

AD-A017 238

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Prepared for:

Office of Naval Research  
Army Research Office - Durham

September 1975

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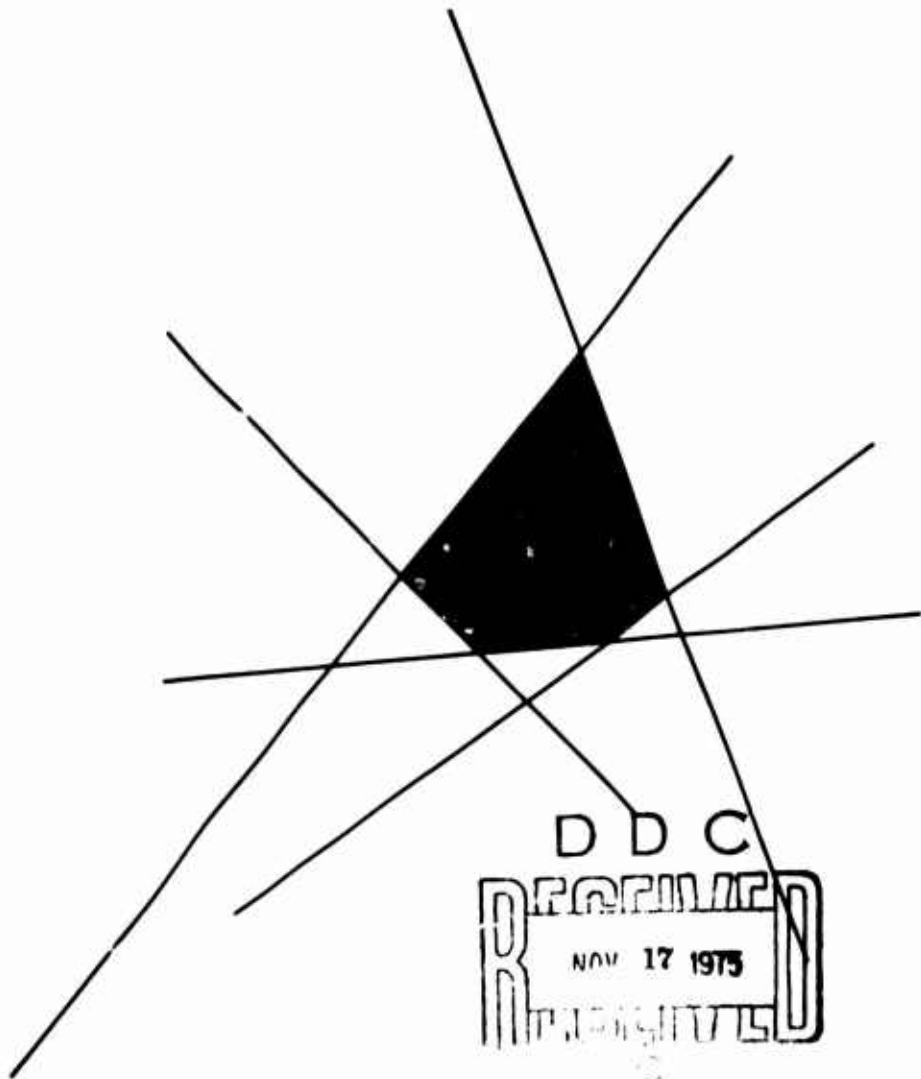
ORC 76-16  
SEPTEMBER 1976

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by

CYRUS DERMAN, GERALD J. LIEBERMAN and SHELDON M. ROSS

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OPTIMAL SYSTEM ALLOCATIONS WITH PENALTY COSTS<sup>†</sup>

Operations Research Center Research Report No. 75-15

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September 1975

U. S. Army Research Office - Durham

DAHC04-75-G-0163

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and Operations Research  
University of California, Berkeley



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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1 REPORT NUMBER ORC 75-15	2 GOVT ACCESSION NO.	3 RECIPIENT'S CATALOG NUMBER
4 TITLE (and Subtitle) OPTIMAL SYSTEM ALLOCATIONS WITH PENALTY COSTS	5 TYPE OF REPORT & PERIOD COVERED Research Report	
	6 PERFORMING ORG REPORT NUMBER	
7 AUTHOR(s) Cyrus Derman, Gerald J. Lieberman and Sheldon M. Ross	8 CONTRACT OR GRANT NUMBER(s) DAHC04-75-G-0163	
9 PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720	10 PROGRAM ELEMENT PROJECT TASK AREA & WORK UNIT NUMBERS P-12549-M	
11 CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office-Durham Box CM, Duke Station Durham, North Carolina 27706	12 REPORT DATE September 1975	
	13 NUMBER OF PAGES 19	
14 MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15 SECURITY CLASS (of this report) Unclassified	
	15a DECLASSIFICATION DOWNGRADING SCHEDULE	
16 DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17 DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18 SUPPLEMENTARY NOTES  Also supported by the Office of Naval Research under Contract N00014-75-G-0781.		
19 KEY WORDS (Continue on reverse side if necessary and identify by block number) Reliability Optimal Allocation Sequential Allocation Penalty Costs		
20 ABSTRACT (Continue on reverse side if necessary and identify by block number)  (SEE ABSTRACT)		

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## ABSTRACT

We have  $N$  stages to sequentially construct  $I$  successful components. At each stage, we allocate a certain amount of money for the construction of a component. If  $y$  is the amount allocated, then the component constructed will be a success with probability  $P(y)$ , where  $P$  is a continuous nondecreasing function satisfying  $P(0) = 0$ . After each component is constructed, we are informed as to whether or not it is successful. If, at the end of the  $N$  stages, we are  $i$  components short, then a final penalty cost  $C(i)$  is incurred. The problem is to, at each stage, determine how much money to allocate so as to minimize the total expected cost (construction cost plus penalty cost) incurred. This model is similar to that presented in [1] except for the cost function.

In Section 2, we obtain the structure of the optimal policy. In particular, if  $C(i+1) - C(i) \leq C(i+2) - C(i+1)$ , and if  $y_n(i)$  denotes the optimal value to allocate when  $i$  components are needed with  $n$  stages remaining, then  $y_n(i)$  is nondecreasing in  $i$  and nonincreasing in  $n$ .

In Section 3, we consider the special case of  $C(i) = iA$ . If we now write  $y_n^A(i)$  to indicate the dependence on  $A$ , we show that  $y_n^A(i)$  is nondecreasing in  $A$  for fixed  $n$  and  $i$ . We also find that the necessary and sufficient condition under which it is never optimal to construct components (buy them instead at a fixed price of

A per component) is given by  $A \leq \inf_y y/P(y)$  . We then

show that whenever  $i \geq n$  and  $A > \inf_y y/P(y)$  , then  $y_n(i)$

is just the largest value of  $y$  that minimizes

$y + (1 - P(y))A$  . Finally, we investigate special forms for

$P(y)$  , namely,  $P(y) = \min(y,1)$  for  $y \leq 1$  , and

$P(y) = 1 - e^{-y}$  .

## OPTIMAL SYSTEM ALLOCATIONS WITH PENALTY COSTS

by

C. Derman, G. J. Lieberman and S. M. Ross

### 1. Introduction

We have  $N$  stages to sequentially construct  $I$  successful components. At each stage we allocate a certain amount of money for the construction of a component. If  $y$  is the amount allocated then the component constructed will be a success with probability  $P(y)$ , where  $P$  is a continuous nondecreasing function satisfying  $P(0) = 0$ . After each component is constructed we are informed as to whether or not it is successful. If, at the end of the  $N$  stages, we are  $i$  components short then a final penalty cost  $C(i)$  is incurred. The problem is to, at each stage, determine how much money to allocate so as to minimize the total expected cost (construction cost plus penalty cost) incurred. This model is similar to that presented in [1] except for the cost function.

In Section 2, we obtain the structure of the optimal policy. In particular if  $C(i+1) - C(i) \leq C(i+2) - C(i+1)$ , and if  $y_n(i)$  denotes the optimal value to allocate when  $i$  components are needed with  $n$  stages remaining, then  $y_n(i)$  is nondecreasing in  $i$  and nonincreasing in  $n$ .



In Section 3, we consider the special case of  $C(i) = iA$ .

If we now write  $y_n^A(i)$  to indicate the dependence on  $A$ , we show that  $y_n^A(i)$  is nondecreasing in  $A$  for fixed  $n$  and  $i$ . We also find the necessary and sufficient condition under which it is never optimal to construct components (buy them instead at a fixed price of  $A$  per component) is given by

$$A \leq \inf_{y > 0} y/P(y).$$

We then show that whenever  $i \geq n$  and  $A > \inf_y y/P(y)$ , then  $y_n^A(i)$  is just the largest value of  $y$  that minimizes  $y + (1 - P(y))A$ .

Finally, we investigate special forms for  $P(y)$ , namely,  $P(y) = \min(y, 1)$  for  $y \leq 1$ , and  $P(y) = 1 - e^{-y}$ .

## 2. Structure of the Optimal Policy

If we let  $V_n(i)$  denote the minimal expected additional costs incurred if there are  $n$  stages to go and we still need an additional  $i$  successful components, then  $V_n(i)$  satisfies the optimality equations

$$V_0(i) = C(i), \quad i \geq 0$$

$$V_n(i) = \min_{C(i) \geq y \geq 0} \{y + (1 - P(y)) V_{n-1}(i) + P(y) V_{n-1}(i-1)\},$$

$$i \geq 0, n \geq 1$$

$$V_n(0) = 0.$$

If we let  $y_n(i)$  denote the largest value of  $y$  that minimizes the above then the policy that allocates  $y_n(i)$  when  $i$  components are needed with  $n$  stages remaining is an optimal policy.

Lemma 1: If  $y(c)$  represents the largest value of  $y$  that minimizes  $y + (1 - P(y))c$ , then  $y(c)$  is a nondecreasing function of  $c$ .

Proof: Let  $y_1 = y(c_1)$  and let  $c_2 = c_1 + \epsilon$ ,  $\epsilon > 0$ . Now

$$y_1 + (1 - P(y_1))c_1 \leq y + (1 - P(y))c_1,$$

or

$$y_1 + (1 - P(y_1))c_2 - (1 - P(y_1))\epsilon \leq y + (1 - P(y))c_2 - (1 - P(y))\epsilon.$$

or

$$\begin{aligned} y_1 + (1 - P(y_1))c_2 &\leq y + (1 - P(y))c_2 + \epsilon(P(y) - P(y_1)) \\ &\leq y + (1 - P(y))c_2 \quad \text{whenever } y \leq y_1, \end{aligned}$$

implying that  $y(c_2) \geq y_1$ . ||

We shall assume throughout that  $C(i)$  is a nondecreasing function of  $i$ , with  $C(0) = 0$ .

Lemma 2:  $V_n(i)$  is a nonincreasing function of  $n$  and a nondecreasing function of  $i$ .

Proof: Follows immediately from the definition of  $V_n(i)$  and the monotonicity of  $C(i)$ . ||

Lemma 3: If

$$C(i+1) - C(i) \leq C(i+2) - C(i+1), \quad i = 0, 1, \dots$$

then the following inequalities are satisfied:

$$A_{i,n}: \quad V_{n+1}(i+1) - V_{n+1}(i) \leq V_n(i+1) - V_n(i), \quad i, n \geq 0$$

$$B_{i,n}: \quad V_{n+1}(i) - V_n(i) \leq V_{n+2}(i) - V_{n+1}(i), \quad i, n \geq 0$$

$$C_{i,n}: \quad V_n(i+1) - V_n(i) \leq V_n(i+2) - V_n(i+1), \quad i, n \geq 0.$$

Proof: The proof is by induction on  $k \equiv n+i$ . As the inequalities  $A_{i,n}$ ,  $B_{i,n}$ ,  $C_{i,n}$  are true when  $k = 0$  assume they are true whenever  $n+i < k$ . Now suppose  $n+i = k$ . We first show that  $A_{i,n}$  is true. As  $A_{i,n}$  is clearly valid when  $i = 0$ , suppose that  $i > 0$ . Now for some  $\bar{y}$

$$V_{n+1}(i) = \bar{y} + P(\bar{y}) V_n(i-1) + (1 - P(\bar{y})) V_n(i)$$

and thus

$$V_{n+1}(i) - V_n(i) = \bar{y} + P(\bar{y}) [V_n(i-1) - V_n(i)].$$

Also,

$$V_{n+1}(i+1) \leq \bar{y} + P(\bar{y}) V_n(i) + (1 - P(\bar{y})) V_n(i+1)$$

and thus

$$V_{n+1}(i+1) - V_n(i+1) \leq \bar{y} + P(\bar{y}) [V_n(i) - V_n(i+1)].$$

Thus  $A_{i,n}$  will follow if we can show that

$$V_n(i) - V_n(i+1) \leq V_n(i-1) - V_n(i) .$$

However this is just the inequality  $C_{i-1,n}$  which is true when  $i+n = k$  by the induction hypothesis.

To prove  $B_{i,n}$  we first note that for some  $\bar{y}$

$$V_{n+2}(i) = \bar{y} + P(\bar{y}) V_{n+1}(i-1) + (1 - P(\bar{y})) V_{n+1}(i)$$

and thus

$$V_{n+2}(i) - V_{n+1}(i) = \bar{y} + P(\bar{y}) [V_{n+1}(i-1) - V_{n+1}(i)] .$$

Also,

$$V_{n+1}(i) \leq \bar{y} + P(\bar{y}) V_n(i-1) + (1 - P(\bar{y})) V_n(i)$$

implying that

$$V_{n+1}(i) - V_n(i) \leq \bar{y} + P(\bar{y}) [V_n(i-1) - V_n(i)] .$$

Therefore  $B_{i,n}$  will follow if we could prove that

$$V_n(i-1) - V_n(i) \leq V_{n+1}(i-1) - V_{n+1}(i) .$$

However the above is just the inequality  $C_{i-1,n}$ , which is thus true by the induction hypothesis.

To prove  $C_{i,n}$  we first note that  $B_{i+1,n-1}$  states that

$$2V_n(i+1) \leq V_{n-1}(i+1) + V_{n+1}(i+1)$$

and thus  $C_{i,n}$  will follow if we can show that

$$V_{n-1}(i+1) + V_{n+1}(i+1) \leq V_n(i) + V_n(i+2). \quad (1)$$

Now for some  $\bar{y}$

$$V_n(i+2) - V_{n-1}(i+1) = \bar{y} + (1 - P(\bar{y}))[V_{n-1}(i+2) - V_{n-1}(i+1)].$$

Furthermore,

$$V_{n+1}(i+1) - V_n(i) \leq \bar{y} + (1 - P(\bar{y}))[V_n(i+1) - V_n(i)].$$

Thus (1) (and  $C_{i,n}$ ) will follow if we can show that

$$V_n(i+1) - V_n(i) \leq V_{n-1}(i+2) - V_{n-1}(i+1). \quad (2)$$

Now, from  $A_{i,n-1}$  it follows that

$$V_n(i+1) - V_n(i) \leq V_{n-1}(i+1) - V_{n-1}(i)$$

and thus it suffices to show that

$$V_{n-1}(i+1) - V_{n-1}(i) \leq V_{n-1}(i+2) - V_{n-1}(i+1) .$$

However the above is just the inequality  $C_{i,n-1}$  which is true by the induction hypothesis. ||

Theorem 1: If

$$C(i+1) - C(i) \leq C(i+2) - C(i+1)$$

then

$y_n(i)$  is nondecreasing in  $i$  and nonincreasing in  $n$ .

Proof: As

$$\begin{aligned} V_n(i) &= \min_y \{y + P(y) V_{n-1}(i-1) + (1 - P(y)) V_{n-1}(i)\} \\ &= \min_y \{y + (1 - P(y)) [V_{n-1}(i) - V_{n-1}(i-1)]\} + V_{n-1}(i-1) \end{aligned}$$

the result follows from Lemma 1 and the inequalities  $C_{i,n}$  and  $A_{i,n}$  of Lemma 3. ||

Remarks:

- (1) The results go through in an identical manner when one is only allowed to allocate amounts of money within a given class  $\mathcal{O}$  as long as  $\inf_{y \in \mathcal{O}} \{y + (1 - P(y))c\}$  is attained for all  $c \geq 0$ .

(ii) The structural result presented in Theorem 1 can be used to reduce the computations needed for obtaining the optimal policy.

For instance we can write

$$V_n(i) = \min_{\substack{y_{n-1}(i) \leq y \leq y_n(i-1) \\ y = 0}} (y + P(y)V_{n-1}(i-1) + (1-P(y))V_{n-1}(i)) .$$

(iii) When  $P(y)$  is concave it follows that

$$y + P(y)V_{n-1}(i-1) + (1 - P(y))V_{n-1}(i)$$

is convex and thus its negative is unimodal. This fact can be used to reduce computations in the obvious manner.

### 3. The Special Case $C(i) = iA$

An important special case under which Theorem 1 is valid is when  $C(i) = iA$ . Such a penalty cost function would be valid if at the end of the problem we were forced to buy, at a price  $A$  per component, additional components to make up for those we are short.

It seems reasonable that, as a function of  $A$ , the optimal amounts to allocate should be nondecreasing. That is, writing  $V_n^A(i)$ ,  $y_n^A(i)$  to indicate the dependence on  $A$ , then it seems intuitive that  $y_n^A(i)$  should be nondecreasing in  $A$ . We now prove this.

Theorem 2:  $v_n^A(i)$  is nondecreasing in  $A$  for fixed  $n, i$ .

Proof: We will prove the theorem by first proving that

$$\begin{aligned} v_n^A(i) - v_{n-1}^A(i) & \text{ is nonincreasing in } A, \\ \text{and} & \\ v_n^A(i) - v_n^A(i-1) & \text{ is nondecreasing in } A. \end{aligned} \tag{3}$$

As both of the above statements are easily verified when  $n = 1$ , assume they are true when  $n = k$ . Now

$$v_{k+1}^A(i) = \min_{y \geq 0} \{y + P(y)[v_k^A(i-1) - v_k^A(i)]\} + v_k^A(i)$$

implying, by the induction hypothesis, that

$$v_{k+1}^A(i) - v_k^A(i) \text{ is nonincreasing in } A. \tag{4}$$

Hence it remains to show that  $v_{k+1}^A(i) - v_{k+1}^A(i-1)$  is nondecreasing in  $A$ . Now,

$$v_{k+1}^A(i) = \min_y \{y + (1 - P(y))[v_k^A(i) - v_k^A(i-1)]\} + v_k^A(i-1)$$

and thus, by the induction hypothesis, it follows that

$$v_{k+1}^A(i) - v_k^A(i-1) \text{ is nondecreasing in } A,$$

which implies from (4) that

$$v_{k+1}^A(i) - v_{k+1}^A(i-1) = v_{k+1}^A(i) - v_k^A(i-1) + v_k^A(i-1) - v_{k+1}^A(i-1)$$



is nondecreasing in  $A$ . Thus the statements (3) are proven by induction. The theorem now follows from Lemma 1 and the second inequality in (3) since

$$V_n^A(i) = \min_{y \geq 0} (y + (1 - P(y))(V_{n-1}^A(i) - V_{n-1}^A(i-1))) + V_{n-1}^A(i-1). \quad ||$$

Another question of interest is when it is optimal to always invest 0. That is, under what conditions on  $P(y)$  is it optimal never to attempt to construct components yourself but rather to buy them at a fixed price of  $A$  per component? If we let  $U_n(i)$  denote the return from the policy of always investing 0 then  $U_n(i)$  will satisfy the optimality equation if

$$U_n(i) \leq y + P(y) U_{n-1}(i-1) + (1 - P(y)) U_{n-1}(i) \quad \text{for all } y > 0$$

which, since  $U_n(i) = iA$  is equivalent to

$$iA \leq y + P(y)(i-1)A + (1 - P(y))iA \quad \text{for all } y > 0$$

or, equivalently, if

$$A \leq \frac{y}{P(y)} \quad \text{for all } y > 0$$

or

$$A \leq \inf_{y > 0} \frac{y}{P(y)}. \quad (5)$$

Since, in finite stage dynamic programming problems, a policy is optimal if and only if its return satisfies the optimality equation it follows that (5) is the necessary and sufficient condition for it to be optimal to always invest 0.

When  $i \geq n$  and  $k > \inf_y y/P(y)$  then

$y_n(i)$  = largest value of  $y$  that minimizes  $y + (1 - P(y))A$ .

The above follows by noting that when  $i \geq n$  the problem is equivalent to  $n$  separate problems each of which is such that if you invest  $y$  and are successful then your cost is  $y$  and if you are unsuccessful then your cost is  $y + A$ .

Remarks:

(a) In the special case

$$P(y) = \min(y, 1), \quad y \leq 1$$

it is easy to verify that

$$y_n(i) = \begin{cases} 0 & \text{if } A \leq 1 \\ 1 & \text{if } A > 1 \end{cases} .$$

(b) In the special case

$$P(y) = 1 - e^{-y}$$

it can be verified that the  $V_n(i)$  and  $y_n(i)$  can be recursively computed according to

$$V_n(i) = 1 + y_n(i) + V_{n-1}(i-1), \quad i \geq 1, n \geq 1$$

where

$$y_n(i) = \log[V_{n-1}(i) - V_{n-1}(i-1)], \quad i \geq 1, n \geq 1$$

using the boundary condition

$$V_n(0) = 0 .$$

#### Reference

- [1] Derman, C., Lieberman, G.J. and Ross, S.M., "Optimal Allocations in the Construction of k-Out-of-n Reliability Systems," Management Science, 21, No. 3, November 1974.