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INTEGRATED AEROSPACE ENGINE MANAGEMENT. FOUNDATIONS IN ESTIMATION AND PREDICTION OF ENGINE REMOVALS

Laurence L. George

Technology, Incorporated

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Aerospace engine management is an exceedingly complicated problem with important consequences. Analytical management techniques may have potential application in a future, integrated engine management system. Fundamental to any such system is information about engine lives and the number of engine replacements required to meet flying hour program requirements. In this report, the engine management problem is described in the context of a production system producing flying hours. This perspective shows the

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20 ABSTRACT (Continued)

necessity for engine life information since the engines produce flying hours. The maximum likelihood estimator of a multi-risk engine life cumulative distribution function with inspections has been derived. It may be an improvement over the actuarial method now used, and information about usage removals and inspection removals is also available from the maximum likelihood estimator. An hierarchical sequence of families of distributions has been constructed for ease of sequential likelihood ratio testing for more information about the engine life distributions. A simulation model of the engine replacement process has been constructed that will obtain predictions of the number of replacements necessary to meet flying hour program requirements. Antithetic variates appear to reduce the variance of the engine replacement estimator in the simulation model. Last is an annotated bibliography of articles that appear relevant to the engine management problem.

PREFACE

The research for this report was performed at the U. S. A. F. Aerospace Research Laboratories while in the capacity of a Technology Incorporated Visiting Research Associate under contract F33615-73-C-4155. The work done between May 15, 1974 and August 22, 1974 at the Applied Mathematics Laboratory (LB), Wright-Patterson AFB, Ohio 45433, under the direction of Dr. Leon Harter. The author is now an Assistant Professor of Engineering Management at the University of Louisville, Louisville, KY 40208.

This report is the final report although further research is being conducted under contract F33615-75-C-1083. A final report on the latter contract will be completed by April 3, 1976. The project monitor for that contract is Lt. Col. Max Duggins, ARL (LB), Wright-Patterson AFB, Ohio 45433.

I would like to express my gratitude for the opportunity to do the research, for the use of the facilities, and for the advice and the assistance of the Applied Mathematics Laboratory personnel.

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SECTION I

INTEGRATED AEROSPACE ENGINE MANAGEMENT

1. DESCRIPTION OF THIS REPORT

Effective management of high cost equipment requires accurate information about the operating lifetimes of the equipment. There are few items except aerospace engines for which it is worthwhile to collect data on operating or flying time. Since the trouble is taken, the best possible information should be derived from that data, and that information should be used to the fullest extent possible within cost and procedural limitations.

The potential for an integrated system for aerospace engine planning, managing, and decision making may be sufficient to deserve consideration for some changes in procedures. The analytical and statistical aspects of the current procedure, the actuarial technique, and the potential for a future engine management system are described in this section. The remainder of the report is to provide some foundations for the future system, foundations in estimation and prediction of engine removals.

The work statement leading to this report essentially requested more information from less data. That is a challenge to any statistician, and often the way to improve information output is to incorporate newer techniques.

Unfortunately, these advancements may be accompanied by increased data and computation requirements. Not so in the circumstances of this report. In Section II, a maximum likelihood estimator of the distribution function of flying hours till engine removal is derived. This estimator may yield more information from less data than is now needed for the actuarial method, and for less computational effort. In Section III is a plan which may yield still more accuracy.

Also fundamental to an integrated engine management system is a prediction of engine replacement requirements. The simulation procedure for prediction of replacements may be used with the actuarial estimator or with the estimator obtained in Section II or III.

Section V is an annotated bibliography chosen for relevance to engine removal time estimation and to integrated engine management systems in the future.

2. THE ACTUARIAL TECHNIQUE

The statistical procedure now used in aerospace engine management is based on the actuarial technique originally developed for use by life insurance companies. At the time of its adoption by the Air Force, there was little choice of method for estimation of engine lives, and an admirable job was done of adapting the technique to Air Force needs. The actuarial technique is the standard for comparison of estimation techniques, and any proposed technique must provide the same form of information from the same data so there is as little disruption as possible if there is eventual change. Therefore, it is appropriate to describe the actuarial technique and its inherent limitations.

The actuarial technique as applied to aerospace engine systems is used to obtain failure rate information from data on flying hours till engine replacement. This is in addition to accounting and inventory data on engines. The statistical part of the information is summarized in a quarterly report to the Aerospace Engine Life Committee. This Committee monitors the performance of each engine type and its maintenance system of base and depot repair shops and inventories. Extensive reports are provided quarterly to the Air Force Logistics Command, and the information may also be distributed and used in ways unknown to the author.

Data on each engine is in the form of flying hours till engine replacement and the replacement cause. There is also a quarterly report on accumulated flying hours of engines in service and other information of an accounting nature. Let T denote the random variable, flying hours till replacement, for a particular type of engine and a specified set of removal codes. The so-called "removal rates" estimated by the actuarial method are the probabilities of removal in a flying hour interval given the engine has survived till that interval,

$$P\{T \varepsilon [k\Delta t, (k+1)\Delta t] | T > k\Delta t\}, \qquad k=0,1,2,\ldots,m \qquad (1)$$

where Δt is a conveniently chosen interval of from 1/30 to 1/100 of the maximum allowable flying time of an engine. The choice of Δt determines m. A simple estimator is the ratio of the number of engines that are replaced in an interval to the number that survived replacement to the beginning of that interval. The actuarial technique also incorporates information about engines that may have been flown partway through an interval at the time of the quarterly report.

The actuarial technique was originally developed for estimation of human lifetimes, so rather large fluctuations in the estimator of (1) were unsatisfying in appearance. There is no inherent reason why the failure rate of human beings should change drastically from one interval to the next, therefore the ratio estimators of (1) are smoothed by the moving average method. Where data is scarce, the smoothed estimator is extrapolated to the maximum flying time by linear regression. The resulting estimator of the sequence of interval replacement rates (1) is graphed by hand and presented to the Aerospace Engine Life Committee along with several other measures of performance of the engines and the repair system.

The actuarial technique as applied in the Air Force is remarkably well conceived, planned, and executed considering the era in which the procedure was adopted, the late 1950's. Since then, some of the calculation has been programmed for computers, and automated data collection may be implemented for some engine types. But it appears to the author that the procedure and the use of the engine replacement information has not varied significantly since the procedure was first used.

There is only one fundamental criticism of the actuarial technique as applied to aerospace engines. The graphs of the replacement rates are far from smooth. Sharp peaks appear in the neighborhood of periodic inspection times <u>despite</u> the moving average smoothing. There is information in the fact that many engine replacements occur at the times of periodic inspection, and this information is exploited in the estimator derived in Section II.

Other events since the implementation of the actuarial technique increase the pressure for development of new methods to get more information from less data. Advances in statistics and reliability theory have occurred. Proliferation of engine types has resulted in less data for each type, hampering the use of the actuarial technique which depends on having a large amount of data. The future engine management decisions about logistics given a required flying hour program may require more accurate and different information about engine performance. The data exists now; it must be exploited more fully, but without disruption of existing procedures.

3. PROPOSAL FOR AN INTEGRATED ENGINE MANAGEMENT SYSTEM

That part of the Air Force associated with aerospace engines may be considered as a multi-product, multi-echelon production system. The products are flying hours for each engine type. This analogy allows the application of analytical decision making techniques developed for other organizations. The methods of estimation and prediction developed in Sections II, III and IV provide better information fundamental to decision making. That information can penetrate the engine management system along existing paths to the point where it is needed for analytical decision making. Parallel in time to the penetration of better information and the analysis for decision making, a major effort should be made to integrate the decision making policies throughout the Air Force, vertically in the engine management system and horizontally across activities such as civilian and military personnel management, missile system management, etc. What follows is a description of the engine management system considered as an integrated production system.

Aerospace engine management may be analyzed as management of a production system. The product is flying hours. The sales manager, HQ USAF, predicts the required number of flying hours for each command and this becomes a production requirement which is further broken down into engine type flight requirements. Headquarters USAF and the entire logistics system necessary to support the engines constitute the engine management system referred to throughout this report.

H. M. Wagner [1] describes an integrated approach to the management of multi-product, multi-echelon production systems. He focuses on nine aspects of the production system that must be considered if change in any one is contemplated. Translated into the context of the aerospace engine management system, these foci are:

1) forecasting demand for and flight capabilities of the entire system throughout the planning horizon,
2) allocation of the flying hour program among commands, bases, and engine types,
3) purchasing and overhaul of engines and spare parts,
4) inventory levels at every point within the system,
5) future target inventory levels to cover contingencies and anticipated flight requirements,
6) backlog policies to handle shortfalls in flying programs and inventories,
7) the management information system,
8) the contingencies in the flying programs, and
9) the management system, its structure, policies, and measures of effectiveness.

At each of these foci there are decisions that may require information about engine removal times and the number of replacements required to achieve a specified flight plan.

This information should penetrate the logistics system to the point needed and in the form needed. Analysis for decision making will help determine the point of need and the form of the information required for decisions. Eventual integration will take place as relevant information infiltrates the engine management system and as analysis shows the relationship of decisions made anywhere in the engine management system.

There are many ways that information about engine removal times and prediction of replacements enters into analysis for decision making at all nine foci.

1) Clearly, a prediction of the number of replacements necessary to achieve a flying program is relevant to forecasting the capability to meet planned flight requirements. Also, estimates of the engine removal times are relevant because, through interactions between the engine age in its product life cycle and maintenance, inventory, and purchasing policies, it is possible to be left without enough engines to meet contingencies. Risk analysis should determine the probability of meeting contingencies throughout the product life cycle of an engine type.

2) Allocation of the flying program follows from the forecast of requirements and capabilities. Mathematical programming will assist allocation, but many other factors must be recognized before final allocation is made.

3) Purchase of engines and spare parts and overhaul decisions depend on the future flight requirements and the performance of the engines as indicated by the prediction of the number of replacements to meet the requirements.

4) Optimal engine inventory policies may be determined analytically to control stockouts and costs. Parts inventory policies follow from engine policies. The demand for engines as predicted by the simulation model in Section IV or other means is a necessary input for inventory policy analysis.

5) Future target inventory levels are determined indirectly from anticipated flying programs, contingencies, and the capabilities of the engines themselves. Some redistribution may be necessary to meet the targets, and again, mathematical programming techniques can help analysis for redistribution.

6) Alternate policies for making up backorders in inventories and shortfalls in meeting the flying hour program to determine their effect on other parts of the engine management system.

7) The management information system should provide information in the form needed at the point necessary for decision making in foci 1-6. An information system for engine replacements already exists; additional or more accurate information may be necessary at some points but not at other points. The information

gathering system should also be considered to determine whether the effort expended is worthwhile. And work such as that represented by this report should be done to extract more information from less data.

8) Contingencies in the flying program may result from external or internal causes. Internal contingencies may be predicted by risk analysis of policies in foci 1-6.

9) The military command system provides a management structure unmatched by any other production system. This structure has inherent inertia that calls for more justification for change than in a business where the consequences of risk are not so large. Any changes should be accomplished in parallel with existing procedures. The estimation procedure in Section II is designed for use along with the actuarial method until parts of the actuarial method may be replaced. Analysis for decision making may proceed at all levels of the engine management system and, if results are worthwhile, information necessary for the analytical decision making should be made available. Meanwhile, integration of decision making throughout the engine management system is facilitated by the command and communications systems.

This completes a description of the context and anticipated development of integrated engine management. The contributions in the remaining sections are foundations for obtaining better information for use in analytical decision making.

SECTION II

MAXIMUM LIKELIHOOD ESTIMATION OF THE REMOVAL TIME CUMULATIVE DISTRIBUTION FUNCTION

1. INTRODUCTION

The problem is to estimate the distribution function of engine removal times which is assumed to have the form

$$F(t) = 1 - \bar{F}_1(t) \cdot \prod_{i=1}^{i_{[t]}} (1-p_i) \qquad (2)$$

for $t \geq 0$. An estimate is necessary to predict demands for replacement engines. The form of the distribution function is suggested because of the high proportion of removals that occur at inspections. Each p_i is the probability of removal at the time of the i^{th} inspection, $i=1,2,\ldots, k\leq\infty$. The limit $i_{[t]}$ is the index of the last inspection prior to or at time t. The product of $1-p_i$ terms is assumed equal to 1 if there are no inspections at or before t. The tail distribution $\bar{F}_1(t) = 1-F_1(t)$ is the probability distribution of usage removal time. That distribution may be truncated at a maximum operating time, t_{max}, when all engines are removed for overhaul. The model (2) is also appropriate to the life testing situation with all items put on test at one time and a random number of survivors removed from test at known times prior to termination of the test (progressive censoring).

Sample data tell the engine removal time and whether removal occurs during usage, at an inspection, or at the maximum time. (Actually inspection and maximum time removals vary slightly around the scheduled times, but fixed inspection times and t_{max} will be assumed. Estimation of inspection removal times constitutes another problem.) The sample data may be used to construct estimators of $F(t)$ in several ways. The empirical distribution function with jumps inversely proportional to sample size occurring at all removal times may be used to estimate $F(t)$. The sample may be segregated into usage, inspection and maximum time removals; the usage removal times may be used to construct an empirical distribution function for $F_1(t)$, and inspection and maximum time probabilities may be estimated from sample proportions.

The method of maximum likelihood is used to determine an estimator of $F(t)$, denoted $\hat{F}(t)$, which incorporates the information that an engine removed at an inspection or at t_{max} has survived previous usage and vice-versa. That estimator for $F(t)$ is the empirical distribution function, but the estimators of the p_i, $i=1,2,...,k$ and of $F_1(t)$ are <u>not</u> the sample proportions and the empirical distribution function!

The maximum likelihood method insures that estimators will be functions of the sufficient statistics. Asymptotic properties of unbiasedness, efficiency, consistency and normality can probably be proved by verification of regularity conditions for the likelihood function. The sample proportions of survivors which are derived to estimate the

p_i, i=1,2...,k, may also have good small sample properties. The Glivenko-Cantelli theorem may be applied to verify consistency of the estimator for $F_1(t)$, and perhaps the Kolmogorov theorem may be extended to find the asymptotic distribution of the difference between $F_1(t)$ and its estimator. All these results are suggested because of the reasonable form of the estimator of F(t). They have not been proved.

Comparison of $\hat{F}(t)$ to other methods of estimation is in an incomplete state also. However, $\hat{F}(t)$ may prove to be better than the actuarial estimator which breaks $[0, t_{max}]$ into approximately 60 equal intervals and estimates F(t) as piecewise exponential on those intervals.

2. DERIVATION OF THE MAXIMUM LIKELIHOOD ESTIMATOR

The engine removal times $T_1, \ldots, T_N$ in the sample are assumed to be independent and identically distributed according to F(t), (2). In addition, the engine removal code specifies whether the removal was during usage, at inspection, or at maximum time. Define the following sequences:

$\{n_j\}$, the number of removals at the j^{th} inspection and t_{max}, $j=1,2,\ldots,k+1$;

$\{m_j\}$, the number of usage removals between the (j-1)st and the jth inspection; $M_j = \sum_{i=1}^{j} m_j$, $j=1,2,\ldots,k+1$;

$\{t_i\}$, the sequence of usage removal times, $i=1,2,\ldots,M_{k+1}$; and

$\{t_j'\}$, inspection times and t_{max}.

These sequences turn out to be sufficient statistics. The sample size N can be expressed as

$$N = M_{k+1} + \sum_{j=1}^{k+1} n_j$$

in terms of usage removals, inspection removals, and maximum time removals.

The likelihood function for the sample information is

$$\left\{\prod_{i=1}^{M_{k+1}}\left[f_1(t_i)\prod_{j=1}^{j_{[i]}}(1-p_j)\right]\right\}\left\{\prod_{j=1}^{k}\left[p_j\prod_{i=1}^{j-1}(1-p_i)\;(1-F_1(t_j'))\right]^{n_j}\right\}$$

$$\left\{\left[1-F_1(t_{max}^-)\right]\left[\prod_{j=1}^{k}(1-p_j)\right]\right\}^{n_{k+1}}\cdot C$$

with $f_1(t_i)$ denoting the density of $F_1(t)$, discrete or continuous, $t_{max}^- = t_{max}-\varepsilon$ for small $\varepsilon>0$, and the awkward notation $j_{[i]}$ is used to indicate the index of the last inspection prior to the usage removal at t_i. The first term of the likelihood function is the probability of the usage removals, the second term is the probability of inspection removals, and the last is the probability of all the survivals to t_{max}. C is a combinatorial constant.

In order to make the likelihood function positive, all of the $f_1(t_i)$ should be positive, and to maximize it, $1-F_1(t_j')$ and $1-F_1(t_{max}^-)$ should be as small as possible consistent with the constraints on the distribution function $F(t)$; $F(0^-)=0$, $F(\infty)=1$, and $F(t)$ non-decreasing in t. As in the maximum likelihood derivation of the empirical distribution function, all the mass of $F_1(\cdot)$ should be placed at the observations $\{t_i\}$ and t_{max}. Then

$$1-F_1(t_j') = 1-\sum_{i=1}^{M_j} f_1(t_i)$$

and

$$1-F_1(t_{max}^-) = 1-\sum_{i=1}^{M_{k+1}} f_1(t_i)$$

However, not all of the jumps in $\hat{F}_1(\cdot)$ will be of the same size as in the empirical distribution function. The derivative of the natural logarithm of the likelihood function with respect to $f_1(t_i)$ gives

$$\delta \ln L/\delta f(t_i) = 1/f_1(t_i) - f_1(t_i)/[1 - \sum_{i=1}^{M_j} f_1(t_i)]$$

for all $i = M_{j-1} + 1, \ldots M_j$, $j = 1,2,\ldots,k+1$. This indicates that the maximum likelihood estimators of the jumps will be the same in the intervals before the first inspection, between inspections, and after the last inspection. These jump sizes will be denoted by $f_1, f_2, \ldots, f_{k+1}$. The likelihood function now simplifies to

$$\left\{\prod_{i=1}^{k+1}\left[f_i \prod_{j=1}^{i-1}(1-p_j)\right]^{M_i - M_{i-1}}\right\}\left\{\prod_{j=1}^{k+1}\left[P_j \prod_{i=1}^{j-1}(1-p_i)\right]\left[1-\left(\sum_{i=1}^{j} f_i(M_i - M_{i-1})\right)\right]^{n_j}\right\}C$$

It is a fairly long algebraic exercise to show that $\delta \ln L/\delta f_j = 0$, $j=1,2,\ldots k+1$, and $\delta \ln L/\delta p_j = 0$, $j=1,2,\ldots,k$ are satisfied by

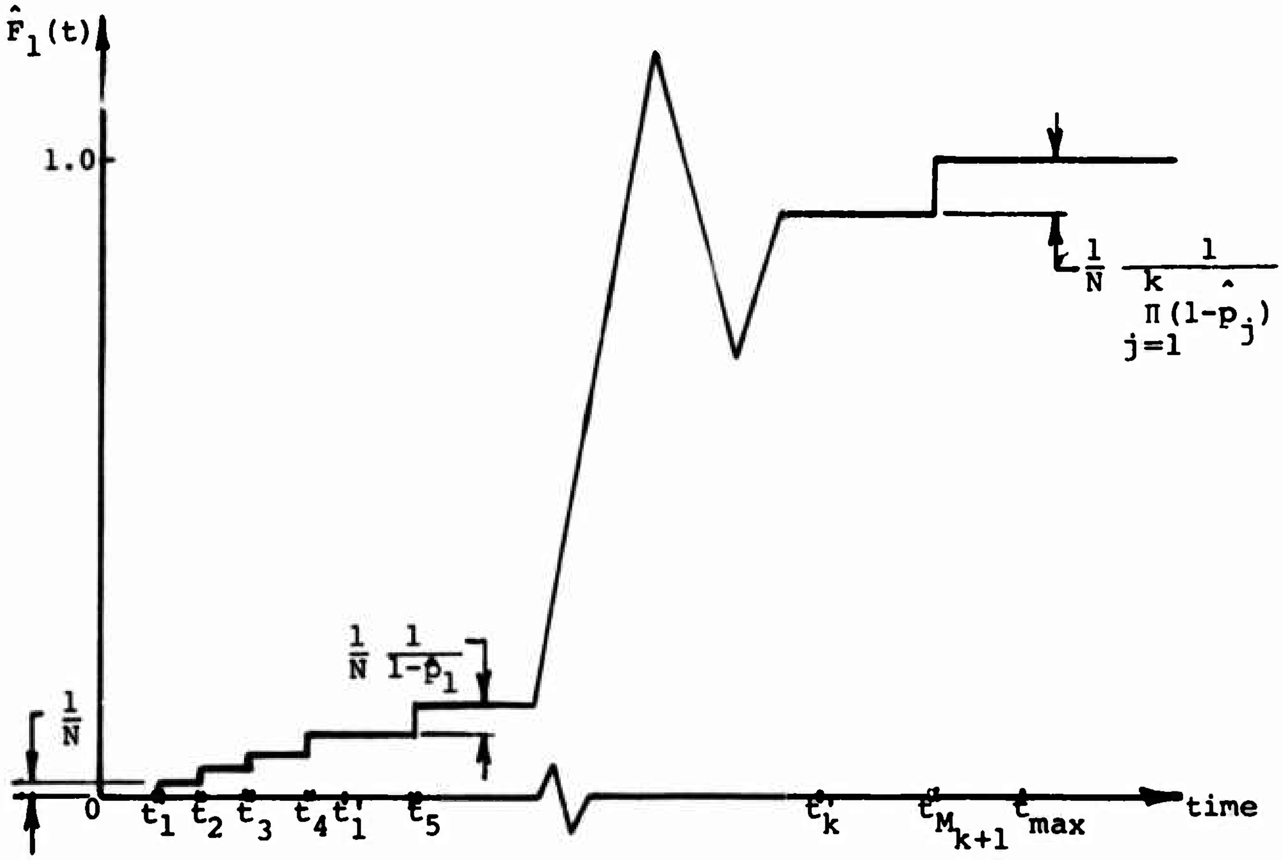
$$\hat{f}_j = N^{-1}\left[\prod_{i=1}^{j-1}(1-\hat{p}_j)\right]^{-1}$$

and

$$\hat{p}_j = n_j\left[N - M_j - \sum_{i=1}^{j-1} n_j\right]^{-1}$$

Insertion of these results into equation (2) shows $\hat{F}(t)$ to be the empirical distribution function!

Although the conditions for maxima on the matrix of second derivatives of the likelihood function have not yet been checked, the reasonableness of the estimators suggests they are maximum likelihood rather than minimum likelihood. These estimators are non-negative and satisfy constraints derived from the constraints on $F(t)$, so they will be presumed to be maximum likelihood estimators. The form of the estimators has some intuitive explanation. The estimators $\hat{p}_j$, $j=1,2,\ldots,k$, are simply the proportions of inspection removals at the j^{th} inspection out of all the engines that survived to that inspection. The $\hat{f}_j$ are the usual jumps of the empirical distribution function, $1/N$, but conditioned on survival through all previous inspections. The estimator of $F_1(t)$ is shown in Figure 1. Notice that the jumps in $\hat{F}_1(t)$ are larger and more sparse in the later inter-inspection intervals because fewer engines have survived inspections.



Other estimators of potential interest are available or are easily obtained from the maximum likelihood estimators.

- The probability that an engine will be removed while in use is

$$\hat{F}_1(t^-_{max}) = \sum_{j=1}^{k+1} \hat{f}_j m_j$$

- The probability of survival to maximum time is

$$1-\hat{F}(t^-_{max}) = n_{k+1}/N$$

- The failure rate function r(t) is obtainable from the relation

$$1-\hat{F}(t) = \exp(-\int_0^t \hat{r}(u)du)$$

- The probability of removal in any interval of size w given survival to time t is

$$\frac{\hat{F}(t+w)-\hat{F}(t)}{1-\hat{F}(t)} = 1-\frac{\hat{\bar{F}}_1(t+w)}{\hat{\bar{F}}_1(t)} - \prod_{i=i_t}^{i_{t+w}} (1-p_i)$$

(These estimates would replace the conditional interval failure probabilities obtained by the actuarial method.) Moments and percentiles of $F(t)$ and $F_1(t)$ are easily obtained.

3. SPECULATION ABOUT THE PROPERTIES OF THE MAXIMUM LIKELIHOOD ESTIMATOR

The only property that can be claimed for the estimator $\hat{F}(t)$ is that it is a function of the sufficient statistics. That is a consequence of the maximum likelihood method used to obtain the estimator, (Lindgren [2] p.225). However, many other desirable large sample properties can be attributed to maximum likelihood estimators. Asymptotic properties of umbiasedness, efficiency, consistency and normality of $\hat{f}_j$ and $\hat{p}_j$, $j=1,2,...,k+1$, follow from verification of regularity conditions on the second derivatives of the distribution F(t), (Wilks [3] chapter 12). This reduces to verification of the same regularity conditions on $F_1(t)$ because $\prod_{i=1}^{i[t]} (1-p_i)$ satisfies the conditions. The user of the model (2) may assume the conditions are satisfied by $F_1(t)$, and the large sample properties are immediate consequences.

The sample proportions of survivors $\hat{P}_j$, j=1,2,...,k are so intuitively appealing that one may expect desirable small sample properties usually attributed to sample proportions--unbiasedness, efficiency, and perhaps others. No investigation of this possibility has been made.

The Glivenko-Cantelli theorem (Gnedenko [4] p.391) states that the empirical distribution function is consistent. The estimators $\hat{f}_i$ appear to be the jumps of the empirical distribution function conditioned on the probability of not being removed at previous inspections. However, that probability is not known but estimated. A proof of consistency of $\hat{F}_1(t)$ would have to deal with that estimate and then apply the Glivenko-Cantelli theorem.

If $F_1(t)$ is assumed to be continuous, the absolute difference between the empirical distribution function and $F_1(t)$ has known asymptotic distribution (Billingsley [5] p.104). This result of Kolmogorov may be extended to the maximum likelihood estimator $\hat{F}_1(t)$ of F(t) if the conditional form of the jumps $\hat{f}_j$, j=1,2,...,k+1 is exploited. There has been no attempt to do this.

It would be of some interest to compare $\hat{F}(t)$ with the empirical distribution function (which is an unbiased function of the maximum likelihood estimator of a distribution function). The efficiency of $\hat{F}(t)$ is probably higher but perhaps not high enough to justify the slight increase in computation required.

In comparison with the conditional interval failure probabilities (called failure rates) estimated by the actuarial method, the corresponding estimators derived from $\hat{F}(t)$ certainly have better large sample properties since they are not constrained to have constant failure rate in each of a fixed set of intervals. Where data is more complete, the intervals between jumps of $\hat{F}_1(t)$ are small, giving more precision during time periods when removals are more likely. Where data are scarce because few removals occurred, $\hat{F}(t)$ is still an improvement over the actuarial estimate which is a linear extrapolation.

The feature of different interval sizes indicates a computational problem that may become serious. The estimate $\hat{F}(t)$ jumps at each observed usage removal and inspection time, and as data increases, some kind of fixed interval grid may have to be established on $[0, t_{max}]$ in order to keep the estimator in a reasonable amount of computer storage. This need creates a new statistical problem of optimal interval size and location.

There is some rate of convergence comparison of interval estimators for restricted families of distributions in chapter 5 of a book by Barlow, Bartholomew, Bremner, and Brunk [6]. Similar comparison may be possible for the empirical distribution function, $\hat{F}(t)$, $\hat{F}(t)$ constrained to optimal intervals, and the estimator given by the actuarial method.

Before any great effort is spent proving these speculations, simulation should be used to compare $\hat{F}_I(t)$ with the empirical distribution function and the actuarial method. If the comparison is favorable to $\hat{F}_I(t)$, it could be used while its more subtle qualities are examined.

SECTION III

AN HIERARCHY OF ENGINE REMOVAL TIME DISTRIBUTIONS

1. DESCRIPTION OF THE HIERARCHY

An hierarchy can be constructed of distribution functions of engine flying hours till removal for each engine type. In Part 1, families of distribution functions will be defined with an hierarchical relationship that lends itself to sequential likelihood ratio tests to determine which family represents engine life data for an engine type. The point of this construction is that estimators of distribution functions for the more restrictive families are more accurate in a statistical sense. Also, updating the official failure rate can be done statistically once the family representing the engine life distributions has been identified. These latter topics are discussed in Parts 2 and 3. Part 4 anticipates some criticism.

Let F(t) denote the cumulative distribution function of the flying hours till removal of a particular type of engine. Script $\mathfrak{F}$ denotes a family of distribution functions. The families proposed for the hierarchy are listed in increasing order of restrictiveness.

$\mathfrak{F}_I$ - all absolutely continuous, discrete, and mixed distributions with support $[0, t_{max}]$, $t_{max} \leq \infty$

$\mathfrak{F}_{II}$ - all distributions of form

$$F(t) = 1 - \bar{F}_1(t) \prod_{i=1}^{i_{[t]}} (1-p_i) \quad 0 \leq t \leq t_{max}$$

where $\bar{F}_1(t)$, the tail c.d.f. of $F_1(t)$, is assumed to be absolutely continuous and $i_{[t]}$ is the index of the last inspection prior to t, $i_{[t]} = 0,1,2,...$

(The product $\prod_{i=1}^{i_{[t]}} (1-p_i)$ is assumed to be 1 for $i_{[t]}=0$.) The discontinuity representing removal of engines at t_{max} is contained in p_i for $i=i_{[t_{max}]}$. Then $F_1(t)$ has as its support $[0, t_{max}]$.

$\mathfrak{F}_{III}$ - Various restricted families of distributions such as IFR (increasing failure rate), IFRA (increasing failure rate on the average) [7], NBU (new better than used), NBUE (new better than used in expected life) [8], DMRL (decreasing mean residual life) [8], and all the familiar distributions of nonnegative random variables,

$\mathfrak{F}_{IV}$ - $F(t) = 1-e^{-rt} (1-p)^{i_{[t]}}$ $0<p<1$, $0 \le t \le \infty$*

The last two families contain distributions for which optimal replacement policies are known [8]. Other families may be of interest such as

$$\mathfrak{F}_V - F(t) = 1-\bar{F}_1(t)\ (1-p)^{i_{[t]}} \qquad 0 \le t \le t_{max},\ 0<p<1 \ ^*$$

The hierarchical relation among these families is

$$\mathfrak{F}_I \supset \mathfrak{F}_{II} \supset \begin{cases} \mathfrak{F}_V \supset \mathfrak{F}_{IV} \\ \mathfrak{F}_{III} \supset \mathfrak{F}_V \supset \mathfrak{F}_{IV} \end{cases}$$

(The inclusion $\mathfrak{F}_{III} \supset \mathfrak{F}_V$ depends on which restrictions in $\mathfrak{F}_{III}$ are chosen.) Some hierarchical relations among the restricted families in $\mathfrak{F}_{III}$ have also been established.

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* These families could be truncated at t_{max} if desired.

2. A SEQUENTIAL LIKELIHOOD RATIO TEST FOR THE APPROPRIATE FAMILY

Since the hierarchy is constructed so that there is containment and since maximum likelihood estimators of F(t) exist for each family, a sequence of likelihood ratio tests may be used to determine which family describes data from an engine type. (The maximum likelihood estimators for $\mathfrak{F}_{II}$ and $\mathfrak{F}_{V}$ are derived in Section II.) Strong preference for family $\mathfrak{F}_{II}$ begs a test to determine, for each engine type, whether $F(t) \varepsilon \mathfrak{F}_{II}$. (Reasons for this preference are discussed in Section II.) This is accomplished by the likelihood ratio test of

$$H_0: \; F(t) \; \varepsilon \; \mathfrak{F}_{II} \text{ vs.}$$

$$H_a: \; F(t) \; \varepsilon \; \mathfrak{F}_{I}$$

The procedure is to accept the hypothesis H_o if the likelihood ratio

$$\Lambda = \frac{\max\limits_{F(\cdot)\varepsilon \mathfrak{F}_{II}} L(T_1,\ldots,T_n | F(\cdot)\varepsilon \mathfrak{F}_{II})}{\max\limits_{F(.)\varepsilon \mathfrak{F}_{I}} L(T_1,\ldots,T_n | F(\cdot)\varepsilon \mathfrak{F}_{I})} \tag{3}$$

is sufficiently large. (The likelihood function $L(\cdot|\cdot)$ is based on a sample $T_1,\ldots,T_n$ of engine removal times given the family containing the underlying distribution function F(t).) If H_o is accepted, then proceed to test

$$H_0: \; F(t) \varepsilon \mathfrak{F}_{III} \text{ vs.}$$

$$H_a: \; F(t) \varepsilon \mathfrak{F}_{II}$$

in the same manner. Continue to test for $\mathfrak{F}_{IV}$ and $\mathfrak{F}_{V}$ until the null hypothesis is rejected or $\mathfrak{F}_{IV}$ is accepted.

The result is that the family $\mathfrak{F}_{I},\ldots,\ \mathfrak{F}_{V}$ may be identified for each engine type, and, as the test proceeds from $\mathfrak{F}_I$ through $\mathfrak{F}_V$, estimators of the underlying life distribution function become successively better in a statistical sense. There are other potential benefits for identifying a family. The family $\mathfrak{F}_{IV}$ of exponential/geometric distributions allows analytical solution of the replacement problem described and simulated in Section IV. Some subsets of family $\mathfrak{F}_{III}$ allow bounds on the solution of the replacement problem simulated in Section IV, [8].

Improvement over the actuarial method is possible for all families $\mathfrak{F}_{I},\ldots\ \mathfrak{F}_{V}$ because the estimators for each family are not restricted to fixed intervals as is the actuarial estimator. E.g., for $\mathfrak{F}_I$ the most common estimator is the empirical d.f. $\hat{F}_n(t)$ with jumps of equal size at each observed engine removal time. However, for large samples the empirical distribution function requires excessive data storage, and some consideration will have to be given to interval estimators, but not necessarily to equal size intervals as in the actuarial method.

3. UPDATING THE OFFICIAL FAILURE RATE

The Aerospace Engine Life Committee periodically decides whether to update the "Offical Failure Rate" curve by visual comparison of the Official Failure Rate curve and the failure rate computed by the actuarial method from current data, because data from current engine operation should be incorporated if it will add to the accuracy of estimates. Also, data from earlier operations should be eliminated if it appears that changes in maintenance, engine modifications, and usage have resulted in a change in the failure rate of an engine type. The decision of the Aerospace Engine Life Committee can be assisted by a statistical comparison within a family of distributions for each engine type.

The statistical procedure requires several comparisons of data from one quarter with data from several other quarters. Some definitions are necessary before description of the procedure. Let $F_1(t)$, $F_2(t),\ldots,$ $F_k(t)$ denote the true underlying distribution functions of engine flying hours at removal in quarters $1, 2,\ldots,k$. Quarter 1 is the first quarter for which data is used in calculation of the Official Failure Rate, and quarter k-1 is the last quarter of data which is used in the calculation. Quarter k is the current quarter with engine removal time data under consideration for incorporation into the Official Failure Rate. Let ${}_jF_{k-1}$ be the underlying c. d. f. of engine removal times throughout quarters $j, j+1,\ldots,k-1$; $j=1,2,\ldots,k-1$. The latter may not be the same as $F_1(t),\ldots,F_{j-1}(t)$ and $F_k(t)$ due to changes in maintenance, usage, or design of the engines.

At the time of data analysis each quarter, a set of hypotheses should be tested as shown in Figure 2. An explanation of the procedure may be helpful. First test whether to incorporate the new data.

$$H_0: \; F_k(t) = {}_1F_{k-1}(t) \text{ for all } t\varepsilon[0, t_{max}] \text{ vs.}$$
$$H_1: \; F_k(t) \neq {}_1F_{k-1}(t) \text{ for some } t\varepsilon[0, t_{max}]$$

If H_o is accepted, then data from the current quarter should be considered for incorporation into the Official Failure Rate, but not before this next test to determine whether the oldest data continue to be used.

$$H_2: \; F_1(t) = {}_2F_k(t) \text{ for all } t\varepsilon[0, t_{max}] \text{ vs.}$$
$$H_3: \; F_1(t) \neq {}_2F_k(t) \text{ for some } t\varepsilon[0, t_{max}]$$

If hypothesis H_2 is accepted, the estimator of engine removal time distribution should be revised to include data from all k quarters.

If either hypothesis rejected, additional study is necessary to determine which data should be used to estimate the engine removal time distribution and Official Failure Rate. If H_o is rejected, the data for quarter k seems to have come from a population of engines that does not resemble engines in quarters 1,2,...,k-1 in its flying hour life. Further tests should be made to compare $F_k(t)$ with ${}_jF_{k-1}(t)$ for j = 2,3,...,k-1.

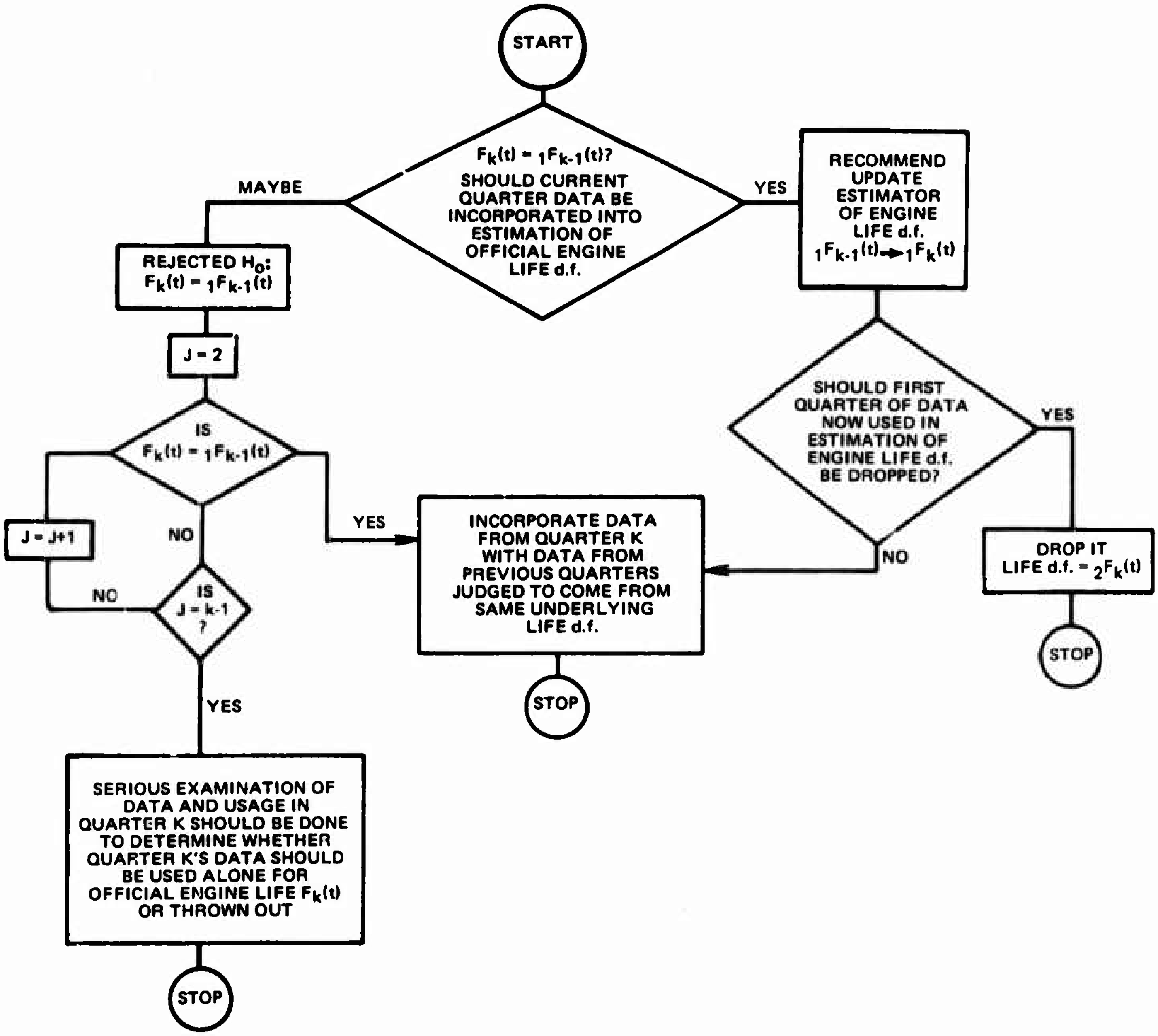

Figure 2. Statistical Updating Procedure

If F_k passes statistical comparison with ${}_jF_{k-1}(t)$ for some j, the official engine life c.d.f. should be updated to an estimate ${}_j\hat{F}_k$ including data from the current quarter. If not, then the official engine life data should be based on the current quarter, but not until there is an explanation for the apparent change and an assessment of whether the current quarter usage is representative of the future. This decision should remain the responsibility of the Aerospace Engine Life Committee. It should also be recognized that all other statistical conclusions in this section are only recommendations. These recommendations will be accompanied by a statistical level of confidence, a measure of the possible error in the recommendation.

As indicated in Part 2 of this section, increased statistical accuracy of the recommendations is the result of the hierarchical relationship among families $\mathfrak{F}_I, \ldots, \mathfrak{F}_V$. Once the family has been established as in Part 2, comparison of distribution functions within each family as required in this part may be carried out with the aid of existing statistical tests. For example:

$\mathfrak{F}_I$: Kolmogorov - Smirnov test,

$\mathfrak{F}_{II}$: Kolmogorov - Smirnov test on $F_1(\cdot)$ and multinominal test on p_i, $i=1,2,\ldots,i_{[t_{max}]}$,

$\mathfrak{T}_{III}$: Likelihood ratio tests which exist for certain restricted families (Barlow [7]),

$\mathfrak{T}_{IV}$: Tests of the exponential failure rate r and the survival probability p, and

$\mathfrak{T}_{V}$: Same as $\mathfrak{T}_{II}$.

4. CAUTION ON THE USES OF THIS SECTION

The major limitation is that further development is needed. This development should proceed in conjunction with the use of the procedures in this section in management decision making. Statistics can only specify the probability of error in an hypothesis test. Management decision makers needs to determine what tests are necessary and an acceptable level of risk. This section is proposed in anticipation of decision making needs and for use in obtaining better estimates within the existing data structure.

Fortunately, most of the statistical work remaining to be done is development, rather than high risk basic research. Maximum likelihood estimators of F(t) exist for all families. (See Section II for estimators of $\mathcal{F}_{II}$ and $\mathcal{F}_{V}$; most of the restricted distributions in $\mathcal{F}_{IV}$ also have estimators.) The remaining work is to develop sequential likelihood ratio tests and other comparison tests that are convenient for use.

SECTION IV

FORMULATION AND SIMULATION OF THE ENGINE REPLACEMENT PROBLEM

1. INTRODUCTION

Stripped of all complications, engine inventory management requires reorder quantities of each engine type each quarter based on the flying hour program for that quarter. The reorder quantity is equal to the number of replacements taken from inventory. The number of replacements is the number of engines that are removed but not reinstalled, so a prediction of the number of removals as a function of the flying program is required for inventory management. That is the engine replacement problem to be studied in this section.

Even under the simplification that there is only one single-engine plane at a base, the distribution of the number of removals cannot be simply computed as a function of the removal time distribution and the flying program. The reason is that the replacement engines are not necessarily new but have already accumulated a known amount of flying time. The only exception occurs when the removal time distribution has the "memoryless form of the exponential and geometric distributions,

$$F(t) = 1-e^{-\lambda t}(1-p)^{i_{[t]}} \qquad \lambda,\ t\geq 0, \quad 0\leq p<1 \qquad (4)$$

where $i_{[t]}$ denotes the index of the last inspection, if any, prior to or at time t. In this case the replacement process becomes a renewal process for which many results are known (Feller [9] and Cox [10]).

Despite obstacles to solution, formulation of the single engine replacement problem is of value to point out analytical limitations, to suggest approximations, and to identify the mathematical model that can be simulated. The single engine simulation can be adapted to multi-plane, multi-engine simulation with complications sufficient to defy analysis. Computer models for simulation of the engine replacement problem are given in Part 3 for single quarter and multiple quarter planning periods. If the sequence of replacement engines (with known accumulated flying hours) is random, the conditional distribution of the number of engines in stock with less than a certain age, given the number of replacements, has binomial form. This distribution may be used to simulate a random replacement sequence. Further combinatorial analysis may yield this distribution if there are many operating engines. However, engine replacement may be simulated regardless of pending analysis, and the event by event type computer model proposed is much quicker in execution than interval by interval simulation based on the actuarial method.

Another reason for the event type computer model is that it may be easily modified to incorporate antithetic random variables. These negatively correlated random variables are used to reduce the variance of means and percentiles in simulation of waiting times in single server queuing systems

(Mitchell [11]). The same technique and proof can be applied to simulation of the mean value function of a renewal process. Although data are not very encouraging, the use of antithetic variates may reduce the variance of the sample mean in simulation of the engine replacement problem.

Further analytical and empirical study of the engine replacement prediction problem is recommended in conjunction with analysis of the buffer quantity to be reordered each quarter. Approximations should be compared since some are being used without any current justification. The event by event simulation technique should be incorporated since it saves computer time in what promises to be a large scale simulation. Also there should be further investigation of the application of antithetic variates to simulation of the number of engine replacements.

2. FORMULATION OF THE ENGINE REPLACEMENT PROBLEM

Fundamental to the engine replacement problem is the random process $N(t)$ representing the number of engine replacements on a single engine plane necessary to complete t flying hours. (Modifications can be made for a multi-engine problem with a random sequence of replacements.) The distribution $F(t)$, $t \geq 0$, of flying hours between removals is assumed to be known. The currently installed engine has already accumulated t_o flying hours and the replacement engines have

$t_1, t_2, \ldots t_s$, $s \leq \infty$ flying hours. They replace the original engine in that order. Denote their residual lives given ages $t_0, t_1, \ldots t_s$ by $R(t_0), \ldots, R(t_s)$. These random variables are assumed independent and have the distribution

$$P[R(t_i) \leq v \,|\, \text{removal time} > t_i] = \frac{F(t_i+v)-F(t_i)}{1-F(t_i)} \tag{5}$$

for $v>0$, and $i=0,1,2,\ldots,s$.

The relation between partial sums of residual lives and the number of replacements in a time interval t may be used to obtain the distribution of $N(t)$. If an engine is not removed in $[0,t]$, then $R(t_0)>t$, and no replacement occurs. If $R(t_0) \in [0,t]$ and the second engine survives, then $R(t_0)+R(t_1)>t$, and only one replacement occurs. The general relation between $N(t)$ and partial sums of $R(t_i)$ may be expressed in terms of the equivalent events:

$$\begin{aligned}
N(t) &= 0 \Longleftrightarrow R(t_0)>t \\
N(t) &= 1 \Longleftrightarrow R(t_0) \leq t < R(t_0)+R(t_1) \\
&\;\vdots \\
N(t) &= n \Longleftrightarrow \sum_{i=0}^{n-1} R(t_i) \leq t < \sum_{i=0}^{n} R(t_i) \\
&\;\vdots \\
N(t) &= s+1 \Longleftrightarrow \sum_{i=0}^{s} R(t_i) \leq t
\end{aligned} \tag{6}$$

Therefore, the probability distribution of $N(t)$ is known if

$$P\left[\sum_{i=0}^{n-1} R(t_i) \leq t < \sum_{i=0}^{n} R(t_i)\right] = P[N(t)=n]$$

can be determined for all n. That requires the distribution of partial sums of residual lives given the ages,

$$\sum_{i=0}^{n} R(t_i),\ n=1,2,\ldots,s\leq\infty$$

The density of the sum of two independent non-negative random variables is obtained by the convolution formula

$$\frac{\delta}{\delta t} P[X_1+X_2\leq t] = \int_0^t f_1\ (t-u)dF_2(u) \qquad (7)$$

where $f_1(\cdot)$ is the density of X_1 and $F_2(\cdot)$ is the c. d. f. of X_2. If X_1 and X_2 have the same distribution, the convolution is of simpler form, but it is still inconvenient to compute except for special cases. If X_1 and X_2 are two residual lives, Equation (7) becomes the convolution of two distributions (5) with possibly different ages. There is no computationally convenient representation for such convolutions even though the integral formula is easily expressed as

$$\frac{\delta}{\delta t} P[R(t_o) + R(t_1) \leq t] = \int_0^t \left[\frac{f(t_0+u)}{1-F(t_0)}\right]\left[\frac{f(t-u+t_1)}{1-F(t_1)}\right] du$$

where $f(\cdot)$ denotes the density of $F(\cdot)$, assumed to exist for this illustration.

Several approximations may be adequate for determination of the reorder quantity in certain situations. One of them is in current use, and the others warrant investigation also.

1) Assume that every engine lasts its expected residual life, and compute the corresponding $N(t)$. If the engine ages and the sequence of replacements are known, then $N(t)$ is determined. This method is now used with some modifications.

2) If the flying program represented by t is small relative to the residual life expected for the currently installed engine, an adequate approximation to the distribution of N(t) may be obtained by numerical convolution for the first few values of P[N(t)=n].

3) If the value of t is large relative to the residual lives of the first engine and its replacements, the central limit theorem for approximately identically distributed random variables (Gnedenko [12] chapter 8) may yield an adequate normal approximation for the partial sums of residual lives.

4) If the distribution of removal times F(t) is sufficiently similar to the exponential and geometric distributions (4), it may be practically memoryless. The replacement process then becomes a renewal process. Analysis of exponential and geometric renewal processes is essentially complete.

5) If instead of using the known accumulated flying hours for each replacement engine, the equilibrium age distribution is assumed for each replacement, then the generating function of the Laplace transform of the distribution of N(t)

$$G(z,\psi) = \sum_{n=0}^{s} z^n \int_0^\infty e^{-\psi t} P[N(t)=n]dt$$

for $\Psi \geq 0$, $0<z<1$ may be found. The formula for $G(z,\Psi)$ is given by Cox [10] on page 38. If this yields an adequate approximation to the distribution of $N(t)$, then it may not be necessary to incorporate the actual ages into prediction of $N(t)$. However there is a notorious problem of inverting the generating function of a Laplace transform to obtain the distribution function of a random process.

6) Simulation of values of $N(t)$ can be done given the distribution $F(t)$ of removal times, ages t_0, $t_1, \ldots, t_s$, the residual life distribution (5), and the relations (3). This method is strongly recommended. It will be described in part 3 of this section.

Given the distributions of $N(t)$ conditioned on all possible sequences of replacements $t_1, \ldots, t_s$, the distribution of $N(t)$ for a random replacement policy may be determined. An ingredient necessary for calculating the probability of every possible replacement sequence is the age distribution of the inventory after a specified number of replacements have occurred. Each replacement is a random selection from the inventory with age distribution which is given next.

Let $D_0(x;t)$ be a step function that tells the proportion of engines initially in stock with accumulated flying hours less than or equal to t, $x=0,1,2,\ldots,s$, $t \leq 0$. $D_1(x;t)$ gives the same information after the first replacement has been made, except that $x=x(t)$ is the value of a random function of t because it is not known which engine is chosen from stock.

$D_y(x;t)$ is the probability function after the yth replacement $y=0,1,\ldots,s<\infty$.

Since the original inventory of engines had accumulated flying hours $t_1, t_2,\ldots,t_s$, the probability function $D_y(x;t)$ will change only at those times. Let t_x represent the time value $t_x \varepsilon [t_i;$ $i=1,2,\ldots,s]$ for which exactly x engines in the original stock had t_x or fewer accumulated flying hours. Then $D_y(x,t)$ can be represented in binomial form

$$D_y(x-z;\ t_x) = \binom{z}{y} \left(\frac{x}{s}\right)^z \left(1-\frac{x}{s}\right)^{y-z}$$

for $t_x \varepsilon \{t_i;\ i=1,2,\ldots,s\}$ and $z=0,1,2,\ldots,\min(x,y)$. A few examples may be sufficient to convince the skeptic. Let $t_1 \leq t_2 \leq \ldots \leq t_s$ and examine the function $D_1(x;t)$ for the engine age distribution after the first replacement is installed:

$D_1(0;t_1) = \frac{1}{s}$ is the probability the newest engine was installed,

$D_1(1;t_1) = 1-\frac{1}{s}$ if the newest engine was not installed;

all other $D_1(x;\ t_1)$ are zero. The general form of $D_1(x;t)$ is:

$D_1(x-1;\ t_x) = x/s$ if an engine of age $\leq t_x$ was chosen as replacement, and

$D_1(x;\ t_x) = 1-x/s$ if not.

Additional combinatorial work will be necessary to obtain the distribution of the number of replacements for a fleet with more than one engine in operational status supplied from the same inventory of spares. The formidable nature of the problem is sufficient reason to suggest simulation.

3. SIMULATION OF THE ENGINE REPLACEMENT SYSTEM

It is necessary to simulate the engine replacement system in order to obtain some information about the distribution of the number of replacements for a specified flying program. In the absence of any solution to the problem formulated in part 2 of this section, simulation will provide data for comparison with approximations and bounds, and it will also provide a prediction method that can be incorporated into engine management procedures.

The first flow chart, Figure 3, outlines a program to simulate operation of a single-engine plane for a specified number of flying hours and a specified sequence of replacement engines with known accumulated operating times. Incorporating random selection from the stock of replacement engines is easily done. Simulation of a multi-engine fleet is next illustrated. It is assumed that each operating engine must fly an equal part of the flying program, but this assumption is easily removed. The flying hour program for each engine type is given on a quarterly basis. For long range planning, a simulation is necessary for many quarters. The second flow chart, Figure 4, describes an event by event type of simulation that is being incorporated into a master program by L. A. Coco of AFLC/MMOMA.

The program in Figure 1 requires as input accumulated flying times for the installed engine t_o and for all spares $t_1, \ldots, t_s$, the removal time c. d. f. $F(t)$, the flight plan D, and the run size K. Each cycle simulates the

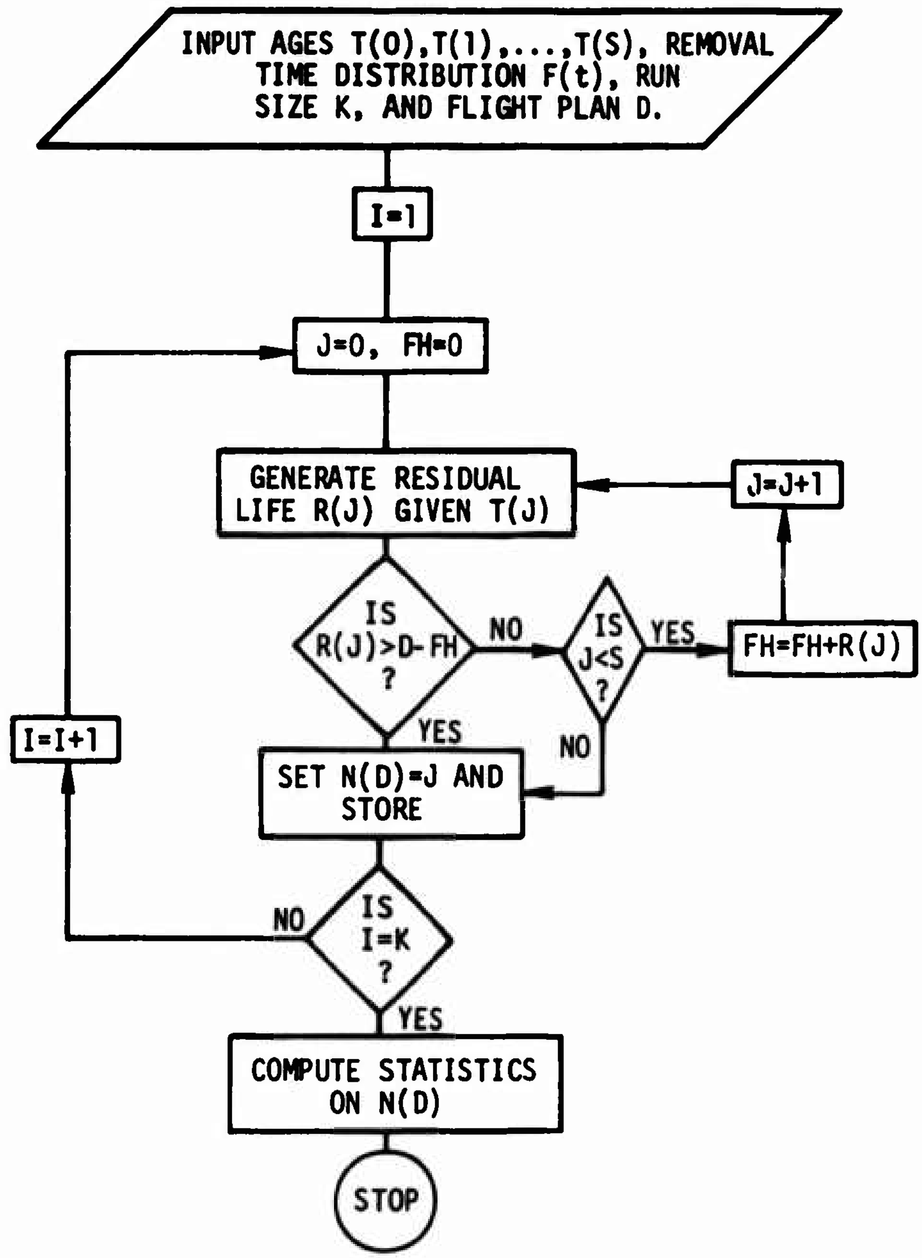

Figure 3. Simulation Flowchart of the Single-Engine Replacement Problem

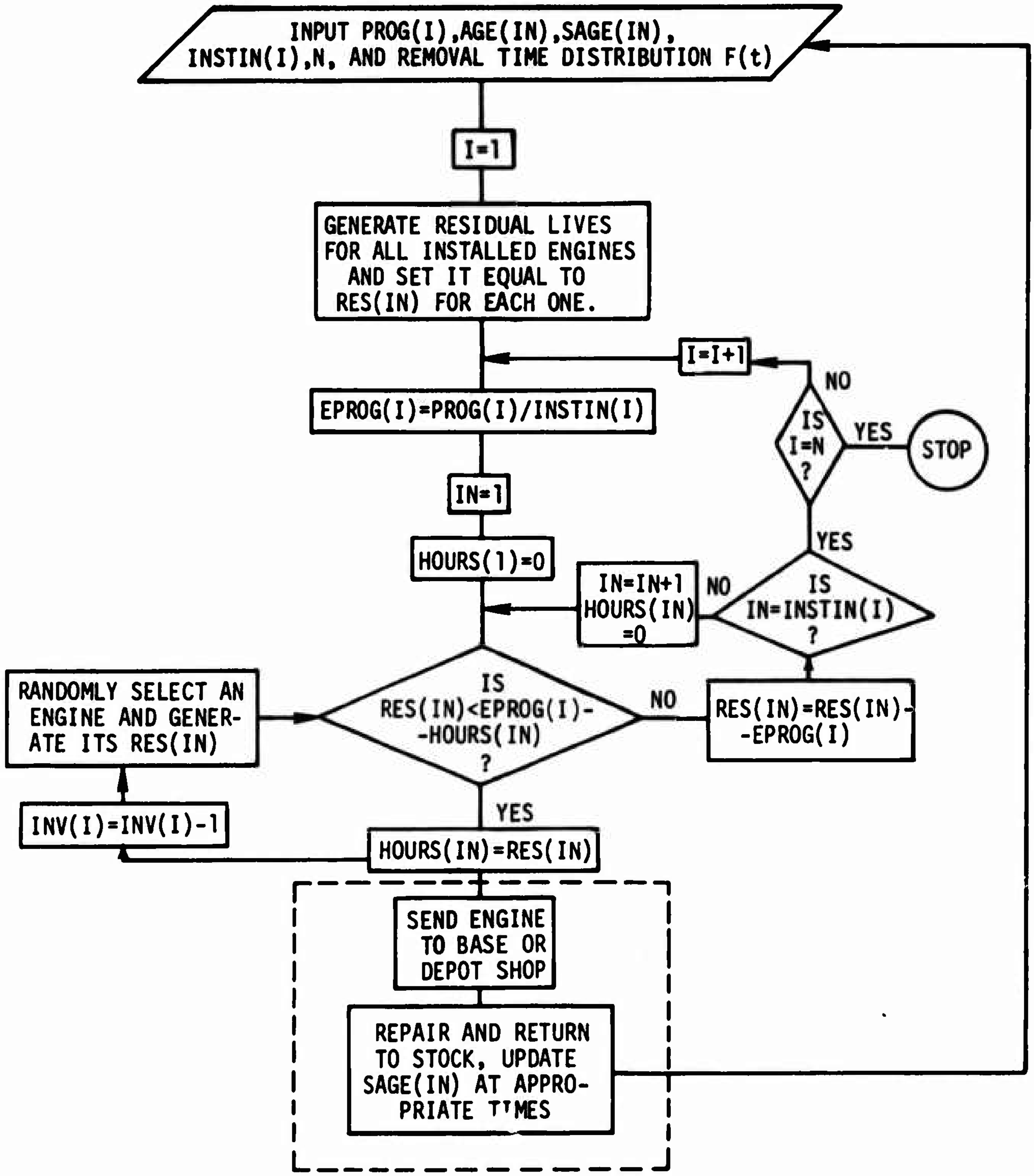

Figure 4. Simulation of Fleet Operation

flying program and generates a value, N(D), for the number of replacements required to achieve the flying program. Residual lives $R(t_i)$ are generated from the distribution(5) by any method, (see Fishman [13] chapter 8). Statistics computed on the output values of N(D) depend on the application, but there are standard computer programs for most needs. If desired, flight plan D and ages t_0, $t_1, \ldots, t_s$ could be randomly generated as part of the program. A random choice of replacement engines from stock is easily incorporated.

A more complicated program is required to simulate multi-period operation of a multi-engine fleet. Flexibility is provided to allow incorporation of a different flight plan and changing inventory and age distribution for each period. Each cycle through the program generates the number of replacements required to satisfy the flight plan for each period. The flow-chart in Figure 4 shows just one such cycle. The master program which will use Figure 4 generates transfers to and from the inventory and ages of engines put into inventory.

Variables used in the program are defined as follows:

N $\equiv$ the number of quarters to be simulated;

PROG(I) $\equiv$ flying program for quarter, I=1,2,...,N;

AGE(IN) $\equiv$ age of installed engine, IN = 1,2,..., total numbers of engines installed;

SAGE(JN) $\equiv$ age of spare engine, JN = 1,2,..., upper bound on the number of spares available;

RES(IN) $\equiv$ residual life of an installed engine after completion of quarterly flying program;

EPROG(I) ≡ flying hour program for each engine and quarter;

RES ≡ residual life of an engine (RES + AGE(IN) tells the total number of flying hours on an engine before removal);

INSTIN(I) ≡ the number of installed engines in quarter I;

INV(I) ≡ the number of spares in quarter I.

The routine in the dotted box is provided by the master program. A varying maximum operating time for engines may be incorporated into the program.

The event by event type of simulation will save execution time compared with the interval by interval simulation of each engine now used in the master program. In that program, each quarter is divided into many equal size intervals and each engine is simulated to see whether it survives successive intervals. While this method conveniently incorporates actuarial failure rates, it is not necessary. Survival probabilities obtained by the actuarial method may be converted into the c. d. f. F(t) required as input in Figure 4. An alternative is to use the estimator derived in Section V.

4. APPLICATION OF ANTITHETIC VARIATES TO THE REPLACEMENT PROBLEM SIMULATION

The complicated program required to simulate engine management systems may strain the capacity of available computers. One method to reduce variability of output for a fixed run size or to reduce run size for fixed level of variability is the method of antithetic variates (Hammersley and Handscomb [14]). This method has long been known in statistical sample design but has received little use in simulation. The method will be illustrated in simulation of the mean value of random variable. Application of the technique to simulation of the replacement problem of part 3 is easy to do. It has been tested for a renewal and a replacement process with uniform and exponential removal times with the same mean. In most tests, anitthetic variates gave a reduction in the variance of the mean number of removals. An outline of a proof is given that this is true for all distributions when simulating the renewal process and the replacement process.

The simple program flowchart in Figure 5 is designed to estimate the expected value of a random variable with c.d.f. $F(x)$. The sample mean $\bar{X} = \sum_{i=1}^{N} X_i/N$ is an unbiased, consistent, and otherwise well behaved estimator of the expected value of a random variable. Its variance is

$$\text{Var } \bar{X} = \left\{ \sum_{i=1}^{N} \text{Var } X_i + 2 \sum_{i<j} \text{Cov } (X_i, X_j) \right\} / N^2 \qquad (8)$$

The covariance term enters into the calculation if the random variables $X_i, \ldots, X_N$ are dependent. The "trick" of the antithetic variate method is to generate the sequence so that $\text{Cov } (X_i, X_j) \leq 0$ for all i and j. In some cases, exponential and uniform, the use of negatively correlated random number sequences for generation of $X_1, \ldots, X_N$ will yield negatively correlated X_i (Fishman [13] p. 320).

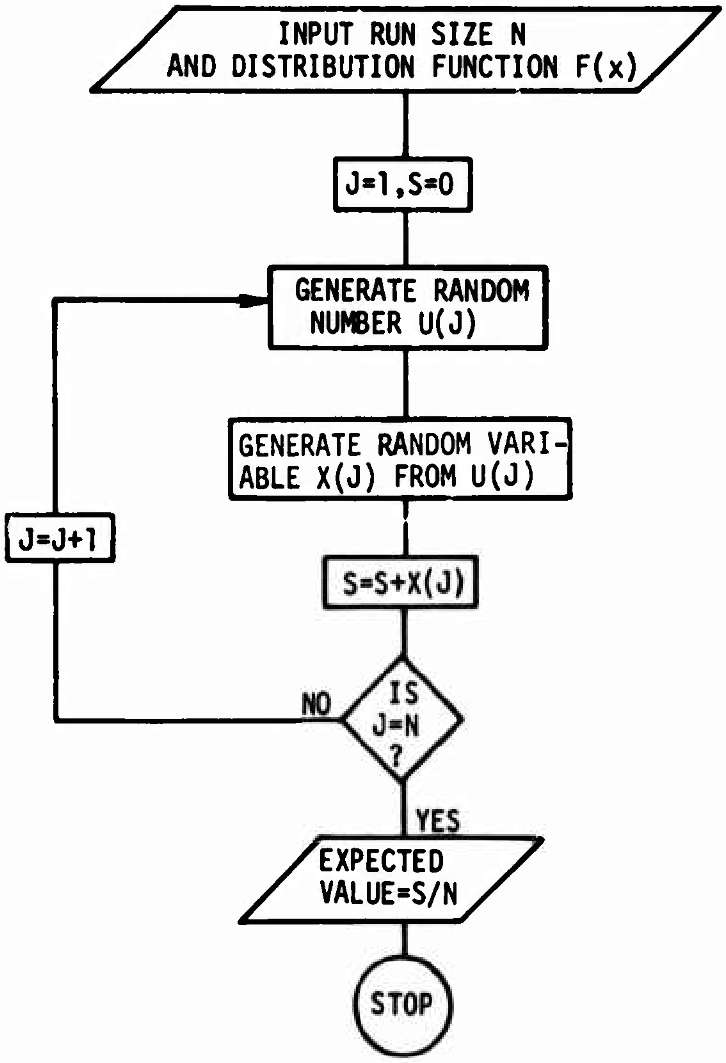

Figure 5. Simulation of the Expected Value of a Random Variable

Two methods for creating negative correlation among random numbers are to use the complement, 1-random number, and to generate half the required set of random numbers and reverse the sequence for use in the second half. The method of complements flowchart is given in Figure 6. Both methods were tested, and the former seems more promising for simulation of renewal and replacement processes. One reason for this preference is that many random numbers must be generated to give one value of the number of removals according to the relations (6). Reversing the sequence of random numbers used to generate the $R(t_i)$ in the next run may give independent values of $N(t)$ if n is small relative to the number of random numbers generated. For example, suppose a test program generates 50 random numbers for each run, $U(1)$, $U(2)$,...$U(50)$, which are used to generate 50 residual lives. If t is small, then n may also be small, say 5. On the second run, the sequence of random numbers will be used in reverse order, $U(50)$, $U(49)$,...,$U(1)$, and $N(t)$ will probably involve only the first few values of the reversed sequence, independent of $U(1)$, $U(2)$,...,$U(5)$.

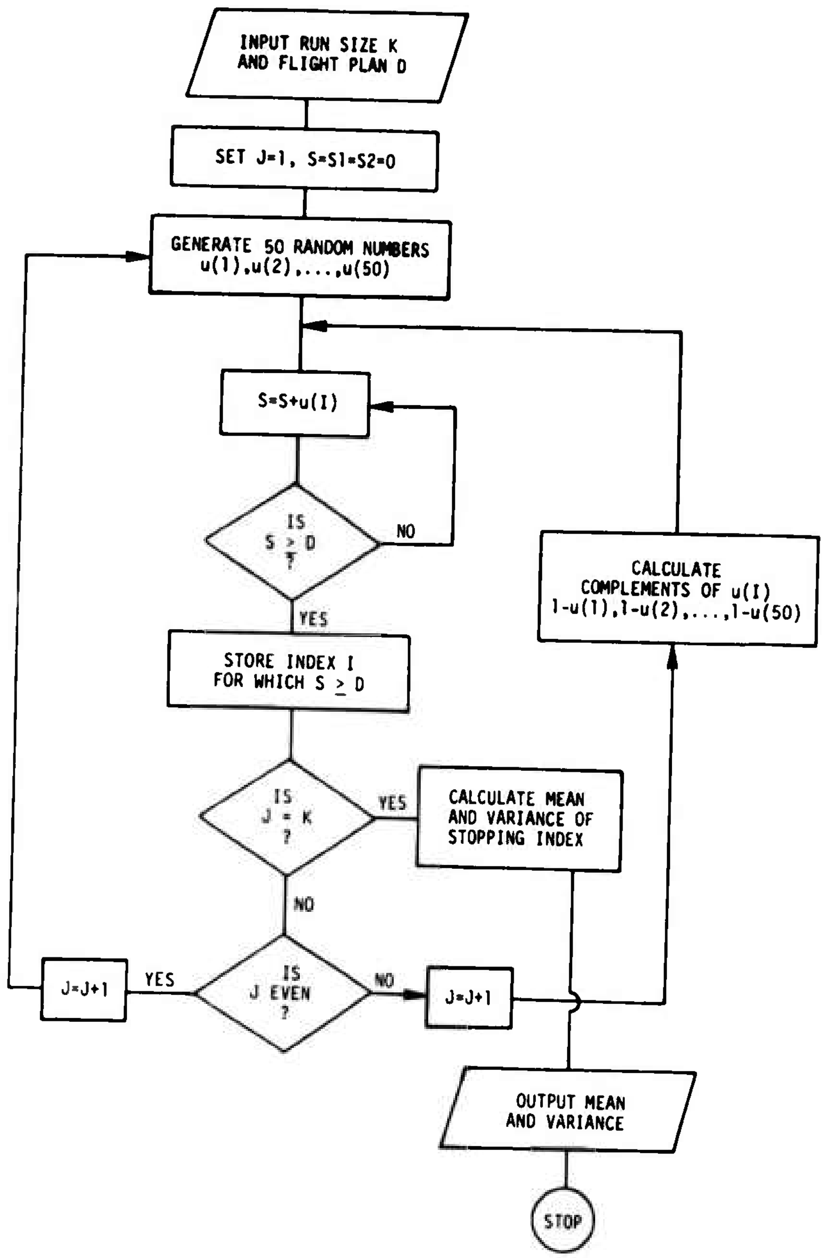

Figure 6. Simulation of a Uniformly Distributed Renewal Process with Antithetic Variates

The proof of the variance reduction property of complementary antithetic variates will only be sketched because a search is now under way for a more elegant proof. The proof outlined here applies the results of Mitchell, [11]. The statement to be proved is:

> The use of complimentary antithetic variates reduces the variance of the sample mean in the simulation of the mean value function $E\,N(t)$ of a renewal process or a replacement process.

The method of proof is to use Michell's propositions P1-P5 to show negative correlation between counting functions in antithetic simulations of a replacement process for a fixed length of time t.

The method of proof is to construct a sequence of random variables $\{N_j(t);\ j\geq 1\}$ for fixed t that satisfy Mitchell's propositions. The propositions are slightly restated here for brevity and for this application.

P1. $N_1(t) = 0$ with probability one.

P2. There exists a sequence of functions $\{f_n\}$ such that f_n maps the unit n dimensional hypercube into the non-negative real line.

P3. $N_j(t)$ is independent of U_i for $i\geq j$.

P4. f_n is non-increasing in U_i for all j, $1\leq j\leq n$.

P5. The stationary distribution of N_j exists.

Let X_i be a sequence of independent, identically distributed non-negative random variables generated from a c.d.f. $F(x)$ by inverse transformations of a sequence of uniform random variables $\{U_i\}$. Let $N_j(t)$ be the function that counts renewals in the following way

$$N_1(t) = 0$$

$$N_2(t) = \begin{cases} 0 & \text{if } X_1 > t \\ 1 & \text{if } X_1 \leq t \end{cases}$$

$$N_3(t) = \begin{cases} 0 & \text{if } X_1 > t \\ 1 & \text{if } X_1 \leq t \cap X_1 + X_2 > t \\ 2 & \text{if } X_1 + X_2 \leq t \end{cases}$$

$$N_4(t) = \begin{cases} 0 & \text{if } X_1 > t \\ 1 & \text{if } X_1 \leq t \cap X_1 + X_2 > t \\ 2 & \text{if } X_1 + X_2 \leq t \cap X_1 + X_2 + X_3 > t \\ 3 & \text{if } X_1 + X_2 + X_3 \leq t \end{cases}$$

etc.

By construction, the sequence $\{N_j(t)\}$ satisfies P1, P2, and P3. Since the X_j are generated by the inverse transformation $F^{-1}(U_j)$, non-decreasing in U_j, the $N_j(t)$ are non-increasing in U_i, $i \leq j$. In the context of the engine replacement process, longer engine lives (larger X_i) require fewer replacements $(N_j(t)$. Proposition P5, the existence of a stationarity distribution, is satisfied because $N_j(t)$ converges to the renewal or replacement counting function $N(t)$ as j increases. The $N_j(t)$ represent the number of replacements if there are only j spare engines, and $N(t)$ is the number of replacements if there are unlimited spares. Satisfaction of propositions P1 - P5 is sufficient for proof of the variance reduction property.

SECTION V

AN ANNOTATED BIBLIOGRAPHY OF SOME POTENTIALLY USEFUL REFERENCES

1. CRITERIA FOR INCLUSION IN THE BIBLIOGRAPHY.

The Articles and Research Reports described here were selected for their relevance to three levels of engine management.

1) Estimation of the distribution function of flying hours to engine replacement and the related problem of predicting the number of replacements required to meet the flying program. (These are the problems studied in Sections II, III, and IV.)

2) The logistics problems of inventory, repair, distribution, and replacement of engines of a particular type.

3) The Air Force wide engine management policies and planning.

General results and applications are balanced to give a perspective for the future. More recent results are cited unless there appears to be potentially useful work that has not been applied to engine management.

This bibliography is not claimed to be comprehensive. It does not include applications of traditional statistics to stochastic processes, and several potentially useful statistical techniques, such as time series analysis and spectral analysis, have been omitted. This is because the author feels that their potential use lies furthe[r i]n the future than the use of the articles cited here and the results of Sections II, III, and IV. Also deliberately omitted is the literature on system reliability under assumptions about the structure of the system or the underlying distribution function of lifetimes. These results are fairly well recognized, so there is little need for citation. On the other hand, this Bibliography includes articles from 1974 Department of Defense bibliographic searches on maintainability and applications of stochastic processes to engine management, so it is a fairly thorough indication of the state of the art familiar to the armed forces and deemed relevant by the author.

2. GENERAL REFERENCES

In this part are collected general references of potential use in engine management problems. Two specific problems were in mind during the selection of these articles, estimation of engine flying hours till removal and prediction of the number of replacements required to meet the flying program. However, the generality of the references means that other applications are possible, if not likely. First cited are articles on the estimation of stochastic processes under some assumptions about the processes. Second are articles that may collectively be described as curve fitting techniques with potential for applications to stochastic processes.

Consider one engine mounting position on an operational aircraft. A sequence of engine replacements occurs in that position, and data for each engine is recorded about the flying hours till removal and the removal cause. If all replacement engines are new, or as good as new, the stochastic process counting replacements as a function of flying hours may be a "renewal process". If replacement engines have already accumulated some flying time, the stochastic process will be referred to as a replacement process. There are several techniques for estimating characteristics of renewal processes that are potentially useful for replacement processes. The references focus on estimation from data on the number of replacements. This is in contrast to the actuarial technique and the estimator derived in Section II which estimate flying hours till replacement directly from flying hours data. It is possible to estimate the flying hours distribution from replacement counts, and collection of replacement counts is easier than collection of flying hours data.

A major figure in the development of statistics for renewal processes is D. R. Cox. His article "On the Estimation of the Intensity Function of a Stationary Point Process", 1965 [15], deserves citation because of its potential for application. The article provides an estimator of the renewal intensity function which can be integrated to obtain an estimate of the expected number of replacements for any number of flying hours.

The article, "The Determination of the Distribution Function of a Renewal Process from a Single Realization Observed with Some Error" by P. Bártfai, 1966 [16], is cited because of the interesting possibility of providing estimates from data that contains errors.

Two reference books follow the early article by Cox. First published was The Statistical Analysis of Series of Events by D. R. Cox and P. A. W. Lewis, 1966 [17]. Later, Lewis was editor of articles presented at a symposium. The title of that collection is Stochastic Point Processes, 1972 [18]. It contains some articles of statistical nature that may be useful in engine management. In "Multivariate Point Processes", 1972 [19], Cox and Lewis provide means for characterizing certain kinds of multivariate point processes and for estimating them. This last article may be useful if it is necessary to analyze the engine replacement process as a multivariate process. For instance such questions as the relation between calendar age and flying hours and the relation between flying hours and maintenance level require bivariate techniques.

Until this point in the bibliography, methods of estimation that make assumptions about the underlying distribution functions have been avoided. It has been observed that the effect of periodic inspection and other factors result in an engine flying hour distribution of sufficiently complicated nature to be excluded from commonly estimated families of distributions. However, there is another approach which may be of some use, and not without precedent in estimation of engine removal rates. That approach is to make some assumptions about the smoothness of the distribution function or its associated failure rate function. (One precedent is that the engine failure rate data now calculated is smoothed by moving average, extrapolated by linear regression, and then presented to the Aerospace Engine Life Committee in graphical form with data points connected by hand.) There is no functional relation between the smoothness assumptions and the underlying replacement process. It is a matter of satisfaction and convenience for calculation.

S. A. Krane presented the article "Analysis of Survival Data by Regression Techniques", 1963 [20], which described estimation of the integral of the failure rate function as a polynomial. The technique is for large samples and restricts the polynomial coefficients to be non-negative. L. J. Bain considers the same problem and hypothesis testing in "Analysis for the Linear Failure Rate Life Testing Distribution", 1974 [21].

Spline functions (segments of polynomials) may also be used to estimate the failure rate function or its integral. The article "Spline Functions in Data Analysis", 1974 [22], by S. Wold and the program "FITLOS, a FORTRAN Program for Fitting Low Order Polynomial Splines by the Method of Least Squares" by Patricia Smith, 1971 [23], will provide assistance in curve fitting. The use of spline functions is appropriate only when there is a large amount of data.

3. APPLICATIONS TO ENGINE MANAGEMENT PROBLEMS

The article "Classes of Distributions Applicable in Replacement with Renewal Theory Implications" by A. W. Marshall and Frank Proschan, 1972 [8], indicates the limitations of our knowledge about optimal replacement policies for items with life distributions in restricted classes. These limitations lead to the need for simulation of the engine management system to determine whether policies are satisfactory. (The simulation proposed in Section IV is one such attempt.) However, there have been several analyses of moderately complex machine-repair and logistics systems that may be of value as approximations for engine management planning and evaluation.

The thesis "The Statistical Analysis of Semi-Markov Processes with Applications to Queuing Problems" by T. L. Goyal, 1970 [24], is cited here because many of the subsequent citations are special cases of Semi-Markov processes. This thesis (and similar articles) provides the statistics to estimate parameters of models in the articles to follow.

The next three citations are related to replacement or renewal processes with inspection such as the engine replacement process. J. M. Cockerham, "Maintainability Analyses and Prediction Technique", 1967 [25], provides an analysis of an "on-off" process with several modes of "off" depending on whether an item was removed from service during operation or at an inspection. The results are limited to exponential lifetimes. G. L. Lind, "A General Procedure for Maintainability Prediction", 1968 [26], gives an estimation procedure for the life distribution for a system of components of which some fail in service and some have failure detected at the times of periodic inspection. As in the Cockerham article, the lifetime distribution of components is assumed to be exponential and the system is assumed to be parallel-series in construction, i.e. there are spares for some of the components. The article, "Adaptive Statistical Procedures in Reliability and Maintenance Problems" by J. L. Gastwirth and J. H. Venter, 1964 [27], is representative of research on inspection plans. The authors assume that the lifetime distribution is expontial and derive an adaptive procedure to estimate simultaneously the unknown parameter of the exponential distribution and the optimal inspection times. The adaptive feature is relevant for engine types just going into service for which there is little data. The article is also worthwhile because it indicates that inspection times as well as maximum operating times are subject to optimization as part of maintenance policies.

The three articles cited in the previous paragraph have application to the engines themselves rather than the supply, repair, and distribution system. As mentioned repeatedly in Section I, any policies regarding engine management must be integrated with policies for operation of the entire engine logistics system and with Air Force flying hour predictions. Articles cited next present a broader view of the engine management system.

The article, "A Decision Making Model for Applications of Queuing Theory" by R. T. Nelson, 1970 [28], is not the only article to recognize the value of queuing analyses is for decision making. However, it describes a queuing model within a decision structure that could be a simplified representation of future engine management systems.

"Generalized Maintainability Method (GMM)" by R. M. Fraser, 1971 [29], is an extremely detailed procedure that may be applied to the estimation of engine repair times.

Several branches of the Department of Defense have supported research on "multi-echelon" logistics systems. A multi-echelon system represents the engine repair system which has repair and inventories at bases and at a depot. The final two citations are the latest studies of multi-echelon systems.

The Aerospace Research Laboratories technical report by H. P. Galliher and R. C. Wilson", 1974 [30], attempts to justify the exponential assumption for the engine flying hour distribution. It also includes simulation of several variations on "METRIC" inventory policies. (METRIC, Multi-Echelon Technique for Recoverable Inventory Control, is a reorder policy that places an order whenever an item is removed from inventory.) This report is a study of the same logistics system constituting part of the engine management system described in Section I.

Last, but not least, is the thesis "A Queuing Theoretic Analysis of Logistics Repair Models with Spare Units" by Francois Lureau, 1974 [31]. It represents the state of analysis for METRIC type logistics systems. Most analyses assume events occur as Poisson processes, and this is not necessarily true. It should be indicated that even though engine flying hours till replacement may be approximately exponentially distributed, this does not mean that the calendar time till replacement has an exponential distribution. Remember that engine life is measured in flying hours and is related to calendar time only through the flying hour program, whereas all other activities in the engine management system take place in calendar time. Despite this warning, the analyses in the Lureau thesis provide approximate results, and the Poisson type METRIC logistics system analyses may be incorporated into simulation models by the "control variates" technique, Fishman [13], to sharpen the accuracy of engine logistics systems studies.

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