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# QUANTITY ADJUSTMENTS IN RESOURCE ALLOCATION: A STATISTICAL INTERPRETATION

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13. ABSTRACT

One method of successive approximations to a constrained optimum maintains feasibility while adjusting the decision variables along the gradient of the Lagrangian. Then the adjustments can be found as the residuals in the regression of the partial derivatives of the objective function on the partial derivatives of the constraint functions. Implications for decentralization are discussed.

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# QUANTITY ADJUSTMENTS IN RESOURCE ALLOCATION: A STATISTICAL INTERPRETATION

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# QUANTITY ADJUSTMENTS IN RESOURCE ALLOCATION:

# A STATISTICAL INTERPRETATION

Kenneth J. Arrow[1]

## 1. Introduction and Summary.

Resource allocation is part of the general theory of constrained optima. Any method of successive approximation seeks to approximate a solution of the Lagrangian conditions (if we ignore non-negativities and the possibility of slack in the constraints).

The following notation is used:

(N.1) $x$ is a column vector of $n$ decision variables;

$f(x)$ is the objective function, to be maximized;

$g(x)$ is a column vector function defining constraints, specifically, $g(x) = 0$.

$f_x$ is the gradient of $x$, the row vector with components $\partial f/\partial j$;

$g_x$ is the matrix of gradients of the constraint functions, with components $(\partial g_i/\partial x_j)$;

primes denote transpose.

Then the optimization problem is,

(1) maximize $f(x)$ subject to $g(x) = 0$.

If the matrix $g_x$ has full row rank, then the solution to (1) satisfies the Lagrangian conditions, namely, there exists a row vector $p$ such that,

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(2) $f_x + pg_x = 0$,

(3) $g(x) = 0$.

In the standard discussion of decentralized resource allocation, attention is concentrated upon adjustments in the Lagrange parameters, p. At each stage, an approximation to p is given. Then x is chosen to satisfy (2); this can be interpreted as choosing x to maximize,

(4) $L = f(x) + p\, g(x)$,

if f(x) and g(x) are assumed concave. However, unless p is already that associated with the constrained optimum, (3) will not be satisfied. The deviation of g(x) from 0 is used to guide changes in p. A specific adjustment process in differential equation form is suggested by interpreting $g_i(x)$ as the excess supply of primary factor i when the productive activities are determined by the decision variables x. Then we wish to lower $p_i$ if $g_i(x) > 0$ and raise it otherwise; specifically, the adjustment process might take the form,

(5) $\dot{p} = -\, g(x)$,

where the dot denotes differentiation with respect to time.

This process will in fact converge to the constrained optimum under suitable hypotheses, which we will not investigate here [2, 70-71, 84-85]. The idea is standard in the theory of market socialism. It is usually defended on the grounds that, not only does it converge, but it is also informationally economical. At each stage, the decision on x requires knowledge only of the gradients of f(x) and g(x) (which can be interpreted as marginal productivities and marginal input requirements). The decision to adjust p, in turn, requires only the simple reflection of the x-decision on resource limitations through $g(x)$.

Marglin [6] challenged the view that price adjustments have any unique virtues. He considered a very simple case, with one resource: decision variables were taken to be the allocations of the resource to different uses, so that,

(6) $g(x) = r - \sum_j x_j$,

where r is the total resource availability, and $\partial f/\partial x_j$ can be interpreted as the marginal productivity of the resource in its $j^{th}$ use. In the price adjustment process, satisfaction of (2) implies that all the marginal productivities are equal throughout the adjustment process. Marglin suggested instead that at each stage the allocation x be chosen so as to be feasible (to satisfy (3) ). Then, if the allocation is not optimal, (2) will not be satisfied. He suggested that each $x_j$ be adjusted so as to increase L, i.e.,

(7) $\dot{x} = L'_x$,

where L is defined by (4); in computing L as a function of x, p is to be so chosen that feasibility is maintained when x is adjusted in accordance with (7).

In his special case, Marglin argued that the proposed quantity adjustment system is guaranteed to converge and that the amount of information transmitted at each stage is comparable to that in the price adjustment system.

One interesting implication of the Marglin process is the adjustment equations can be stated in statistical terminology.

Specifically, (7) turns out to say that $x_j$ should be adjusted in proportion to the difference between the marginal productivity of the resource in its $j^{th}$ use and the average marginal productivity of the resource in all uses. Further, the rate of increase of the objective function is proportional to the variance of the marginal productivities, which, naturally, falls to zero when (2) is satisfied.

Do these conclusions generalize to the case of many resources? In particular, what is the generalization of the "statistical" interpretation of the Marglin process?

Actually, the notion of quantity adjustments had appeared earlier in studies of methods of approximating constrained optima; see Forsythe [4] and Arrow and Solow [3, Section 3]. Their interest lay rather in the fact that convergence was valid under less stringent conditions than in questions of informational economy. However, the results developed earlier can be reinterpreted to give rise to a generalized statistical interpretation.

Specifically, the tentative prices and the quantity adjustments in a quantity-adjustment process can be thought of as determined by a regression. Each "observation" is taken to correspond to one component of the decision vector. For the $j^{th}$ observation, the value of the dependent variable is taken to be $\partial f/\partial x_j$, while the value of the $i^{th}$ independent variable is $\partial g_i/\partial x_j$. I.e., given any tentative values for the decision variables, the marginal gains to the different decision variables are regressed against the marginal inputs. The regression coefficients can then be interpreted as the (tentative) prices, while the residuals in the regression are the rate of adjustment of the decision variables. Finally, the rate of growth of the objective function is precisely the square of the standard error of estimate multipled by the number of decision variables.

In section 2, the Marglin model is reviewed in the present language. In section 3, the generalization to any number of resources is given, and the results in the preceding paragraph proved. In section 4, some comments are made relating the quantity adjustment process to decentralization and informational economy.

## 2. The Marglin Quantity Adjustment Process.

We reexamine Marglin's model in somewhat more general form. He assumed that $f(x)$ was additively separable, an issue important for decentralization (see section 4 below) but notnecessary to his main results.

If $g(x)$ has the special form (6) and if we insist that the resource allocation be feasible at every moment of the adjustment process, i.e., that (3) hold throughout, then we are requiring that,

$$\sum_j x_j(t) \equiv r. \tag{8}$$

This condition will hold if and only if the following two statements are valid:

(9) $\sum_j x_j(0) = r;$

(10) $\sum_j \dot{x}_j(t) \equiv 0.$

From (6), the Lagrangian can be written,

(11) $L(x, p) = f(x) + p\,(r - \sum_j x_j),$

where p is now a scalar, so that,

$$\partial L/\partial x_j = (\partial f/\partial x_j) - p,$$

and the adjustment process for any component $x_j$ is defined by,

(12) $\dot{x}_j = (\partial f/\partial x_j) - p.$

To make sure that (10) holds, p has to be selected appropriately at any time t. Substitute (12) into (10), and solve for p.

(13) $p = \sum_j (\partial f/\partial x_j)/n,$

i.e., p is the average marginal productivity of the resource in all uses.

Then (12) asserts that the rate of change of the resource allocation to any use is the difference between its marginal productivity in that use and the average over all uses.

We will also compute the rate of growth of the objective function itself.

$$\dot{f} = \sum_j (\partial f/\partial x_j)[(\partial f/\partial x_j) - p] = n\, s^2,$$

where $s^2$ is the sample variance of the marginal productivities about their mean.

So long as the Lagrange condition (2) is not satisfied, the marginal productivities will not all be equal. Hence $s^2$ will be positive, and therefore so will $\dot{f}$. It is clear, then, that the process can only come into equilibrium at a point where (2) is satisfied as well as (3). Since the path is a path of resource allocations, it must be bounded and therefore must have a limit point. It is easy to see that $\dot{f} = 0$ at any limit point, and from this it can be shown that an adjustment path starting from any initial point which is feasible, i.e., satisfies (9), will converge to a point satisfying (2) and (3).

Remark: The adjustment process (7) is arbitrary with regard to the choice of adjustment speeds. The rate of change of any particular $x_j$ could be thought of as proportional to $\partial L/\partial x_j$, rather than equal to it. However, in that case, a suitable change of units in measuring $x_j$ will restore the form given.

3. The General Case Without Non-negativity or Slack.

Let us revert to the general constrained maximization problem. We follow the discussion in [3, section 3] but reinterpret the results.

We now wish to require that (3), the feasibility condition, hold throughout the adjustment process and therefore as an identity in time.

(14) $g[x(t)] \equiv 0.$

(14) will hold for all t if and only if (a) it holds for t = 0, and (b) its derivative with respect to time is identically zero.

(15) $g[x(0)] = 0;$

(16) $d\ g[x(t)]/dt \equiv 0.$

By the chain rule, (16) becomes,

(17) $g_x \dot{x} \equiv 0.$

From (4), the definition of L,

$L_x = f_x + p\, g_x.$

Hence, the adjustment process for the resource allocation (7) is,

(18) $\dot{x} = f_x' + g_x' \, p'.$

The vector p is to be chosen, at any time t, so that (17) holds. Write (18) as,

(19) $f_x' = - g_x' \, p' + \dot{x}.$

We are, then, seeking a linear combination of the columns of a matrix, $- g_x'$, such that the difference between a given vector, $f_x'$, and the linear combination is orthogonal to every column of the given matrix (note that the rows of $g_x$ are the columns of $g_x'$). This is precisely the defining characteristic of the vector of regression coefficients estimated from a sample, where the columns of the matrix represent different independent variables and the given vector represents the dependent variable.

In more detail, let a regression of y be fitted to variables $z_1,\ldots,z_m$. Let $u_j$ be the residual in the $j^{th}$ observation. Then the linear regression model asserts that, for each j (=1,...,n),

$$y_j = \sum_i \beta_i \, z_{ji} + u_j,$$

where $\beta_i$ is the regression coefficient of $z_i$, $z_{ji}$ is the $j^{th}$ observation on the independent variable $z_i$, and $u_j$ is an error term. Let $b_i$ be the least squares estimate of $\beta_i$ and $v_j$ the $j^{th}$ estimated residual.

Then, by definition of estimated residual,

$$y_j = \sum_i b_i z_{ji} + v_j,$$

or, in matrix-vector notation,

(20) $y = Z\,b + v.$

The estimates b satisfy the normal equations,

$$Z'Z\,b = Z'y,$$

which can be written,

$$Z'(y-Zb) = 0,$$

or, from (20),

(21) $Z'\,v = 0.$

The analogy is now obvious. In (20) and (21) replace y by $f_x'$, Z by $-\,g_x'$, b by p', and v by $\dot{x}$; then (20) translates into (19) and (21) into (17) (after multiplying by -1).

Hence, at any stage t, there is an approximation, x(t), to the optimal allocation. At this value of the decision vector, compute the marginal benefit vector, $f_x'$, and the marginal input vectors for all inputs forming the matrix $-\,g_x'$. Take the regression, across decision variables, of marginal benefits on marginal inputs. The estimated regression coefficients are the approximation at stage t to the resource prices; the calculated residuals are the rates of adjustment of the individual decision variables.

Further, we can easily relate the rate of increase of the objective function to the standard error of the residuals. With the aid of (17) and (19), we have,

$$\dot{f} = f_x \dot{x} = (\dot{x}' - p\, g_x)\, \dot{x} = |\dot{x}|^2 - p\, g_x \dot{x} = |\dot{x}|^2 = n\, s_E^2,$$

where,

$$s_E = [(\sum_j \dot{x}_j^2)/n]^{1/2},$$

is the standard error of estimate (since the regression has no constant term, the deviations are taken from zero rather than from the sample mean).

As in the simple Marglin case, the objective function continues to increase so long as the regression does not fit perfectly. The path cannot come to an equilibrium unless the Lagrange conditions (2) are staisfied. Suppose the adjustment path is bounded. Then by standard use of Lyapunov's second method (see [5, pp. 7-9] or [1, Chapter 11, section 4]), with f(x) as the Lyapunov function, x(t) must converge to a limit at which condition (2) holds; (3) has been required to hold for all points on the path. Under suitably concavity conditions (or even quasi-concavity conditions), conditions (2-3) are sufficient as well as necessary for a constrained optimization.

When will the adjustment path be bounded? Let,

$$F = \{ x \mid f(x) \geqq f[x(0)] \}.$$

Since f[x(t)] is increasing x(t) must belong to F for all t. Hence, the boundedness of F is sufficient for that of the path x(t).

Alternatively, it has been insured by construction that x(t) is feasible for all t. If the set of feasible resource allocations is bounded, then again the path must be bounded.

<u>Theorem</u>. Let $g_x$ have full row rank. Then the quantity adjustment process defined as a path x(t), p(t) satisfying the conditions,

(a) $g[x(t)] \equiv 0$,

(b) $\dot{x} = L_x'$,

where $L = f(x) + p\,g(x)$, is well defined if the initial point satisfies the condition $g[x(0)] = 0$. If, for each $x = x(t)$, the regression across decision variables of the components of the gradient of f on the corresponding components of the gradients of the constraint functions $g_i(x)$ $(i=1,\ldots,m)$ is taken, then the estimated regression coefficients are the components of p(t), and the estimated residuals are the components of $\dot{x}$. If $s_E$ is the standard error of estimate (about zero), then $\dot{f} = n\,s_E^2$.

If either the set $\{x \quad f(x) \geq f[x(0)]\}$ or the feasible set, $\{x \quad g(x) = 0\}$, is bounded, then the path converges to a point that satisfies the Lagrangian condition, $L_x = 0$, as well as the feasibility condition, $g(x) = 0$.

4. <u>Observations on Decentralization, Information, and Computation.</u>

Let us take the case most favorable to the possibility of decentralization, that in which both the objective function and the constraint functions are additively separable, i.e.,

(22) $f(x) = \sum_j f^j(x_j)$, $g(x) = \sum_j g^j(x_j)$.

Here, $x_j$ might be interpreted as an activity level, and, for given j, the functions $f^j(x_j)$ and $g_i^j(x_j)$ $(i=1,\ldots,m)$ define the final output and intermediate outputs (or inputs, with sign reversed) of a nonlinear activity. In that case, the information in the $j^{th}$ "observation", i.e., $\partial f/\partial x_j$ and $\partial g_i/\partial x_j$ $(i=1,...,m)$ is solely a function of $x_j$ and hence can be determined by the $j^{th}$ activity manager without other information. Therefore, the information can be transmitted to the central authority. Indeed, in some sense, the information transmitted is less expensive than the demands and supplies needed under a price adjustment mechanism, for the latter requires optimization and hence global knowledge by the activity manager, while the former requires only information on the production structure of the $j^{th}$ activity in the neighborhood of the present point.

Hence, from the information point of view, Marglin's thesis is valid in the more general case. The information to be transmitted by the activity managers is not greater and may even be less in the quantity adjustment process than in the price adjustment process.

But a different valuation must be made when we consider computing costs at the center. In the price adjustment model, all that is needed is aggregate excess demand; this is computed by simply adding up the excess demands of the individual activities. In the quantity adjustment model, per contra, the central authority has to fit a regression, a much more complicated operation. Indeed, it involves, among other steps, the inversion of a matrix whose order equals the number of resources. The Marglin model, which involves only one resource, thus gives an unrepresentatively favorable impression of the computational problem, since the regression estimation reduces to computing mean.

It should also be noted that any commodity which enters into the production of another commodity is a "resource" from this point of view; that is, the resources which are constrained include both primary resources and intermediate goods. Thus, the number of resources is apt to be almost the same as the number of commodities.

These cursory remarks do leave some issues unresolved. For example, if the production structure is marked by constant coefficients (as in a Leontief structure) then the inversion need only be done once, not repeated at each iteration. It is clear that we need a more sophisticated theory of computational and informational efficiency, in which _a priori_ knowledge of production and utility structures is used to reduce the need for calculation. But if we stick to the conventional rules for evaluating alternative optimal resource allocation mechanisms, in which the central authorities know no more of the activity structures than what is transmitted to them, the quantity adjustment process appears to be inferior in terms of the computational load on the center, though not in terms of the costs of information transmission.

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