is given in Appendix B. It is shown there that the resulting probability distribution gives the correct behavior in the limits $\sigma_1^2 \ll 1$ and $\sigma_1^2 \to \infty$; for intermediate values of σ_1^2, further investigation is required.

Because it is exceedingly difficult to derive the probability distribution of the intensity fluctuations from first principles (such as the characteristic functional method[27]), Wang and Strohbehn[101-103] have used a novel technique in which they guess a possible probability distribution and then examine its consequences. They first[102] assumed a log-normal intensity distribution and then proved that over part of the propagation path the probability density cannot be log-normal; they also ruled out a Rice distribution in the same region. Further investigations[103] showed that, although the intensity probability density is not exactly log-normal, small perturbations on the log-normal distribution lead to results which are consistent with measured data. Other possible distributions are presently under consideration.[104a, b] In fact, recently it has been demonstrated[104b] that a three-term truncated log-normal is an excellent approximation to the irradiance statistics for finite σ_1^2.

In summary, Eq. (71) is a good approximation to the probability distribution of the intensity fluctuations for $\sigma_1^2 < 0.3$ and for $25 \leq \sigma_1^2 \leq 100$. For other ranges of σ_1^2, either a perturbed log-normal or a distribution such as given in Eq. (B4) of Appendix B may be a good appropriation. Further work on this problem is clearly needed.

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101. Strohbehn, J., and Wang, T. (1972) Simplified equation for amplitude scintillations in a turbulent atmosphere, J. Opt. Soc. Amer. 62:1061-1068.

102. Wang, R., and Strohbehn, J. (1974) Log-normal paradox in atmospheric scintillations, J. Opt. Soc. Amer. 64:583-591

103. Wang, R., and Strohbehn, J. (1974) Perturbed log-normal distribution of irradiance fluctuations, J. Opt. Soc. Amer. 64:994-999.

104. (a) Strohbehn, J., Wang, T., and Speck, J. (1975) On the probability distribution of line-of-sight fluctuations of optical signals, Radio Science 10:59-70.
(b) Davidson, F., and Gonzalez-del-Valle, A. (1975) Measurements of three-parameter log-normally distributed optical field irradiance fluctuations in a turbulent medium, J. Opt. Soc. Amer. 65:655-662.

7. ANGLE OF ARRIVAL FLUCTUATIONS

A wave propagating in vacuum has a uniform wavefront; however, because different portions of the wavefront experience different phase shifts, a signal propagating in a random medium has random surfaces of constant phase, such as shown in Figure 10. This phase distortion leads to fluctuations in the angle of arrival α of the wavefront; these are the cause of image jitter in a telescope, an effect which is well known to astronomers. In this section, we will first discuss the mean-square angle of arrival. We will next discuss the frequency spectrum of the angle of arrival fluctuations and, finally we will discuss methods for compensating for these fluctuations.

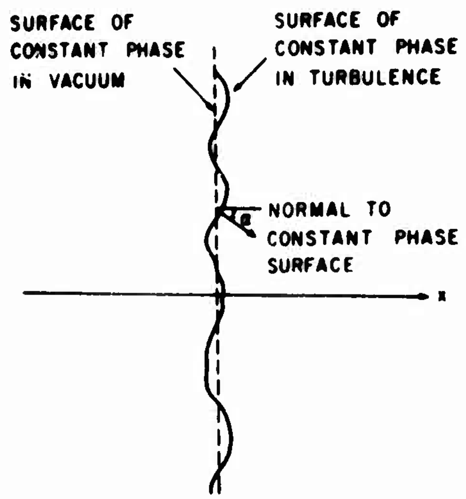

Figure 10. Typical Surface of Constant Phase for a Wave in a Random Medium. The angle α is the local angle of arrival

7.1 Mean Square Angle of Arrival

Consider the receiving aperture of diameter D' shown in Figure 4; the phase difference ΔS across this aperture can be approximated by

$$\Delta S \simeq k D' \sin \alpha \simeq k D' \alpha \quad .$$

Therefore, the mean square angle-of-arrival fluctuation can be written as* (see Figure 10)

$$\langle \alpha^2 \rangle \simeq \frac{\langle \Delta S^2 \rangle}{k^2 D'^2} \simeq \frac{d_s(x, 0, D')}{k^2 D'^2} \; , \tag{73}$$

where d_s is known as the phase structure function and is defined as

$$d_s(x, \underline{\rho}_1, \underline{\rho}_2) \equiv \langle | S_1(x, \underline{\rho}_1) - S_1(x, \underline{\rho}_2) |^2 \rangle \; . \tag{74}$$

From Eq. (73), it is clear that in order to compute the angle of arrival fluctuations we must first compute the phase structure function; for the case when $\sigma_1^2 < 0.3$ this can readily be done by employing[69, 73, 89] the Rytov method, which was described in Section 2.2. The result for a laser beam with a field distribution at $x = 0$ given by Eq. (36) is

$$\begin{aligned} d_s(x, \underline{\rho}_1, \underline{\rho}_2) = 2\pi^2 \int_0^x d\eta \int_0^\infty \kappa \, d\kappa \, \Phi_n(\eta, \kappa) \Bigg[\Big\{ & J_0(i2\kappa |\underline{\rho}_1| \gamma_2) \\ & + J_0(i2\kappa |\underline{\rho}_2| \gamma_2) - J_0(\kappa P^*) - J_0(\kappa P) \Big\} |H|^2 \\ & - \Big\{ 1 - J_0(\kappa Q) \Big\} H^2 - \Big\{ 1 - J_0(\kappa Q^*) \Big\} (H^*)^2 \Bigg] \; , \end{aligned} \tag{75}$$

where H, P, Q and γ_2 have been defined previously in Section 5.1. Because the diameter of the beam at the receiver location is often much larger than the diameter of the receiver, it is usually acceptable to approximate Eq. (75) by its plane wave limit. This is

$$d_s(x, \hat{\underline{\rho}}) \simeq \begin{cases} \frac{1}{2} d_1(\hat{\rho}) & \ell_0 << \hat{\rho} << (\lambda x)^{1/2} \\ d_1(\hat{\rho}) & \hat{\rho} >> (\lambda x)^{1/2} \; , \end{cases} \tag{76}$$

---

*Because $\langle S_1 \rangle = 0$, it is clear that $\langle \alpha \rangle = 0$. When $\langle \alpha^2 \rangle$ is defined as in Eq. (73), it is a measure of the "average" angle of arrival. That is, if $I_{D'}(\alpha)$ is the distribution of the angular fluctuations over an aperture of diameter D', then Eq. (73) is a measure of $\langle (\int \alpha I_{D'}(\alpha) \, d\alpha / \int I_{D'}(\alpha) \, d\alpha)^2 \rangle$. This is especially clear when D' is large because $\Delta S/kD'$ does not give the local normal to the phase surface, but rather the average normal.

where $d_1(\hat{\underline{\rho}})$ is known as the plane-wave structure function and for $\ell_o << |\hat{\underline{\rho}}| << L_o$ is given by*

$$d_1(\hat{\underline{\rho}}) = 2.92\, |\hat{\underline{\rho}}|^{5/3}\, k^2 \int_0^x C_n^2\,(x')\, dx' \,. \tag{77}$$

If we use Eq. (77) in (73), we have the result for a plane wave in nearly-homogeneous turbulence

$$\langle \alpha^2 \rangle \simeq \begin{Bmatrix} 1.46 \\ \\ 2.92 \end{Bmatrix} \frac{\int_0^x dx'\, C_n^2\,(x')}{D'^{1/3}} \qquad \begin{matrix} \ell_o << D' << (\lambda x)^{1/2} \\ \\ L_o > D' >> (\lambda x)^{1/2} \end{matrix} \quad . \tag{78}$$

Although Eqs. (75), (76) and (78) are strictly valid only for weak turbulence, it can be demonstrated[77] that they are also approximately valid in strong turbulence. In particular, Eq. (64) can be used to derive the phase structure function for a plane wave when $\sigma_1^2 >> 1$. The result is

$$d_s(x, \hat{\underline{\rho}}) = \begin{cases} \frac{3}{4}\, d_1(\hat{\rho}) & \ell_o << \hat{\rho} << \rho_p \\ \\ d_1(\hat{\rho}) & \hat{\rho} >> \rho_p \end{cases} \quad , \tag{79}$$

where ρ_p was defined in Eq. (51). Therefore, in strong turbulence the mean square angle of arrival for a plane wave is

*For $\hat{\rho} \lesssim L_o$, we have:

$$d_1(\hat{\rho}) = 2.92\, |\hat{\underline{\rho}}|^{5/3}\, k^2 \left[1 - 0.805 \left|\frac{\hat{\rho}}{L_o}\right|^{1/3}\right] \int_0^x C_n^2\,(x')\, dx',$$

whereas for $\hat{\rho} < \ell_o$ we have

$$d_1(\hat{\rho}) = 2.45\, k^2\, \ell_o^{-1/3}\, |\hat{\rho}|^2 \int_0^x C_n^2\,(x')\, dx'.$$

$$\langle \alpha^2 \rangle \simeq \left\{ \begin{array}{c} 2.19 \\ \\ 2.92 \end{array} \right\} \frac{\int_0^x dx' \, C_n^2(x')}{D'^{1/3}} \qquad \begin{array}{c} \ell_o << D' << \rho_p \\ \\ L_o >> D' >> \rho_p \end{array} \; . \tag{80}$$

If we compare Eqs. (78) and (80) we see that, except for a slight difference in the value of the numerical coefficient, the results are identical. This same conclusion can be shown to hold for a spherical wave, and we can infer that it also holds for an arbitrary beam, except possibly in the focal plane. This conclusion explains why predictions made using the weak turbulence theory were able to give[98] good agreement with experimental results taken for $\sigma_1^2 >> 1$.

Therefore, we may conclude that the weak turbulence results of Eqs. (75) - (78) can also be applied to give an estimate of the mean square angle of arrival in strong turbulence; the results should be accurate to within a numerical coefficient of order unity.

7.2 Angle of Arrival Spectrum

The spectrum $W_\alpha(\omega)$ of the angle of arrival of fluctuations can be obtained by evaluating

$$W_\alpha(\omega) = \frac{1}{k^2 D'^2} \int_0^\infty d\tau \cos \omega\tau \; \langle \Delta S(t) \, \Delta S(t+\tau) \rangle \; , \tag{81}$$

where $\Delta S(t) = S(x, 0, t) - S(x, \underline{\rho}, t)$ and t is the time variable. If we assume that the turbulent flow is frozen, as we discussed in Section 2.1, then the product $\langle \Delta S(t) \, \Delta S(t+\tau) \rangle$ can be rewritten as

$$\langle [S_1(x, 0, t) - S_1(x, \underline{\rho}, t)] \; [S_1(x, -\underline{V}\tau, t) - S_1(x, \underline{\rho} - \underline{V}\tau, t)] \rangle \; , \tag{82}$$

where $\underline{V}$ is the flow velocity of the turbulence in the direction transverse to the direction of propagation. If Eq. (82) is used in (81), we find for a plane wave that

$$W_\alpha(\omega) = \frac{1}{2k^2 D'^2} \int_0^\infty d\tau \cos \omega\tau \; [d_s(x, \underline{\rho} - \underline{V}\tau) + d_s(x, \underline{\rho} + \underline{V}\tau) - 2 d_s(x, \underline{V}\tau)] \, . \tag{83}$$

If Eq. (76) is used for the phase structure function, it is found that for a plane wave[1, 2, 91, 105] in homogeneous turbulence

$$W_\alpha(\omega) = \begin{Bmatrix} 0.0326 \\ \\ 0.0652 \end{Bmatrix} \frac{V^{5/3} C_n^2 x}{D'^2} \left[1 - \cos\left(\frac{\omega D'}{V}\right)\right] \frac{(2\pi/\omega)^{8/3}}{\left[1 + \left(\frac{1.07\,V}{\omega L_o}\right)^2\right]^{4/3}} \quad \begin{matrix} \ell_o << D' << (\lambda x)^{1/2} \\ \\ D' >> (\lambda x)^{1/2} \end{matrix} \tag{84}$$

For a spherical wave propagating in homogeneous turbulence, the angle-of-arrival spectrum is given by[90, 91]

$$W_\alpha(\omega) = \begin{Bmatrix} 0.0326 \\ \\ 0.0652 \end{Bmatrix} \frac{V^{5/3} C_n^2 x}{D'^2} \left[1 - \frac{\sin\left(\frac{\omega D'}{V}\right)}{\left(\frac{\omega D'}{V}\right)}\right] \frac{(2\pi/\omega)^{8/3}}{\left[1 + \left(\frac{1.07\,V}{\omega L_o}\right)^2\right]^{4/3}} \quad \begin{matrix} \ell_o << D' << (\lambda x)^{1/2} \\ \\ D' >> (\lambda x)^{1/2} \end{matrix} \tag{85}$$

The validity of these expressions has been verified in detail by measurements.[96, 98, 106-108] A plot of the normalized angle-of-arrival spectrum is shown in Figure 11. Note that the spectrum decays sharply (as $\omega^{-8/3}$) when $(\omega D'/V) >> 1$. From Figure 11, we also observe that if $D' << L_o$ nearly all of the angle of arrival fluctuations will have frequencies* $f = \omega/2\pi$ in the interval

$$\frac{0.01\ V}{2\pi D'} \lesssim f \lesssim \frac{10\ V}{2\pi D'} \ . \tag{86}$$

In Section 7.3, where we briefly discuss adaptive methods to compensate for angle of arrival fluctuations, we will better understand the importance of the results in Eqs. (84) and (86).

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*In practical systems, angle of arrival scintillations are generally slower than amplitude scintillations. For example, in weak turbulence the ratio of the width of the angle of arrival spectrum to that of the log-amplitude spectrum is $(\lambda x)^{1/2}/D'$, which is usually small because $D' \sim 1$ meter.

105. Woo, R., and Ishimaru, A. (1974) Effects of turbulence in a planetary atmosphere on radio occultation, IEEE Trans. Ant. and Prop. AP-21: 566-573.

106. Clifford, S., Bouricius, G., Ochs, G., and Ackley, M. (1971) Phase variations in atmospheric optical propagation, J. Opt. Soc. Amer. 61:1279-1284.

107. Ar sen'yan et al, T. (1972) Interferometric investigation of phase fluctuations of coherent optical radiation in the atmosphere, Radiophysics and Quantum Electronics 15:937-940.

108. Bertolotti, M., Carnevale, M., Muzii, L., and Sette, D. (1974) Atmospheric turbulence effects on the phase of laser beams, Appl. Optics 13:1582-1585.

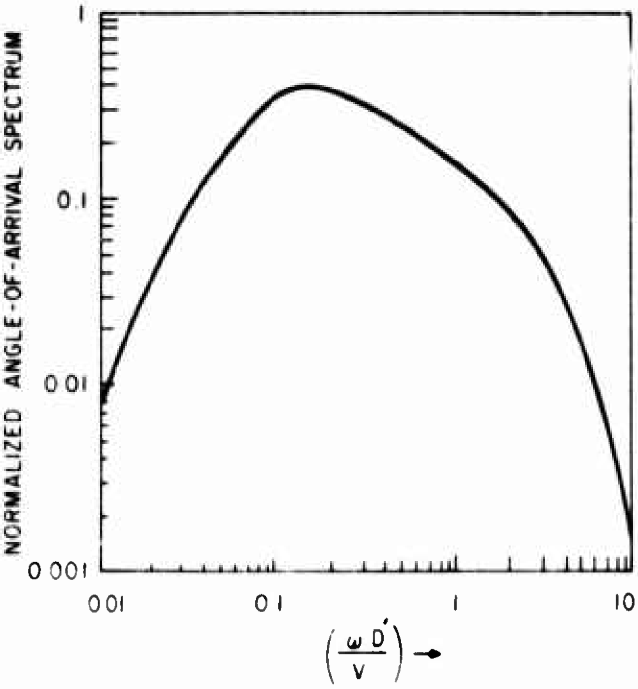

Figure 11. Angle of Arrival Spectrum for a Spherical Wave Received by an Aperture with $D/L_0 = 0.1$

7.3 Adaptive Methods

Because angle-of-arrival fluctuations are the principal limitation on the effectiveness of an optical communications system, Fried and Yura[109] have suggested some methods for their compensation using adaptive techniques. One such technique for improved spacecraft-to-ground communications through the turbulent atmosphere involves, placing a laser on the spacecraft which sends a pilot tone to the ground. This provides, because of atmospheric reciprocity, sufficient information for pointing the ground transmitter (by spatial modulation). In particular, the optimum[110] ground transmitter signal is obtained by simply reversing the direction of propagation of the received pilot signal. In order for the system to work properly, however, the response time of the ground receiver transmitter system must be short enough to respond to the temporal fluctuations in the angle of arrival; this is why we studied the angle-of-arrival spectrum in

109. Fried, D., and Yura, H. (1972) Telescope-performance reciptocity for propagation in a turbulent medium, J. Opt. Soc. Amer. 62:600-602.

110. Shapiro, J. (1971) Optimal power transfer through atmospheric turbulence using state knowledge, IEEE Trans. Comm. Tech. COM-19:410-414.

Section 7.2, since we learned there that the ground receiver/transmitter system must be able to respond in times $t_R \sim 1/f \sim 2\pi D'/(10V)$. For the case of a segmented adaptive receiver, D' would correspond to the segment size.

In order to quantitatively study the operation of an adaptive communications system, let us suppose that the transmitting aperture is located in the $x = 0$ plane and that the receiving aperture and a beacon (which emits the pilot tone used for compensation of the transmitting aperture) are co-located at the position x. If we assume that the field on the transmitter is $u_0(\underline{\rho})$ and that the receiver is a circular aperture of diameter $D' < \rho_0$, where ρ_0 is the spherical-wave coherence length given in Eq. (38), we have from Eq. (26) that the field at the receiver can be approximated by

$$u(x, \underline{\rho}_1) = \left(\frac{k}{2\pi i x}\right) \int_{-\infty}^{\infty} d^2\rho_1' \, u_0(\underline{\rho}_1') \exp\left[\frac{ik}{2x}(\underline{\rho}_1 - \underline{\rho}_1')^2 + \chi(0, \underline{\rho}_1') + iS(0, \underline{\rho}_1')\right] \tag{87}$$

where the complex phase ψ has been decomposed into a log-amplitude χ and a phase S. We now assume that it is possible to instantaneously measure the amplitude and phase fluctuations induced on the pilot tone from the beacon by the turbulent medium. We commented in Section 2.4 that the atmosphere is reciprocal. We can, therefore, use these measurements to adjust the amplitude and phase of the transmitter field, u_0, to compensate for the effect of the turbulence. In particular we choose (in the Fraunhofer zone)

$$u_0(\underline{\rho}_1') = \frac{u_0(\underline{\rho}_1') \exp\left[-\chi(0, \underline{\rho}_1') - iS(0, \underline{\rho}_1') - \frac{ik\rho_1'^2}{2x}\right]}{\left(\iint e^{2\chi(0, \underline{\rho}')} d^2\rho'\right)^{1/2}} \tag{88}$$

where $u_0(\underline{\rho}_1')$ is the signal we desire to transmit if there were no turbulent atmosphere between the transmitter and receiver. If Eq. (88) is used in (87), it is easy to show that the received power is, on the average, precisely what would be obtained in vacuum; therefore, when both the amplitude and phase of the transmitted field are compensated the turbulence does not cause any loss in the received power.

Next, let us suppose that it is only possible to measure and compensate for phase fluctuations induced on the pilot tone by the turbulence. For example, this might be done by point-to-point tracking and compensating for the angle of arrival fluctuations.[111] In this case, the field which would be impressed on the transmitting aperture is

111. Dunphy, J., and Kerr, J. (1974) Atmospheric beam wander cancellation by a fast-tracking transmitter, J. Opt. Soc. Amer. 64:1015-1016.

$$u_o(\underline{\rho}_1') = u_o(\underline{\rho}_1') \exp[-iS(0, \underline{\rho}_1')] \ . \tag{89}$$

If we use Eq. (89) in (87), we find that the (long-term) ensemble averaged power received is

$$P_{rec} = \left(\frac{k}{2\pi x}\right)^2 \iint_{-\infty}^{\infty} d^2\rho_1 \, W(\underline{\rho}_1) \iint_{-\infty}^{\infty} d^2\rho_1' \iint_{-\infty}^{\infty} d^2\rho_2' \, u_o(\underline{\rho}_1') \, u_o^*(\underline{\rho}_2')$$

$$\times \exp\left[i\frac{k}{2x}(\underline{\rho}_1 - \underline{\rho}_1')^2 - i\frac{k}{2x}(\underline{\rho}_2 - \underline{\rho}_2')^2 - \frac{1}{2} d_\chi(0, \underline{\rho}_1' - \underline{\rho}_2')\right] , \tag{90}$$

where $W(\underline{\rho}_1) = 1$ for $\rho_1 \leq D'/2$ and is equal to zero otherwise, and $d_\chi(\underline{\rho}_1 - \underline{\rho}_2, \underline{\rho}_1' - \underline{\rho}_2') \equiv \langle [\chi(\underline{\rho}_1, \underline{\rho}_1') - \chi(\underline{\rho}_2, \underline{\rho}_2')]^2 \rangle$ is the two-source log-amplitude structure function for a spherical wave. For $\sigma_1^2 < 0.3$, it can be shown that[58]

$$d_{\chi,s}(\underline{\alpha}, \underline{\beta}) = (2\pi k)^2 \int_0^x d\eta \int_0^\infty \kappa d\kappa \, \Phi_n(\eta, \kappa) \left[1 \mp \cos^2 \frac{\kappa^2 \eta(x-\eta)}{kx}\right]$$

$$\times \left\{1 - J_o\left(\kappa \left|\frac{\eta}{x}\underline{\alpha} + \underline{\beta}\left(1 - \frac{\eta}{x}\right)\right|\right)\right\} \ , \tag{91}$$

where $d_s(\underline{\rho}_1 - \underline{\rho}_2, \underline{\rho}_1' - \underline{\rho}_2') \equiv \langle [S(\underline{\rho}_1, \underline{\rho}_1') - S(\underline{\rho}_2, \underline{\rho}_2')]^2 \rangle$. Because $d_\chi \leq 4\sigma_\chi^2$, where $\sigma_\chi^2 = B_\chi(x, 0, 0)$ is the variance of the log-amplitude fluctuations, it is possible to obtain a lower bound on the received power. To do this, let us suppose that $\hat{u}_o(\underline{\rho}_1') = \bar{u}_o(\underline{\rho}_1') \exp[-ik\rho_1'^2/2x]$ where $\bar{u}_o$ is a real function. We also assume that $kDD'/2x < 1$, where D is the diameter of the transmitting aperture. It is then easy to show that

$$P_{vac} \geq P_{rec} \geq e^{-2\sigma_\chi^2} P_{vac} \tag{92}$$

where P_{vac} is the power which would be received in vacuum. Experimental measurements indicate that σ_χ^2 is always less than 0.6; therefore, in an adaptive communications system with point-to-point angle of arrival compensation only, $P_{rec} \geq 0.3 \, P_{vac}$. This conclusion has been verified by the measurements of Dunphy and Kerr.[111]

Before leaving our discussion of adaptive systems, it is appropriate to also briefly discuss imaging systems. A great deal of effort has been devoted to this problem;[112] two of the most promising systems for atmospheric turbulence

112. Huang, T., Schreiber, W., and Tretiak, O. (1971) Image processing, Proc. IEEE 59:1586-1609.

compensation are intensity interferometry and predetection image compensation. In intensity interferometry, it is found[113-116] that the autocorrelation of the received intensity pattern (sometimes called the speckle pattern) is proportional to the square of the spatial Fourier transform of the short-term intensity distribution of the source. Because intensity interferometry eliminates phase fluctuations and since the amplitude fluctuations saturate, as we saw in Section 5, this method of imaging is rather insensitive to atmospheric turbulence (it is, however, quite sensitive to system noise).

In order to discuss pre-detection image compensation, let us consider the system in Figure 12. From Eq. (26), we have that the field $u(\underline{\rho}_1)$ is

$$u(\underline{\rho}_1) = \iint d^2\rho \; u_o(\underline{\rho}) \, h(\underline{\rho}_1, \underline{\rho}) \quad , \tag{93}$$

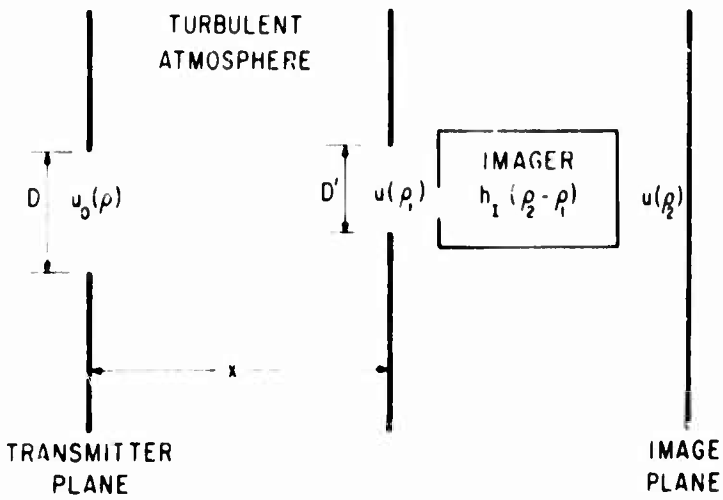

Figure 12. System Geometry for an Adaptive Imager

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113. Gezari, D., Lebeyrie, A., and Stachnik, R. (1972) Speckle interferometry: diffraction limited measurement of nine stars with the 200 inch telescope, Astrophys J. 173:L1-L5.

114. Dietz, P., and Carlson, F. (1973) Intensity interferometry in the spatial domain, J. Opt. Soc. Amer. 63:274-280.

115. Dietz, P. (1975) Image information by means of speckle-pattern processing, J. Opt. Soc. Amer. 65:279-285.

116. (a) Korff, D. (1973) Analysis of a method for obtaining near-diffraction-limited information in the presence of atmospheric turbulence, J. Opt. Soc. Amer. 63:971-980.
(b) Roddier, C., and Roddier, F. (1975) Influence of exposure time on spectral properties of turbulence-degraded astronomical images, J. Opt. Soc. Amer. 65:664-667.

and the received image field can be written neglecting noise, as

$$u_I(\underline{\rho}_2) = \iint d^2\rho_1 \; u(\underline{\rho}_1) \, h_I \, (\underline{\rho}_2 - \underline{\rho}_1) \quad , \tag{94}$$

where h_I is the (as yet unspecified) spatial impulse response of the optical filter. Over distances Δ such that the atmosphere is isoplanatic,

$$h(\rho_1 - \Delta, \rho - \Delta) = h(\underline{\rho}_1, \underline{\rho}) \, . \tag{95}$$

The isoplanatic distance has not been precisely defined; a sufficient condition for Eq. (95) to hold is that $|\underline{\Delta}| < \rho_o$ where ρ_o is given by Eq. (38). The necessary conditions for Eq. (95) to hold are still somewhat vague. If we use Eq. (95) in (93), with $\underline{\Delta}$ set equal to $\underline{\rho}$ we can write Eq. (93) as a convolution integral

$$u(\underline{\rho}_1) = \iint d^2\rho \; u_o(\underline{\rho}) \; h(\underline{\rho}_1 - \underline{\rho}, 0) \tag{96a}$$

on equivalently

$$\hat{U}_1(\bar{f}) = \hat{U}_o(\bar{f}) \, \hat{H}(\bar{f}) \quad , \tag{96b}$$

where $\hat{U}_1$, $\hat{U}_o$ and $\hat{H}$ are the spatial Fourier transforms of u, u_o and h, respectively, and $\bar{f}$ is the spatial frequency. Because Eq. (94) is also a convolution, we can then combine Eqs. (94) and (96) to write for the Fourier transform $\bar{U}_I(\bar{f})$ of u_I

$$\hat{U}_I(\bar{f}) = \hat{U}_o(\bar{f}) \, \hat{H}(f) \, \hat{H}_I(\bar{f}) \quad . \tag{97}$$

From Eq. (97), we see that in the absence of noise the optimum[117, 118] image filter H_I is

$$\hat{H}_I(\bar{f}) = \frac{1}{\hat{H}(\bar{f})} \quad . \tag{98}$$

Therefore, if we could continuously measure $h(\underline{\rho}_1, \underline{\rho})$ and we can assume isoplanacity, the optimum image filter would be obtained by continously adapting H_I in such a fashion that $H_I = (H)^{-1}$. Unfortunately, when noise is present this system is

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117. Horner, J. (1970) Optical restoration of images blurred by atmospheric turbulence using optimum filter theory, Appl. Optics 9:167-171.

118. Horner, J. (1969) Optical spatial filtering with least mean-square error filter, J. Opt. Soc. Amer. 59:553-558.

no longer optimum, and it can be shown[119] that the variance $\langle \int [u_I - u_o]^2 d^2\rho \rangle$ may be quite large.

One simple adaptive imaging system which approaches the optimum system in noise is the transmitted reference channel-matched filter.[119] In this system a point source is placed at the location (x = 0) of the transmitter, and this signal is used to obtain short-term measurements of h $(\underline{\rho}_1, 0)$ in the x-plane. The channel-matched filter has $h_I(\underline{\rho}_2 - \underline{\rho}_1) = h^*(\underline{\rho}_1 - \underline{\rho}_2, 0)$. If we use this result in Eq. (94), combine Eqs. (93) and (94), and then use Eq. (31) to evaluate $\langle hh^* \rangle$, we find that the (long-term) ensemble-averaged field $\langle u_I \rangle$ is

$$\langle u_I(\underline{\rho}_2)\rangle \, e^{i\frac{k}{2x}\rho_2^2} = \frac{kD'}{4\pi x} \iint_{-\infty}^{\infty} d^2\rho \, u_o(\underline{\rho}) \, e^{i\frac{k}{2x}\rho^2} \, \frac{J_1\left(\frac{kD'}{2x}|\underline{\rho} - \underline{\rho}_2|\right)}{|\underline{\rho} - \underline{\rho}_2|}$$

$$\times \exp\left\{ -\frac{\pi k^2}{4} \int_0^x dx' \, H\left[x', \underline{\rho}_2 + \left(1 - \frac{x'}{x}\right)(\underline{\rho} - \underline{\rho}_2)\right] \right\} \qquad (99)$$

From Eq. (99), it can be shown that if D' > D this system approaches vacuum-limited diffraction as $\pi DD'/4\lambda x \rightarrow \infty$.

8. SHORT PULSE PROPAGATION

The discussion in the preceding sections is applicable to continuous-wave signals or pulses longer than about 100 picoseconds. For the very short pulses proposed for high-data-rate communications systems, the dispersion due to the turbulent atmosphere must be included. That is, temporal fluctuations due to eddy motion plus random delays incurred by the scattered radiation in propagating from the turbulent eddies to the receiver can cause pulse distortion and broadening.

In order to study this problem quantitatively, let us write the field in the turbulent medium as

$$U(x, \underline{\rho}, t) = \int_{-\infty}^{\infty} A_o(\omega) \, u(x, \underline{\rho}, \omega) \, e^{i\omega\left(t - \frac{x}{c}\right)} \, d\omega \quad , \qquad (100)$$

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119. Shapiro, J. (1974) Optimum adaptive imaging through atmospheric turbulence, Appl. Optics 13:2609-2613.

where $A_o(\omega)$ is the complex frequency spectrum of the pulse at $x = 0$, c is the speed of light, and $u(x, \underline{\rho}, \omega)$ is the solution of Eq. (12) for a specific radian frequency ω. If the pulse has a carrier frequency ω_o, it is convenient to rewrite Eq. (100) as

$$U(x, \underline{\rho}, t) = e^{i\omega_o(t - \frac{x}{c})} \int_{-\infty}^{\infty} d\omega \, A_o(\omega + \omega_o) \, u(x, \underline{\rho}, \omega + \omega_o) \, e^{i\omega(t - \frac{x}{c})} \quad . \tag{101}$$

If we use Eq. (101), we can readily write the ensemble averaged signal intensity $\langle I(x, \underline{\rho}, t) \rangle$ at $(x, \underline{\rho})$ as

$$\langle I(x, \underline{\rho}, t) \rangle = \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \, A_o(\omega - \omega_1) \, A_o^*(\omega - \omega_2) \, \tilde{\Gamma}(x, \underline{\rho}_1, \underline{\rho}_2, \omega_1, \omega_2)$$

$$\times \exp\left[i(\omega_1 - \omega_2)(t - \frac{x}{c}) \right] , \tag{102}$$

where $\tilde{\Gamma}(x, \underline{\rho}_1, \underline{\rho}_2, \omega_1, \omega_2) = \langle u(x, \underline{\rho}_1, \omega_1 + \omega_o) \, u^*(x, \underline{\rho}_2, \omega_2 + \omega_o) \rangle$. From Eq. (102), we can see that the intensity of a very short pulse can be determined if we know* the function $\tilde{\Gamma}$. The equation satisfied by $\tilde{\Gamma}$ can be determined by a straightforward extension of the methods given in Section 2.3 For frozen-flow turbulence, the result is[120-122]

$$(2i \frac{\partial}{\partial x} + \frac{1}{k_1} T_1' - \frac{1}{k_2} T_2') \tilde{\Gamma} = \frac{i}{4} \left[(k_1^2 + k_2^2) A(x, 0) - 2k_1 k_2 A(x, \underline{\rho}_1 - \underline{\rho}_2) \tilde{\Gamma} \right], \tag{103}$$

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*Of course, once we know $\tilde{\Gamma}$, the generalized mutual coherence function $\tilde{H}(x, \underline{\rho}_1, \underline{\rho}_2, t_1, t_2) = \langle u(x, \underline{\rho}_1, t_1) u^*(x, \underline{\rho}_2, t_2) \rangle$ follows immediately from

$$\tilde{H}(x, \underline{\rho}_1, \underline{\rho}_2, t_1, t_2) = \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \, e^{i\omega_1(t_1 - \frac{x}{c})} \, e^{-i\omega_2(t_2 - \frac{x}{c})} \tilde{\Gamma}(x, \underline{\rho}_1, \underline{\rho}_2, \omega_1, \omega_2)$$

120. Erhukumov, L., Zarnitsyna, I., and Kirsh, P. (1973) Selective properties and the form of a signal passed through a statistically inhomogeneous arbitrary-thickness layer, Radiofizika (Russian) 16:573-580.

121. Sreenivashiah, I., amd Ishimaru, A. (1974) Plane Wave Pulse Propagation Through Atmospheric Turbulence at mm and Optical Wavelengths, Dept. of Electrical Engineering, University of Washington, Research Report AFCRL-TR-74-0205.

122. Liu, C., Wernik, A., and Yeh, K. (1974) Propagation of pulse trains through a random medium IEEE Trans. Ant. and Prop. AP-22:624-627.

where $k_1 = \omega_1/c$, $k_2 = \omega_2/c$, $\underline{\rho}_1' = \underline{\rho}_1 - \underline{V}t$, $\underline{\rho}_2' = \underline{\rho}_2 - \underline{V}t$,

$$T_j' = \frac{\partial^2}{\partial y_j'^2} + \frac{\partial^2}{\partial z_j'^2}$$

and $A(x, \underline{\rho})$ is defined in Eq. (14). In writing Eq. (103), it has been assumed that $A(x, \underline{\rho})$ is independent of ω. The conditions governing the validity of Eq. (103) are those given in Eq. (19); consequently, this result is valid in either strong or weak turbulence. The general solution to Eq. (103) is not available, although solutions are available for some special cases.[121, 122] In addition, some solutions for $\tilde{\Gamma}$ have also been obtained in the weak turbulence limit using other approaches such as the Rytov method,[123-126] and related techniques.[127]

The solution to Eq. (103) has been obtained[121] in the limiting case of a plane wave propagating in nonflowing ($\underline{V} = 0$) homogeneous turbulence. The result is

$$\tilde{\Gamma}(x, \omega_1, \omega_2) = \left[1 + \frac{i(k_1 - k_2)}{4} C_o x^2\right] \exp\left[-\frac{1}{8}(k_1 - k_2)^2 A(x, 0)\, x\right], \tag{104}$$

where $A(x, 0) = 3.12\, C_n^2 L_o^{5/3}$ and $C_o = 3.626\, C_n^2 \ell_o^{1/3}$. By using Eq. (104) in (102) and then taking the inverse Fourier transform, we can show that the frequency spectrum of the ensemble averaged intensity is

$$\langle I(x, \omega) \rangle = \left[1 + \frac{i\omega}{4c} C_o x^2\right] \exp\left[-\frac{\omega^2}{8c^2} A(x, 0)\, x\right] I_o(\omega), \tag{105}$$

where $I_o(\omega)$ is the frequency spectrum of the pulse at $x = 0$.

By studying Eq. (105), it is evident that if the bandwidth Ω of the pulse is such that

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123. Su, H., and Plonus, M. (1971) Optical-pulse propagation in a turbulent medium, J. Opt. Soc. Amer. 61:256-260.

124. Plonus, M., Su, H., and Gardner, C. (1972) Correlation and structure functions for pulse propagation in a turbulent atmosphere, IEEE Trans. Ant. and Prop. AP-20:801-805.

125. Gardner, C., and Plonus, M. (1974) Optical pulses in atmospheric turbulence, J. Opt. Soc. Amer. 64:68-77.

126. Ishimaru, A. (1972) Temporal frequency spectra of multifrequency waves in turbulent atmosphere, IEEE Trans. Ant. and Propagation, AP-20:10-19.

127. Fried, D. (1971) Spectral and angular covariance of scintillation for propagation in a randomly inhomogeneous medium, Appl. Optics 10:721-731.

$$\frac{\Omega}{4c} C_o x^2 = \frac{0.91\, \Omega\, C_n^2\, x^2}{c\, \ell_o^{1/3}} << 1 \tag{106}$$

and

$$\frac{\Omega^2}{8c^2} \langle A(x, 0) \rangle = \frac{0.39\, \Omega^2\, C_n^2\, L_o^{5/3}\, x}{c^2} << 1 \tag{107}$$

the spectrum $\langle I(x, \omega) \rangle$ of a short pulse in a turbulent medium will be approximately equal to the spectrum $I_o(\omega)$ of the incident pulse. Consequently, when the conditions in Eqs. (106) and (107) hold, there is neither broadening nor distortion of the pulse. A plot of the conditions in Eqs. (106) and (107) for a typical value of C_n^2 in the lower atmosphere* on a clear day[128] is given in Figure 13.

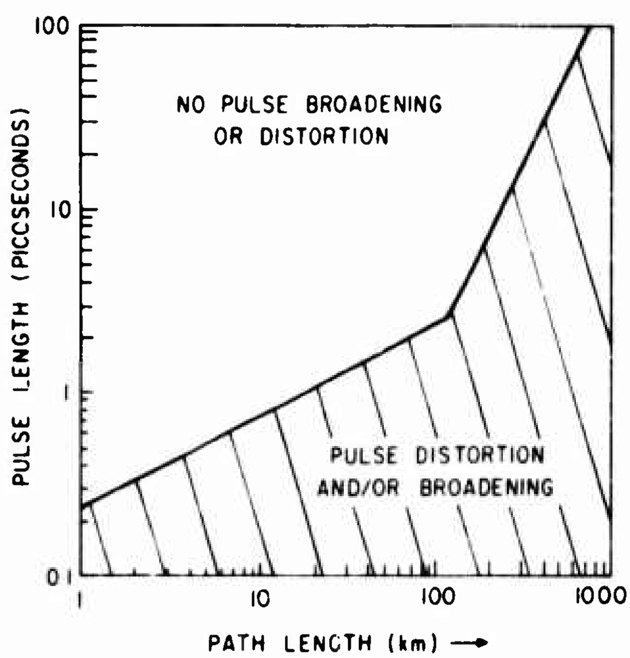

Figure 13. Plot of the Conditions Expressed in Eqs. (106) and (107) for $C_n^2 = 6.4 \times 10^{-15}\ m^{-2/3}$, $L_o = 100$ m and $\ell_o = 0.001$ m

*We have also assumed that Ω is not so large that there is overlap of the pulse spectrum with any of the atmospheric absorption lines.

128. Brookner, E. (1971) Improved model for structure constant variations with altitude, Appl. Optics 10:1960-1962.

9. MISCELLANEOUS RELATED RESULTS

In this section, we will survey some additional results which are not important enough to warrant fuller coverage, but are of interest in some applications.

9.1 Phase-Amplitude Scintillations

We have already discussed intensity fluctuations and related quantities such as amplitude fluctuations $\langle \chi_1(x, \rho_1)\, \chi_1(x, \rho_2) \rangle$ and phase fluctuations $\langle S(x, \rho_1)\, S(x, \rho_2)\rangle$, but have not considered the phase-amplitude correlation function $\langle \chi_1(x, \rho_1)\, S_1(x, \rho_2)\rangle$ since it is of lesser practical importance than the other quantities. For weak turbulence, this quantity has been evaluated by the Rytov method and studied in some detail;[1] however, for strong turbulence it has been shown[129] that results for $\langle \chi_1 S_1 \rangle$ computed using the Rytov method are incorrect. Some attempts have been made to calculate this in strong turbulence, but the only useful results so far obtained are applicable only in the geometric optics limit ($x < k\, \ell_o^2$). In this case, it has been shown[130, 131] that, for a plane wave

$$\langle SI \rangle = -\frac{kx^2}{16}\left[\nabla_\rho^2\, A(x, \rho)\right]_{\rho = 0}, \tag{108}$$

where $A(x, \rho)$ was given previously in Eq. (14) and $I = \exp(2\chi)$. This expression is valid provided $2.94\, \sigma_I^2\, (k\ell_o^2/x)^{-7/6}\, (x/\ell_o)^{-2} << 1$, which is easily satisfied even in strong turbulence ($\sigma_I^2 >> 1$).

9.2 Fluctuations Behind a Turbulent Layer

In addition to obtaining a knowledge of the field statistics inside a turbulent medium, it is also important in some applications to know the statistical properties of the field after it has exited from the turbulent medium. One specific application of such a result is in determining the statistics in the exit plane of a refracting telescope, as illustrated in Figure 14. The equations satisfied by the second and fourth* moments of the field behind the objective lens can be obtained directly

*As we pointed out in Section 2, the first and other odd moments of the field are negligible at optical wavelengths; we therefore consider only the even moments.

129. Klyatskin, V. (1972) On amplitude-phase fluctuations of a plane light wave in a turbulent medium, Radiophysics and Quantum Electronics 15:406-409.

130. Zavorotnyi, V., and Klyatskin, V. (1972) The geometric-optics approximation and amplitude-phase functuations of a plane light wave in a randomly inhomogeneous medium, Radiophysics and Quantum Electronics 15:684-688.

131. Klyatskin, V., and Tatarskii, V. (1972) Statistical theory of light propagation in a turbulent medium, Radiophysics and Quantum Electronics 15:1095-1112.

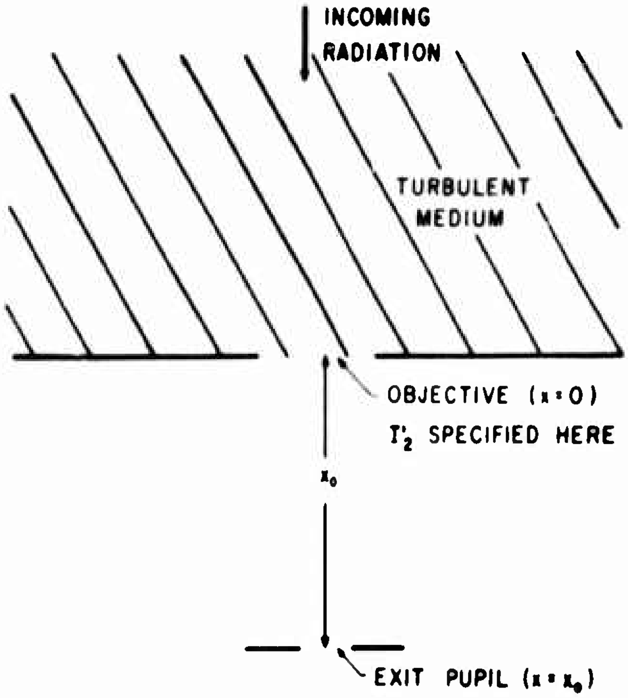

Figure 14. Geometry for Calculating the Fluctuations in the Signal Received Beyond a Turbulent Layer

from Eqs. (20) and (24) by setting $A(x, \underline{\rho}) = 0$. If we assume that the field in the objective plane can be approximated by a plane wave, it is possible to solve Eqs. (20) and (24) in a straight-forward fashion. For example, suppose the second moment of the field in the objective plane is $\Gamma_2\,(x = 0, \underline{\rho}) = \langle u(0, \underline{\rho})\, u^*(0, \underline{\rho}_2) \rangle$ where $\underline{\rho} = \underline{\rho}_1 - \underline{\rho}_2$. Then it can be shown[132, 133] that in the pupil plane

$$\Gamma_2(x_0, \underline{\rho}) = \frac{1}{2\pi} \int_0^\infty \kappa\, d\kappa\; J_0(\kappa\rho)\, \hat{\Gamma}_2(0, \kappa)\, \exp\left[\frac{i\kappa^2 x_0}{k}\right], \qquad (109)$$

where $\hat{\Gamma}_2(0, \underline{\kappa})$ is the Fourier transform of $\Gamma_2(0, \underline{\rho})$. Equation (24) can be solved if we assume that the four points $\underline{\rho}_1, \underline{\rho}_2, \underline{\rho}_3$ and $\underline{\rho}_4$ are located at the corners of a parallelogram such that $\underline{\rho}_1 - \underline{\rho}_2 = \underline{\rho}_4 - \underline{\rho}_3$; in this case, Γ_4 is a function only of

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132. Torrieri, D., and Taylor, L. (1971) Propagation of random electromagnetic fields, J. Math. Phys. 12:1211-1214.

133. LeVine, D. (1972) The propagation of stochastic waves in uniform media, IEEE Trans. Ant. and Prop. AP-20:809-811.

$\xi_1 = \underline{\rho}_1 - \underline{\rho}_2$ and $\xi_2 = \underline{\rho}_2 - \underline{\rho}_3$. If the fourth moment of the field in the objective plane is $\Gamma_4(x = 0, \underline{\xi}_1, \underline{\xi}_2)$, it can be shown[134-136] that in the pupil plane

$$\Gamma_4(x_0, \underline{\xi}_1, \underline{\xi}_2) = \left(\frac{k}{2\pi x_0}\right)^2 \iint_{-\infty}^{\infty} d^2\alpha\, d^2\beta\, \Gamma_4(0, \underline{\alpha}, \underline{\beta}) \exp\left[-i \frac{k}{x_0} (\underline{\xi}_1' - \underline{\alpha}) \cdot (\underline{\xi}_2 - \underline{\beta})\right]. \quad (110)$$

Mercier[137] has studied the higher order field statistics for the case when the objective field $u(0, \underline{\rho}_1)$ can be written as $\exp[i\theta(\underline{\rho}_1)]$ which is equivalent to a random phase screen. Assuming x_0 is sufficiently large and that the correlation function for the phase fluctuations is gaussian, he demonstrates that $A_0 \quad (uu^*)^{1/2}$ is Rice distributed. (In the limit of large phase fluctuations, the distribution approaches Rayleigh.) Related results have recently been obtained by Rumsey[138] and Furuhama[139] for more realistic phase correlation functions.

9.3 Nonforward Scatter

The nature of the field scattered by a random medium is quite often used to probe the properties of that medium. For example, radar backscatter has long been used to prove the characteristics of the clear atmosphere,[140-142] and microwave and laser scattering is commonly used as a diagnostic for

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134. Taylor, L. (1972) Scintillation of randomized electromagnetic fields, J. Math. Phys. 13:590-595.
135. Liu, C., Wernik, A., Yeh, K., and Youakim, M. (1974) Effects of multiple scattering of scintillation of transionospheric radio signals, Radio Science 9:599-607.
136. Beran, M., and Whitman, A. (1974) Free-space propagation of irradiance fluctuations and the fourth-order coherence function, J. Opt. Soc. Amer. 64:1636-1640.
137. Mercier, R. (1962) Diffraction by a screen causing large random phase fluctuations, Proc. Cambridge Phil. Soc. 51:382-400.
138. Rumsey, V. (1975) Scintillations due to a concentrated layer with a power-law turbulence spectrum, Radio Science 10:107-114.
139. Furuhama, Y. (1975) Probability distribution of irradiance fluctuation propagating through the turbulent slabs, Digest of 1975 URSI Meeting, page 14.
140. Hardy, K., and Katz, I. (1969) Probing the clear atmosphere with high power, high resolution radars, Proc IEEE 57:468-480.
141. Booker, H., and Gordon, W. (1950) A theory of radioscattering in the troposphere, Proc. IRE 38:401-412.
142. Villars, F., and Weisskopf, V. (1954) The scattering of electromagnetic waves by turbulent atmospheric fluctuations, Phys. Rev. 94:232-240.

laboratory plasmas.[143-145] In order to calculate the relationship between the scattered intensity and the index-of-refraction fluctuations in the scattering medium, the Born approximation is commonly used.[1, 9, 140-142, 146] This approximation is obtained by assuming that the field incident on the random medium is scattered only by a single turbulent eddy, so that multiple scattering effects are ignored. For this case, it is found that the scattering cross section per unit volume of the turbulent medium is

$$\sigma_V(n_i - n_s) = 2\pi k^4 \sin^2 \phi_o \, \Phi_n \, [\, x, \underline{\kappa} = k(\hat{n}_i - \hat{n}_s)\,] \, , \tag{111}$$

where $\hat{n}_i$ is a unit vector in the direction of propagation of the incident wave as indicated in Figure 15, $\hat{n}_s$ is a unit vector pointing from the scattered to the receiver, ϕ_o is the angle between $\hat{n}_s$ and the direction of polarization of the field of the incident wave, and Φ_n is the three-dimensional spectrum of the index-of-refraction fluctuations which was defined in Eq. (2). If the receiver is a distance s from the scattered, the received intensity $\langle I_b \rangle$ is then given by

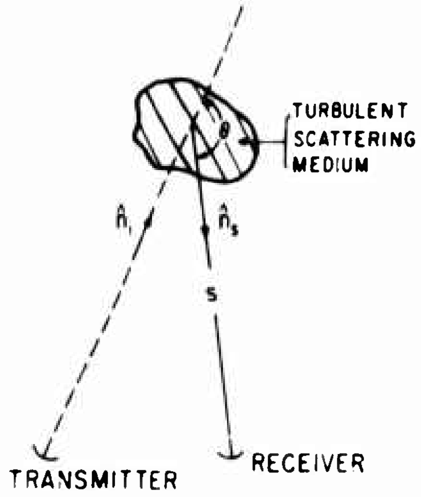

Figure 15. Scatter of an Electromagnetic Wave by a Turbulent Region of Volume V_s

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143. Sheffield, J. (1975) Plasma Scattering of Electromagnetic Radiation, Academic Press, New York.

144. Wort, D. (1966) Microwave transmission through turbulent plasma, Plasma Physics (J. Nuclear Energy, Part C) 8:79-93.

145. Wort, D. (1969) Microwave scattering by turbulent plasma, J. Phys. A. (General Physics), Series 2 2:75-86.

146. Wheelon, A. (1959) Radio-wave scattering by tropospheric irregularities J. Res. NBS 63D:205-233.

$$\langle I_b \rangle = \frac{\iiint\limits_{V_s} \sigma_V \, dx\, dy\, dz}{s^2} \cdot I_{inc} \quad , \tag{112}$$

where the integral in Eq. (112) is over the volume V_s of the scatterer, and I_{inc} is the intensity of the incident wave at the position of the scatterer. The Born approximation may also be employed to calculate the frequency spectrum[1, 147] of the scattered radiation, and its mutual coherence function.[148, 149]

Equation (111), and other results obtained from the Born approximation, are valid as long as the size of the turbulent scatterer is small enough to ignore multiple scattering effects. In particular, it can be demonstrated[150] that multiple scattering may be ignored as long as the characteristic dimension X of the scatterer is such that

$$\frac{X}{\ell_t} = \pi k^4 X \int_0^\pi d\theta \sin\theta\,(1 + \cos^2)\theta\;\Phi_n\,[\underline{\kappa} = k(\hat{n}_i - \hat{n}_s)] \;<\; 1 \,, \tag{113}$$

where θ is the angle between n_i and n_s. For the spectrum given in Eq. (3), the condition in Eq. (113) becomes $0.78\, k^2 L_o^{5/3} C_n^{\,2} X < 1$. In order to include multiple scattering, an approach based on transport theory may be used.[21, 22, 150-155] By using energy conservation, it can be shown that the

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147. Brown, E. (1974) Turbulent spectral broadening of backscattered acoustic pulses, J. Acoust. Soc. Amer. 56:1398-1408.
148. Pieroni, L., and Bremmer, H. (1970) Mutual coherence function of light scattered by a turbulent medium, J. Opt. Soc. Amer. 60:936-947.
149. Denison, N., and Tamoikin, V. (1971) Correlation theory of the backscatter of radiowaves, Radiophysics and Quantum Electronics 14:1045-1048.
150. Salpeter, E., and Trieman, S. (1964) Backscatter of electromagnetic radiation from a turbulent plasma, J. Geophys. Res. 69:869-881.
151. Bugnolo, D. (1961) Radio star scintillation and multiple scattering in the ionosphere, IRE Trans. Antennas and Prop. AP-9:89-96.
152. Watson, K. (1969) Multiple scattering of electromagnetic waves in an underdense plasma, J. Math. Phys. 10:688-702.
153. Stott, P. (1968) A transport theory for the multiple scattering of electromagnetic waves by a turbulent plasma, J. Phys. A 1:675-689.
154. Feinstein, D., and Granatstein, V. (1969) Scalar radiative transport model for microwave scattering from a turbulent plasma, Phys. Fluids 12:2658-2668.
155. Ishimaru, A. (1975) Correlation functions of a wave in a random distribution of stationary and moving scatterers, Radio Science 10:45-52.

ensemble averaged intensity $\langle I(\underline{r}, n)\rangle$ at position $\underline{r}$ propagating in the direction of the unit vector n satisfies

$$(\hat{n} \cdot \nabla_r + \frac{1}{\ell_t}) \langle \hat{I}(\underline{r}, \hat{n}) \rangle = \iint d\Omega' \, \sigma_V (\hat{n} - \hat{n}') \langle \hat{I}(\underline{r}, \hat{n}) \rangle \quad , \tag{114}$$

where ℓ_t is defined in Eq. (113), $\sigma_V(\hat{n}-\hat{n}')$ was given in Eq. (111) and $d\Omega'$ is the element of solid angle. For the limiting case of nearly forward scatter Eq. (114) can be shown[1, 156-159] to be equivalent to Eq. (20). That is, for narrow-angle beams transport theory and the Markov approximation are equivalent. The solution of Eq. (114) can be used to calculate the radiation scattered in any direction, except backscatter ($\theta = \pi$); for that case, the solution of Eq. (114) must be modified as pointed out by Watson and deWolf.[152, 160]

A general solution to Eq. (114) is not presently available, although some results have been obtained for special cases.[161-163] One case of particular interest is the backscatter of a plane wave which is normally incident on a turbulent slab of thickness L_s. In this case, it is found[22, 161] that if $\lambda \ll L_o$ and $L_s \gg kL_o^2$, the backscattered intensity is

$$\langle I_b \rangle = I_{inc} \, L_s \, \sigma_V(\theta = \pi) \left[2 - \frac{1 - e^{-2q}}{2q} \right] , \tag{115}$$

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156. Fante, R., and Poirier, J. (1973) Mutual coherence function of a finite optical beam in a turbulent medium, Appl. Optics 12:2247.
157. Fante, R. (1974) Intensity of a focused beam in a turbulent medium, Proc. IEEE 62:1400-1402.
158. Barabanenkov, Y., and Finkel'berg, D. (1968) Radiation transport equation for correlated scatterers, Soviet Phys. JETP 26:587-591.
159. Ovchinnikov, G., and Tatarskii, V. (1972) On the problem of the relationship between coherence theory and the radiation-transfer equation, Radiophysics and Quantum Electronics 15:1087-1089.
160. deWolf, D. (1972) Discussion of radiative-transfer methods applied to electromagnetic reflection from turbulent plasma, IEEE Trans. Antennas and Propagation AP-20:805-807.
161. deWolf, D. (1971) Electromagnetic reflection from an extended turbulent medium: cumulative-forward scatter single-backscatter approximation, IEEE Trans. Antennas and Propagation AP-19:254-262.
162. Feinstein, D., Butler, F., Prech, N., and Leonard, A. (1972) Radiative transport of electromagnetic propagation in isotropic plasma turbulence, Phys. Fluids 15:1641-1651.
163. Granatstein, V. (1972) Multiple scatter of laser light from a turbid medium, Appl. Optics 11:1217-1224.

where I_{inc} is the intensity of the incident radiation and $q = (L_s/\ell_t) = 0.78 C_n^2 L_o^{5/3} k^2 L_s$. We note that if $q = (L_s/\ell_t) << 1$, Eq. (115) reduces to $\langle I_0 \rangle = I_{inc} L_s \sigma_V$, which is the single-scatter result. The term in square brackets is a measure of the effect of multiple scattering. For $q \to \infty$, we note that the scattered intensity including multiple scattering is twice the intensity calculated neglecting multiple scatter.

10. CONCLUDING REMARKS

In this report, we have presented a review of some important results on the propagation of low-power electromagnetic waves through large-scale isotropic turbulence; the results are equally valid for microwave or optical frequencies, provided $k\ell_o > 1$. We have chosen to stress the electromagnetic aspects of the propagation; consequently, we have not presented results for the index-of-refraction fluctuations in the earth's atmosphere in any detail. The reader interested in such data can consult references 128 and 164-169. We have also tended to avoid discussing in detail the systems applications of the results presented; an idea of their application in optical communications may be found in references 170-171.

Throughout our discussion, we have ignored the effect of any absorption, such as would be due to water, vapor clouds, etc. That is, for the case of atmospheric propagation we have assumed that the only effect is due to clear-air turbulence; in general, this is not the case. Absorption is readily included in our analysis. For example, if $\alpha_0(x, \omega)$ is the net absorption coefficient then we can include the

164. Brookner, E. (1970) Atmospheric propagation and communication channel model for laser wavelengths, IEEE Trans. Comm. Tech. COM-18: 396-416.

165. Hufnagel, R. (1966) Restoration of Atmospherically Degraded Images, Woods Hole Summer Study, Vol. 2.

166. Lawrence, R., Ochs, G., and Clifford, S. (1970) Measurements of atmospheric turbulence relevant to optical propagation, J. Opt. Soc. Amer. 60:826-830.

167. Strohbehn, J. (1973) Optical and millimeter line-of-sight propagation effects in the turbulent atmosphere, Boundary Layer Meteorol. 4:397-422.

168. Hufnagel, R. (1974) Variations of atmospheric turbulence, OSA Topical Meeting on Propagation Through Turbulence, Paper No. WaI-1.

169. Dolukhov, M. (1971) Fluctuation Processes in Radiowave Propagation (Russian) Svyaz Press, Moscow.

170. Kennedy, R. (1970) Communication through optical scattering channels: an introduction, Proc IEEE 58:1651-1655.

171. Pratt, W. (1969) Laser Communications Systems, John Wiley and Sons, Inc. New York.

effect of absorption in our solution for Γ_2 simply by multiplying Eq. (21) by $\exp\left[-\int_0^x \alpha_o(x',\omega)\,dx'\right]$.

Finally, we comment that, in our view, the most significant recent advance in the field of propagation in turbulence has been the development of analytical and physical models which are capable of explaining and predicting the saturation of the intensity scintillations in strong turbulence. Especially important in this regard are the analytical models presented in references 32 and 78 and the physical models developed in references 70-72 and 172. The aforementioned physical models show clearly the relative importance of refraction, diffraction, coherence loss, and turbulent eddy size in producing intensity scintillation.

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172. Clifford, S., and Yura, H. (1974) Equivalence of two theories of strong optical scintillation, J. Opt. Soc. Amer. 64:1641-1643.

References

1. Tatarskii, V. (1971) The Effect of the Turbulent Atmosphere on Wave Propagation, U.S. Dept. of Commerce, Springfield, Virginia.

2. Lawrence, R., and Strohbehn, J. (1970) A survey of clear-air propagation effects relevent to optical communications, Proc IEEE 58:1523-1545.

3. Barabanenkov, Y., Kravtsov, Y., Rytov, S., and Tatarskii, V. (1971) Status of the theory of propagation of waves in randomly inhomogeneous media, Soviet Physics Usp. 13:551-580.

4. Strohbehn, J. (1968) Line of sight wave propagation through the turbulent atmosphere, Proc. IEEE 56:1301-1318.

5. Houbolt, J. (1973) Atmospheric turbulence, AIAA Journal 11:421-437.

6. Fox, J., and Rungaldier, H. (1972) Electron density fluctuation measurements in projectile wakes, AIAA Journal 10:790-795.

7. Fox, J., and Rungaldier, H. (1973) Turbulent scattering spectra of electron density fluctuations in Mach 16 projectile wakes, AIAA Journal 11:1059-1064.

8. Bergmann, P. (1946) Propagation of radiation in a medium with random inhomogeneities, Phys. Rev. 70:486-492.

9. Tatarskii, V. (1967) Wave Propagation in a Turbulent Medium, Dover Publications, Inc., New York.

10. Chernov, L. (1967) Wave Propagation in a Random Medium, Dover Publications, Inc., New York.

11. Strohbehn, J., and Clifford, S. (1967) Polarization and angle-of-arrival fluctuations for a plane wave propagated through a turbulent medium, IEEE Trans. Ant. and Prop. AP-15:416-421.

12. Collett, E., and Alferness, R. (1972) Depolarization of a laser beam in a turbulent medium, J. Opt. Soc. Am. 62:529-533.

13. Gracheva, M., Gurvich, A., and Kallistrova, M. (1970) Dispersion of strong atmospheric fluctuations in the intensity of laser radiation, Radiophysics and Quantum Electronics, 13:40-42.

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14. Lee, R., and Harp, J. (1969) Weak scattering in random media, with applications to remote probing, Proc. IEEE 57:375-406.

15. Brown, W. (1972) Moment equations for waves propagated in random media, J. Opt. Soc. Amer. 62:45-54.

16. Brown, W. (1972) Fourth moment of a wave propagating in a random medium, J. Opt. Soc. Amer. 62:966-971.

17. deWolf, D. (1973) Strong irradiance fluctuations in turbulent air: plane waves, J. Opt. Soc. Amer. 63:171-179.

18. deWolf, D. (1973) Strong irradiance fluctuations in turbulent air II: spherical waves, J. Opt. Soc. Amer. 63:1249-1253.

19. deWolf, D. (1974) Strong irradiance fluctuations in turbulent air III: diffraction cutoff, J. Opt. Soc. Amer. 64:360-365.

20. Sancer, M., and Varvatsis, A. (1970) Saturation calculation for light propagation in the turbulent atmosphere, J. Opt. Soc. Amer. 60:654-659.

21. Dolin, L. (1964) Propagation of a narrow light beam in a random medium, Radiofizika (Russian) 7:380-391.

22. Fante, R. (1973) Propagation of electromagnetic waves through a turbulent plasma using transport theory, IEEE Trans. Ant. and Prop. AP-21:750-755.

23. Beran, M. (1970) Propagation of a finte beam in a random medium, J. Opt. Soc. Amer. 60:518-521.

24. Beran, M., and Ho, T. (1969) Propagation of the fourth-order coherence function in a random medium, J. Opt. Soc. Amer. 59:1134-1138.

25. Molyneux, J. (1971) Propagation of N^{th} order coherence functions in a random medium, J. Opt. Soc. Amer. 61:369-377.

26. Lutomirski, R., and Yura, H. (1971) Propagation of a finite optical beam in an inhomogeneous medium, Appl. Optics 10:1652-1658.

27. Furutsu, K. (1972) Statistical theory of wave propagation in a random medium and the irradiance distribution function, J. Opt. Soc. Amer. 62:240-254.

28. Tatarskii, V. (1969) Light propagation in a medium with random refractive index inhomogeneities in the Markov approximation, Soviet Phys. JETP 29:1133-1138.

29. Klyatskin, V. (1970) Applicability of the approximation of a Markov random process in problems related to the propagation of light in a medium with random inhomogeneities, Soviet Phys. JETP 30:520-523.

30. Klyatskin, V., and Tatarskii, V. (1970) The parabolic equation approximation for propagation of waves in a medium with random inhomogeneities, Soviet Physics JETP 31:335-339.

31. (a) Novikov, E. (1965) Functionals and the method of random forces in the theory of turbulence, Soviet Physics JETP 20:1290-1294.
(b) Furutsu, K. (1963) On the statistical theory of electromagnetic waves in a fluctuating medium, J. Res NBS 67D:303-310.

32. Prokhorov, A., Bunkin, F., Gochelashvily, K., and Shishov, V. (1975) Laser irradiance propagation in turbulent media, Proc. IEEE 63:790-811.

33. Feizulin, Z., and Kravtsov, Y. (1967) Broadening of a laser beam in a turbulent medium, Radiophysics and Quantum Electronics 10:33-35.

34. Born, M., and Wolf, E. (1965) Principles of Optics, Pergamon Press, Oxford, England.

35. Shapiro, J. (1971) Reciprocity of the turbulent atmosphere, J. Opt. Soc. Amer. 61:492-495.

36. Yura, H. (1973) Short term average optical-beam spread in a turbulent medium, J. Opt. Soc. Amer. 63:567-572.

37. Fante, R. (1974) Mutual coherence function and frequency spectrum of a laser beam propagating through atmospheric turbulence, J. Opt. Soc. Amer. 64:592-598.

38. Poirier, J., and Korff, D. (1972) Beam spreading in a turbulent medium, J. Opt. Soc. Amer. 62:893-898.

39. Bunkin, F., and Gochelashvili, K. (1970) Spreading of a light beam in a turbulent medium, Radiophysics and Quantum Electronics 13:811-821.

40. Kallistrova, M., and Pokasov, V. (1971) Defocusing and fluctuations of the displacement of a focused laser beam in the atmosphere, Radiophysics and Quantum Electronics 14:940-945.

41. Dowling, J., and Livingston, P. (1973) Behavior of focused beams in atmospheric turbulence: measurements and connents on the theory, J. Opt. Soc. Amer. 63:846-858.

42. Wesley, M., and Derzko, Z. (1975) Atmospheric turbulence parameters from visual resolution, Appl. Optics 14:847-853.

43. Fried, D. (1966) Optical resolution through a randomly inhomogeneous medium for very long and very short exposures, J. Opt. Soc. Amer. 56:1372-1379.

44. Andreev, G., and Gel'fer, E. (1971) Angular random walks of the center of gravity of the cross section of a diverging light beam, Radiophysics and Quantum Electronics 14:1145-1147.

45. Kon, A. (1970) Focusing of light in a turbulent medium, Radiophysics and Quantum Electronics 13:43-50.

46. Gel'fer, E., Kravtsov, V., and Finkel'shtein, S. (1972) Measurement of the light intensity on the axis at the center of gravity of a focused light beam, Radiophysics and Quantum Electronics 15:696-699.

47. Chiba, T. (1971) Spot dancing of the laser beam propagated through the turbulent atmosphere, Appl. Optics 10:2456-2461.

48. Gel'fer, E., Kon, A., and Cheremukhin, A. (1973) Correlation of the shift of the center of gravity of a focused light beam in a turbulent medium, Radiophysics and Quantum Electronics 16:182-187.

49. Fried, D. (1973) Statistics of laser beam fade induced by pointing jitter, Appl. Optics, 12:422-423.

50. Titterton, P. (1973) Power reduction and fluctuations caused by narrow laser beam motion in the far field, Appl. Optics 12:423-425.

51. Klyatskin, V., and Kon, A. (1972) On the displacement of spatially-bounded light beams in a turbulent medium in the Markovian-random-process approximation, Radiophysics and Quantum Electronics 15:1056-1061.

52. Starobinets, E. (1972) The average illumination and intensity fluctuations at the focus of a light beam in a turbulent atmosphere, Radiophysics and Quantum Electronics 15:738-742.

53. Mironov, V. and Khmelevstov, S. (1972) Broadening of a laser beam propagating in a turbulent atmosphere along inclined routes, Radiophysics and Quantum Electronics 15:567-571.

54. Kallistrova, M., and Kon, A. (1972) On the effect of the size of optical systems on the definition of light beams in a turbulent atmosphere, Radiophysics and Quantum Electronics 15:545-549.

55. Bremmer, H. (1973) General remarks concerning theories dealing with scattering and diffraction in random media, Radio Science 8:511-534.

56. Fried, D., (1967) Optical heterodyne detection of an atmospherically distorted signal wavefront, Proc IEEE 55:57-67.

57. Hufnagel, R., and Stanley, N. (1964) Modulation transfer function associated with image transmission through turbulent media, J. Opt. Soc. Am. 54:52-61.

58. Kon, A., and Feizulin, V. (1970) Fluctuations in the parameters of spherical waves propagating in a turbulent atmosphere, Radiophysics and Quantum Electronics 13:51-53.

59. Yura, H. (1972) Mutual coherence function of a finite cross section optical beam propagating in a turbulent medium, Appl. Optics 11:1399-1406.

60. (a) Gilmartin, T., and Holtz, J. (1974) Focused beam and atmospheric coherence measurements at 10.6 and 0.63 micrometers, Appl. Optics 13:1906-1912.
(b) Dainty, J., and Scanlon, R. (1975) Measurements of the atmospheric transfer function at Mauna Kea, Hawaii, Monthly Not. Royal Astr. Soc. 170:519-532.

61. Gurvich, A., Time, N., Turevtseva, L., and Turchin, V. (1974) Reproduction of temperature pulsation spectra from optical measurements in the atmosphere Izv. Akad. Sci. USSR, Fiz. Atmos. Okeana (Russian) 10:484-492.

62. Dunphy, J., and Kerr, J. (1973) Scintillation measurements for large integrated-path turbulence, J. Opt. Soc. Am. 63:981-986.

63. Gracheva, M., Gurvich, A., Kashkarov, S., and Pokasov, V. (1973) Similarity Correlations and their Experimental Verification in the case of Strong Intensity Fluctuations of Laser Radiation (Russian). English Translation No. LRG-73-T-28 available from Aerospace Corp., Library Services, P.O. Box 92957, Los Angeles, California.

64. S'edin, V., Khmelevstov, S., and Tsvyk, R. (1972) Intensity fluctuations in a focused light beam that has passed through a stratum of turbulent atmosphere, Radiophysics and Quantum Electronics 15:612-613.

65. Dietz, P., and Wright, N. (1969) Saturation of scintillation magnitude in near-earth optical propagation, J. Opt. Soc. Amer. 59:527-535.

66. Kerr, J. (1972) Experiments on turbulence characteristics and multiwavelength scintillation phenomena, J. Opt. Soc. Amer. 62:1040-1051.

67. (a) S'edin, V., Khmelevstov, S., and Nevol'sin, M. (1970) Intensity fluctuations in a pulsed laser beam propagating up to 9.8 km in the atmosphere, Radiophysics and Quantum Electronics, 13:32-35.
(b) Khmelevstov, S. (1973) Propagation of laser radiation in a turbulent atmosphere, Appl. Optics 12:2421-2432.

68. Young, A. (1970) Saturation of scintillation, J. Opt. Soc. Amer. 60:1495-1501.

69. Ishimaru, A. (1969) Fluctuations of a beam propagating through a locally homogeneous medium, Radio Science 4:295-305.

70. Clifford, S., Ochs, G., and Lawrence, R. (1974) Saturation of optical scintillations by strong turbulence, J. Opt. Soc. Amer. 64:148-154.

71. Yura, H. (1974) Physical model for strong optical-amplitude fluctuations in a turbulent medium, J. Opt. Soc. Amer. 64:59-67.

72. deWolf, D. (1974) Waves in turbulent air: a phenomenological model, Proc. IEEE 62:1523-1529.

73. Gochelashvili, K. (1971) Saturation of the fluctuations of focused radiation in a turbulent medium, Radiophysics and Quantum Electronics 14:470-473.

74. Dagkesamanskaya, I., and Shishov, V. (1970) Strong intensity fluctuations during wave propagation in statistically homogeneous and isotropic media, Radiophysics and Quantum Electronics 13:9-12.

75. Shishov, V. (1972) Strong fluctuations of the intensity of a plane wave propagating in a random refractive medium, Soviet Phys. JEPT 34:744-748.

76. Shishov, V. (1972) Strong fluctuations of the intensity of a spherical wave propagating in a randomly refractive medium, Radiophysics and Quantum Electronics 15:689-695.

77. Fante, R. (1975) Some new results on propagation of electromagnetic waves in strongly turbulent media, IEEE Trans. Ant. and Prop. AP-23:382-385.

78. Fante, R. (1975) Electric field spectrum and intensity covariance of a wave in a random medium, Radio Science 10:77-85.

79. Gochelashvili, K., and Shishov, V. (1974) Saturated intensity fluctuations of laser radiation in a turbulent medium Zh. Eksp. Teor. Fiz. (Russian) 66:1237-1247.

80. Fante, R. (1975) Irradiance scintillations: comparison of theory with experiment, J. Opt. Soc. Amer. 65:548-550.

81. Gochelashvili, K., and Shishov, V. (1971) Laser beam scintillation beyond a turbulent layer, Optica Acta 18:313-320.

82. Fante, R. (1975) Some results for the variance of the irradiance of a finite beam in a random medium, Opt. Soc. Amer. 65:608-610.

83. Gochelashvili, K., and Shishov, V. (1972) Focused irradiance fluctuations beyond a layer of turbulent atmosphere, Optica Acta 19:327-332.

84. Banakh, V., Krekov, G., Mironov, V., Khmelevtsov, S., and Tsvyk, S. (1974) Focused-laser-beam scintillations in the turbulent atmosphere, J. Opt. Soc. Amer. 64:516-518.

85. (a) Gurvich, A., and Tatarskii, V. (1975) Coherence and intensity fluctuations of light in the turbulent atmosphere, Radio Science 10:3-14.
(b) Gurvich, A., and Tatarskii, V. (1973) The frequency spectra of strong fluctuations of laser radiation in a turbulent atmosphere, Radiophysics and Quantum Electronics 16:701-704.

86. Homstad, G., Strohbehn, J., Berger, R., and Heneghan, J. (1974) Aperture-Averaging effects for weak scintillations, J. Opt. Soc. Amer. 64:162-165.

87. Kerr, J., and Dunphy, J. (1973) Experimental effects of finite transmitter apertures on scintillations, J. Opt. Soc. Amer. 63:1-7.

88. Yura, H. (1974) Temporal-frequency spectrum of an optical wave propagating under saturation conditions, J. Opt. Soc. Amer. 64:357-360.

89. Ishimaru, A. (1969) Fluctuations of a focused beam wave for atmospheric turbulence probing, Proc. IEEE 57:407-414.

90. Clifford, S. (1971) Temporal-frequency spectra for a spherical wave propagating through atmospheric turbulence, J. Opt. Soc. Amer. 61:1285-1292.

91. Reinhardt, G., and Collins, S. (1972) Outer-scale effects in turbulence-degraded light beam spectra, J. Opt. Soc. Amer. 62:1526-1528.

92. Time, N. (1971) The spectrum of the amplitude fluctuations in a bounded light beam, Radiophysics and Quantum Electronics 14:936-939.

93. Strohbehn, J. (1974) Covariance functions and spectra for waves propagating in a turbulent medium to or from moving vehicles, IEEE Trans. Ant. and Prop. AP-22:303-311.

94. Livingston, P., Dietz, P., and Alcaraz, A. (1970) Light propagation through a turbulent atmosphere: measurement of the optical filter function, J. Opt. Soc. Amer. 60:925-935.

95. Lawrence, R., Ochs, G., and Clifford, S. (1972) Use of scintillations to measure average wind across a light beam, Appl. Optics 11:239-243.

96. Mandics, P., Lee, R., and Waterman, A. (1973) Spectra of short-term fluctuations of line-of-sight signals: electromagnetic and acoustic, Radio Science 8:185-201.

97. Gurvich, A., and Pokasov, V. (1973) Frequency spectra of strong laser radiation fluctuations in the turbulent atmosphere Izv. Vyssh. Vcheb. Zaved., Radiofiz. (Russian) 16:913-917.

98. Gurvich, A., Kallistrova, M., and Time, N. (1968) Fluctuation in the parameters of a light wave from a laser during propagation in the atmosphere, Radiophysics and Quantum Electronics 11:771-776.

99. Beckmann, P. (1967) Probability in Communication Engineering, Harcourt, Brace and World, Inc. New York.

100. deWolf, D. (1968) Saturation of irradiance fluctuations due to turbulent atmosphere, J. Opt. Soc. Amer. 58:461-466.

101. Strohbehn, J., and Wang, T. (1972) Simplified equation for amplitude scintillations in a turbulent atmosphere, J. Opt. Soc. Amer. 62:1061-1068.

102. Wang, R., and Strohbehn, J. (1974) Log-normal paradox in atmospheric scintillations, J. Opt. Soc. Amer. 64:583-591.

103. Wang, T., and Strohbehn, J. (1974) Perturbed log-normal distribution of irradiance fluctuations, J. Opt. Soc. Amer. 64:994-999.

104. (a) Strohbehn, J., Wang, T., and Speck, J. (1975) On the probability distribution of line-of-sight fluctuations of optical signals, Radio Science 10:59-70.
(b) Davidson, F., and Gonzalez-del-Valle, A. (1975) Measurements of three-parameter log-normally distributed optical field irradiance fluctuations in a turbulent medium, J. Opt. Soc. Amer. 65:655-662.

105. Woo, R., and Ishimaru, A. (1974) Effects of turbulence in a planetary atmosphere on radio occultation, IEEE Trans. Ant. and Prop. AP-21: 566-573.

106. Clifford, S., Bouricius, G., Ochs, G., and Ackley, M. (1971) Phase variations in atmospheric optical propagation, J. Opt. Soc. Amer. 61:1279-1284.

107. Ar sen'yan et al, T. (1972) Interferometric investigation of phase fluctuations of coherent optical radiation in the atmosphere, Radiophysics and Quantum Electronics 15:937-940.

108. Bertolotti, M., Carnevale, M., Muzii, L., and Sette, D. (1974) Atmospheric turbulence effects on the phase of laser beams, Appl. Optics 13:1582-1585.

109. Fried, D., and Yura, H. (1972) Telescope-performance reciprocity for propagation in a turbulent medium, J. Opt. Soc. Amer. 62:600-602.

110. Shapiro, J. (1971) Optimal power transfer through atmospheric turbulence using state knowledge, IEEE Trans. Comm. Tech. COM-19:410-414.

111. Dunphy, J., and Kerr, J. (1974) Atmospheric beam wander cancellation by a fast-tracking transmitter, J. Opt. Soc. Amer. 64:1015-1016.

112. Huang, T., Schreiber, W., and Tretiak. O. (1971) Image processing, Proc. IEEE 59:1586-1609.

113. Gezari, D., Lebeyrie, A., and Stachnik, R. (1972) Speckle interferometry: diffraction limited measurement of nine stars with the 200 inch telescope, Astrophys J. 173:L1-L5.

114. Dietz, P., and Carlson, F. (1973) Intensity interferometry in the spatial domain, J. Opt. Soc. Amer. 63:274-280.

115. Dietz, P. (1975) Image information by means of speckle-pattern processing, J. Opt. Soc. Amer. 65:279-285.

116. (a) Korff, D. (1973) Analysis of a method for obtaining near-diffraction-limited information in the presence of atmospheric turbulence, J. Opt. Soc. Amer. 63:971-980.
(b) Roddier, C., and Roddier, F. (1975) Influence of exposure time on spectral properties of turbulence-degraded astronomical images, J. Opt. Soc. Amer. 65:664-667.

117. Horner, J. (1970) Optical restoration of images blurred by atmospheric turbulence using optimum filter theory, Appl. Optics 9:167-171.

118. Horner, J. (1969) Optical spatial filtering with least mean-square error filter, J. Opt. Soc. Amer. 59:553-558.

119. Shapiro, J. (1974) Optimum adaptive imaging through atmospheric turbulence, Appl. Optics 13:2609-2613.

120. Erhukumov, L., Zarnitsyna, I., and Kirsh, P. (1973) Selective properties and the form of a signal passed through a statistically inhomogeneous arbitrary-thickness layer, Radiofizika (Russian) 16:573-580.

121. Sreenivasiah, I., and Ishimaru, A. (1974) Plane Wave Pulse Propagation Through Atmospheric Turbulence at mm and Optical Wavelengths, Dept. of Electrical Engineering, University of Washington, Research Report AFCRL-TR-74-0205.

122. Liu, C., Wernik, A., and Yeh, K. (1974) Propagation of pulse trains through a random medium IEEE Trans. Ant. and Prop. AP-22:624-627.

123. Su, H., and Plonus, M. (1971) Optical-pulse propagation in a turbulent medium, J. Opt. Soc. Amer. 61:256-260.

124. Plonus, M., Su, H., and Gardner, C. (1972) Correlation and structure functions for pulse propagation in a turbulent atmosphere, IEEE Trans. Ant. and Prop. AP-20:801-805.

125. Gardner, C., and Plonus, M. (1974) Optical pulses in atmospheric turbulence, J. Opt. Soc. Amer. 64:68-77.

126. Ishimaru, A. (1972) Temporal frequency spectra of multifrequency waves in turbulent atmosphere, IEEE Trans. Ant. and Propagation, AP-20:10-19.

127. Fried, D. (1971) Spectral and angular covariance of scintillation for propagation in a randomly inhomogeneous medium, Appl. Optics 10:721-731.

128. Brookner, E. (1971) Improved model for structure constant variations with altitude, Appl. Optics 10:1960-1962.

129. Klyatskin, V. (1972) On amplitude-phase fluctuations of a plane light wave in a turbulent medium, Radiophysics and Quantum Electronics 15:406-409.

130. Zavorotnyi, V., and Klyatskin, V. (1972) The geometric-optics approximation and amplitude-phase fluctuations of a plane light wave in a randomly inhomogeneous medium, Radiophysics and Quantum Electronics 15:684-688.

131. Klyatskin, V., and Tatarskii, V. (1972) Statistical theory of light propagation in a turbulent medium, Radiophysics and Quantum Electronics 15:1095-1112.

132. Torrieri, D., and Taylor, L. (1971) Propagation of random electromagnetic fields, J. Math. Phys. 12:1211-1214.

133. LeVine, D. (1972) The propagation of stochastic waves in uniform media, IEEE Trans. Ant. and Prop. AP-20:809-811.

134. Taylor, L. (1972) Scintillation of randomized electromagnetic fields, J. Math. Phys. 13:590-595.

135. Liu, C., Wernik, A., Yeh, K., and Youakim, M. (1974) Effects of multiple scattering on scintillation of transionospheric radio signals, Radio Science 9:599-607.

136. Beran, M., and Whitman, A. (1974) Free-space propagation of irradiance fluctuations and the fourth-order coherence function, J. Opt. Soc. Amer. 64:1636-1640.

137. Mercier, R. (1962) Diffraction by a screen causing large random phase fluctuations, Proc. Cambridge Phil. Soc. 51:382-400.

138. Rumsey, V. (1975) Scintillations due to a concentrated layer with a power-law turbulence spectrum, Radio Science 10:107-114.

139. Furuhama, Y. (1975) Probability distribution of irradiance fluctuation propagating through the turbulent slabs, Digest of 1975 URSI Meeting, page 14.

140. Hardy, K., and Katz, I. (1969) Probing the clear atmosphere with high power, high resolution radars, Proc IEEE 57:468-480.

141. Booker, H., and Gordon, W. (1950) A theory of radioscattering in the troposphere, Proc. IRE 38:401-412.

142. Villars, F., and Weisskopf, V. (1954) The scattering of electromagnetic waves by turbulent atmospheric fluctuations, Phys. Rev. 94:232-240.

143. Sheffield, J. (1975) Plasma Scattering of Electromagnetic Radiation, Academic Press, New York.

144. Wort, D. (1966) Microwave transmission through turbulent plasma, Plasma Physics (J. Nuclear Energy, Part C) 8:79-93.

145. Wort, D. (1969) Microwave scattering by turbulent plasma, J. Phys. A (General Physics), Series 2 2:75-86.

146. Wheelon, A. (1959) Radio-wave scattering by tropospheric irregularities, J. Res. NBS 63D:205-233.

147. Brown, E. (1974) Turbulent spectral broadening of backscattered acoustic pulses, J. Acoust. Soc. Amer. 56:1398-1408.

148. Pieroni, L., and Bremmer, H. (1970) Mutual coherence function of light scattered by a turbulent medium, J. Opt. Soc. Amer. 60:936-947.

149. Denison, N., and Tamoikin, V. (1971) Correlation theory of the backscatter of radiowaves, Radiophysics and Quantum Electronics 14:1045-1048.

150. Salpeter, E., and Trieman, S. (1964) Backscatter of electromagnetic radiation from a turbulent plasma, J. Geophys. Res. 69:869-881.

151. Bugnolo, D. (1961) Radio star scintillation and multiple scattering in the ionosphere, IRE Trans. Antennas and Prop. AP-9:89-96.

152. Watson, K. (1969) Multiple scattering of electromagnetic waves in an underdense plasma, J. Math. Phys. 10:688-702.

153. Stott, P. (1968) A transport theory for the multiple scattering of electromagnetic waves by a turbulent plasma, J. Phys. A 1:675-689.

154. Feinstein, D., and Granatstein, V. (1969) Scalar radiative transport model for microwave scattering from a turbulent plasma, Phys. Fluids 12:2658-2668.

155. Ishimaru, A. (1975) Correlation functions of a wave in a random distribution of stationary and moving scatterers, Radio Science 10:45-52.

156. Fante, R., and Poirier, J. (1973) Mutual coherence function of a finite optical beam in a turbulent medium, Appl. Optics 12:2247.

157. Fante, R. (1974) Intensity of a focused beam in a turbulent medium, Proc. IEEE 62:1400-1402.

158. Barabanenkov, Y., and Finkel'berg, D. (1968) Radi on transport equation for correlated scatterers, Soviet Phys. JETP 26:587-591.

159. Ovchinnikov, G., and Tatarskii, V. (1972) On the problem of the relationship between coherence theory and the radiation-transfer equation, Radiophysics and Quantum Electronics 15:1087-1089.

160. deWolf, D. (1972) Discussion of radiative-transfer methods applied to electromagnetic reflection from turbulent plasma, IEEE Trans. Antennas and Propagation AP-20:805-807.

161. deWolf, D. (1971) Electromagnetic reflection from an extended turbulent medium: cumulative-forward scatter single-backscatter approximation, IEEE Trans. Antennas and Propagation AP-19:254-262.

162. Feinstein, D., Butler, F., Prech, N., and Leonard, A. (1972) Radiative transport of electromagnetic propagation in isotropic plasma turbulence, Phys. Fluids 15:1641-1651.

163. Granatstein, V. (1972) Multiple scatter of laser light from a turbid medium, Appl. Optics 11:1217-1224.

164. Brookner, E. (1970) [illegible]mospheric propagation and communication channel model for laser [illegible]engths, IEEE Trans. Comm. Tech. COM-18: 396-416.

165. Hufnagel, R. (1[illegible] [illegible]tion of Atmospherically Degraded Images, Woods Hole Summer [illegible] 2.

166. Lawrence, R., Oc[illegible], [illegible], and Clifford, S. (1970) Measurements of atmospheric turbulence relevant to optical propagation, J. Opt. Soc. Amer. 60:826-830.

167. Strohbehn, J. (1973) Optical and millimeter line-of-sight propagation effects in the turbulent atmosphere, Boundary Layer Meteorol. 4:397-422.

168. Hufnagel, R. (1974) Variations of atmospheric turbulence, OSA Topical Meeting on Propagation Through Turbulence, Paper No. Wa1-1.

169. Dolukhov, M. (1971) Fluctuation Processes in Radiowave Propagation (Russian) Svyaz Press, Moscow.

170. Kennedy, R. (1970) Communication through optical scattering channels: an introduction, Proc IEEE 58:1651-1655.

171. Pratt, W. (1969) Laser Communications Systems, John Wiley and Sons, Inc. New York.

172. Clifford, S., and Yura, H. (1974) Equivalence of two theories of strong optical scintillation [illegible] Opt. Soc. Amer. 64:1641-1643.

Bibliography

SECTION 2

Beran, M. (1975) Coherence equations governing propagation through random media, Radio Science 10:15-21.

Beran, M., and McCoy, J. (1974) Propagation through an anisotropic random medium, J. Math Phys. 15:1901-1912.

Chang, F., Mott, H., and Webb, W. (1973) Depolarization of an uncollimated beam in a turbulent medium, J. Franklin Inst. 296:347-358.

Kon, A., and Tatarskii, V. (1972) On the theory of propagation of partially coherent light beams in a turbulent atmosphere, Radiophysics and Quantum Electronics 15:1187-1192.

SECTION 3

Cook, R. (1975) Beam wander in a turbulent medium, J. Opt. Soc. Amer. 65: 942-948.

SECTION 5

Gracheva, M., Gurvich, A., and Khrupin, A. (1974) Light intensity correlation functions in the turbulent atmosphere, Izv. Vyssh. Ucheb. Zaved, Radiofizika, (Russian) 17:155-157.

Modesitt, G. (1971) Moliere approximation for wave propagation in turbulent media, J. Opt. Soc. Amer. 61:797-799.

Shishov, V. (1973) Frequency correlation of scintillation, Radiophysics and Quantum Electronics 16:319-327.

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SECTION 6

Gracheva, M., Gurvich, A., Lomadze, S., Pokasov, V., and Khrupin, A. (1974) Probability distribution of strong light intensity fluctuations in the atmosphere, Izv. Vysch. Ucheb. Zaved., Radiofizika, (Russian) 17:105-112.

SECTION 7

Fried, D. (1975) Differential angle of arrival; theory, evaluation, and measurement feasibility, Radio Science 10:71-76.

Lowry, J., Wolf, J., and Carter, J. (1974) Acquisition and Tracking Assembly, McDonnell Douglas Technical report for Air Force Avionics Laboratory, AFAL-TR-73-380.

Lukin, V., Pokasov, V., and Khmelevstov, S. (1972) Investigation of the time characteristics of fluctuations of the phases of optical waves propagating in the bottom layer of the atmosphere, Radiophysics and Quantum Electronics 15: 1426-1430.

Lutomirski, R., and Buser, R. (1974) Phase difference and angle-of-arrival fluctuations in tracking a moving point source, Appl. Optics 13:2869-2873.

Lutomirski, R., and Warren, R. (1975) Atmospheric distortions in a retro-reflected laser signal, Appl. Optics 14:840-846.

Lutomirski, R., and Yura, H. (1974) Imaging of extended objects through a turbulent atmosphere, Appl. Optics 13:431-437.

Pearson et al, J. (1974) Atmospheric turbulence compensation using coherent optical adaptive techniques, Paper ThB5-1, OSA Topical Meeting on Propagation Through Turbulence, Boulder, Colorado.

Slepian, D. (1967) Linear least-squares filtering of distorted images, J. Opt. Soc. Amer. 57:918-922.

Yura, H. (1973) Holography in a random spatially inhomogeneous medium, Appl. Optics 12:1188-1192.

SECTION 8

Gardner, C., and Plonus, M. (1975) The effects of atmospheric turbulence on the propagation of pulsed laser beams, Radio Science 10:129-137.

SECTION 9

Apresyan, L. (1973) The radiative-transfer equation with allowance for longitudinal waves, Radiophysics and Quantum Electronics 16:348-356.

Barabanenkov, Y., Vinogradov, A., Kravtsov, Y., and Tatarskii, V. (1972) Application of the theory of multiple scattering of waves to the derivation of the radiation transfer equation for a statistically inhomogeneous medium, Radiophysics and Quantum Electronics 15:1420-1425.

Martens, C., and Jen, N. (1975) Electromagnetic wave scattering from a turbulent plasma, Radio Science 10:221-228.

Besieris, I. (1975) Long-range electromagnetic random wave propagation using the parabolic equation method, Digest of the 1975 URSI Meeting, p 15.

Appendix A

The intensity I of a wave is related to its log-amplitude χ_1 via

$$I = \exp\,[\,2(\chi_o + \chi_1)\,] \quad , \tag{A1}$$

where χ_o and χ_1 are defined in Section 2.1. If χ_1 is a gaussian random variable, it is readily demonstrated[1] that

$$\langle I(x, \underline{\rho}_1) \rangle = \exp \left\{ 2\chi_o(x, \underline{\rho}_1) + 2\langle \chi_1^2(x, \underline{\rho}_1) \rangle \right\} \quad . \tag{A2}$$

In deriving Eq. (A2), we have used the fact that $\langle \chi_1 \rangle = 0$; this point is evident from Eq. (10) since $\langle n_1 \rangle = 0$. Similarly, for $\langle I(x, \underline{\rho}_1) I(x, \underline{\rho}_2) \rangle$ we get

$$\langle I(x, \underline{\rho}_1) I(x, \underline{\rho}_2) \rangle = \exp \Big\{ 2\chi_o(x, \underline{\rho}_1) + 2\chi_o(x, \underline{\rho}_2) + 2\langle \chi_1^2(x, \underline{\rho}_1) \rangle + 2\langle \chi_1^2(x, \underline{\rho}_2) \rangle + 4\langle \chi_1(x, \underline{\rho}_1)\,\chi_1(x, \underline{\rho}_2) \rangle \Big\} \tag{A3}$$

If we now use Eqs. (A2) and (A3) to form the quantity in Eq. (58), we can readily show that the result is Eq. (59).

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1. Beckmann, P. (1967) Probability in Communication Engineering, Harcourt, Brace and World, Inc., New York.

Appendix B

In the physical model proposed by de Wolf,[1] the field at the location (x, 0) of the receiver consists of two components; one is the component which is forward scattered by the eddies on the propagation axis, which we denote by $A \exp(i\phi)$, where the phase ϕ is assumed to be gaussian distributed and the amplitude A satisfies the log-normal distribution

$$p(A) = \frac{1}{(2\pi)^{1/2}\,\sigma_0 \hat{A}} \exp\left[-\frac{(\ln \hat{A} - \gamma_0)^2}{2\sigma_0^2}\right]. \tag{B1}$$

In weak turbulence, nearly all the received signal is that scattered by the on-axis eddies, and therefore Eq. (B1) is a good approximation for the probability distribution of the field amplitude. However, as the length of the propagation path is increased, and $\sigma_1^2 = 1.23\, k^{7/6} C_n^2\, x^{11/6}$ becomes comparable with or greater than unity, there are no longer any axial eddies large enough to strictly forward scatter the energy, and nearly all the received signal at (x, 0) is due to energy which is scattered to (x, 0) by off-axis eddies. The field at (x, 0) due to all the different off-axis components can be denoted by $Z e^{i\theta}$; because these contributions are all statistically independent, then by the central limit theorem Z is Rayleigh distributed according to

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1. deWolf, D. (1974) Waves in turbulent air: a phenomenological model, Proc. IEEE 62:1523-1529.

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$$p(Z) = \frac{2Z}{\langle Z^2 \rangle} \exp -\left[\frac{Z^2}{\langle Z^2 \rangle}\right] . \tag{B2}$$

We note that $\langle Z^2 \rangle$ is a function of σ_1^2; $\langle Z^2 \rangle$ is quite small for $\sigma_1^2 << 1$, but is sizable for $\sigma_1^2 >> 1$. An explicit relationship between $\langle Z^2 \rangle$ and σ_1^2 has not yet been determined.

The total field at (x, 0) is the sum of the two components discussed above, and can be written as*

$$u e^{i\eta} = Z e^{i\theta} + A e^{i\phi} . \tag{B3}$$

It can be readily demonstrated[2] that the probability density for the amplitude u of the total field at (x, 0) then satisfies

$$p(\hat{u}) = \frac{2\hat{u}}{(2\pi)^{1/2}\sigma_0 \langle Z^2 \rangle} \int_0^{\infty} \frac{d\hat{A}}{\hat{A}} I_0\left(\frac{2\hat{u}\hat{A}}{\langle Z^2 \rangle}\right) \exp\left\{-\frac{(\ell n \hat{A} - \gamma_0)^2}{2\sigma_0^2} - \frac{\hat{u}^2 + \hat{A}^2}{\langle Z^2 \rangle}\right\}, \tag{B4}$$

where $I_0(\ldots)$ is the zeroth order modified Bessel function. We can note that in the limit of very small $\langle Z^2 \rangle$ or correspondingly very small σ_1^2, Eq. (B4) approaches

$$p(\hat{u}) \simeq \frac{1}{(2\pi)^{1/2}\sigma_0 \hat{u}} \exp -\left[\frac{(\ell n \hat{u} - \gamma_0)^2}{2\sigma_0^2}\right] , \tag{B5}$$

as expected. For large values of $\langle Z^2 \rangle$ the result in Eq. (B4) approaches the Rayleigh distribution

$$p(\hat{u}) \simeq \frac{2\hat{u}}{\langle Z^2 \rangle} \exp\left[-\frac{\hat{u}^2}{\langle Z^2 \rangle}\right] , \tag{B6}$$

as expected[1] for $\sigma_1^2 \to \infty$.

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*In writing Eq. (B3), we have neglected the contribution from the coherent component of the electric field. This is generally negligible for optical frequencies propagating in the atmosphere.

2. Beckmann, P. (1967) Probability in Communication Engineering, Harcourt, Brace and World, Inc., New York.

Symbols

A = defined in Eq. (14)

b_I = normalized covariance of the intensity fluctuations - defined in Eq. (58)

B_S = covariance of the phase fluctuations - defined in Eq. (60)

B_χ = covariance of the log-amplitude fluctuations - defined in Eq. (57)

c = speed of light

C_n^2 = index-of-refraction structure constant - see Eq. (3)

d_S = phase structure function - defined in Eq. (74)

d_t = plane-wave structure function - defined in Eq. (77)

D = transmitting aperture diameter

D' = receiving aperture diameter

E = electric field strength

f = frequency

F = initial radius of curvature of the beam

G = reduction in intensity fluctuations due to aperture averaging

h = linear system spatial impulse response - see Eq. (26)

H = defined in Eqs. (22) and (23)

i = $(-1)^{1/2}$

I = intensity or irradiance

$\langle I \rangle$	long-term averaged intensity
$\langle I_s \rangle$	short-term averaged intensity
k	signal wavenumber $2\pi/\lambda$
ℓ_o	inner scale size of the turbulent eddies - see Eq. (3)
L_o	outer scale size of the turbulent eddies - see Eq. (3)
L	smaller of ρ_o and D
n_1	fluctuation in the index of refraction
$\underline{r}$	position (x, y, z)
R	$\hat{\rho}/(\lambda x)^{1/2}$
S	phase of the electromagnetic wave
S_1	random part of the phase
t	time
u	$E\, e^{-ikx}$
u^*	complex congugate of u
u_o	u in the plane $x = 0$
$\underline{V}$	transverse wind velocity
W_I	frequency spectrum of the intensity fluctuations
W_α	frequency spectrum of the angle-of-arrival fluctuations
W_χ	frequency spectrum of the log-amplitude fluctuations
x	distance measured in direction of propagation
x_o	distance from objective to pupil plane in Figure 14
y	distance in plane transverse to direction of propagation
z	distance in plane transverse to direction of propagation
α	angle of arrival of signal wavefront - see Figure 10
Γ_2	second moment of the electric field $\langle u(x, \underline{\rho}_1)\, u^*(x, \underline{\rho}_2) \rangle$
Γ_4	fourth moment of the electric field $\langle u(x, \underline{\rho}_1)\, u^*(x, \underline{\rho}_2)\, u(x, \underline{\rho}_3)\, u^*(x, \underline{\rho}_4) \rangle$
Γ_s	two-source spherical wave coherence function
λ	signal wavelength
$\underline{\rho}$	distance transverse to the x-axis (y, z)
ρ_b	beam coherence length (long-term) - defined in Eq. (50)

ρ_c	deflection of the beam centroid - see Figure 1
ρ_L	long term beam radius - see Figure 2
ρ_0	spherical wave coherence length - defined in Eq. (38)
ρ_p	plane-wave coherence length - defined in Eq. (51)
ρ_s	short term beam radius - see Figure 1
$\hat{\underline{\rho}}$	$\underline{\rho}_1 - \underline{\rho}_2$
σ_I^2	normalized variance of the intensity fluctuations
σ_V	Born-approximation cross section of a turbulent scatterer - see Eq. (111)
σ_χ^2	variance of the log-amplitude fluctuations
σ_1^2	$1.23\, k^{7/6} C_n^2 x^{11/6}$
Φ_n	wavenumber spectrum of the index-of-refraction fluctuations - see Eqs. (2) and (3)
χ	log-amplitude - $\mathrm{Re}(\psi)$
χ_1	fluctuation of the log amplitude
ψ	$\ln(E)$
ω	$2\pi f$ radian frequency
Ω	bandwidth of electromagnetic pulse