(2)

NPS55SY75071

NAVAL POSTGRADUATE SCHOOL

Monterey, California

MODELING A RANDOM SEARCH

by

Bruno O. Shubert

July 1975

Approved for public release; distribution unlimited.

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral Isham Linder
Superintendent

Jack R. Borsting
Provost

Reproduction of all or part of this report is authorized.

This report was prepared by:

Bruno O. Shubert, Associate Professor
Department of Operations Research and
Administrative Sciences

Reviewed by:

David A. Schrady, Chairman
Department of Operations Research
and Administrative Sciences

Released by:

Robert Fossum
Dean of Research

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. ~~REPORT NUMBER~~ (14) NPS55SY75071	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. ~~TITLE (and Subtitle)~~ (6) MODELING A RANDOM SEARCH	5. ~~TYPE OF REPORT & PERIOD~~ COVERED (9) Technical Report
	6. ~~PERFORMING ORG. REPORT~~ NUMBER
7. AUTHOR(s) (10) Bruno O. Shubert	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept. of OR/AS Code 55Sy Naval Postgraduate School Monterey, CA 93940	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (12) 23p.
11. CONTROLLING OFFICE NAME AND ADDRESS	12. ~~REPORT DATE~~ (11) Jul[redacted]75
	13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Random Search

Wiener Process

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A model for random search is considered which differs from the classical random search by modeling the random paths of the searcher and/or target as those of a Wiener process. The quantity of interest is the distribution of detection time for a cookie cutter mode of detection. Only one-dimensional search is considered here.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

251450

1. Introduction

This report is a preliminary study of some models for random search. It begins by reexamining the, by now, classical formula of Koopman for the probability of detection. It is shown that the assumptions under which the formula was derived are incompatible with the natural requirement that the searcher's trajectory be continuous. To remedy the situation a different model of random motion is needed. The main theme of this report is to model such a motion as a continuous version of a random walk - the Wiener process. With this modification a new formula for the detection time distribution is obtained. This is done both for a stationary target and, under some simplifying assumptions, also for a moving target.

This work is preliminary in the sense that only one-dimensional search is considered. Although search in a narrow straight may be regarded as one-dimensional, this restriction clearly limits the application of our model at its present state. The model itself, however, readily extends to two dimensions; it is its mathematical analysis that becomes more involved. Thus, such a restriction seemed necessary for the initial phase of research.

2. Stationary Target

In this section we intend to reconsider the basic formula for random search

$$p = 1 - e^{-\frac{WL}{A}}, \tag{1}$$

where p is the probability of detection, A is the area in which the target is located, W is the effective search width and L is the total length of the observer's path. This formula was originally derived by Koopman under the following assumptions [1], page 28:

(A1) "The target's position is uniformly distributed in A."

(A2) "The observer's path is random in A in the sense that it can be thought of as having its different (not too near) portions placed independently of one another in A."

(A3) "On any portion of the path which is small relatively to the total length of path but decidedly larger than the range of possible detection, the observer always detects the target within the lateral range W/2 on either side of the path and never beyond."

Koopman then proceeds by first dividing the observer's path into n segments of length L/n and using (A1) and (A3) to conclude that the probability of detecting the target in a particular segment is LW/nA. He then employs (A2) to reason that for sufficiently large n the events {detection in k-th segment}, $k = 1, 2, \ldots$ should be independent, whence the formula immediately follows by letting n increase to infinity,

$$1 - p = \lim_{n\to\infty}\left(1 - \frac{LW}{nA}\right)^n = e^{-\frac{WL}{A}}.$$

Let us now reexamine the assumptions in a little more detail.

Assumption (A1) seems quite reasonable inasmuch as it expresses a complete lack of prior knowledge of the target location (Principle of Insufficient Reason). However, it is not clear whether the target is stationary during the search or whether it moves. In the latter case, what should be assumed about its motion? We will return to this question shortly.

Assumption (A3) concerns a mode of detection - the so-called cookie cutter detection - and can be rephrased more simply as:

> (A3') Detection occurs as soon as the distance between the target and the observer decreases to or below some positive, constant $C = W/2$.

Clearly, this is a matter of postulating a mode of detection, (A3) or (A3') representing the simplest case.

Assumption (A2) is the crucial one, inasmuch as it defines the random path of the observer. Unfortunately, it is stated rather vaguely. To try to understand it, let us denote by Y_t, $t \geq 0$ the observer's location at time t, the time $t = 0$ being the beginning of search. (A2) seems to say that Y_t and $Y_{t+\tau}$ should be independent, at least for larger $\tau > 0$. If the observer moves with a constant speed v, dividing its path into n segments of length L/n is then equivalent to dividing the time interval $[0,T=L/v]$ into n equal subintervals of length T/n. Calling t_k a point in the k-th subinterval, let us say the midpoint $t_k = \frac{T}{n}(k+\frac{1}{2})$, assumption (A2) requires that the random variables Y_{t_k}, $k = 0, \ldots, n-1$ be independent as long as T/n is not too small (cf. "not too near portions" in (A2)).

Let X_t, $t \geq 0$ be the location of the target at time t. Now, if either

$$X_{t_k} = X_0 \sim \text{ uniform in } A \text{ (stationary target)}$$

or

X_{t_k}, $k = 0, \ldots, n-1$, independent, identically distributed uniformly in A (moving target),

then using (A3)

$$P(\text{no detection along a path of length } L)$$

$$= P\left(\| Y_t - X_t \| > \frac{W}{2} \text{ for all } 0 \leq t \leq T \right)$$

$$\doteq P(X_{t_k} \notin S_k \text{ for } k = 0, \ldots, n-1)$$

$$= \left(1 - \frac{LW}{nA}\right)^n , \text{ where } S_k \text{ is a}$$

rectangle of length L/n, width W and center at Y_{t_k}. Notice that the approximate equality above holds by (A3) for large n, i.e., for small T/n. On the other hand, the Y_{t_k}'s are assumed independent provided T/n is <u>not too small</u>. But this prevents us from taking the limit $n \to \infty$! Thus, we are forced to drop the restriction on "not too near portions" from (A2) and assume that Y_t and $Y_{t+\tau}$ are independent for all $\tau > 0$, no matter how small.

But, as is well known, this is incompatible with the obvious requirement that the <u>observer's path be continuous</u>. Hence, if we indeed wish to model a search for a stationary or moving target by a moving craft, we have no choice but to abandon (A2) altogether and define the observer's random motion in a different way.

Again, let Y_t, $t \geq 0$ be the location of the observer at time t. Since the motion is to be random, $\{Y_t, t \geq 0\}$ should be defined as a stochastic process. Our first requirement is that almost all sample paths of Y_t be continuous. This eliminates processes obeying the original assumption (A2) (white noise process) but still leaves quite a large class to choose from. To retain most of the flavor of the original (A2), let us visualize for a while that the search region A has been partitioned into small cells ΔA and that the time interval $[0,T]$ has also been divided into small subintervals Δt, and let us replace (A2) by:

> (A2') If at time t the observer is in a cell ΔA then at $t + \Delta t$ he is <u>equally likely</u> to be in any of the cells <u>adjacent</u> to ΔA.

In other words, we now assume that the observer performs a symmetric Bernoulli random walk on the partition of A. Regarding this as a discrete approximation to a time-continuous motion in a continuum, we obtain our fundamental assumption:

> A random motion is to be modeled as a symmetric (i.e., zero drift) Wiener process.

It may still be objected that no moving craft can in reality follow a sample path of a Wiener process. This is true since the sample paths, although being continuous with probability one, are almost surely nowhere differentiable. Nevertheless, we are at least closer to reality than we were with discontinuous paths.

In the remainder of this section, we derive a new formula for the probability of detection using our modified model. As mentioned in the introduction, we restrict ourselves here to the case where the search takes place in a one-dimensional region. Note that although the basic formula (1) was derived in [1] for planar regions, its derivation, the problem with assumption (A2), and in fact the entire discussion so far, applies to any number of dimensions.

For one-dimensional search, the region A will be an interval $[-b,a]$, $a > 0$, $b > 0$. We consider first a stationary target $X_t = X_0$ uniformly distributed over $[-b,a]$ with the observer's location Y_t being a Wiener process with drift $\mu = 0$ and variance parameter $\sigma^2 > 0$. Without loss of generality, we choose $Y_0 = 0$, the observer at the origin at the beginning of search. Since the observer clearly should not leave the search region $[-b,a]$, we consider the enpoints $-b,a$ to act as reflecting barriers. Let T_{det} be the time when the target is detected, and let $T_x = \min\{t\geq 0: Y_t = x\}$ be the first time the process Y_t reaches level x. With $c > 0$ being the detection distance as in (A3) we have for $X_0 = x$ the relation

$$T_{det} = \begin{cases} T_{x-c} & \text{if} \quad c < x \leq a, \\ 0 & \text{if} \quad |x| \leq c, \\ T_{x+c} & \text{if} \quad -b \leq x < -c. \end{cases} \tag{2}$$

Typically, we assume that $c << a+b$. Our goal is to compute the probability of detection by the time t, $p = P(T_{det} \leq t)$, which corresponds to the right-hand side of (1) with $L = vt$. Now according to (2)

$$p = F_{T_{det}}(t) = \frac{1}{a+b} \int_c^a P(T_{x-c} \leq t)dx + \frac{1}{a+b} \int_{-b}^{-c} P(T_{x+c} \leq t)dx, \tag{3}$$

Let

$$\gamma(u,x) = \int_0^\infty e^{-ut} P(T_x \in dt)$$

be Laplace transform of the distribution of T_x. By taking Laplace transform of the diffusion equation for the Wiener process and solving the resulting second-order differential equation, we obtain (see [2], p. 233 for details)

$$\gamma(u,x) = \begin{cases} \dfrac{\cosh \frac{b}{\sigma}\sqrt{2u}}{\cosh \frac{b+x}{\sigma}\sqrt{2u}} & \text{if } 0 < x \le a, \\[2ex] \dfrac{\cosh \frac{a}{\sigma}\sqrt{2u}}{\cosh \frac{a-x}{\sigma}\sqrt{2u}} & \text{if } -b \le x < 0. \end{cases} \tag{4}$$

Hence from (3) and (4)

$$\int_0^\infty e^{-ut} F_{T_{det}}(dt) = \frac{1}{a+b}\int_0^{a-c} \gamma(u,x)dx + \frac{1}{a+b}\int_{-b+c}^{0} \gamma(u,x)dx$$

$$= \frac{1}{a+b}\left[\cosh \frac{b}{\sigma}\sqrt{2u}\int_0^{a-c} \frac{dx}{\cosh \frac{b+x}{\sigma}\sqrt{2u}} + \cosh \frac{a}{\sigma}\sqrt{2u}\int_0^{b-c} \frac{dx}{\cosh \frac{a+x}{\sigma}\sqrt{2u}}\right]. \tag{5}$$

Next, using the expansion

$$\frac{1}{\cosh z} = 2\sum_{n=0}^{\infty} (-1)^n e^{-(2n+1)z}, \quad z > 0 ,$$

and $$\cosh \frac{b}{\sigma}\sqrt{2u} = \frac{1}{2}\left[e^{\frac{b}{\sigma}\sqrt{2u}} + e^{-\frac{b}{\sigma}\sqrt{2u}}\right],$$

we have

$$\cosh \frac{b}{\sigma}\sqrt{2u}\int_0^{a-c} \frac{dx}{\cosh \frac{b+x}{\sigma}\sqrt{2u}} = \frac{\sigma}{\sqrt{2u}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[e^{-(2n+1)\frac{b}{\sigma}\sqrt{2u}}\right.$$

$$+ e^{-2n\frac{b}{\sigma}\sqrt{2u}} - e^{-((2n+1)(a+b-c)-b)\frac{1}{\sigma}\sqrt{2u}} \tag{6}$$

$$\left. - e^{-((2n+1)(a+b-c)+b)\frac{1}{\sigma}\sqrt{2u}}\right],$$

and similarly for the second term in (5). However, the inverse Laplace transform to

$$\frac{e^{-K\sqrt{2u}}}{\sqrt{2u}}, \quad K \geq 0, \quad \text{is} \quad \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{K^2}{2t}},$$

and hence inverting (6) we obtain for the density $f_{T_{det}}(t) = \frac{d}{dt}F_{T_{det}}(t)$ of the detection time the expression:

$$f_{T_{det}}(t) = \frac{2\sigma}{a+b}\frac{1}{\sqrt{2\pi t}}\left\{1 - \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)}\left[e^{-\frac{2n^2a^2}{t\sigma^2}}\right.\right.$$

$$\left. + e^{-\frac{2n^2b^2}{t\sigma^2}}\right] - \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\, e^{-(2n+1)^2(a+b-c)^2\frac{1}{2t\sigma^2}}$$

$$\left[e^{-\frac{a^2}{2t\sigma^2}}\cosh\frac{(2n+1)a(a+b-c)}{t\sigma^2}\right. \tag{7}$$

$$\left.\left. + e^{-\frac{b^2}{2t\sigma^2}}\cosh\frac{(2n+1)b(a+b-c)}{t\sigma^2}\right]\right\}.$$

Notice that the infinite series inside curly brackets converge uniformly for $t \geq 0$, and very rapidly. Thus, the value of the density $f_{T_{det}}(t)$ can be computed approximately for each fixed t by taking only the first few terms of the series. Rather than doing that, we prefer to look at the asymptotic behavior of the density for small t, that is at the beginning of search. It is easily seen that as $t \to 0+$

$$f_{T_{det}}(t) \sim \frac{2\sigma}{a+b} \frac{1}{\sqrt{2\pi t}}, \tag{8}$$

and

$$F_{T_{det}}(t) \sim \frac{2\sigma}{a+b} \sqrt{\frac{2t}{\pi}} \tag{9}$$

Thus, at the beginning of search the probability of detection by the time t increases like $\sqrt{t}$, in contrast to the corresponding asymptotic behavior obtained from (1), where the increase is linear.

<u>Remark</u>: For a uniformly distributed stationary target the distribution function $F_{det}(t)$ can also be derived from the range R_t of the Wiener process. The range

$$R_t = \max_{s \leq t} Y_t - \min_{s \leq t} Y_t$$

is the length of the interval visited by the process Y_t up to the time t. Clearly

$$\begin{aligned} F_{det}(t) &= E(P(T_{det} \leq t \mid R_t)) = E\left(\frac{[R_t - 2c]^+}{a+b}\right) \\ &\doteq \frac{1}{a+b} E(R_t) \quad \text{for} \quad c << a+b \end{aligned} \tag{10}$$

For $t \to 0+$ we expect $E(R_t)$ to be almost as if the reflecting barriers were absent, in which case it is shown in [3] (for $\sigma = 1$) that

$$E(R_t) = \sqrt{\frac{8t}{\pi}} \quad \text{so that}$$

$$F_{det}(t) \sim \frac{2}{a+b}\sqrt{\frac{2t}{\pi}} \quad \text{in agreement with (9).}$$

Next, let us investigate the behavior of $F_{det}(t)$ for large t, that is when the region $[-b,a]$ is becoming saturated with search. Performing the integration in equation (5) we have

$$\int_0^\infty e^{-ut} F_{T_{det}}(dt) = \frac{2\sigma}{a+b}\frac{1}{\sqrt{2u}}\left[\cosh\frac{b}{\sigma}\sqrt{2u}\right.$$

$$\arctan\frac{e^{(a+b-c)\sqrt{2u}} - e^{b\sqrt{2u}}}{1+e^{(a+2b-c)\sqrt{2u}}} + \cosh\frac{a}{\sigma}\sqrt{2u} \tag{11}$$

$$\left.\arctan\frac{e^{(a+b-c)\sqrt{2u}} - e^{a\sqrt{2u}}}{1+e^{(2a+b-c)\sqrt{2u}}}\right].$$

Calling, temporarily, the right-hand side of (11) $\omega(u)$ it can be easily verified that for each $u > 0$

$$\frac{\omega(\tau u)}{\omega(\tau)} \to 1 \quad \text{as} \quad \tau \to 0+ .$$

Hence, by Tauberian theorem of Feller ([4], p. 443)

$$F_{T_{det}}(t) \sim \omega(\tfrac{1}{t}) \quad \text{as} \quad t \to \infty$$

Since, for instance $\omega(\frac{1}{t}) \sim e^{-(a+b)\sqrt{\frac{2}{t}}}$,

we can write $F_{T_{det}}(t) \sim e^{-(a+b)\sqrt{\frac{2}{t}}}$ as $t \to \infty$,

which is, however, not very informative since, in fact,

$$\omega(\frac{1}{t}) \sim e^{-\frac{K}{t^{\alpha}}} \sim 1 - e^{-|K|t^{\alpha}}$$

for any $\alpha > 0$ and any K.

Let us now compute the expected value of the detection time. From (2)

$$\begin{aligned} E(T_{det}) &= \int_{c}^{a} E(T_{x-c})\,\frac{dx}{a+b} + \int_{-b}^{-c} E(T_{x+c})\,\frac{dx}{a+b} \\ &= \int_{0}^{a-c} E(T_x)\,\frac{dx}{a+b} + \int_{-b+c}^{0} E(T_x)\,\frac{dx}{a+b}\,. \end{aligned} \tag{12}$$

Since
$$E(T_x) = -\frac{\partial}{\partial u}\gamma(u,x)\Big|_{u=0}$$

we obtain

$$E(T_x) = \begin{cases} \dfrac{x}{\sigma^2}(x+2b) & \text{if} \quad 0 < x \le a\,, \\ \dfrac{x}{\sigma^2}(x-2a) & \text{if} \quad -b \le x < 0\,. \end{cases} \tag{13}$$

Substituting into (12) and integrating gives the result

$$E(T_{det}) = \frac{1}{3\sigma^2(a+b)} \left| (a-c)^3 + 3b(a-c)^2 + 3a(b-c)^2 + (b-c)^3 \right| \doteq \frac{(a+b)^2}{3\sigma^2} \quad \text{for} \quad c \ll a+b \ . \tag{14}$$

Thus, the mean detection time increases approximately as a square of the length of search region. Higher moments of the detection time can be obtair d in similar fashion.

3. Moving Target

In the previous section, the target was assumed to be stationary. Here, we would like to consider the case when the target is also undergoing random motion over the search region. An obvious extension of the model discussed previously would be just to add the assumption that the target's position X_t, $t \geq 0$ is also a Wiener process with drift $\mu_X = 0$ and variance parameter $\sigma_x^2 > 0$. The initial position X_0 could still be assumed uniformly distributed over the search region.

Since by (A3') detection occurs as soon as $|X_t - Y_t| \leq c$ and since $Z_t = X_t - Y_t$ is again a Wiener process with drift $\mu_Z = 0$, variance parameter $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$, and initial distribution same as that of X_0 (we still assume $Y_0 = 0$), the detection time T_{det} is now simply the time the process Z_t first enters the interval $[-c,c]$. Unfortunately, the presence of a reflecting barrier at $-b$ and a in the original model results in rather complicated boundary conditions.

We will, therefore, restrict ourselves to the case when there are no reflecting barriers. For instance, we may assume that the initial distribution of X_0 is uniform over some interval $[-b_0, a_0]$ such that $a_0 << a$ and $b_0 << b$. For small t the distribution of detection time should be approximately the same as if the barriers were absent.

Let T_x be the first time a Wiener process W_t with, $W_0 = 0$, drift $\mu = 0$ and variance parameter $\sigma^2 > 0$ crosses level x. As is well known, the density of T_x is inverse Gaussian, namely,

$$f_{T_x}(t) = \frac{|x|}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{x^2}{2\sigma^2 t}}, \ t > 0, \ -\infty < x < +\infty . \tag{15}$$

Since for $Z_0 = z_0$ we have again

$$T_{det} = \begin{cases} T_{z_0-c} & \text{if } c < z_0 \le a_0 , \\ 0 & \text{if } |z_0| \le c , \\ T_{z_0+c} & \text{if } -b_0 \le z_0 < -c , \end{cases}$$

we can write for the density of the detection time

$$f_{T_{det}}(t) = \frac{1}{a_0+b_0}\left[\int_0^{a_0-c} \frac{x}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{x^2}{2\sigma^2 t}} dx\right.$$

$$\left. + \int_0^{b_0-c} \frac{x}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{x^2}{2\sigma^2 t}} dx \right] = \frac{2\sigma}{a_0+b_0} \frac{1}{\sqrt{2\pi t}}$$

$$\left[1 - \frac{1}{2}\left(e^{-\frac{(a_0-c)^2}{2\sigma^2 t}} + e^{-\frac{(b_0-c)^2}{2\sigma^2 t}}\right)\right],$$

where $\sigma = \sigma_Z = \sqrt{\sigma_X^2+\sigma_Y^2}$. Notice that for $t \to 0+$ $F_{T_{det}}(t) \sim \frac{2\sigma_Z}{a_0+b_0}\sqrt{\frac{2t}{\pi}}$ as for a stationary target, the only difference being in the variance parameter.

Next, consider the case when the search region is the entire line (no reflecting barriers) and the initial distribution of X_0 is normal $N(0,\tau\sigma_X^2)$, $\tau > 0$. This model may correspond to the case where the searcher arrives at time $t = 0$ at the place where the target was τ time units before. The density is now

$$f_{T_{det}}(t) = 2\int_c^\infty \frac{x-c}{\sqrt{2\pi\sigma_Z^2 t^3}} e^{-\frac{(x-c)^2}{2\sigma_Z^2 t}} \cdot \frac{1}{\sqrt{2\pi\tau\sigma_X^2}} e^{-\frac{x^2}{2\tau\sigma_X^2}}\, dx$$

$$\doteq \frac{1}{\pi\sqrt{t}} \frac{\sigma_X \sqrt{\sigma_X^2+\sigma_Y^2}}{(t+\tau)\sigma_X^2 + t\sigma_Y^2} \quad \text{for} \quad c \approx 0 \ .$$

For the asymptotic behavior we find, readily, that

$$F_{T_{det}}(t) \sim \frac{2\sigma_Z}{\pi\tau\sigma_X}\sqrt{t} \quad \text{as} \quad t \to 0+$$

and

$$1 - F_{T_{det}}(t) \sim \frac{\sigma_X}{2\pi\sigma_Z}\frac{1}{\sqrt{t}} \quad \text{as} \quad t \to \infty \ .$$

However, in the absence of reflecting barriers, the mean detection time $E(T_{det})$ is infinite, in fact, for any initial distribution of X_0 since the inverse Gaussian distribution (15) has infinite mean for all $x \neq 0$.

In view of this fact, it may be interesting to ask whether there is continuous motion of the searcher with bounded speed which would result in a finite mean detection time. More precisely, we assume that the target's motion is still a Wiener process X_t, $t \geq 0$ with drift $\mu = 0$, variance parameter $\sigma_X^2 > 0$, and the initial distribution X_0 normal $N(0,\tau\sigma_X^2)$. The search region is the entire real line, and the detection mode as in (A3'). However, we now allow the observer to choose any motion Y_t, $t \geq 0$, deterministic or random, with $Y_0 = 0$ as long as his speed between turns remains bounded, i.e., $\left|\frac{dY_t}{dt}\right| \leq v$ whenever the derivative exists.

It seems reasonable to assume that the searcher may wish to move with maximum speed v between turns. At the same time he must plan on turning indefinitely since if he ever (before detection) stopped turning and proceeded straight, the mean detection time would still be infinite. Hence, we will assume that he performs a zigzag motion sweeping with speed v around 0 with increasing amplitude. A typical path Y_t of this kind of motion is depicted in Figure 1.

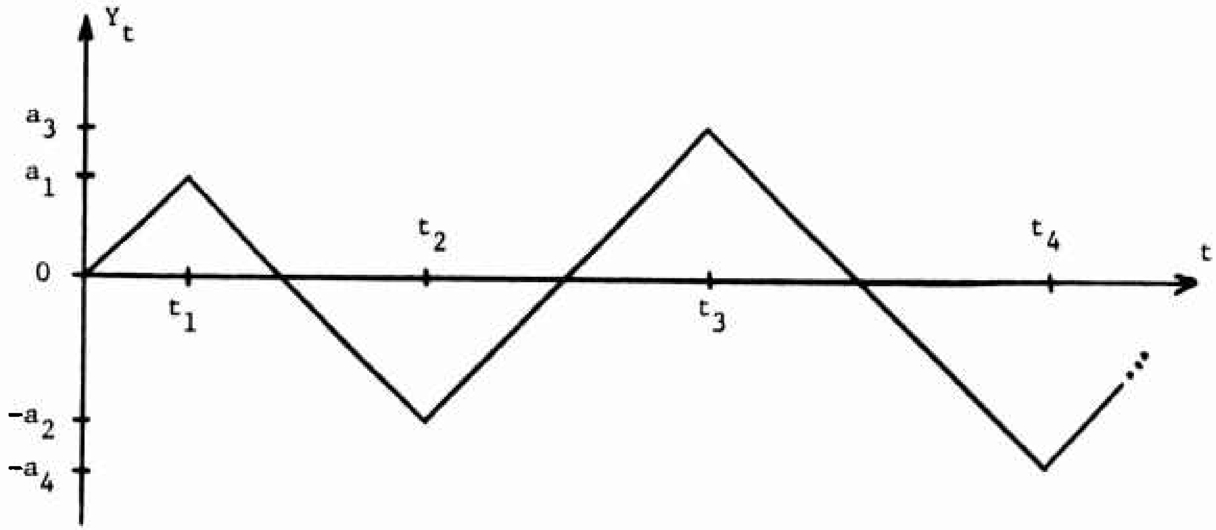

FIGURE 1.

Here $t_1, t_2, \ldots,$ are turning times and $a_n = |Y_{t_n}|$ are sweep amplitudes. Clearly, with $t_0 = a_0 = 0$, $\Delta t_n = t_{n+1} - t_n$, $n = 0, 1, \ldots,$ we have

$$v\Delta t_n = a_{n+1} + a_n \tag{16}$$

We now show, following a suggestion of A. Washburn, that with

$$a_n = \alpha^n, \; n = 1, 2, \ldots, \; \alpha > 1 \;, \tag{17}$$

we have indeed $E(T_{det}) < \infty$. To begin, let us write

$$E(T_{det}) = \int_0^\infty E(T_{det}|X_0=x) \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx$$
$$+ \int_{-\infty}^0 E(T_{det}|X_0=x) \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx \,, \tag{18}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} .$$

Now

$$\int_0^\infty E(T_{det}|X_0=x) \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx$$

$$= \int_0^\infty \int_0^\infty P(T_{det}>t|X_0=x) \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dt\,dx$$

$$\le \int_0^\infty \left[t_1 + \sum_{n=1}^\infty P(T_{det}>t_{2n-1}|X_0=x)\Delta t_{2n-1}\right] \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx \tag{19}$$

$$\le \frac{t_1}{2} + \sum_{n=1}^\infty \int_0^\infty P(X_{t_{2n-1}}>a_{2n-1}|X_0=x)\Delta t_{2n-1} \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx$$

since clearly $x > 0 \Rightarrow P(T_{det}>t_{2n-1}|X_0=x) \le P(X_{t_{2n-1}}>a_{2n-1}+c|X_0=x) \le P(X_{t_{2n-1}}>a_{2n-1}|X_0=x)$. Next, for $a \ge x$

$$P(X_t>a|X_0=x) = \int_{a-x}^\infty \frac{1}{\sigma\sqrt{t}} \phi\left(\frac{z}{\sigma\sqrt{t}}\right) dz \le \frac{\sigma\sqrt{t}}{a-x} \phi\left(\frac{a-x}{\sigma\sqrt{t}}\right)$$

so that

$$\int_0^\infty P(X_{t_{2n-1}} > a_{2n-1} | X_0 = x)\, \Delta t_{2n-1} \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx$$

$$\leq \int_0^{\frac{1}{2}a_{2n-1}} \frac{\sqrt{t_{2n-1}}}{a_{2n-1}-x} \phi\left(\frac{a_{2n-1}-x}{\sigma\sqrt{t_{2n-1}}}\right) \frac{\Delta t_{2n-1}}{\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx$$

$$+ \int_{\frac{1}{2}a_{2n-1}}^\infty \frac{\Delta t_{2n-1}}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx = \phi\left(\frac{a_{2n-1}}{\sigma\sqrt{t_{2n-1}+\tau}}\right) \Delta t_{2n-1} \int_0^{\frac{1}{2}a_{2n-1}} \frac{\sqrt{t_{2n-1}}}{a_{2n-1}-x}$$

$$\phi\left(\frac{x - \frac{a_{2n-1}}{\tau+t_{2n-1}}}{\sqrt{\frac{t_{2n-1}}{\tau+t_{2n-1}}}}\right) dx + \Delta t_{2n-1} \int_{\frac{1}{2}a_{2n-1}}^\infty \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx.$$

Now

$$\int_{\frac{1}{2}a_{2n-1}}^\infty \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{x}{\sigma\sqrt{\tau}}\right) dx \leq \frac{2\sigma\sqrt{\tau}}{a_{2n-1}} \phi\left(\frac{a_{2n-1}}{2\sigma\sqrt{\tau}}\right)$$

and since $0 < x < \frac{1}{2}a_{2n-1} \Rightarrow \frac{1}{a_{2n-1}} < \frac{1}{a_{2n-1}-x} < \frac{2}{a_{2n-1}}$

$$\int_0^{\frac{1}{2}a_{2n-1}} \frac{\sqrt{t_{2n-1}}}{a_{2n-1}-x} \phi\left(\frac{x - \frac{a_{2n-1}}{\tau+t_{2n-1}}}{\sqrt{\frac{t_{2n-1}}{\tau+t_{2n-1}}}}\right) \leq \frac{\sqrt{t_{2n-1}}}{a_{2n-1}} 2 \max\{1, \tau^{-1/2}\} .$$

Substituting into (19) we obtain the bound

$$\int_0^\infty E(T_{det}|X_0=x)\,\frac{1}{\sigma\sqrt{\tau}}\,\phi\left(\frac{x}{\sigma\sqrt{\tau}}\right)dx \le \frac{t_1}{2} + \sum_{n=1}^{\infty}$$

$$2\max\{1,\tau^{-1/2}\}\,\frac{\Delta t_{2n-1}\sqrt{t_{2n-1}}}{a_{2n-1}}\,\phi\left(\frac{a_{2n-1}}{\sigma\sqrt{t_{2n-1}+\tau}}\right)$$

$$+\sum_{n=1}^{\infty}\frac{2\sigma\sqrt{\tau}}{a_{2n-1}}\,\phi\left(\frac{a_{2n-1}}{2\sigma\sqrt{\tau}}\right).$$

Now with a_n as in (17), the latter series obviously converges and so does the former, since by (16)

$$\frac{\Delta t_{2n-1}\sqrt{t_{2n-1}}}{a_{2n-1}} = \mathcal{O}(\alpha^n)\ ,\quad \frac{a_{2n-1}}{\sigma\sqrt{t_{2n-1}+\tau}} = \mathcal{O}(\alpha^n)\ ,$$

and $\sum_{n=1}^{\infty}\alpha^n e^{-\alpha^{2n}}$ is a convergent series. Hence, the first integral in (18) is finite. The second integral is handled in the same fashion.

It would be of considerable interest to determine the motion of the searcher, constrained to be continuous and of bounded speed, which actually minimizes the mean detection time. Although one may conjecture that the optimum path will be as in Figure 1, the optimal choice of the turning times t_n remains an open problem.

REFERENCES

[1] B. O. Koopman, Search and screening, OEG Report No. 56, Washington, D.C., 1946.

[2] D. R. Cox, H. D. Miller, The theory of stochastic processes, Methuen & Co., London, 1965.

[3] D. A. Darling, A. J. F. Siegert, The first passage problem for a continuous Markov process, Annals Math. Statist., 24 (1953), pp. 624-639.

[4] W. Feller, An introduction to probability theory and its applications, Vol. II, John Wiley, 1966.

INITIAL DISTRIBUTION LIST

	Copies
Defense Documentation Center Cameron Station Alexandria, Virginia 22314	12
Dean of Research Code 023 Naval Postgraduate School Monterey, California 93940	2
Library Code 0212 Naval Postgraduate School Monterey, California 93940	2
Library, Code 55 Naval Postgraduate School Monterey, California 93940	2
Prof. Gaver	1
Prof. Forrest	1
Prof. Washburn	1
Prof. Schrady	1
Prof. Shubert Department of Operations Research and Administrative Sciences Naval Postgraduate School Monterey, California 93940	10
Commanding Officer Submarine Development Group Two Groton, Connecticut 06340 ATTN: W. Hayne	1
A. P. Borden, Director Naval Warfare Analysis Group Center for Naval Analysis 1401 Wilson Boulevard Arlington, Virginia 22209	1
A. G. Cicolani Strategic Systems Project Office Code SP2021 Crystal Mall #3 1931 Jefferson Davis Highway Arlington, Virginia 20376	1

Bernard J. Buc
Operations Research, Inc. 1
1400 Spring Street
Silver Spring, Maryland 20910

T. C. Varley
Office of Naval Research 1
Code ONR-434
800 N. Quincy Street
Arlington, Virginia 22217

J. G. Smith
Office of Naval Research 1
Code ONR-616
800 N. Quincy Street
Arlington, Virginia 22217