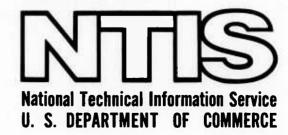AD-A011 674

ALMOST PERIODIC SOLUTIONS TO DIFFERENCE EQUATIONS

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Almost Periodic Solutions
to Difference Equations

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The theory of Massera and Schäffer relating the existence of unique almost periodic solutions of an inhomogeneous linear equation to an exponential dichotomy for the homogeneous equation has been completely extended to discretizations by a strongly stable difference scheme. In addition it has been shown that the

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almost periodic sequence solution will converge to the differential equation solution at a rate $O(k^p)$, where p is the accuracy of the scheme, uniformly in t, if the coefficients are sufficiently smooth.

The preceding theory has also been applied to a class of exponentially stable partial differential equations to which one can apply the Hille-Yoshida Theorem. It is possible to prove the existence of unique almost periodic solutions of the inhomogeneous equation which can be approximated by almost periodic sequences which are the solutions to appropriate discretizations. Two methods of discretizations are discussed; the strongly stable scheme described above and the Lax-Wendroff scheme.

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Acknowledgements

Abstract

The theory of Massera and Schäffer relating the existence of unique almost periodic solutions of an inhomogeneous linear equation to an exponential dichotomy for the homogeneous equation has been completely extended to discretizations by a strongly stable difference scheme. In addition it has been shown that the almost periodic sequence solution will converge to the differential equation solution at a rate $O(k^p)$ where p is the accuracy of the scheme, uniformly in t, if the coefficients are sufficiently smooth.

The preceding theory has also been applied to a class of exponentially stable partial differential equations to which one can apply the Hille-Yoshida Theorem. It is possible to prove the existence of unique almost periodic solutions of the inhomogeneous equation which can be approximated by almost periodic sequences which are the solutions to appropriate discretizations. Two methods of discretizations are discussed; the strongly stable scheme described above and the Lax-Wendroff scheme.

Introduction

This work extends certain facets of the theory of Massera and Schäffer [1] (in the future referred to as M&S) relating properties of the solution of an inhomogeneous ordinary differential equation (ODE) to properties of the solutions of the homogeneous system. In Part I we extend this theory to difference approximations of the ODE. In Part II we use the ODE theory, exclusively in the stable case, together with the Hille-Yoshida Theorem to obtain results for partial differential equations.

The study of admissibility theory and dichotomy theory for difference equations was first done by Coffman and Schäffer [2]. This work differs from their work in that we are concerned with the preservation of admissibility and dichotomy properties by a difference equation which is used to approximate an ODE system which has certain of these properties.

First let us review the theory of M&S. Consider the equations

$$(0.1) \qquad \dot{y} = A(t)y$$

$$(0.2) \qquad \dot{y} = A(t)y + f(t)$$

Here the independent variable t ranges over the whole real line and for each t the vector $y(t)$ lies in a Banach space E (which may be infinite dimensional). $A(t)$, for each fixed t, belongs to the space of bounded operators on E

-1-

which we will denote by L(E,E). Let B and D be two
Banach spaces of a function from the real line into E.
We say the pair (B,D) is admissible for (0.2) if

      a) $\forall f \in B$ $\exists$ a unique solution y (to (0.2)) $\in$ D, and
(0.3)

      b) $\|y\|_D \leq K \|f\|_B$.

This definition is more restrictive than that of M&S. We are
not going to deal with all the subtleties of their theory,
but just with certain important parts. It is pointed out,
however, that (0.3b) is actually superfluous (see M&S,
Chapter 5) but we include it in the definition of admissi-
bility for simplicity.

      Observe that (0.3b) states that if we write the y
given in (0.3a) as

(0.4) $y = C(f)$

then C, which is obviously a linear operator from B into D,
is also bounded. We will be concerned almost exclusively
with the case that $D = B = \tilde{A}(E)$ where $\tilde{A}$ is the space of
almost periodic (AP) functions with range in E. The range
in this notation will generally be omitted and we will simply
write $\tilde{A}$, etc., if no confusion might arise. At times we
will also be concerned with the cases $B = D = L_\infty$ or
$B = D = C_\infty$ which we define as the space of bounded continu-
ous functions with range in E. Observe that A and C_∞

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are closed subspaces of L_∞ and we have the inclusions

$$(0.5) \qquad \tilde{A} \subset C_\infty \subset L_\infty$$

$A(t)$ will always be an AP or L_∞ operator function unless stated otherwise.

Associated with the concept of admissibility we define the concept of an exponential dichotomy for (0.1). Specifically (0.1) has an exponential dichotomy if there exist projections P_1, $P_2 = I - P_1$ such that if $Y(t)$ is the fundamental operator solution to (0.1) ($Y(0) = I$, see M&S, Chapter 3), the following estimates hold for some $\alpha > 0$.

a) $\qquad \| Y(t) P_1 Y^{-1}(s) \| \leq K e^{-\alpha(t-s)}$, $\qquad t \leq s$

(0.6)

b) $\qquad \| Y(t) P_2 Y^{-1}(s) \| \leq K e^{-\alpha(s-t)}$, $\qquad s \geq t$.

Note that this is not the general definition of an exponential dichotomy given in M&S (Chapter 8) but it is equivalent to their definition when E is finite dimensional. In the case that E is infinite dimensional we will in general only be concerned with the stable case; that is, $P_1 = I$, $P_2 = 0$.

Observe that P_1 is merely the projection on E_+, the space of initial data of solutions to (0.1) which are bounded for $t \in [0, \infty)$. To see this note that if y_0 were in the range of P_2 and if the corresponding solution $y(s)$ were bounded for $s \geq 0$ then, by setting $t = 0$ in (0.6b), we obtain

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$$(0.7) \qquad y_0 = Y^{-1}(s)y(s) = P_2 Y^{-1}(s)y(s)$$

and $y_0 = 0$ follows from letting $s \to \infty$. Similarly P_2 is just the projection on E_-, the space of initial data which are bounded for $t \in (-\infty, 0]$. An exponential dichotomy merely states that $E = E_+ \oplus E_-$ and that the solutions, in addition to being bounded, decay exponentially.

Now it is a fundamental result of M&S that if $A(t) \in \tilde{A}$ then $(\tilde{A}, \tilde{A})$ is admissible for (0.2) iff (0.1) has an exponential dichotomy. The preceding statement is also true if $\tilde{A}$ is replaced everywhere by L_∞. We prove here the easy part of the result; namely if we have an exponential dichotomy, then we have admissibility. If $f \in L_\infty$ define

$$(0.8) \qquad y(t) = C(f) = \int_{-\infty}^{\infty} G(t,s)\ f(s)\ ds$$

where

$$(0.9) \qquad \begin{aligned} G(t,s) &= Y(t)P_1 Y^{-1}(s)\ , & t \geq s \\ &= -Y(t)P_2 Y^{-1}(s)\ , & s \geq t\ . \end{aligned}$$

Observe that the estimate

$$(0.10) \qquad \| G(t,s) \| \leq Ke^{-\alpha|t-s|}$$

holds by the definition of an exponential dichotomy.

Using (0.10) we see that the integral in (0.8) exists, that y satisfies (0.2), and that y is the unique L_∞ solution to (0.2) and in fact

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(0.11) $$\|y\|_\infty \leq \frac{2K}{\alpha} \|f\|_\infty.$$

So that C, defined in (0.8), is a bounded operator from $L_\infty(E) \to L_\infty(E)$. To see that $C: \tilde{A} \to \tilde{A}$ we merely let T be a common ε-almost period for f and A. Then $w(t) = y(t+T) - y(t)$ is the unique L_∞ solution to (0.2) with inhomogeneous term

(0.12) $[A(t+T)-A(t)]y(t+T) + f(t+T) - f(t) = O(\varepsilon)$

We then use (0.11) to obtain

(0.13) $\|(y(t+T) - y(t))\|_\infty = O(\varepsilon)$.

The proof of the converse is more difficult and can be found in M&S, Chapter 10. A simplified proof valid only in the finite dimensional case is given in the Appendix.

In Section 1 we will define almost periodic sequences, which will be the type of solution we will be searching for. Section 2 is the most important of this work. Here, after discussing the properties of the strongly stable difference schemes we will be using, we will introduce a transformation which will separate out the roots of the scheme inside the unit circle and permit us to work with the Euler 1-step scheme. This technique was originally developed by Engquist [3] although the author was not aware of his work when the formulation given in Section 2 was developed. Engquist's results will be discussed more thoroughly at the end of Section 3.

In Section 3 we settle the question of admissibility when the homogeneous system is exponentially stable. Using an inequality which is an exact discrete analogue to the Gronwall inequality, we will show that the homogeneous difference equation is also exponentially stable. From there it will be a simple matter to obtain, for any sufficiently small time step k, the existence of an almost periodic sequence as a solution to the inhomogeneous difference equation. Furthermore the sequences converge uniformly to the unique AP solution to the ODE with a uniform error $O(k^p)$ where p is the order of accuracy of the scheme. We will also show that this solution can in fact be calculated, i.e. it is stable under roundoff errors and errors in initial data.

In Sections 4 and 5 we deal with the case that the homogeneous system has a general exponential dichotomy. We will show that the corresponding inhomogeneous difference equation also has an exponential dichotomy. This is only of theoretical interest as the solution will no longer be stable under roundoff errors or errors in initial data. In Section 6 we will deal with some miscellaneous topics, especially the convergence of the mean value of the AP sequence to the mean value of the AP solution, and also the weakly nonlinear case.

In Part II we extend this theory to a simple class of partial differential equations which can be written as an evolution equation

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$$(0.14) \qquad \dot{y} = [B-\delta]y + f$$

where B is an unbounded operator which satisfies the
conditions of the Hille-Yoshida Theorem and $\delta(t)$ is an
AP function such that the homogeneous system is exponentially
stable. We can obtain a unique AP solution to (0.14) by
using the formula which would be valid if B were bounded
and then showing that under mild restrictions on f the
resultant function does in fact satisfy (0.14).

In Section 8 we introduce a family of bounded operators
B_h which are spatial discretizations to B. We construct
functions y_h which are the unique AP solutions to

$$(0.15) \qquad \dot{y}_h = [B_h-\delta]y_h + f \ .$$

We will give conditions to insure that

$$(0.16) \qquad \| (y-y_h) \|_\infty = O(h^j) \ ,$$

where j is the order of the approximation of B_h to B.
In Section 9 we apply the theory of Part I to (0.15) to
obtain an AP sequence $y_{n;h}$ which approximates y_h.
We have however the unfortunate restriction

$$(0.17) \qquad \frac{k}{h^{2m}} = O(1)$$

where $\|B_h\| = O(1/h^m)$. A more favorable result

$$(0.18) \qquad \frac{k}{h^m} = O(1)$$

is obtained in Section 10 when using the Lax-Wendroff scheme.

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In Section 11 we will consider the extension of these results in the case that the operator B is perturbed by some bounded AP perturbation D(t).

Finally in the Appendix we will give a proof of the basic ODE theorem that admissibility is equivalent to the existence of an exponential dichotomy. The proof is valid only in the finite dimensional case but is simpler than the proof given in M&S and is also simpler than a finite dimensional proof to be found in Coppel [4].

1. Almost Periodic Sequences

Our first task is to introduce the discrete analogue of an AP function. Following Corduneanu [5] we define an almost periodic (AP) sequence a_n with range in E as follows:

a_n is AP iff given $\varepsilon > 0$ there exists a length $L(\varepsilon)$ (a positive integer) such that in any sequence of L consecutive integers there exists an N such that

$$(1.1) \qquad \qquad \| a_{n+N} - a_n \|_\infty < \varepsilon$$

where the sup in (1.1) is taken over n. As shown in Corduneanu (page 45) this is equivalent to normality i.e. given any sequence of integers N_i the sequence $b_{n;i} = a_{n+N_i}$ will have a uniformly convergent subsequence. Although the proof given in Corduneanu is stated for scalar valued sequences this proof is obviously valid if the range is any Banach space E.

If we define the space $L_\infty^n(E)$ as the Banach space of bounded sequences with range in E then the AP sequences $\tilde{A}^n(E)$ form a closed subspace of $L_\infty^n(E)$. As usual the argument E will be omitted when no confusion can arise.

We point out that normality can be used, exactly as in the continuous case, to show that for any finite set of AP sequences $a_n^1, \ldots, a_n^r$ with range in possibly different spaces $E^1, \ldots, E^r$ and for any $\varepsilon > 0$, there is always a length $L(\varepsilon)$

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such that in any interval of length L we can find a common ϵ-almost period.

Finally we note that if $f(t)$ is an AP function then the sequence $f_n = f(nk)$ is an AP sequence for any real k. The converse is also true as shown by Corduneanu (page 47) but we shall not use that.

2. Properties of the Difference Scheme

We consider linear ℓ-step, strongly stable schemes described as follows.

$$(2.1) \qquad \sum_{j=0}^{\ell} \alpha_j y_{n+j} = k \sum_{j=0}^{\ell} \beta_j \dot{y}_{n+j} = k \sum_{j=0}^{\ell} \beta_j [A_{n+j} y_{n+j} + f_{n+j}]$$

Here we assume that we are discretizing the ODE

$$(2.2) \qquad \dot{y} = A(t)y + f(t)$$

and $y_n = y(nk)$, $A_n = A(nk)$, $f_n = f(nk)$ where k is the time step.

Associated with (2.1) we have the polynomials

$$(2.3) \qquad p(x) = \sum_{j=0}^{\ell} \alpha_j x^j , \qquad \sigma(x) = \sum_{j=0}^{\ell} \beta_j x^j .$$

It is well known (see Dählquist [6] or Heinrici [7]) that consistency implies that $x = 1$ is a simple root of $p(x) = 0$ and that

$$(2.4) \qquad p'(1) = \sigma(1) = 1$$

where we have normalized the coefficients so that the common value in (2.4) is one.

Strong stability of the scheme is achieved by restricting the size of the other ℓ-1 roots of $p(x) = 0$. Specifically, if we number these roots x_u , $u = 1,\ldots,\ell,$ then we require

that there be a positive number $\theta < 1$, such that

$$(2.5) \qquad x_1 = 1 , \qquad |x_u| < \theta < 1 , \qquad u = 2,\dots,\ell .$$

Finally, we make two further assumptions.

$$(2.6) \qquad\qquad x_u \text{ distinct} , \qquad u = 2,\dots,\ell ,$$

$$(2.7) \qquad\qquad x_u \neq 0 , \qquad u = 2,\dots,\ell .$$

(2.7) is necessary because to get an AP solution, the difference equation should be solved backwards and forwards. This assumption can be removed in the important case that the homogeneous ODE system is exponentially stable. Condition (2.6) can be removed in all cases and is included here only to simplify the following proofs. The removal of these conditions will be discussed in Section 6.

Now in working with a multistep scheme, the standard procedure is to convert it into a one-step scheme. To do this we define the space $E_\ell = E \times E \times \dots \times E$. (We will usually write vectors in E_ℓ in column vector form.) We give E_ℓ the norm inherited from this definition, namely if $w \in E_\ell$ and

$$(2.8) \qquad\qquad w = \begin{pmatrix} y^1 \\ \vdots \\ y^\ell \end{pmatrix}$$

then

$$(2.9) \qquad\qquad \| w \|^\ell = \max_i \| y^i \|_E .$$

Here we have explicitly indicated the E norm in (2.9).

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We now consider the discretization of the linear inhomogeneous system (2.2). If $w_n = \begin{pmatrix} y_{n+\ell-1} \\ \vdots \\ y_n \end{pmatrix}$ then we get

(2.10) $$w_{n+1} = C_n w_n + k \tilde{f}_n .$$

Here if we define $z_n = k A_n$,

(2.11) $C_n = \bar{C}(z_n, \ldots, z_{n+\ell}) =$

$$(\alpha_\ell I - \beta_\ell z_{n+\ell})^{-1} (\beta_{\ell-1} z_{n+\ell-1} - \alpha_{\ell-1} I) \cdots (\alpha_\ell I - \beta_\ell z_{n+\ell})^{-1} (\beta_0 z_n - \alpha_0 I)$$

$$\begin{pmatrix} I & 0 & \cdots & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I & \ddots & \vdots & & & & \vdots \\ \vdots & & \ddots & I & & & & \vdots \\ 0 & & & & \cdot & & & 0 \\ \vdots & & & & & \cdot & & \vdots \\ 0 & \cdots & 0 & \cdots & & 0 & I & 0 \end{pmatrix}$$

$$\tilde{f}_n = \begin{pmatrix} (\alpha_\ell I - \beta_\ell z_{n+\ell})^{-1} \sum_{j=0}^{\ell} \beta_j f_{n+j} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Observe that $C_n \in L(E_\ell, E_\ell)$ and will be just an $m\ell \times m\ell$ matrix in the case that E is an m-dimensional space (in which case it is called the companion matrix). Note also that since

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A is uniformly bounded in t, the implicit term $(\alpha_\ell I - \beta_\ell z_{n+\ell})$ can be inverted, for k sufficiently small, uniformly in n. We finally point out that if A and f are AP (L_∞) then C_n and $\tilde{f}_n$ are AP (L_∞^n) and also that $\bar{C}$ given in (2.11) is a smooth function (in the Frechet sense) of its $\ell+1$ arguments.

Now with A a constant and $z = kA \in L(E,E)$ we consider for small $\|z\|$ the homogeneous difference scheme

(2.12) $w_{n+1} = U(z)w_n$

$$U(z) = (\alpha_\ell I - \beta_\ell z)^{-1}(\beta_{\ell-1} z - \alpha_{\ell-1} I) \cdots (\alpha_\ell I - \beta_\ell z)^{-1}(\beta_0 z - \alpha_0 I)$$

$$\begin{pmatrix} I & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot \cdot & \cdot \cdot & \cdot & & \cdot \\ \cdot & & \cdot & I & 0 & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & I & 0 \end{pmatrix}$$

$U(z)$ is a mapping, defined for small $\|z\|$, from $L(E,E)$ into $L(E_\ell, E_\ell)$. Observe the following properties of $U(z)$.

(2.13) (a) $U(z)$ is a smooth (in the Frechet sense) function of z

   (b) $U(0)$ has eigenvalues exactly x_u with eigenspaces E_u, where

$$E_u = \left\{ \begin{pmatrix} x_u^{\ell-1} y \\ \vdots \\ x_u y \\ y \end{pmatrix} \right\}, \quad y \in E$$

   (c) $E = \bigoplus \sum_{u=1}^{\ell} E_u.$

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Now (a) and (b) should be clear. To see (c), note that the spaces E_u are closed and also that any two have only zero in common. Let $w = \begin{matrix} x_1 \\ \vdots \\ x_\ell \end{matrix}$ and suppose w has an expansion

(2.14)
$$w = \sum_{u=1}^{\ell} \begin{pmatrix} x_u^{\ell-1} y_u \\ \vdots \\ x_u y_u \\ y_u \end{pmatrix}$$

Then if we define the vector $\bar{w}$ to be $\begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix}$ we can write

(2.15)
$$w = V \bar{w}$$

$$V = \begin{matrix} x_1^{\ell-1} & \cdots & x_\ell^{\ell-1} \\ \vdots & & \vdots \\ x_1 & \cdots & x_\ell \\ 1 & \cdots & 1 \end{matrix}$$

i.e. V is just the Vandermonde matrix associated with the distinct numbers $x_1, \ldots, x_\ell$. (2.13c) now follows directly from the invertibility of V.

The representations (2.14) and (2.15) enable us to define an equivalent norm on the space E_ℓ. Specifically if w is expressed as in (2.14) we define

(2.16)
$$\|w\|^u = \max_u \|y_u\|_E$$

That $\|\ \|^u$ is equivalent to $\|\ \|^\ell$ follows immediately from the representation (2.15) and the invertibility of V. Note that the equivalence of these norms implies the equivalence of the operator norms they induce on $L(E_\ell, E_\ell)$. Of course

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this paragraph is superfluous in the case that E is finite
dimensional.

For future use we point out that (2.13b,c) imply
that $U^{-1}(z)$ exists for small $\|z\|$ and is smooth in z.

Now define the space $E^1 = \oplus \sum\limits_{u>1} E_u$. We have $E_\ell = E_1 \oplus E^1$
and we note that E_1 is canonically isomorphic to E. At times
we will identify E_1 with E but this should not cause any
confusion. With respect to the decomposition $E = E_1 \oplus E^1$ we
see that $U(0)$ is in block diagonal form; symbolically,

$$(2.17) \qquad U(0) = \begin{pmatrix} D(0) & 0 \\ 0 & B(0) \end{pmatrix}$$

here $D(0): E_1 \to E_1$ and is the identity, while $B(0): E^1 \to E^1$
and $\|B(0)\|^u < \theta$. This is the operator norm induced on
$L(E_\ell, E_\ell)$ by the $\|\ \|^u$ norm on E_ℓ and follows from the fact
that $B(0)$ is just multiplication by x_u on the space E_u.

We can now state the fundamental theorem of this section.

Theorem 1. For small $\|z\|$ there exists an operator
$T(z): L(E,E) \to L(E_\ell, E_\ell)$ such that

(a) $T(0) = I$ (Identity on E_ℓ)

(b) $T(z), T^{-1}(z)$ are smooth in z

(c) $L(z) = T^{-1}(z) U(z) T(z)$ is in block diagonal form
with respect to the decomposition $E_\ell = E_1 \oplus E^1$.

(d) Writing L(z) symbolically as

$$L(z) = \begin{pmatrix} D(z) & 0 \\ 0 & B(z) \end{pmatrix}$$

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then $B(z)\big|_{z=0} = B(0)$ (from (2.17)) and $\|B(z)\|^u < \theta$ while $D(z)$ has an expansion

$$(2.18) \qquad\qquad D(z) = I + z + O(\|z\|^2)$$

where E_1 is identified with E.

Theorem 1 is basically trivial and the proof involves familiar arguments. First we note that $U(z)$ is smoothly invertible for small $\|z\|$. (The restriction "for small $\|z\|$" will not be stated explicitly in the future.)

Next we observe that $(wI - U(z))^{-1}$ exists for w in a small annulus around the circle $|w| = \theta$ and this holds uniformly in z. This follows from (2.13b) and the geometric series.

Now define the projections

$$(2.19) \quad P(z) = \frac{1}{2\pi i} \int_{|w|=\theta} (wI-U(z))^{-1} \, dw , \quad Q(z) = I - P(z) .$$

That P and Q are projections is a familiar result which follows from the resolvent identity,

$$(w_1 I - U(z))^{-1} - (w_2 I - U(z))^{-1}$$
$$(2.20)$$
$$= (w_2 - w_1)(w_1 I - U(z))^{-1}(w_2 I - U(z))^{-1} ,$$

calculating P^2 by integrating around two slightly different circles, and interchanging the order of integration. It follows from the construction that $P(z)$ and $Q(z)$ commute with $U(z)$ and that they are smooth functions of z.

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It is clear by applying $P(0)$ to an arbitrary vector $w \in E_\ell$ and using the expansion given in (2.15) that $P(0)$ is exactly the projection onto E^1 along E_1 and $Q(0)$ is the projection onto E_1 along E^1.

Now define

$$(2.21) \qquad T(z) = P(z)P(0) + Q(z)Q(0) \ .$$

Clearly $T(0) = I$ and $T(z)$ is smooth whence $T^{-1}(z)$ exists and is also smooth. Note that this holds in either of the norms $\| \ \|^u$ or $\| \ \|^\ell$ on $L(E_\ell, E_\ell)$. Let

$$(2.22) \qquad L(z) = T^{-1}(z)U(z)T(z) \ .$$

We claim L is in block diagonal form with respect to the decomposition $E_\ell = E_1 \oplus E^1$. This follows directly from the fact that $P(z)$ and $Q(z)$ commute with $U(z)$. In fact let $x \in E_1$; then

$$(2.23) \quad U(z)T(z)x = U(z)Q(z)x = Q(z)U(z)x = w \quad \text{(say)}.$$

Now if $y = T^{-1}(z)w$ and $y = y_1 + y^1$ with $y_1 \in E_1$ and $y^1 \in E^1$ we have

$$(2.24) \quad w = Q(z)U(z)x = T(z)y = Q(z)y_1 + P(z)y^1$$

whence

$$(2.25) \qquad P(z)y^1 = T(z)y^1 = 0$$

and so $y^1 = 0$ by the invertibility of $T(z)$. Thus $L(z)$ maps $E_1 \to E_1$ and a similar argument shows $L(z): E^1 \to E^1$.

If we write (using the notation of (2.17)) $L(z)$ as

$$(2.26) \quad L(z) = \begin{matrix} D(z) & 0 \\ 0 & B(z) \end{matrix} \quad \begin{matrix} D(z): E_1 \to E_1 \\ B(z): E^1 \to E^1 \end{matrix}$$

then we see that $D(z)$ and $B(z)$ are smooth in z; $D(0) = I$ and $\|B(0)\|^u < \theta$. It follows immediately by continuity that

$$(2.27) \quad \|B(z)\|^u < \theta$$

for small $\|z\|$. Now (2.27) will imply that given $z_1, \ldots, z_N$ with $\|z_i\|$ small then

$$(2.28) \quad \| \prod_{i=1}^{N} B(z_i) \|^\ell < K \, \theta^N$$

for a constant K independent of the z_i and N.

It only remains to study the first order structure of $D(z)$ in order to obtain (2.18). Expanding $D(z)$ about $z = 0$ we can write

$$(2.29) \quad D(z) = I + C(z) + O(\|z\|^2)$$

where C is the Frechet derivative of $D(z)$ at $z = 0$ and is a bounded linear map from $L(E,E)$ into $L(E_1,E_1)$ which we identify with $L(E,E)$. It is necessary to show that C is the identity, and as one might expect this will follow very easily from consistency.

Let A be an arbitrary element in $L(E,E)$ and let $z = kA$, $A' = C(A)$. For $y \in E$ and identifying E_1 with E, we

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calculate the following limit

$$(2.30) \qquad \lim_{\substack{k \to 0 \\ nk = t}} D(kA)^n y = (I + kA' + O(k^2))^n y$$

for any fixed $t > 0$.

As one would expect this limit is $e^{tA'} y$ because the difference scheme expressed in (2.30) is consistent with the ODE $\dot{y} = A'y$.

More precisely since

$$(2.31) \qquad \| D(kA) \| \leq 1 + kR$$

for small k, where R is some fixed constant, it follows that

$$(2.32) \qquad \| D(kA)^N \| \leq e^{RT}, \qquad Nk \leq T, \quad 0 \leq t \leq T.$$

Let x_n solve the difference equation

$$(2.33) \qquad \begin{aligned} x_{n+1} &= D(kA) x_n \\ x_0 &= y \quad ; \end{aligned}$$

i.e. $x_n = D(kA)^n y$. Now if $\bar{x}_n = e^{A'kn} y$ then $\bar{x}_n$ satisfies the perturbed difference equation

$$(2.34) \qquad \bar{x}_{n+1} = [D(kA) + O(k^2)] \bar{x}_n$$

where the term $O(k^2)$ is uniform for $nk \leq T$. Equation (2.34) follows directly from the fact that $\bar{x}(t)$ satisfies the ODE

$$(2.35) \qquad \dot{\bar{x}} = A' \bar{x}.$$

Letting $w_n = \bar{x}_n - x_n$ we can show easily enough that

$$(2.36) \qquad \|w_n\| = O(k) , \qquad nk \le T .$$

In fact from (2.33) and (2.34) we obtain

$$(2.37) \qquad w_n = k^2 \sum_{j=1}^{n} D(kA)^{n-j} O(1) w_{j-1} .$$

If we use (2.32) to bound the powers of $D(kA)$ and thus $\|x_n\|$, we then obtain (2.36) and letting $k \to 0$ we obtain

$$(2.38) \qquad \lim_{\substack{n \to \infty \\ nk=t}} D(kA)^n y = e^{tA'} y$$

Recalling the identification of E_1 with E we define the vector $\bar{y} = \begin{pmatrix} y \\ \vdots \\ y \end{pmatrix}$ whence $\bar{y} \in E_1$.

Thus (2.38) can be expressed

$$(2.39) \qquad \lim_{\substack{k \to 0 \\ nk = t}} L(kA)^n \bar{y} = \begin{pmatrix} e^{A't} y \\ \vdots \\ e^{A't} y \end{pmatrix}$$

Now we are going to show that $A' = A$ by calculating this limit directly from the definition of $L(kA)$. In fact since $\|U(0)\|^u = 1$ it follows that $\|U(kA)^n\|^u$ is uniformly bounded for $nk \le T$ (and of course this is also true if we replace $\| \ \|^u$ by $\| \ \|^\ell$). Then writing

$$(2.40) \qquad L(kA)^n \bar{y} = T^{-1}(kA) U(kA)^n T(kA) \bar{y} ,$$

and letting $k \to 0$ such that $nk = t$, using the continuity of $T(z)$ and the fact that $T(0) = I$ together with the

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uniform boundedness of the powers $U(kA)^n$, we are left with

$$
(2.41) \qquad \lim_{\substack{k \to 0 \\ nk = t}} L(kA)^n \, \bar{y} \; = \; \lim_{\substack{k \to 0 \\ nk = t}} U(kA)^n \, \bar{y} \; = \; \begin{matrix} e^{At}y \\ \vdots \\ e^{At}y \end{matrix}
$$

the last equality follows from the fact that a consistent

and stable difference scheme is convergent.

Since A was arbitrary, (2.41) together with (2.39) shows

that $A = A'$, which proves that the map C is the identity

and this completes the proof of Theorem 1.

3. The Exponentially Stable Case

We now assume that the homogeneous system (0.1) is exponentially stable. Thus, if $Y(t)$ is the fundamental operator solution to (0.1) $(Y(0) = I)$, then the estimate

$$(3.1) \qquad \| Y(t) Y^{-1}(s) \| \leq K e^{-\alpha(t-s)} , \qquad t \geq s ,$$

will hold with positive constants K and α. In the sequel we will use K as a generic positive constant so it will appear in contexts other than (3.1).

We wish to prove a similar estimate for the homogeneous difference equation (see (2.10), (2.11))

$$(3.2) \qquad w_{n+1} = C_n w_n .$$

Now C_n is invertible for all n by (2.7). Then (3.2) has a fundamental solution W_n $(W_0 = I)$ which is simply

$$W_n = \prod_{i=0}^{n-1} C_i , \qquad n > 0 ,$$

$$(3.3)$$

$$W_{-n} = \prod_{i=-1}^{-n} C_i^{-1}, \qquad n > 0 .$$

It can easily be verified that the unique solution to (3.2), w_n, given initial data w_j is simply

$$(3.4) \qquad w_n = W_n W_j^{-1} w_j .$$

Note that for $n > j$, $W_n W_j^{-1}$ is simply

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$$(3.5) \qquad\qquad W_n W_j^{-1} = \prod_{i=j}^{n-1} C_i , \qquad\qquad n > j .$$

In the stable case we will only use $W_n W_j^{-1}$ for $n \geq j$ and we can see that the invertibility of C_i is not required in this case and assumption (2.7) is therefore not required here.

We intend to show that for small k (this will not be stated explicitly in the future) (3.1) implies

$$(3.6) \qquad\qquad \| W_n W_j^{-1} \| \leq K e^{-\alpha_1 k(n-j)} , \qquad\qquad n \geq j ,$$

where K is used as a generic constant and can be taken independently of k, while $\alpha_1 = \alpha + O(k)$. Thus since $\alpha > 0$, α_1 can also be taken independent of k for small k.

To attain (3.6) we suppose $A(t)$ is C^1. All derivatives that are assumed will always be required to be AP or L_∞. This will not be stated explicitly in the future. The smoothness condition on A will be removed in Section 6.

Expanding A_{n+j} about A_n we can write (3.2) as

$$(3.7) \qquad\qquad w_{n+1} = U(kA_n) w_n + k^2 O(1) w_n .$$

Here the $O(1)$ term is uniform in n and the operator U is given by (2.12). We are going to show that perturbations of the type $k^2 O(1)$ preserve exponential stability. We thus consider the unperturbed system

$$(3.8) \qquad\qquad \bar{w}_{n+1} = U(kA_n) \bar{w}_n$$

If we let $\bar{v}_n = T(kA_n)\bar{w}_n$, where the operator T is introduced from Theorem 1, and note that by the smoothness of T and A we have $T(kA_{n+1}) = T(kA_n) + k^2 O(1)$, we see that we should consider the difference equation

(3.9)
$$v_{n+1} = L(kA_n)v_n$$

where the operator L has been studied in Theorem 1.

If V_n denotes the fundamental solution to (3.9) then since L is block diagonal we see that V_n itself is also block diagonal. If we write V_n in the form

(3.10)
$$V_n = \begin{pmatrix} V_n^1 & 0 \\ 0 & V_n^2 \end{pmatrix}$$

we can see that the second part, the contribution from the roots inside the unit circle, causes no difficulty. In fact for $n > j$ we have

$$V_n^2 V_j^{2^{-1}} = \prod_{i=j}^{n-1} B(kA_i)$$

(3.11)

$$\| V_n^2 V_j^{2^{-1}} \| \leq K \, \theta^{n-j} \, ,$$

where we have used (2.28) and K as a generic constant. Since θ is a fixed number less than 1 we see immediately that (3.6) holds for V_n^2 where α_1 can in fact be taken equal to α for small k.

We must now deal with V_n^1. Neglecting perturbations of $k^2 O(1)$ and identifying the space E_1 with E we

consider the difference equation

$$(3.12) \qquad x_{n+1} = (I + kA_n)x_n$$

i.e. we have reduced the problem to the study of the Euler 1-step scheme.

We will compare X_n , the fundamental solution to (3.12), with Y_n $(= Y(nk))$ where Y is the fundamental solution to the homogeneous equation (0.1).

Let $H(t,s) = Y(t)Y^{-1}(s)$. As a function of t, H satisfies the equations

$$(3.13) \qquad \begin{aligned} \dot{H} &= A(t)H \quad , \\ \ddot{H} &= A^2(t)H + \dot{A}(t)H, \\ H(t,t) &= I \quad . \end{aligned}$$

Integrating (3.13) we obtain

$$(3.14) \qquad \begin{aligned} H((n+1)k,nk) &= I + k\int_0^1 \dot{H}(nk+\theta k,nk)\, d\theta \\ &= I + kA(nk) + k^2\int_0^1\int_0^1 \theta\, d\theta\, d\phi\, \ddot{H}(nk + \phi\theta k,\, nk) \quad . \end{aligned}$$

Now since A is bounded it is a standard result (M&S, Theorem 3.1 C), that

$$(3.15) \qquad \|H(t,s)\| \leq K_1(R) \quad \text{for} \quad |t-s| \leq R \ ,$$

where R is any positive number. This does not require exponential stability and in fact is an immediate consequence of Gronwall's inequality. It now follows from the second

equation in (3.13) that the coefficient of k^2 in (3.14) is bounded uniformly in n and k. This implies that Y_n is the fundamental solution operator to an equation

$$(3.16) \quad Y_{n+1} = [I+kD_n(k)]Y_n = [I+kA_n + k^2O(1)]Y_n$$

where the O(1) term is uniform in n. This means that we can regard the x_n equation (3.12) as a perturbation of the Y_n equation. We point out that the reduction we have obtained does not use exponential stability and so is valid in the case that (0.1) has a general exponential dichotomy. If we use the stability, however, we see that Y_n satisfies the estimate

$$(3.17) \quad \| Y_n Y_j^{-1} \| \le K \, e^{-\alpha k(n-j)}, \qquad n \ge j \, .$$

We must now prove the proposition that exponential stability is preserved under k^2 perturbations.

Theorem 2.   Consider two difference equations

$$(3.18) \qquad \tilde{y}_{n+1} = R(n,k)\tilde{y}_n$$

$$(3.19) \qquad \tilde{x}_{n+1} = R(n,k)\tilde{x}_n + kS(n,k)\tilde{x}_n$$

defined for small k. Suppose that (3.18) is exponentially stable, that is there exists constants K and α independent of k such that

$$(3.20) \qquad \| \tilde{Y}_n \tilde{Y}_j^{-1} \| \le K \, e^{-\alpha k(n-j)}, \qquad n \ge j \, ,$$

where $\tilde{Y}_n$ is the fundamental solution to (3.18). We point out that we do not require $R(n,k)$ to be invertible since we only study the solution $\tilde{Y}_n \tilde{Y}_j^{-1}$ for $n \geq j$. In that case $\tilde{Y}_n$ is not invertible and we must use (3.5) in place of $\tilde{Y}_n \tilde{Y}_j^{-1}$, but we will retain this notation for simplicity.

Under this hypothesis there exists an ε_0 such that if for small k,

(3.21) $$\| S(n,k) \|_\infty \leq \varepsilon \leq \varepsilon_0$$

where the sup in (3.21) is over n, then (3.19) is exponentially stable and if $\tilde{X}_n$ is the fundamental solution to (3.19) (the dependence on k has been suppressed) the estimate

(3.22) $$\| \tilde{X}_n \tilde{X}_j^{-1} \| \leq K_1 e^{-\alpha_1 k(n-j)}$$

where K_1 can be taken independent of k and $\alpha_1 = \alpha + O(\varepsilon)$.

Before proving this we note that the case where $S(n,k) = O(k)$ is automatically covered. The more general formulation will be used in Section 6 and more importantly in Part II. Note also that it is a discrete analogue to the Gronwall inequality and the proof is in fact immediately suggested by the proof of the Gronwall inequality. This theorem is equivalent to a lemma of Engquist itself based on a theorem of Strang [8], but the proof given here is simpler than Strang's proof and much more suggestive of the Gronwall inequality.

To prove Theorem 2 we first note that for $n > j$ we have

(3.23) $$\tilde{X}_n \tilde{X}_j^{-1} = \tilde{Y}_n \tilde{Y}_j^{-1} + k \sum_{\ell=j+1}^{n} \tilde{Y}_n \tilde{Y}_j^{-1} S_{\ell-1} \tilde{X}_{\ell-1} \tilde{X}_j^{-1}$$

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This analogue of the variation of constants formula can be verified immediately. Using (3.20) we get

$$(3.24) \quad \| X_n X_j^{-1} \|$$

$$\leq K e^{-\alpha k(n-j)} + kK e^{\alpha k} \varepsilon \sum_{\ell=j}^{n-1} e^{-\alpha k(n-\ell)} \| \tilde{X}_\ell \tilde{X}_j^{-1} \| \ .$$

Here K is the constant of (3.20). If we redefine K as $K e^{\alpha k}$ (for small k) and define $v_{n,j} = e^{\alpha kn} \| \tilde{X}_n \tilde{X}_j^{-1} \|$ we can write

$$(3.25) \quad v_{n,j} \leq K + kK\varepsilon \sum_{\ell=j}^{n-1} v_{\ell,j} \equiv s_{n-1,j} \ , \quad n > j \ .$$

From (3.25) we obtain

$$(3.26) \quad v_{n,j} = \frac{s_{n,j} - s_{n-1,j}}{kK\varepsilon} \leq s_{n-1,j}$$

whence

$$(3.27) \quad s_{n,j} \leq (1+\varepsilon Kk)^{n-j} s_{j,j} \leq K_1 e^{\varepsilon Kk(n-j)} \ ,$$

where K_1 is defined so that $s_{j,j} \leq K_1$ independent of k and j (from (3.25) specialized to $n = j+1$ and the fact that $v_{j,j} = 1$).

Inequality (3.27) together with (3.25) and the definition of $v_{n,j}$ yields (3.22) immediately and thus completes the proof of Theorem 2.

Returning to our specific case, it should be clear that after several applications of Theorem 2 we can prove exponential stability for $\bar{v}_n$ defined as $T(kAn)\bar{w}_n$ (see (3.8)

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and the following paragraph). Since in terms of fundamental solution operators we have

(3.28)
$$\bar{W}_n \bar{W}_j^{-1} = T^{-1}(kAn)\bar{V}_n\bar{V}_j^{-1}T(kA_j)$$

we obtain exponential stability for (3.8) and another application of Theorem 2 yields exponential stability for the full homogeneous system (3.2). It is a simple matter to go from exponential stability to admissibility for the inhomogeneous difference equation

(3.29)
$$w_{n+1} = C_n w_n + kg_n$$

We first define admissibility in the obvious way.

The pair (L_∞^n, L_∞^n) will be admissible for (3.29), for small k, iff for any sequence g_n in L_∞^n (3.29) has a unique solution w_n in L_∞^n and this assignment is a bounded mapping, i.e. there exists a K (independent of k) such that

(3.30)
$$\|w_n\|_\infty \leq K\|g_n\|_\infty \ .$$

An entirely analogous definition holds for the admissibility of the pair $(\tilde{A}^n, \tilde{A}^n)$. As in the ODE case we will show (L_∞^n, L_∞^n) is admissible and then show that the solution is AP if the coefficients are.

To show (L_∞^n, L_∞^n) admissibility we merely write the solution

(3.31)
$$w_n = k \sum_{j=-\infty}^{n} W_n W_j^{-1} g_{j-1} \ .$$

Exponential stability implies the convergence of this series,

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which can be immediately verified to be a solution of (3.29).
To show $w_n \in L_\infty^n$ we estimate, using (3.6)

$$(3.32) \quad \|w_n\|_\infty \le \|g_n\|_\infty Kk \sum_{j=0}^\infty e^{-\alpha jk} = \|g_n\|_\infty K \frac{k}{1-e^{-\alpha k}} \le \|g_n\|_\infty \bar{K}$$

(using $x/(1-e^{-x}) \to 1$ as $x \to 0$) and this yields (3.30) after
a redefinition of K.

To show uniqueness is trivial, if w_n were a bounded
solution to the homogeneous equation (3.2) we would have
for $n \ge j$,

$$(3.33) \qquad\qquad w_n = W_n W_j^{-1} w_j$$

and $w_n = 0$ follows immediately upon letting $j \to -\infty$.

It remains to show that $(\tilde{A}^n, \tilde{A}^n)$ is admissible. In
fact if N is a common ε-almost period for C_n and g_n ,
we would have $w_{n+N} - w_n$ to be the unique L_∞ solution
to an inhomogeneous equation with inhomogeneous term

$$(3.34) \qquad\qquad \frac{1}{k} [C_{n+N} - C_n]w_{n+N} + g_{n+N} - g_n$$

and for fixed $k > 0$ the almost periodicity of w_n is a
consequence of (3.30).

We can remove the factor $1/k$ in the denominator by using
the expression (2.11) for C_n ,

$$(3.35) \qquad\qquad C_n = \bar{C}(kA_n, \ldots, kA_{n+k}) ,$$

where $\bar{C}$ is a smooth function of its arguments. Expanding
$C_{n+N} - C_n$ using (3.35) we see that if N is a common ε-almost

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period for A_n and f_n, then N will be an $O(\varepsilon)$-almost period of w_n, with the bound in the $O(\varepsilon)$ term independent of k. Note that these two paragraphs follow only from (L_∞^n, L_∞^n) admissibility of (3.29) and do not depend on the stability of (0.1).

The convergence of w_n to the solution y_n also follows very easily. If y is the unique AP solution to (0.2) we have

$$(3.36) \qquad \sum_j \alpha_j y_{n+j} - k \sum_j \beta_j \dot{y}_{n+j}$$

$$= k \, [\, \sum_j \alpha_j j \int_0^1 (\dot{y}(nk+\theta kj) - \dot{y}(nk)) \, d\theta$$

$$- \sum_j \beta_j [\dot{y}(nk+kj) - \dot{y}(nk)]] \; = \; k \, g(k,nk) \; .$$

(The summation is from 0 to ℓ in the above.) Here $g(k,t)$ is AP in t for each k and

$$(3.37) \qquad \qquad \| g(k,t) \|_\infty = o(1) \; , \qquad k \to 0 \; ;$$

(3.37) follows from the uniform continuity of AP functions on the whole real axis.

Now if $\bar{w}_n = \begin{pmatrix} y_{n+\ell-1} \\ \vdots \\ y_n \end{pmatrix}$ then (3.36) implies that $\bar{w}_n$ is the unique AP solution to the same equation as w_n except for an error,

$$(3.38) \qquad \qquad k \begin{pmatrix} g(k,nk) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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and (3.37) together with (3.30) yield

(3.39) $$\| w_n - \bar{w}_n \|_\infty = o(1) , \qquad k \to 0 .$$

We can obtain finer convergence results by imposing some smoothness conditions on $A(t)$ and $f(t)$. In fact if the scheme has order of accuracy p, and A and f have $p+1$ derivatives, then y also has $p+1$ derivatives. It is shown in Henrici (p. 247) that

(3.40) $$\| \sum_j \alpha_j y_{n+j} - k \sum_j \beta_j \dot{y}_{n+j} \| \leq k^{p+1} G \| y^{p+1} \|_\infty ,$$

where the constant G depends only on the scheme. (3.40) together with (3.30) yield

(3.41) $$\| w_n - \bar{w}_n \|_\infty = O(k^p) .$$

We point out that these convergence arguments are valid whenever we have admissibility and do nct require stability. Before leaving the stable case we would like to discuss problems relating to the computability of the solution w_n.

Consider first errors in initial data. Suppose we solve the exact difference equation (3.29) but use as initial data $w_0^*(k) = w_0(k) + e(k)$ where $w_0(k)$ is the exact initial data for the AP sequence solution w_n and $e(k)$ is bounded for small k, i.e. $\| e(k) \| \leq e_0$. Now the solution w_n^*, which we solve for, will be

(3.42) $$w_n^* = w_n + W_n e(k) ,$$

and we have

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$$(3.43) \qquad \| w_n^* - w_n \| = O(e^{-\alpha k n}) \ , \qquad n \geq 0 \ .$$

Now in general we cannot expect $e(k) \to 0$ as $k \to 0$. This would certainly occur if we could obtain initial data consistent with the initial data of the solution y to the ODE. However this solution is known only by an improper integral involving not only the given forcing function $f(s)$ but also the fundamental solution $Y(t)$ to the homogeneous system which we could not expect to know explicitly unless A were a constant. Thus the most we can assert is that, using fixed initial data (say $w_0^*(k) = 0$), if we integrate over a sufficiently long interval we will get the solution w_n up to some exponentially decaying error.

Now let us consider round-off error. If we could postulate that we solved a perturbed equation

$$(3.44) \qquad w_{n+1}^{**} = C_n w_n^{**} + k \tilde{f}_n + O(k^{r+1}) \ , \quad n \geq 0, \ r > 0 \ ,$$

where the error is uniform in n then we could relate this solution to w_n^* by

$$(3.45) \qquad w_n^{**} = w_n^* + k \sum_{j=1}^{n} W_n W_j^{-1} O(k^r) \ , \quad n > 0 \ ,$$

and we can estimate the error simply,

$$(3.46) \quad \| k \sum_{j=1}^{n} W_n W_j^{-1} O(k^r) \| \leq k^r K k \sum_{r=0}^{\infty} e^{-\alpha k r} = O(k^r) \ .$$

Let us further examine the validity of (3.44). If we include rounding errors in forming the companion matrix (or operator), together with the error of taking only a finite

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expansion of the term $(\alpha_\ell I - \beta_\ell k A_{n+\ell})^{-1}$ in the implicit

case (see (2.11)), then we could assume that the linear

part of (3.44) would be replaced by

$$(3.47) \qquad C_n^{**} = C_n + O(k^{r+1}) \ .$$

If we also assume an error $O(k^{r+1}\|w_n\|)$ in forming the

product $C_n w_n$ we see we should postulate a system

$$(3.48) \, w_{n+1}^{**} = C_n w_n^{**} + k g_n + O(k^{r+1}) + O(k^{r+1}\|w_n^{**}\|)$$

and in order to justify (3.44) we must show that the solution

to (3.48) will be bounded for $n \geq 0$.

We can write this solution as

$$w_n^{**} = W_n w_0^{**} + k \sum_{j=1}^{n} W_n W_j^{-1} g_{j-1}$$

(3.49)

$$+ k[\sum_{j=1}^{n} W_n W_j^{-1} (O(k^r) + O(k^r\|w_{j-1}^{**}\|))]$$

Now if we use the fact that

$$(3.50) \qquad k \sum_{r=0}^{\infty} e^{-\alpha k r} = O(1)$$

and define $h_n = \max_{j=0,\ldots,n} \|w_j^{**}\|$ we can estimate h_n by

$$(3.51) \qquad h_n = O(1) + k^r h_n O(1)$$

where the $O(1)$ terms are independent of n. This implies,

for small k, the boundedness of $\|w_n^{**}\|$ and so justifies (3.44).

The techniques used in Section 2 of block diagonalizating

the companion matrix were first developed by Engquist

although I was not aware of his work when my formulation was developed.

Engquist's procedure may appear to be more complicated than the procedure presented here, but this is because he proves a more general result; namely that the companion matrix can be block diagonalized to within an error of arbitrary order (say $O(k^{p_1})$). (He also does not restrict himself to strongly stable schemes, but this imposes stability requirements on several additional homogeneous systems.) If one were to consider his procedure restricted to a block diagonalization up to $O(k^2)$ then, while the two formulations differ in the lines of approach, they are equally simple.

Engquist also shows that the upper block will agree with the Taylor series expansion of (0.1), up to terms of order p, if $p < p_1$ is the accuracy of the scheme. The proof of the first order structure of the upper block (the term $D(z)$ in Section 2) is, I believe, somewhat simpler than his proof restricted to the first order term.

I would like to point out that in the case we are considering (an exponentially stable ODE and a strongly stable scheme) the extra fineness of the block diagonalization being carried to order $p+1$ will only give the advantage of the homogeneous system having a stability exponent $-\alpha+O(k^p)$ instead of $-\alpha+O(k)$. In particular in Engquist's study of the uniform convergence (for $t \geq 0$) of the solutions of the homogeneous equation to the solution to the ODE (see his

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Theorem 3, p. 24-27 and note that what he uses as α is

what we call $-\alpha$)   one would still obtain uniform convergence

of $O(k^p)$ (neglecting round-off and errors in initial data),

however the decay exponent would be $-\alpha+O(k)$ instead of

$-\alpha+O(k^p)$.

4. Admissibility in the Case of a
General Exponential Dichotomy

Here we are concerned with admissibility properties
in the case that the homogeneous equation (0.1) has a
general exponential dichotomy as described by (0.6) with
$P_2 \neq 0$. The results will be valid in the infinite dimen-
sional case, although in this case one must bear in mind
that the situation described by equations (0.6) is not
the most general form of an exponential dichotomy.

First of all consider arbitrary homogeneous and
inhomogeneous difference equations

$$(4.1) \qquad\qquad w_{n+1} = C_n(k) w_n$$

$$(4.2) \qquad\qquad w_{n+1} = C_n(k) w_n + k g_n .$$

Here C_n is defined for $k \in (0, k_0]$ and is AP in n for
fixed k, and is now assumed invertible so that a fundamental
solution operator W_n, as described by (3.3), exists and
is invertible for all n.

Associated with (4.2) we have variation of constants
formulas in both the forward and backward directions,

$$\text{(a)} \qquad w_n = W_n w_0 + k \sum_{j=1}^{n} W_n W_j^{-1} g_{j-1} , \qquad n > 0$$

(4.5)

$$\text{(b)} \qquad w_n = W_n w_0 - k \sum_{j=n+1}^{0} W_n W_j^{-1} g_{j-1} , \qquad n < 0$$

as can be easily verified.

Now we will define an exponential dichotomy for (4.1) as follows. Suppose there exist projections $\bar{P}_1(k)$, $\bar{P}_2(k) = I - \bar{P}_1(k)$, such that the estimates

$$\| W_n \bar{P}_1 W_j^{-1} \| \leq K e^{-\alpha k(n-j)} , \qquad n \geq j ,$$

(4.6)

$$\| W_n \bar{P}_2 W_j^{-1} \| \leq K e^{-\alpha k(j-n)} , \qquad j \geq n ,$$

hold with K and α independent of k. The only difference between this and the ODE case is that the projections may depend on k.

If we have an exponential dichotomy then (L_∞^n, L_∞^n) is admissible for (4.2). In fact the unique bounded solution w_n is

(4.7)
$$w_n(k) = k \sum_{j=-\infty}^{\infty} G_j^n g_{j-1}$$

where

(4.8)
$$G_j^n = W_n \bar{P}_1 W_j^{-1} , \qquad n \geq j$$
$$= -W_n \bar{P}_2 W_j^{-1} , \qquad j \geq n+1 .$$

Equation (4.7) is of course suggested by the ODE case and is easily verified using (4.6). The admissibility bound (3.30) can be derived exactly as in the stable case as can $(\tilde{A}^n, \tilde{A}^n)$ admissibility.

We next observe that (L_∞^n, L_∞^n) (or $(\tilde{A}^n, \tilde{A}^n)$) admissibility is preserved under perturbations of the linear term. In fact if we had a system

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(4.9) $w_{n+1} = C_n(k)w_n + kS_n(k)w_n + kg_n$

where, for small k,

(4.10) $\| S_n(k) \|_\infty \leq \varepsilon_0$

and ε_0 is some number to be determined, then the unique L_∞^n
solution can be found by defining $w_{n;0} \equiv 0$ and $w_{n;i+1}$ to be
the unique L_∞^n solution to

(4.11) $w_{n+1;i+1} = C_n w_{n,i+1} + kS_n w_{n,i} + kg_n$

(for simplicity the dependence of C_n and S_n on k has been
suppressed). If $\varepsilon_0 K_1 < 1$, where K_1 is the admissibility
bound for (4.2), the contracting mapping principle establishes
(L_∞^n, L_∞^n) admissibility with admissibility bound

(4.12) $K_2 = \dfrac{K_1}{(1-\varepsilon_0 K_1)}$

Of course we will get AP solutions if $D_n + kS_n$ and g_n are AP.

     Now these two principles certainly settle the question
of admissibility when the homogeneous system has a general
exponential dichotomy. Referring to the reduction obtained
in Section 3 we see that after a nonsingular change of
dependent variable, $v_n = T(kA_n)w_n$, v_n satisfies the
equation (see (3.8) ff.);

(4.13) $v_{n+1} = \begin{pmatrix} I+D_n(k) & 0 \\ 0 & B_n \end{pmatrix} v_n + k^2 \, O(1) v_n + k\tilde{g}_n$.

Here $\tilde{g}_n = T(kA_{n+1})g_n$. By the boundedness of T and T^{-1},

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admissibility for (4.13) is equivalent to admissibility for (4.2). But (4.13) expresses the homogeneous term as a perturbation of the system

$$(4.14) \qquad X_{n+1} = \begin{pmatrix} I+D_n & 0 \\ 0 & B_n \end{pmatrix} X_n$$

with fundamental solution

$$X_n = \begin{pmatrix} Y_n & 0 \\ 0 & \prod\limits_{i=1}^{n-1} B_i \end{pmatrix} \quad , \quad n > 0 \; ,$$

(4.15)

$$X_{-n} = \begin{pmatrix} Y_{-n} & 0 \\ 0 & \prod\limits_{i=0}^{-n+1} B_i^{-1} \end{pmatrix} \quad , \quad n > 0 \; ,$$

and this has an exponential dichotomy with projections independent of k and in fact (after identifying E_1 with E)

$$\bar{P}_1 = P_1 \oplus P(0)$$

(4.16)

$$\bar{P}_2 = P_2$$

Here P_1 and P_2 are defined in (0.6) while $P(0)$ is the projection onto E^1 along E, (see (2.19) ff).

We thus obtain admissibility in this case and we point out that the convergence proofs given in Section 3 did not require stability and remain valid here.

5. Existence of an Exponential Dichotomy for the Homogeneous Linear Equation

We will prove here that admissibility for the inhomogeneous difference equation

$$(5.1) \qquad w_{n+1} = C_n w_n + k g_n$$

implies an exponential dichotomy for the homogeneous equation

$$(5.2) \qquad w_{n+1} = C_n w_n$$

The proof is very similar to a proof of the analogous property for the ODE case given in the Appendix. It is a strictly finite dimensional proof and it may then be simpler to think in terms of matrices rather than linear operators. We can then regard $T(z)$ as a similarity transformation which follows an initial transformation putting $U(0)$ into block diagonal form. The matrix $L(z)$ is strictly in block diagonal form and it is no longer necessary for us to continually make the qualification "identifying E_1 with E."

If we let $v_n = T(kA_n) w_n$, (5.1) is transformed into (see (4.13))

$$(5.3) \quad v_{n+1} = \begin{pmatrix} I+kD_n & 0 \\ 0 & B_n \end{pmatrix} v_n + k^2 O(1) v_n + k g_n .$$

(For simplicity we will write the inhomogeneous term in (5.13) as g_n rather than $\tilde{g}_n$.)

If we call the leading matrix in (5.3) $R_n(k)$, then the unperturbed linear system

(5.4)
$$x_{n+1} = R_n x_n$$

has an exponential dichotomy with projections $\bar{P}_1$ and $\bar{P}_2$, independent of k (see (4.15) and (4.16)), and we must exhibit an exponential dichotomy for the homogeneous version of (5.3).

(5.5)
$$v_{n+1} = R_n v_n + k^2 O(1) v_n \,.$$

By admissibility (5.5) can have no L_∞^n solution. Let $S_+(k)$ and $S_-(k)$ be the subspaces of initial data which give rise to solutions bounded for positive and negative n respectively. We have $S_+ \cap S_- = \{0\}$ (suppresssing the k dependence). If we let S be any complementary space we have

(5.6)
$$E_\ell = S_+ \oplus S_- \oplus S$$

with associated projections

(5.7)
$$I = P_1 + P_2 + P_3$$

Our first task is to show $S = \{0\}$ i.e. $P_3 = 0$.

Define $\tilde{L}_\infty^n$ to be the submanifold of L_∞^n of sequences with only finitely many nonzero components. Let $g_n \in \tilde{L}_\infty^n$ and let v_n be the corresponding unique L_∞^n solution to (5.3). Then using (4.5a) we see that for large positive n we have

(5.8)
$$v_n = V_n [v_0 + k \sum_{j=1}^{\infty} V_j^{-1} g_{j-1}]$$

and since v_n is bounded the vector in the brackets lies in S_+ i.e.

(a) $\qquad P_2 v_0 = - k \sum\limits_{j=1}^{\infty} P_2 \, v_j^{-1} \, g_{j-1}$

(5.9)

(b) $\qquad P_3 v_0 = - k \sum\limits_{j=1}^{\infty} P_3 \, v_j^{-1} \, g_{j-1}$

Similarly looking at v_n for $n \to -\infty$ and using (4.5b) we obtain

(a) $\qquad P_1 v_0 = k \sum\limits_{j=-\infty}^{0} P_1 \, v_j^{-1} \, g_{j-1}$

(5.10)

(b) $\qquad P_3 v_0 = k \sum\limits_{j=-\infty}^{0} P_3 \, v_j^{-1} \, g_{j-1}$

Now if we simply set

$$g_0 = V_1 P_3 z$$

(5.11)

$$g_i = 0 \qquad , \; i \neq 0 \; ,$$

where z is an arbitrary vector in E_ℓ we see immediately that $P_3 = 0$. We can also see that for any given $g_n \in \tilde{L}_\infty^n$ the unique L_∞^n solution v_n is given by

(5.12) $\qquad v_n = k \sum\limits_{j=-\infty}^{\infty} G_j^n \, g_{j-1}$

where G_j^n is given by (see (4.8))

$$G_j^n = V_n \, P_1 \, V_j^{-1} \, , \qquad n \geq j$$

(5.13)

$$= -V_n \, P_2 \, V_j^{-1} \, , \qquad j \geq n+1 \; .$$

It is of course G_j^n which we wish to prove is exponentially damped.

Let $\bar{G}_j^n$ be the Green's function associated with (5.4). We know

$$(5.14) \qquad \| \bar{G}_j^n \| \leq K \, e^{-\alpha k |n-j|}$$

Now define $H_j^n = G_j^n - \bar{G}_j^n$. If $g_j \in \tilde{L}_\infty^n$ and v_n the corresponding solution to (5.3), while x_n is the L_∞^n solution to the inhomogeneous version of (5.4),

$$(5.15) \qquad x_{n+1} = R_n x_n + k g_n \,.$$

Then $u_n = v_n - x_n$ is the unique bounded solution to

$$(5.16) \qquad u_{n+1} = R_n u_n + k^2 \, O(1) \, v_n \,.$$

We thus have

$$(5.17) \quad u_n = k \sum_{j=-\infty}^{\infty} H_j^n \, g_{j-1} = k^2 \sum_{j=-\infty}^{\infty} \bar{G}_j^n \, O(1) \, v_{j-1} \,.$$

If we fix i and let $g_{i-1} = z$, $g_j = 0$ for $j \neq i-1$ where z is an arbitrary vector in E_ℓ , we then obtain, using (5.14) and the admissibility bound for (5.3),

$$(5.18) \qquad \| H_j^n \| \leq K_1 \,, \qquad \text{for all } n, i$$

where K_1 is some constant independent of k. (The restriction "for small k" is always understood.) The use of (5.14) then implies the boundedness of G_j^n i.e.

$$(5.19) \qquad \| G_j^n \| \leq K_2 \,,$$

where K_2 is independent of k.

We will now show that

$$(5.20) \qquad \| P_1(k) - \bar{P}_1 \| = O(k) \ .$$

Observe that $\bar{P}_1$ is independent of k. To obtain (5.20) we note that $P_1 - \bar{P}_1 = H_0^0$. If we define $g_{i-1} = z$, $g_j = 0$ ($j \neq i$) where z is an arbitrary vector we then have from (5.12)

$$(5.21) \qquad v_{j-1} = k \ G_i^{j-1} \ g_{i-1} = k \ G_i^{j-1} \ z \ .$$

Substituting into (5.17) we obtain

$$(5.22) \qquad u_n = k H_i^n z = k^3 \left[\sum_{j=-\infty}^{\infty} \bar{G}_j^n \ O(1) \ G^{j-1} \right] z$$

Now using (5.19) and (5.14) we see the sum in brackets in (5.22) is $O(1/k)$ whence we conclude that

$$(5.23) \qquad \| H_i^n \| = O(k)$$

and setting $n = i = 0$ we obtain (5.20).

Note that this proves that the ranks of the stable and unstable manifolds are unchanged for small k.

We can now show quite easily that (5.19) can be replaced by an exponential decay factor.

For a certain small positive ε consider new systems with the linear part of (5.3) and (5.4) multiplied by $e^{-\varepsilon k}$ i.e.

$$(5.24) \qquad v_{n+1}^{\varepsilon} = e^{-\varepsilon k} [R_n + k^2 \ O(1)] v_n^{\varepsilon}$$

$$(5.25) \qquad x_{n+1}^{\varepsilon} = e^{-\varepsilon k} \ R_n \ x_n^{\varepsilon} \ .$$

Now by Section 4 (see (4.9) ff.) the inhomogeneous versions of (5.24) and (5.25) will both have (L_∞^n, L_∞^n) admissible, if ε is small enough. But the new fundamental solutions are

$$(5.26) \qquad V_n^\varepsilon = e^{-\varepsilon n k} V_n$$

$$(5.27) \qquad X_n^\varepsilon = e^{-\varepsilon n k} X_n$$

and clearly (5.25) will have an exponential dichotomy with projections $\bar{P}_1$ and $\bar{P}_2$ if ε is small enough.

Now if $P_1(k)$ and $P_2(k)$ are the corresponding stable and unstable projections for (5.24) while $G_{j;\varepsilon}^n$ is the Green's function for (5.24), the same analysis that led to (5.19) will yield for some constant K_3 ,

$$(5.28) \qquad \| G_{j;\varepsilon}^n \| \leq K_3$$

In particular if $j \geq n+1$, (5.28) reduces to

$$(5.29) \qquad \| V_n \, P_2^\varepsilon \, V_j^{-1} \| \leq K_3 \, e^{-\varepsilon k(j-n)}$$

and one part of the definition of an exponential dichotomy for (5.5) will have been shown if we can show $P_1^\varepsilon = P_1$, $P_2^\varepsilon = P_2$. But this is a simple consequence of (5.20). In fact if S_+^ε , S_-^ε are the new stable and unstable manifolds for (5.24) then (5.26) yields

$$(5.30) \quad \begin{array}{ll} \text{(a)} & S_+ \subset S_+^\varepsilon \\ \text{(b)} & S_-^\varepsilon \subset S_- \end{array}$$

But (5.20) shows that the ranks of P_1^ε and P_1 are equal

(being equal to the rank of $\bar{P}_1$) and similarly for P_2^{ε} and P_2. This shows that the inclusions in (5.30) are equalities and that

(5.31)
    (a) $P_1^{\varepsilon} = P_1$

    (b) $P_2^{\varepsilon} = P_2$

Thus one part of the requirements of an exponential dichotomy for (5.5) has been shown and the other part will follow on replacing $-\varepsilon$ by $+\varepsilon$.

Lastly we point out that an exponential dichotomy for (5.5) implies one for (5.1) since in terms of fundamental solutions we have

$$(5.32) \qquad W_n = T^{-1}(kA_n)V_n\, T(kA_0)$$

whence the projections $P_1'(k)$, $P_2'(k)$ for (5.2) are related to P_1 and P_2 by a similarity transformation

(5.33)
    (a) $P_1'(k) = T^{-1}(kA_0)P_1(k)T(kA_0)$

    (b) $P_2'(k) = T^{-1}(kA_0)P_2(k)T(kA_0)$.

6. Miscellaneous Results

This section is concerned with certain generalizations of the theory developed in the preceding section. We first remove certain restrictions which had been imposed previously in order to make the exposition clearer. In Section 6A we will remove the restriction that A be C^1 while in Section 6B we remove the restrictions (2.6) and (2.7) on the roots of the polynomial $p(x) = 0$ which lie inside the unit circle.

We then consider certain trivial extensions of the theory. In Section 6C we remark on the general L_∞ (non-AP) case. In Section 6D we consider the convergence of the mean value of our sequence solution to the mean value of the solution of the ODE and finally in Section 6E we consider the weakly nonlinear case.

6A. Removal of the Differentiability Condition on A

First consider the case when the homogeneous equation

$$(6.1) \qquad \qquad \dot{y} = Ay$$

is exponentially stable; i.e.

$$(6.2) \qquad \| Y(t) Y^{-1}(s) \| \leq K e^{-\alpha(t-s)}, \qquad t \geq s,$$

where $Y(t)$ is the fundamental solution to (6.1). We assume the A is AP but is not C^1. Now Theorem 2 does not require a perturbation $O(k^2)$ but is certainly valid for a perturbation $k \, o(1)$ $(k \rightarrow 0)$. This leads us to expect that the requirement that A be C^1 can be replaced by the uniform continuity of AP functions on the whole real axis. This is indeed the case as one can verify with little difficulty.

In this subsection, we will merely trace through the proofs in Section 3 and indicate what changes must be made if A is not C^1.

Discretizing (6.1) we obtain the homogeneous difference equation

$$(6.3) \qquad \qquad w_{n+1} = C_n w_n$$

where C_n is given in (2.11). Using the uniform continuity of A we see that (3.7) can be replaced by

$$(6.4) \qquad w_{n+1} = U(k \, A_n) w_n + k \, o(1) w_n$$

where the term $o(1)$ is uniform in n (this will not be stated

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explicitly in the future). If we also use the uniform continuity of A to obtain

(6.5) $T(k\, A_{n+1}) = T(k\, A_n) + k\, o(1)$

we see that Theorem 2 will yield exponential stability for (6.3) provided we can show exponential stability for the system

(6.6) $v_{n+1} = \begin{pmatrix} I+kA_n & 0 \\ 0 & B_n \end{pmatrix} v_n$

The lower block causes no difficulty and we are left with the system

(6.7) $x_{n+1} = [I + k\, A_n]x_n.$

Finally we can show that (6.7) is exponentially stable by using Theorem 2 to compare the fundamental solution X_n with Y_n (= $Y(nk)$). This requires some modification, as the proof leading to (3.16) used the differentiability of A (see (3.13) ff).

Following the notation of Section 3, we define $H(t,s) = Y(t)Y^{-1}(s)$. We can then write (compare with (3.14))

(6.8) $H((n+1)k,\, nk) = I + k\dot{H}(nk,nk) + k \int_0^1 d\theta\, [\dot{H}(nk+\theta k,nk)$
$- \dot{H}(nk,nk)]$

$= I + kA_n + k \int_0^1 d\theta\ [A(nk+\theta k)-A_n]\ H(nk+\theta k,nk)$

$+ k \int_0^1 d\theta\ A_n[H(nk+\theta k,nk) - H(nk,nk)]\ .$

Finally we see that if we use (6.2) together with the uniform continuity of A on the whole real axis and the equation

(6.9) $$\dot{H}(t,s) = A(t)H(t)$$

then (6.8) can be rewritten as

(6.10) $$Y_{n+1} = [I + kA_n + k\, o(1)]\, Y_n$$

and Theorem 2 is immediately applicable.

We thus have exponential stability for (6.3) and hence $(\tilde{A}^n, \tilde{A}^n)$ admissibility for the inhomogeneous version of (6.3).

Finally we point out that the results of Section 4, regarding admissibility in the case that (6.1) has a general exponential dichotomy is equally valid if A is not C^1, as the contracting mapping principle (see (4.9) ff) would certainly be applicable if the perturbation is $k\, o(1)$. The results in Section 5, however, use crucially the differentiability of A (in the argument involving the deduction of (5.18) from (5.17)) and it has not been possible to extend this result when A is not C^1.

6B. Removal of Restrictions on the Roots of $p(x) = 0$

In Section 2 we prescribed two conditions on the roots
of $p(x) = 0$ which lie inside the unit circle; namely

(6.11) x_u distinct , $u = 2,\ldots,\ell$,

(6.12) $x_u \neq 0$, $u = 2,\ldots,\ell$.

(Recall that $x_1 = 1$ and $|x_u| < \theta < 1$ for $u > 1$). We will
now remove these restrictions.

It should be clear by now that in the exponentially
stable case (6.12) is unnecessary. In fact the fundamental
solution to the homogeneous equation is used only for $n \geq j$,
and the equation (3.5) shows that the invertibility of the
linear term C_i is not required for $n \geq j$. Of course the
notation $W_n W_j^{-1}$ is no longer accurate, but except for this
detail the results of Sections 2 and 3 are valid without
assuming (6.12).

We deal next with the restriction (6.11). (Note that
the root $x_1 = 1$ is always simple.) The removal of (6.11)
in the finite dimensional case is trivial.   In fact suppose
the root x_r has multiplicity $q > 1$; i.e.

(6.13) $p^1(x_r) = \ldots = p^{q-1}(x_r) = 0$

                  $p^q(x_r) \neq 0$.

It is shown in Henrici (p. 214) that the effect of (6.13)
is that the eigenvalue x_r of $U(0)$ (see (2.12)) will now have

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nontrivial Jordan blocks. Thus U(0) can still be put in block diagonal form (see (2.17))

$$(6.14) \qquad U(0) = \begin{pmatrix} I & 0 \\ 0 & B(0) \end{pmatrix}$$

where B(0) is no longer diagonalizable, but has all of its eigenvalues bounded by θ in the norm. Now since B(0) can be put in Jordan normal form with ε instead of 1 on the superdiagonal, for any $\varepsilon > 0$, we can certainly choose a matrix norm such that

$$(6.15) \qquad \| B(0) \| < \theta$$

Since all norms are equivalent on a finite dimensional space, the proof of Theorem 1 can now proceed exactly as in Section 2.

The case when E is infinite dimensional can be handled in exactly the same manner; however since we can no longer appeal to the theorems of linear algebra, we will have to carry out the proof in more detail.

It is shown in Henrici (p. 214), that in the scalar case a basis for the generalized eigenspace E_r corresponding to the eigenvalue x_r is $\{\bar{e}_{r,j}\}_{j=1,\ldots,q}$ where

$$(6.16) \qquad \bar{e}_{r,1} = \begin{pmatrix} x_r^{\ell-1} \\ \vdots \\ 1 \end{pmatrix} ,$$

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$$\bar{e}_{r,j} = \begin{pmatrix} \prod\limits_{r=1}^{j-1} (\ell - r) x^{\ell - (j-1)} \\ \vdots \\ \prod\limits_{r=1}^{j-1} (n-r) x^{(n-(j-1))} \\ \vdots \\ \prod\limits_{r=1}^{j-1} (j-r) x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad j > 1.$$

One can verify easily that in fact

$$\text{(a)} \quad U(0) \, \bar{e}_{r,1} = x_r \bar{e}_{r,1}$$

(6.17)

$$\text{(b)} \quad U(0) \, \bar{e}_{r,j} = x_r \, \bar{e}_{r,j} + \bar{e}_{r,j-1} \, , \qquad j > 1.$$

Also since the vectors $\bar{e}_{r,j}$ form a new basis the $\ell \times \ell$ matrix with columns $\bar{e}_{r,j}$ is nonsingular. Now in order to have ε instead of 1 on the superdiagonal, we replace $\bar{e}_{r,j}$ by $e_{r,j}$ where

$$(6.18) \qquad e_{r,j} = \varepsilon^{j-1} \, \bar{e}_{r,j}$$

(see Bellman [9], p. 198). Here $\varepsilon > 0$ is to be specified. The matrix with columns $e_{r,j}$ is still nonsingular but (6.17b) will now be replaced by

$$(6.19) \qquad U(0) \, e_{r,j} = x_r \, e_{r,j} + \varepsilon \, e_{r,j-1}.$$

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Now the extension of this to infinite dimensional space is very simple and follows closely the procedure of Section 2. For any vector $y \in E$ define the vector $e_{r,j}(y) \in E_\ell$ by

$$(6.20) \qquad e_{r,j}(y) = \varepsilon^{j-1} \begin{pmatrix} \prod\limits_{r=1}^{j} (\ell-r) x^{\ell-j} y \\ \vdots \\ \prod\limits_{r=1}^{j} (j+1-r) x \, y \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Equation (6.19) clearly generalizes to

$$U(0) \, e_{r,j}(y) = x_r \, e_{r,j}(y) + \varepsilon \, e_{r,j-1}(y) \, , \quad j > 1$$

(6.21)

$$U(0) \, e_{r,1}(y) = x_r \, e_{r,1}(y) \, .$$

We now define the space $E_{r,j} = \{w \in E_\ell$ such that $w = e_{r,j}(y)$ for some $y \in E\}$. Clearly the space $E_{r,j}$ is closed for all r and j. Furthermore we have

$$(6.22) \qquad\qquad E_\ell = \oplus \sum_{r,j} E_{r,j}$$

This follows by the same argument as that given in Section 2 (see (2.14) ff). In fact if $w = \begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix}$ is a vector in E_ℓ and we had an expansion

$$(6.23) \qquad\qquad \begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} = \sum_{r,j} e_{r,j}(y_{r,j})$$

then the $\{y_{r,j}\}$ would be related to the $\{x_i\}$ by multiplication

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by a nonsingular $\ell \times \ell$ matrix and this establishes (6.22).
It also follows from this that if we define a new norm on
E_ℓ by $\|w\|^u = \max_{r,j} \|y_{r,j}\|_E$ then $\|w\|^u$ is equivalent to $\|w\|^\ell$
$(\max_i \|x_i\|_E)$.

Now if E_1 is defined as $E_{1,1}$ (the root x_1 is simple)
while E^1 is defined as $\oplus \sum_{r>1,j} E_{r,j}$, then, exactly as
in Section 2, we see that $U(0)$ is in block diagonal form
with respect to the decomposition $E = E_1 \oplus E^1$, and we can
write (see (2.17))

$$(6.24) \qquad U(0) = \begin{pmatrix} D(0) & 0 \\ 0 & B(0) \end{pmatrix}$$

where $D(0): E_1 \to E_1$ and is the identity, while $B(0): E^1 \to E^1$
and for ε sufficiently small we have from (6.19)

$$(6.25) \qquad \qquad \|B(0)\|^u < \theta$$

The proof of Theorem 1 can now be carried out exactly
as in Section 2 if we make the final observation that the
spectrum of $U(0)$ is exactly $\{x_u\}$. This should be obvious
and follows from the fact that on each "generalized eigen-
space" $E_r = \oplus \sum_j E_{r,j}$, $U(0) - \lambda I$ (for $\lambda \notin \{x_u\}$) acts as
the matrix

$$(6.26) \qquad \begin{matrix}
x_r-\lambda & \varepsilon & 0 & . & . & . & 0 \\
0 & . & . & & & & . \\
 & & . & . & & & . \\
. & & & . & . & & 0 \\
. & & & & . & . & \\
. & & & & & . & \varepsilon \\
0 & . & . & . & 0 & x_r-\lambda
\end{matrix}$$

and the inverse is obtained merely by inverting this matrix.

6C. Extension to the L_∞ Case

Here we would like to make the simple observation that the difference equation theory, like the ODE theory, is essentially an L_∞ theory which produces AP solutions when the coefficients are AP.

In fact, if we now agree that derivatives are to be understood as L_∞ instead of AP, the theory developed in Sections 3, 4 and 5 is entirely valid in the L_∞ case, with only one unimportant exception. The single exception is the proof given in (3.36) ff of the convergence of the L_∞^n solution w_n to the vector $\bar{w}_n = \begin{pmatrix} y_{n+\ell-1} \\ \vdots \\ y_n \end{pmatrix}$ in the case that the coefficients are not smooth. In fact if we refer to (3.36) and (3.37) we see that (3.37) need not be valid because the function $\dot{y}$ which is C_∞ (we take A and f continuous) need not be uniformly continuous on the whole real axis. In this case we have not been able to show that $w_n \to \bar{w}_n$ uniformly for all n but only uniformly for nk lying in compact intervals.

To see this let us write $w_n(k)$, $\bar{w}_n(k)$ to indicate explicitly the dependence on k. Now if the initial data for $w_n(k)$ is consistent, that is

$$(6.27) \qquad w_0(k) \xrightarrow[k \to 0]{} \begin{pmatrix} y_1 \\ \vdots \\ y_1 \end{pmatrix}$$

for some vector $y_1 \in E$, then it is well known (see Henrici, p. 244) that $w_n(k)$ must converge uniformly on

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compact subsets, to some solution to the equation

(6.28) $$\dot{y} = A(t)y + f \; .$$

But since every solution to (6.28) but one is unbounded, this
limiting solution can only be the unique L_∞ solution $y(t)$. Thus
we must show (6.27) and the proof of this will in fact show
that the vector y_1 in (6.27) is $y(0)$.

We assume at least that A is C^1 (with bounded derivative)
so that the results of Section 5 are applicable. We can then
write

(6.29) $$w_0 - \bar{w}_0 = k \sum_{j=-\infty}^{\infty} G_j^0 \; g(k,(j-1)k)$$

where g is given in (3.37), and G_j^n satisfies the estimate

(6.30) $$\| G_j^n \| \le K_1 \; e^{-\alpha_1 k|n-j|} \; .$$

Observe that $g(k,t) \to 0$ uniformly for t in compact intervals. Let
T be unspecified for the moment and rewrite the sum in (6.29) as

(6.31) $$w_0 - \bar{w}_0 = k \sum_{|jk| \le T} G_j^0 g(k,(j-1)k) + k \sum_{|jk| > T} G_j^0 g(k,(j-1)k) \; .$$

The first term $\to 0$ because $g(k,t) \to 0$ uniformly on the compact
interval $[-T,T]$. The second term is $O(e^{-\alpha_1 T})$ as we can easily
see from (6.30). Thus by first choosing T sufficiently large
to make the second term small, and then choosing k small enough
to make the first term small we see that $\| w_0 - \bar{w}_0 \| \to 0$ $(k \to 0)$ and
this establishes the convergence of w_n to $\bar{w}_n$ uniformly on
compact intervals.

6D. Mean Value Properties

Let $a_n \in \tilde{A}^n$. It is shown in Corduneanu (p. 48) that the limit

$$(6.32) \qquad \lim_{N \to \infty} \frac{1}{N} \sum_{j=n}^{n+N-1} a_j \equiv m(a_n)$$

exists and that this limit is uniform and independent of n, which we henceforth set equal to zero. This is of course the exact analogy to the mean value of an AP function

$$(6.33) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt \equiv \bar{m}(f) \ .$$

Now if w_n is our AP sequence solution, we would like to study the behavior of $m(w_n)$ as $k \to 0$. Define $\bar{w}_n$ as

$$(6.34) \qquad \ddot{w}_n = \begin{pmatrix} y_{n+\ell-1} \\ \vdots \\ y_n \end{pmatrix}$$

where y is the AP solution to the ODE

$$(6.35) \qquad \dot{y} = Ay + f$$

Clearly

$$(6.36) \qquad m(\bar{w}_n) = \begin{pmatrix} m(y_n) \\ \vdots \\ m(y_n) \end{pmatrix}$$

Now since (see (3.39))

$$(6.37) \qquad \| w_n - \bar{w}_n \|_\infty = \underset{k \to 1}{o(1)}$$

we must compare $m(y_n)$ with $m(y)$. We first claim for any AP function g we will have

$$(6.38) \qquad\qquad m(g_n) \xrightarrow[k \to 0]{} \bar{m}(g)$$

where $g_n = g(nk)$. This is obvious because (6.38) holds if g is a trigonometric polynomial and for arbitrary g, we simply approximate by a sequence of trigonometric polynomials.

Now (6.38), together with (6.37) and (6.34), certainly yields

$$(6.39) \qquad\qquad m(w_n) \xrightarrow[k \to 0]{} \begin{pmatrix} \bar{m}(y) \\ \vdots \\ \bar{m}(y) \end{pmatrix}$$

We now concern ourselves with the rates of convergence, assuming smooth coefficients. If A and f are C^{p+1} (so that y is) we can replace (6.37) by (see (3.41))

$$(6.40) \qquad\qquad \| w_n - \bar{w}_n \|_\infty = O(k^p)$$

where p is the order of accuracy of the scheme. It therefore follows that

$$(6.41) \qquad\qquad m(w_n) = \begin{pmatrix} m(y_n) \\ \vdots \\ m(y_n) \end{pmatrix} + O(k^p)$$

and it is only necessary to study the convergence of $m(y_n)$ to $\bar{m}(y)$. In fact for any AP function $y \in C^{r+1}$ we have

$$(6.42) \qquad m(y_n) = \bar{m}(y) + O(k^{r+1})$$

Equation (6.42) is an almost periodic analogue of a theorem of Isaacson and Keller [10] for periodic functions (p. 340) and in fact the proof is very similar, and is a simple consequence of Taylor's theorem. Letting $t_n = nk$ and $s \in [0,1]$ we can write

$$(6.43) \quad y(t_n+sk) = y_n + (sk)y_n^1 + \ldots + \frac{(sk)^r}{r!} y_n^r + O(k^{r+1})$$

where the remainder is uniform in t since y^{r+1} is AP. Integrating (6.43) from $s = 0$ to $s = 1$ we obtain

$$(6.44) \quad \int_{t_n}^{t_{n+1}} y(t) \, dt = k \int_0^1 y(t_n + sk) \, ds$$

$$= k[y_n + \frac{k}{2}y_n' + \ldots \frac{k^r}{(r+1)!} y_n^r + O(k^{r+1})].$$

If we sum (6.44) from $n = 0$ to N, divide by Nk and let $N \to \infty$ we obtain

$$(6.45) \quad \bar{m}(y) = m(y_n) + \frac{k}{2} m(y_n^1) + \ldots \frac{k^r}{(r+1)!} m(y_n^r) + O(k^{r+1})$$

Now if $r = 0$, i.e. y is C^1 the terms involving $m(y_n^1)$ through $m(y_n^r)$ are missing and we obtain (6.42). For $r > 0$ we assume (6.42) holds for $j = 0, \ldots, r-1$. We can then write, as y^1 is C^r,

$$(6.46) \qquad 0 = \bar{m}(y^1) = m(y_n^1) + O(k^r)$$

and similar expressions for $\bar{m}(y^i)$ for $i = 2,\ldots,r$ since $y^i \in C^{(r-i)+1}$ (we have used the obvious fact that the mean value of a derivative is zero), to obtain (6.42). If we now apply this to (6.41) we obtain

$$(6.47) \qquad m(w_n) = \begin{pmatrix} \bar{m}(y) \\ \vdots \\ \bar{m}(y) \end{pmatrix} + O(k^p) \ .$$

6E. Weakly Nonlinear Equations

Consider the equations

$$(6.48) \qquad\qquad \dot{y} = Ay$$

$$(6.49) \qquad\qquad \dot{y} = Ay + f(t)$$

$$(6.50) \qquad\qquad \dot{y} = Ay + h(t,y)$$

If any pair of function spaces (B,D) is admissible for
(6.49) the contracting mapping principle provides a technique
to obtain D-solutions to (6.50) under certain conditions on
the nonlinear term $h(t,y)$, the most important being that
h has a small Lipschitz constant (see Hartman [11],
Chapter 12).

The case of $(\tilde{A},\tilde{A})$ admissibility is particularly simple.
We require that h be almost periodic in t, uniformly for y
in compact subsets (see Hale [12] pp. 113 ff). This simply
insures that $h(t,x(t))$ will be AP for any AP function x.
We also require that

$$(6.51) \qquad \| h(t,x_1) - h(t,x_2) \|_\infty \leq \varepsilon_1 \| x_1 - x_2 \|$$

for any $x_1, x_2 \in E$. The restriction on ε_1 is simply

$$(6.52) \qquad\qquad \varepsilon_1 K_1 < 1$$

where K_1 is the admissibility bound for (6.49) (see (0.3b)).
The unique AP solution to (6.50) will simply be the limit
of the iterates

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(6.53) $y^0 = 0$

y^{i+1} is the unique AP solution to

$\dot{y}^{i+1} = Ay^{i+1} + h(t,y^i)$.

This is of course an immediate consequence of the contracting mapping principle.

The same proof will yield AP sequence solutions to the discretized version of (6.50).

(6.54) $w_{n+1} = C_n w_n + k\, g(nk,k,w_{n+1},w_n)$

Here the nonlinear term is

$$(6.55) \quad g(t,k,\bar{w},w) = \begin{pmatrix} (\alpha_\ell I - k\beta_\ell A(t+\ell k))^{-1}[\beta_\ell h(t,\bar{x}_{\ell-1}) \\ \qquad\qquad + \sum_{j=0}^{\ell-1} \beta_j h(t,x_j)] \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $w = \begin{pmatrix} x_{\ell-1} \\ \vdots \\ x_0 \end{pmatrix}$ and $\bar{w} = \begin{pmatrix} \bar{x}_{\ell-1} \\ \vdots \\ \bar{x}_0 \end{pmatrix}$. Observe that g is,

for fixed k, AP in t uniformly for w and $\bar{w}$ in compact subsets, and that g will have an $O(\varepsilon_1)$ Lipschitz constant which we call ε, i.e.

(6.56) $\|g(t,k,\bar{w}_2,w_2) - g(t,k,\bar{w}_1,w_1)\|_\infty$

$\in \max[\|\bar{w}_2-\bar{w}_1\| , \|w_2-w_1\|]$.

Now if K is the admissibility bound for the linear inhomogeneous version of (6.54) (see (3.30)) and $\mu = \varepsilon K < 1$ then (6.54)

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will have a unique AP solution w_n , which in fact will be
the limit of the iterates $w_{n;i}$ defined by

(6.57) $w_{n;0} = 0$

   $w_{n;i+1}$ is the unique AP solution to

   $w_{n+1;i+1} = C_n w_{n;i+1} + kg(nk,k,w_{n+1;i},w_{n;i})$

This is an immediate consequence of the contracting mapping
principle and we note for future reference that

(6.58) $\| w_n - w_{n;m} \|_\infty = O(\mu^m)$.

   Now the first thing to consider is the convergence of
w_n to $\bar{w}_n$ defined as $\begin{pmatrix} y_{n+\ell-1} \\ \vdots \\ y_n \end{pmatrix}$ where y is the solution to
(6.50) (assuming of course that (6.52) holds). In fact
as was seen in Section 3 $\bar{w}_n$ satisfies (6.54) up to an error
$k \underset{k \to 1}{o(1)}$ $(O(k^{p+1})$ if everything is smooth). Hence the
difference $z_n = \bar{w}_n - w_n$ is the unique AP solution to

(6.59) $z_{n+1} = C_n z_n + k[g(nk,k,w_{n+1} + z_{n+1}, w_n + z_n)$

                $- g(nk,k,w_{n+1},w_n)] + k\, o(1)\ (O(k^{p+1}))$

and we have

(6.60) $\| z_n \|_\infty \leq \mu \| z_n \|_\infty + o(1)\ (O(k^p))$

and since $\mu < 1$ this settles the question of convergence.

   We now examine the question of the computability of
the solution in the exponentially stable case; that is when
we have constants $\bar{K}$ and α such that

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(6.61) $\|W_n W_j^{-1}\| \leq \bar{K} e^{-\alpha k(n-j)}$, $n \geq j$,

where W_n is the fundamental solution to the homogeneous version of (6.54).

To do this we use (6.58) to reduce the problem to a study of the effect of round-off errors and errors in initial data on the solution $w_{n;m}$ for a fixed m. This will be a simple consequence of the stability results of Section 3 (see (3.42) ff) for the inhomogeneous equation.

In fact the results of Section 3 show that at the first iteration we would solve (for $n \geq 0$) for a solution $w_{n;1}^*$ which is related to $w_{n;1}$ by

(6.62) $w_{n;1}^* = w_{n;1} + O(k^r) + O(e^{-\alpha kn})$.

We now consider the effect of this error on the second iterate. Assuming, for the moment, exact calculations and exact initial data, we obtain a solution $w_{n;2}^*$ which solves the equation

(6.63) $w_{n+1;2}^* = C_n w_{n;2}^* + k\ g(nk,k,w_{n+1;1}^*,w_{n;1}^*)$.

If we let $z_{n;i} = w_{n;i}^* - w_{n;i}$ for $i = 1,2$ then we obtain

(6.64) $z_{n+1;2} = C_n z_{n;2} + k[g(nk,k,w_{n+1;1}+ z_{n+1;1},w_{n;1}+z_{n;1})$

$- g(nk,k,w_{n+1;1},w_{n;1})]$

The last term can be written (using (6.62)) as
$k\ O(k^r) + k\ O(e^{-\alpha kn})$ and we have

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$$(6.65) \quad z_{n;2} = k \sum_{j=1}^{n} W_n W_j^{-1} \, O(k^r) + k \sum_{j=1}^{n} W_n W_j^{-1} \, O(e^{-\alpha k j})$$

$$= O(k^r) + O(e^{-\alpha k n} \, kn) = O(k^r) + O(e^{-\alpha_1 k n})$$

where α_1 can be arbitrarily close to α. We have used (6.62), (6.61) and the trivial fact that $t = O(e^{\delta t})$ for any positive δ.

It is now obvious that (6.65) is preserved if we include in the equation for $w_{n;2}^*$ (see (6.63)) the effect of roundoff and errors in the initial data of $w_{n;2}^*$ (compared with the initial data for $w_{n;2}$) and it is also obvious that this process can be repeated for m iterations, for any fixed finite m, and this establishes the computability of the solution w_n.

Part II. Extensions to Partial Differential Equations

We will now use the techniques of Massera and Schäffer, in the simple exponentially stable case, to obtain AP solutions to a certain class of inhomogeneous partial differential equations (PDE's) to which one can apply ODE formalism by use of the Hille Yoshida Theorem. The homogeneous equation has been extensively studied by Krein [1 3], to which we will refer often. A more succinct study of the Hille-Yoshida Theorem can be found in the appendix of Lax and Phillips [14].

Throughout Part II E will denote an infinite dimensional Banach space.

7. Existence Theorems

Consider the linear equation

$$(7.1) \qquad\qquad \dot{y} = By$$

where B is an unbounded operator. The study of the solutions to (7.1) has been answered by the Hille-Yoshida Theorem in the case that the resolvent $R_\lambda(B)$ $(= (\lambda I - B)^{-1})$ satisfies the inequality

$$(7.2) \qquad\qquad \|R_\lambda(B)\| \le \frac{1}{\lambda} \ , \qquad\qquad \lambda > 0 \ .$$

A more general formulation is given in Krein (in particular Section 2, Chapter 1). Krein shows that if:

(7.3) (a)   B is closed

      (b)   B is densely defined

      (c) $\|R_\lambda^n(B)\| \le \dfrac{M}{(\text{Re } \lambda - \omega)^n}$,   R.P. $\lambda > \omega$,

then there exists a semigroup Z_t , $t \ge 0$, such that:

(7.4) (a) Z_t is strongly continuous

      (b) $Z_0 = I$

      (c) $\|Z_t\| \le Me^{\omega t}$

      (d) $\lim\limits_{\Delta \to 0} [Z_\Delta - I]e/\Delta$ converges iff $e \in D_B$
           and in that case it converges to Be

      (e)   if $e \in D_B$ then for $t \ge t_0$ the unique

           solution to (7.1) such that $y(t_0) = e$

           is simply $y(t) = Z_{t-t_0} e$ (uniqueness is

shown in Theorem 2.7, page 47)

(f) $Z_t: D_B \to D_B$ and commutes with B for $t \geq 0$.

For a well posed problem we obviously require $\omega \leq 0$ in (7.4c).
To insure exponential stability we are going to proceed as
if $\omega = 0$ and introduce our own damping term which will be
allowed to depend on t. The reader should note that if
$\omega < 0$ this term is not necessary. It is well known (see
Krein, page 43) that one can always introduce an equivalent
norm so that the factor M in (7.4c) can be replaced by 1,
however we will eventually approximate B by a family of
bounded operators B_h where

$$(7.5) \qquad \| Z_t^h \equiv e^{B_h t} \| \leq M$$

and we may not be able to introduce a norm so that M can
be taken as 1 for all h, and thus we will leave M unspecified.

Now to make the homogeneous equation exponentially
stable we introduce a damping term $-\delta(t)$ and consider
the equation

$$(7.6) \qquad \dot{y} = [B - \delta(t)]y$$

For conditions on δ we first let it be a scalar AP function
such that

$$(7.7) \qquad \text{R.P. } (\bar{m}(\delta)) > 0 .$$

This implies that if $g(t,s) = \exp\left\{- \int_s^t dr \, \delta(r)\right\}$ then

$$(7.8) \qquad |g(t,s)| \leq K \, e^{-\alpha(t-s)} , \qquad t \geq s ,$$

as can easily be seen.

It is also possible to permit δ to be an AP operator function satisfying certain commutivity properties. Specifically if $Y(t)$ is the fundamental solution to

$$(7.9) \qquad \dot{Y} = - \delta Y$$

then

(7.10) (a) $\|Y(t)Y^{-1}(s)\| \leq K e^{-\alpha(t-s)}$, $t \geq s$

   (b) $\delta(t)$, $Y(t)$, $Y^{-1}(t)$ commute with Z_r

      for all t and all $r > 0$

   (c) $\delta(t)$, $Y(t)$, $Y^{-1}(t)$ map D_B into itself

      and commute with B.

These hypotheses permit us to essentially treat δ as a scalar in the following analysis. If we define

$$(7.11) \qquad g(t,s) = Y(t)Y^{-1}(s)$$

then the "fundamental solution" to (7.6) is

$$(7.12) \qquad g(t,s) Z_{t-s} \ , \qquad\qquad t \geq s,$$

and

$$(7.13) \quad \|g(t,s)Z_{t-s}\| \leq KM e^{-\alpha(t-s)} \equiv K e^{-\alpha(t-s)}$$

after a redefinition of K.

Now consider the inhomogeneous equation

$$(7.14) \qquad \dot{y} = [B-\delta]y + f(t)$$

where $f \in \tilde{A}(E)$ or $C_\infty(E)$. We want to study bounded (for all t)

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solutions to (7.14). First we see that any such solution
is unique. In fact if $w(t)$ were a bounded solution to
(7.6) we would have (for $t \geq s$)

(7.15) $w(t) = Z_{t-s} \, g(t,s) \, w(s)$

and $w \equiv 0$ follows immediately on letting $s \to -\infty$.

It is equally simple to see that any C_∞ solution must
be given by

$$(7.16) \quad y(t) = \int_{-\infty}^{t} g(t,s) \, Z_{t-s} \, f(s) \, ds \equiv C(f) \ .$$

This is in fact an immediate consequence of letting $t_0 \to -\infty$
in the variation of constants formula

$$(7.17) \quad y(t) = g(t,t_0) Z_{t-t_0} y(t_0) + \int_{t_0}^{t} ds \, g(t,s) Z_{t-s} f(s)$$

which is proved in Krein (Theorem 6.1, page 129) for the
autonomous case, and it is a trivial matter to see that the
same proof will work if δ is time dependent. Note that (7.17)
need not be a solution to (7.14), but any such solution
must be given by (7.17).

We now consider the operator C defined by (7.16).
C is a bounded operator mapping $L_\infty \to L_\infty$. Boundedness is
an immediate consequence of (7.13) and in fact

$$(7.18) \qquad \|y\|_\infty \leq \bar{K} \ (= \frac{K}{\alpha}) \ \|f\|_\infty$$

The integrand is obviously continuous if $f \in C_\infty$ and the
only point in question is its measurability for a general

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$f \in L_\infty$ (for measurability of functions in a Banach space see Hille and Phillips [15]). However it is shown by Krein, and will be used crucially very shortly, that Z_r is the strong limit, pointwise, of $e^{B_n r}$ where B_n is a bounded operator (Theorem 2.9, page 48). Since $e^{B_n r}$ will be norm continuous this establishes the integrand as the pointwise limit of measurable functions, hence measurable.

We will now show that y as defined in (7.16) is a solution to (7.14). We will make use of the above mentioned approximation theorem of Krein (Theorem 2.9). It is shown that there exists a squence of bounded operators B_n such that

(7.19) (a) $\| (B-B_n)e\| \xrightarrow[n \to \infty]{} 0 \quad (\forall e \in D_B)$

(b) $\| (Z_r - Z_r^n)e\| \xrightarrow[n \to \infty]{} 0 \quad \forall e, \; r \geq 0,$ where $Z_r^n = e^{B_n r}$

(c) $\| Z_r^n \| \leq M, \quad r \geq 0$

(d) $B_n, \; Z_r^n \; (r \geq 0)$ satisfy the same commutivity relations with $\delta(t)$ (7.10b,c) as does B and Z_r.

We only have to point out that (7.19d) follows from the explicit definition of B_n as $-\lambda_n I - \lambda_n^2 R_{\lambda_n}(B)$ where $\lambda_n \to \infty$ (see Krein, page 49). We point out that the operators B_n cannot in general be considered as spatial discretizations. A family of such approximations will be assumed in the following section.

Now consider the operators $C_n(f)$ defined by

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$$(7.20) \qquad y_n = \int_{-\infty}^{t} g(t,s) \ Z_{t-s}^{n} \ f(s) \ ds \equiv C_n(f) \ .$$

It follows immediately that the operators C_n are uniformly bounded operators mapping L_∞ into C_∞ and in fact

$$(7.21) \qquad \| y_n \|_\infty \leq \bar{K} \| f \|_\infty$$

.

where we take the same bound as in (7.18). It is also immediate that y_n is the unique C_∞ solution to

$$(7.22) \qquad \dot{y}_n = [B_n - \delta] y_n + f$$

and y_n is AP if f is.

We are going to show that $C_n \to C$ strongly in a certain subspace of L_∞. Specifically define S to be the subspace spanned by the functions whose range has compact closure. It is well known (see Amerio [16]) that $\tilde{A} \subseteq S$. For $f \in S$ we will have

$$(7.23) \qquad C_n(f) \to C(f)$$

To show (7.23) note that by the uniform boundedness of C_n it is sufficient to show it for a dense subset of S and by the definition of S it is sufficient to take f as

$$(7.24) \qquad f = \sum_{j=1}^{n} \chi_j(t) e_j$$

where $e_j \in E$ and χ_j is the characteristic function of a measurable set. By linearity we only have to work with one term in the sum in (7.24). Thus if I is some measurable

set let $f = \chi_I e$. Then $w_n(t) = y(t) - y_n(t)$ is explicitly

$$(7.25) \qquad w_n(t) = \int_0^\infty \chi(t-r) \, g(t,t-r) \, [Z_r - Z_r^n] e \;\; dr \;.$$

Estimating (7.25) we obtain

$$(7.26) \qquad \|w_n\|_\infty \leq K \int_0^\infty e^{-\alpha r} \|(Z_r - Z_r^n)e\| \;\; dr$$

and (7.23) is an immediate consequence of the Lebesgue dominated convergence theorem.

Observe that (7.23) implies that $C(f)$ is continuous if $f \in S$ and it is AP if f is.

Now consider the almost periodic case. First we point out that if δ and f are C^1 (AP derivatives) then y is also and in fact

$$(7.27) \qquad \dot{y} = C(-\delta y + \dot{f})$$

This is trivial if we observe that

$$(7.28) \qquad \dot{y}_n = C_n(-\delta y_n + \dot{f})$$

and if $\bar{y}$ is defined to be the right-hand side of (7.27) we have

$$(7.29) \qquad \bar{y} - \dot{y}_n = [C - C_n](-\delta y) + C_n(-\delta(y - y_n)) + [C - C_n](\dot{f}) \;,$$

and clearly

$$(7.30) \qquad \|(\bar{y} - \dot{y}_n)\|_\infty \xrightarrow[n \to \infty]{} 0$$

Equation (7.27) is now an immediate consequence of letting

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$n \to \infty$ in the equation

$$(7.31) \qquad y_n(t_1) - y_n(t_2) = \int_{t_1}^{t_2} \dot{y}_n(t) \; dt$$

We next show that y satisfies an integrated version of (7.14). Specifically

$$(7.32) \quad y(t_1) - y(t_2)$$

$$= B \int_{t_1}^{t_2} dt \; y(t) - \int_{t_1}^{t_2} dt \; \delta(t)y(t) + \int_{t_1}^{t_2} dt \; f(t) \; .$$

Since D_B is dense and B is closed it is sufficient to show (7.32) when f is of the form

$$(7.33) \qquad \qquad f = e^{i\lambda t} b \; , \qquad \qquad b \in D_B.$$

Now observe that Bf is AP. It certainly follows that

$$(7.34) \qquad \qquad By = C(Bf) \; .$$

In fact this holds whenever $Bf \in C_\infty$ and is a simple consequence of the closure of B, approximating the integral in (7.16) by Riemann sums over finite intervals and then noting that B applied to each sum is the Riemann sum approximating C(Bf).

Now the same property certainly holds for B_n and C_n i.e.

$$(7.35) \qquad \qquad B_n y_n = C_n(B_n f).$$

Finally since by its construction B_n commutes with Z_r

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(Krein, Theorem 2.9) we can write

(7.36) $$B_n y = C(B_n f) \ .$$

Since y_n satisfies (7.22) we have

(7.37) $y_n(t_1) - y_n(t_2)$

$$= B_n \int_{t_1}^{t_2} dt \ y_n(t) - \int_{t_1}^{t_2} dt \ \delta(t)y_n(t) + \int_{t_1}^{t_2} dt \ f(t)$$

$$= B \int_{t_1}^{t_2} dt \ y(t) + [B_n - B] \int_{t_1}^{t_2} dt \ y(t) + B_n \int_{t_1}^{t_2} dt \ (y_n - y)$$

$$- \int_{t_1}^{t_2} dt \ \delta(t)y_n(t) + \int_{t_1}^{t_2} dt \ f(t) \ .$$

where the fact that $\int_{t_1}^{t_2} dt \ y(t) \in D_B$ is a consequence of
(7.34) and the same type of closure argument as followed
that equation.

If we now let $n \to \infty$ in the last equation in (7.37) we
see that the term $B_n \int_{t_1}^{t_2} dt \ (y_n - y) \xrightarrow[n \to \infty]{} 0$ because

(7.38) $$B_n(y_n - y) = [C_n - C](B_n f) = [C_n - C](Bf) + [C_n - C]((B_n - B)f)$$

and the right-hand side of (7.38) clearly $\to 0$ in the sup norm
as $n \to \infty$. It therefore follows, on letting $n \to \infty$ in (7.37),
that (7.32) does in fact hold for functions of the form (7.33)
and thus for all $f \in \tilde{A}$.

Now if δ and f are C^1 (hence y is) we can show very easily

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that (7.32) implies (7.14). In fact since y is C^1 we can

let $h \to 0$ in the equation

$$(7.39) \quad \frac{y(t+h)-y(t)}{h} = \frac{B \int_t^{t+h} ds \, y(s)}{h} - \frac{\int_t^{t+h} ds \, \delta(s) y(s)}{h} + \frac{\int_t^{t+h} f(s)}{h}$$

where h can be positive or negative, using the closure of B.
Similarly (7.14) will hold if we only require Bf is AP.
In fact since in that case By is AP, B can be brought inside
the integral in (7.32), which can then be differentiated
directly.

We point out that in both these cases both $\dot{y}$ and By are
also AP. For future use we point out, using the integral
equation (7.32) applied to (7.27), that if δ and f are C^2
then y is C^2 and $\dot{y}$ satisfies

$$(7.40) \qquad \ddot{y} = [B-\delta]\dot{y} - \dot{\delta}y + f$$

i.e. (7.14) can be differentiated formally. (Note in parti-
cular that if Bf is AP, $B^2 y$ is AP.) This argument can
obviously be extended; namely if f and δ are C^p then y is C^p
and we have

$$(7.41) \quad y^r = [B-\delta]y^{r-1} + f^{r-1} - (\delta y)^{r-1} + \delta y^{r-1}, \quad r=1,\ldots,p.$$

We will now examine briefly the case that $f \in S$.
We still have $y_n \to y$ but (7.30) need not hold since $\dot{\delta}y$ and $\dot{f}$
need not belong to S. However we see very easily that if
$f \in L_\infty$ and

(7.42)
$$x = C(g)$$

$$x_n = C_n(g)$$

then we will have $x_n(t) \to x(t)$ pointwise and boundedly. In fact if $w_n(t) = x(t) - x_n(t)$, then

$$(7.43) \quad \|w_n(t)\| \leq K \int_0^\infty e^{-\alpha r} \|(Z_r - Z_r^n) \; g(t-r)\| \; dr$$

and the Lebesgue dominated convergence theorem is still applicable and hence (7.31) will still yield (7.27), if we use the bounded convergence theorem and the fact that

$$(7.44) \quad \bar{y}(t) - \dot{y}_n(t) = [C - C_n](-\dot{\delta}y + f)(t) + C_n(-\dot{\delta}(y - y_n))(t) \xrightarrow[n \to \infty]{} 0$$

Thus y has a bounded derivative. Now if we assume that f is of the form

$$(7.45) \quad f = \chi_I b , \qquad\qquad b \in D_B ,$$

then the derivation of (7.32) from (7.37) and (7.38) still follows, if we use the bounded convergence theorem for integrals over finite intervals. Since finite sums of such f's are dense in S, (7.32) holds for all $f \in S$ and if $Bf \in S$ or δ and f are C^1, y will still be the unique C_∞ solution to (7.14). Thus the theory will hold for a certain class of bounded, non AP, inhomogeneous terms, but is not as simple as the AP case and in the future we will restrict ourselves to this case.

Finally we point out that (7.32) can be derived by brute force differentiation of the formula (7.16) but use of the approximating operators B_n permits a cleaner and more straightforward development.

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8. Discretization in Space

The operators B_n which proved so useful in the preceding section need not correspond to a discretization in space. We therefore postulate the existence of a family of bounded operators B_h , where h will generally play the role of a spatial grid size, such that:

(8.1) (a) B_h are defined and bounded for $h \in (0, h_0]$

(b) $\| B_h \| = O(\frac{1}{h^m})$, $m > 0$

(c) B_h generate bounded semi-groups (i.e.
$\| e^{B_h t} \equiv Z_t^h \| \leq M$ for $t \geq 0$)

(d) $e \in D_B$ implies $\| (B - B_h) e \| \xrightarrow[h \to 0]{} 0$

(e)   If δ is not a scalar function, then the commutivity relations (7.10)b,c hold with B and Z_r replaced by B_h and Z_r^h

(f)   If $e \in D_B$ (or any dense manifold in D_B invariant under Z_r for $r \geq 0$) we have $\| B_h Z_r e \| \leq K(I, e)$ for $r \in I$, where I is any compact interval on the nonnegative real axis.

We point out that (8.1f) is designed to insure that $Z_r^h \to Z_r$ strongly, for all $r \geq 0$. This is certainly reasonable since $B_h Z_r e$ converges to $B Z_r e$. It is of course trivial if B_h commutes with Z_r. Also note that the convergence in (8.1d) does not deal with the order of the approximation of B_h to B.

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This will be dealt with by a subsequent assumption.

Before proceeding further, we indicate a simple example of the case we are considering.

(8.2) (a) $E = L_2(0, 2\pi)$

(b) $B = \frac{\partial}{\partial x}$ (with periodic boundary conditions)

(c) $B_h = \frac{u(x+h) - u(x-h)}{2h}$

In this example (8.1f) is unnecessary as B_h in fact commutes with Z_r.

Now associated with B_h we have the equations

(8.3) $$\dot{y} = [B_h - \delta] y$$

(8.4) $$\dot{y} = [B_h - \delta] y + f$$

The homogeneous equation (8.4) has as its fundamental solution $Y_h(t) = g(t,s) e^{B_h t}$, and is exponentially stable, i.e. there exist constants K and α independent of h such that

(8.5) $$\| Y_h(t) Y_h^{-1}(s) \| \le K e^{-\alpha(t-s)}, \qquad t \ge s.$$

Associated with (8.4) we form the operator C_h where

(8.6) $$y_h = C_h(f) = \int_{-\infty}^{t} g(t,s) z_{t-s}^h f(s) \, ds$$

for any AP f, y_h is the unique AP solution to (8.4). Equation (8.5) shows that the operators C_h are bounded uniformly in h, i.e.

(8.7) $$\| y_h \|_\infty \le \bar{K} \| f \|_\infty$$

where $\bar{K}$ is independent of h.

Now our first task is to show that $C_h \to C$ strongly. Using the results of Section 7 (see (7.23) ff) it is sufficient to show that $z_r^h \to z_r$ strongly, for $r \geq 0$. This will be an easy consequence of (8.1f). In fact if $e \in D_B$ (or the manifold described in (8.1f)) and if $y = z_t e$, $y_h = z_t^h e$, then if $w^h = y - y_h$ we will have

$$
\begin{aligned}
\dot{w}^h &= B_h w_h + [B - B_h] y \\
w^h(0) &= 0 .
\end{aligned}
$$
(8.8)

From (8.8) we can write

(8.9)
$$
w^h(t) = \int_0^t ds \; z_{t-s}^h [B - B_h] z_s e
$$

and the fact that $\| w^h(t) \| \to 0$, in fact uniformly on compact intervals, is an immediate consequence of (8.1f) and the Lebesgue bounded convergence theorem. The result for any $e \in E$ follows from the uniform boundedness of the operators z_r^h and the denseness of the manifold of (8.1f).

Now the fact that

(8.10)
$$
\| (y - y_h) \|_\infty \xrightarrow[h \to 0]{} 0
$$

for all $f \in \tilde{A}$ (in fact for all $f \in S$) is not any improvement of the corresponding result for the operators B_n of Section 7. We want to show that the difference in (8.10) approaches zero as fast as some power of h, and to do this we must require that B_h approximate B up to a certain degree of accuracy. Specifically we suppose that there exists an operator L

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(unbounded) with $D_L \subseteq D_B$ and such that if $e \in D_L$ we will
have

(8.11) $\| (B-B_h)e\| \leq \bar{C} \, h^j \|Le\|$ where $j > 0$

and $\bar{C}$ and j are independent of h and e.

    If we have Lf(t) AP we would like to show that (8.11)
implies

(8.12) $\| (y-y_h)\|_\infty = O(h^j)$.

Unfortunately (8.12) requires a further assumption; namely

        (a) $Z_r : D_L \to D_L$
(8.13)

        (b)   If Lf is AP then $\|LZ_r f(t)\|$ will be
                uniformly bounded for $r \geq 0$ and all t.

Assumption (8.13) is of course trivial if L commutes with Z_r.

    It is now a simple matter to show (8.12). In fact
$w_h = y-y_h$ is the unique AP solution to

(8.14) $\dot{w}_h = [B_h - \delta]w_h + (B-B_h)y$

whence by (8.7) we have

(8.15) $\|w_h\|_\infty = O(\| [B-B_h]y\|_\infty)$

and so it is only necessary to study $(B-B_h)y$. But now if
$Bf \in \tilde{A}$, which will generally follow if $Lf \in \tilde{A}$ since L will
be a higher order operator than B, we can write

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$$(8.16) \qquad [B-B_h]y = \int_0^\infty dr \; g(t,t-r) \, [B-B_h] Z_r f(t-r)$$

and clearly

$$(8.17) \qquad \| [B-B_h]y \|_\infty = O(h^j)$$

is an immediate consequence of (8.13).

9. Discretization in Time by the Multistep Linear Method

We will now apply the theory developed in Section 3 to the spatially discretized equation

$$(9.1) \qquad \dot{y}_h = [B_h - \delta] + f .$$

Since B_h is bounded the theory is immediately applicable and we can assert that for k sufficiently small (but depending on h) there will be an AP sequence $w_n(k,h)$ which will converge to $\bar{w}_n(k,h)$ defined by

$$(9.2) \qquad \bar{w}_n(k,h) = \begin{pmatrix} y_h((n+\ell-1)k) \\ \vdots \\ y_h(nk) \end{pmatrix}$$

and if everything is smooth in t we will have

$$(9.3) \qquad \|w_n - \bar{w}_n\|_\infty = O_h(k^P)$$

where the bound in (9.3) may depend on h. It is only left to show that the bound in (9.3) is independent of h and to discuss the relationship of k to h. We will show that what is required is

$$(9.4) \qquad k + \frac{k}{h^{2m}} = O(1)$$

i.e. there exists a constant c independent of k and h, such that if $k + k/h^{2m} < c$ the difference equation corresponding to (9.1) will be exponentially stable with constants independent of k and h.

The quadratic dependence on h in (9.4) is restrictive and will be removed in the following section.

Throughout this section we will use the notation of Sections 2 and 3. We will also take δ to be C^1.

We now fix h and consider the homogeneous difference equation on E_ℓ (see (3.7)),

$$(9.5) \qquad w_{n+1} = U(kB_h - k\delta_n) + k^2 O(1)w_n \ .$$

It is clear that this will hold uniformly in k and h if $k + k/h^m$ is sufficiently small, i.e. if we are in a region

$$(9.6) \qquad k + \frac{k}{h^m} = O(1) \ .$$

Of course restrictions of the form (9.6) are included in (9.4). Also note from the explicit definition of the full linear term in (9.5) (see (2.11)) the last term in (9.5) will be uniform in k and h. For the rest of this section this will be understood for all "O" signs unless stated otherwise.

If $W_n(k,h)$ is the fundamental solution to (9.5) it is clear that all we must show is that in a region of the form (9.4) we will have

$$(9.7) \qquad \| W_n W_j^{-1} \| \le K_1 \, e^{-\alpha_1 k(n-j)} \ , \qquad n \ge j \ ,$$

where the constants K_1 and α_1 are independent of k and h.

Following the procedure of Section 3 we let

$$(9.8) \qquad v_n = T(kB_h - k\delta_h)w_n$$

and we then have in a region of the form (9.6)

$$(9.9) \qquad v_{n+1} = L(kB_n - k\delta_n)v_n + k^2 O(1)v_n$$

Using the structure of the block diagonal operator $L(z)$ derived in Theorem 1 we can write this as

$$(9.10) \quad v_{n+1} = \begin{pmatrix} [I+k(B_n-\delta_n)] & 0 \\ 0 & \bar{B}_n \end{pmatrix} v_n + kO(k + \frac{k}{h^{2m}})v_n$$

Here we have used the notation $\bar{B}_n$ for the lower block (see (2.26)) so as not to confuse it with the operator B_h. Observe that in (9.10) we have the first appearance of the term k/h^{2m}. This is because in order to apply Theorem 2 we must separate out a factor of k in the perturbing term.

Now it certainly follows that in a region of the type (9.6) we have

$$(9.11) \qquad \| \prod_{i=j}^{n-1} \bar{B}_i \| \le K_1 \theta^{n-j}, \qquad n \ge j,$$

where K_1 and θ are independent of h and k. We therefore only have to consider the upper block in the leading term of (9.10) and in order to apply Theorem 2 we are going to compare this with $Y_{n;h} = Y_h(nk)$ where Y_h is the fundamental solution to

$$(9.12) \qquad \dot{Y}_h = [B_h - \delta]Y_h$$

Observe that the estimate

$$(9.13) \qquad \| Y_{n;h} Y_{j;h}^{-1} \| \le K e^{-\alpha k(n-j)}, \qquad n \ge j,$$

holds, where K and α are independent of h.

We will show that $Y_{n;h}$ satisfies the difference equation

(9.14) $Y_{n+1;h} = [I+kD_n(k,h)]Y_{n;h}$

$= [I+k(B_h-\delta_n)]Y_{n;h} + kO(k + \frac{k}{h^{2m}})Y_{n;h}$.

Observe that the last term in (9.14) is of the same order as the perturbation in (9.10).

Equation (9.14) follows easily from the same argument as in Section 3 (see (3.13) and (3.14)) if we simply observe that

(9.15) $(B_h - \delta) = - \dot{\delta}$

and that (3.15) holds in this case for $t \geq s$ by (8.5). We can now apply Theorem 2 comparing the equation

(9.16) $\bar{V}^1_{n+1;h} = [I+k(B_h-\delta_n)]\bar{V}^1_{n;h}$

with (9.14). We simply note that the proof of Theorem 2 indicates that the terms ε_0, α_1 and K_1 (see the statement of Theorem 2) depend only on the constants K and α in (9.13) and are thus independent of both k and h. Another application of Theorem 2 comparing the equation

(9.17) $\bar{V}_{n+1} = \begin{pmatrix} I+k(B_h-\delta_n) & 0 \\ 0 & \bar{B}_n \end{pmatrix}\bar{V}_n$

with (9.10), together with the boundedness of the operators

$T(k(B_h - \delta_n))$, $T^{-1}(k(B_h - \delta_n))$ in the region described by (9.6) will yield (9.7) with the restriction $k + k/h^{2m}$ sufficiently small (i.e. (9.4)).

If we now refer to the proof of admissibility in the stable case given by (3.31) ff we see immediately that (9.7) implies (L_∞^n, L_∞^n) admissibility for the inhomogeneous version (9.5), with an admissibility bound independent of k and h. It is only necessary to show that the convergence of w_n to $\bar{w}_n$ is independent of h (see (9.3)). This will follow from (3.40) if we can show $\|y_h^{p+1}\|_\infty$ will be bounded uniformly in h, assuming that δ and f are smooth.

In fact if δ and f have p+1 derivatives the differentiability properties of y_h are the same as those of y_n introduced in Section 7. In addition, the formula given in (7.28) can obviously be extended (for the operators C_h) to yield

$$(9.18) \qquad y_h^{p+1} = C_h(f^{p+1} + (-\delta y_h)^{p+1} + \delta y_h^{p+1})$$

(note that the argument of C_h in (9.18) does not involve y_h^{p+1}) and the uniform boundedness of the operators C_h immediately yields the uniform boundedness of the derivatives of y_h.

If we now define $\bar{\bar{w}}_n = \begin{pmatrix} y((n+\ell-1)k) \\ \vdots \\ y(nk) \end{pmatrix}$, then (9.3) (uniformly in h) together with (8.12) implies

$$(9.19) \qquad \|w_n - \bar{\bar{w}}_n\|_\infty = O(k^p + h^j) = O(h^{2mp} + h^j)$$

where we have used (9.4) for small k.

10. The Lax-Wendroff Scheme

The term k/h^{2m} in (9.4) can be improved for operators such that the Lax-Wendroff scheme is stable for the homogeneous equation

$$(10.1) \qquad \dot{w} = Bw$$

We will then be able to obtain AP sequence solutions under the more favorable restriction

$$(10.2) \qquad k + \frac{k}{h^m} = O(1) \ .$$

First of all we require that δ and f are C^3. As shown in Section 7 this implies that y is C^3. It implies further that $\ddot{y}$ satisfies

$$(10.3) \qquad \ddot{y} = [B-\delta]\dot{y} - \dot{\delta}y + \dot{f}$$

i.e. that the basic equation

$$(10.4) \qquad \dot{y} = [B-\delta]y + f$$

can be differentiated formally.

We can now apply the Lax-Wendroff scheme to (10.4). Proceeding formally we write, assuming Bf is AP,

$$(10.5) \quad y(t+k) = y(t) + k\dot{y}(t) + \frac{k^2}{2}\ddot{y}(t)$$

$$= y(t) + k[B-\delta]y(t) + \frac{k^2}{2}[B-\delta]^2 y(t)$$

$$+ \frac{k^2}{2}[-\dot{\delta}y + (B-\delta)f + \dot{f}] + kf(t) =$$

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$$= [I+kB + \frac{k^2}{2} B^2]y(t) - k\delta y$$

$$+ k[\frac{k}{2} \delta^2 - \frac{k}{2} \dot{\delta} - k\delta B]y + k[f + \frac{k}{2}(B-\delta)f + \frac{k}{2} \dot{f}]$$

We have written this as an inhomogeneous difference equation in order to apply techniques already developed.

Now in the expression for $\dot{y}$ we approximate B by B_h and in the expression for $\ddot{y}$ approximate B^2 by B^2_{ch} , for a certain constant $c > 0$ to be determined ($c = 1/2$ for the example described in (8.2)). Of course we will have to assume that B^2_h approximates B^2 in the same sense that B_h approximates B, but we will leave the precise hypothesis for later.

Applying these approximations to (10.5) and letting $w_n(k,h)$ stand for the dependent variable we obtain the difference equation

(10.6) $w_{n+1}(k,h) = U(k,h)w_n - k\delta_n w_n + kR(n,k,h)w_n + k\tilde{f}_n$

where

(10.7) (a) $U(k,h) = I + kB_h + \frac{k^2}{2} B^2_{ch}$

(b) $\tilde{f}_n = f_n + \frac{k}{2} [B_{ch} - \delta_n]f_n + \frac{k}{2} \dot{f}_n$

(c) $R(n,k,h) = \frac{k}{2} \delta^2_n - \frac{k}{2} \dot{\delta}_n - k\delta_n B_{ch} = O(k + \frac{k}{h^m})$.

Now $U(k,h)$ represents an approximation to (10.1). We choose c so that for

(10.8) $\frac{k}{h^m} = O(1)$

this discretization is stable, i.e.

$$(10.9) \qquad \| U(k,h)^r \| \leq H , \qquad r \geq 0.$$

We are going to use Theorem 2 to show that if $\| R(n,k,h) \|$ (i.e. $k + k/h^m$) is sufficiently small, then the homogeneous version of (10.6) is exponentially stable; that is if $W_n(k,h)$ is the fundamental solution to the homogeneous equation, then there exists constants K_1 and α_1 independent of k and h such that

$$(10.10) \qquad \| W_n W_j^{-1} \| \leq K_1 e^{-\alpha_1 k(n-j)} , \qquad n \geq j,$$

where for simplicity we have suppressed the dependence of W_n on k and h. Once we have obtained (10.10), (L_∞^n, L_∞^n) (and $(\tilde{A}_n, \tilde{A}_n)$) admissibility will follow in the usual manner.

To obtain (10.10) we observe that since we can allow $k + k/h^m$ to be as small as required, Theorem 2 permits us to neglect the perturbing term R. Also by neglecting a perturbation of the same order as kR we can work with the simpler equation

$$(10.11) \qquad w_{n+1}^1 = [I - k\delta_n] U(kh) w_n^1 .$$

Now consider the function $g(t,s)$ defined in (7.11). Using the same argument given in Section 3 (see (3.16)), we see that $g_n = g(nk,0)$ will satisfy the equation

$$(10.12) \qquad g_{n+1} = [I + kD_n] g_n = [I - k\delta_n + k^2 O(1)] g_n$$

where the "O" sign in (10.12) is independent of n. Thus by

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again neglecting a perturbation of the form $kO(k+k/h^m)$ we obtain finally the equation

(10.13) $$\bar{w}_{n+1} = [I+kD_n] \, U(k,h)\bar{w}_n \; .$$

But the fundamental solution to (10.13) is simply

(10.14) $$\bar{w}_n\bar{w}_j^{-1} = g(nk,jk)U(k,h)^{n-j}$$

as one can verify immediately, using the commutivity properties of g_n with B_h (see (8.1e)) in the case that δ is an operator. Now the estimate

(10.15) $$\|\bar{w}_n\bar{w}_j^{-1}\| \le \bar{K} \, e^{-\bar{\alpha}k(n-j)} \; ; \qquad n \ge j,$$

follows from (7.10a) and (10.9) and we can now apply Theorem 2, noting, again that the terms K_1, α_1, ε_0 depend only on the terms $\bar{K}$ and $\bar{\alpha}$ (see the hypothesis of Theorem 2), to obtain (10.10).

We now have (L_n^∞, L_n^∞) (or $(\tilde{A}_n, \tilde{A}_n)$) admissibility for (10.6) and it is only necessary to study the convergence in the case that the inhomogeneous term is given by (10.7b). To do this we must now assume that B_h^2 approximates B^2; namely that there exists an operator $\tilde{L}$ such that if $e \in D_{\tilde{L}} \cap D_{B^2}$ then

(10.16) $$\|(B^2 - B_h^2)e\| \le C_2 h^j \|\tilde{L}e\|$$

We also assume that $\tilde{L}$ satisfies the same assumption as L (see (8.13)) so that if Lf and $\tilde{L}f$ are AP we can conclude

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(using the same argument as in (8.16) and (8.17))

(10.17) (a) $\| (B-B_h)y \|_\infty = O(h^j)$

   (b) $\| (B^2-B_h^2)y \|_\infty = O(h^j)$.

If $w_n(k,h)$ is the unique AP solution to (10.6), we can now show that w_n converges to y_n uniformly in n.

   In fact since y is C^3, y_n satisfies (10.5) up to an error $O(k^3)$. If we then make use of (10.17) we see that y_n will satisfy (10.6) with an error $kO(h^j+k^2)$, where this is uniform in n. Then as the admissibility bound of (10.6) is independent of h (by (10.10)), we can conclude

(10.18) $\| w_n-y_n \|_\infty = O(h^j+k^2) = O(h^j+h^{2m})$

where we have used (10.2).

   For the example described in (8.2) we will have

(10.19) $L = \dfrac{\partial^3}{\partial x^3}$ and $\tilde{L} = \dfrac{\partial^4}{\partial x^4}$, $c = \frac{1}{2}$, m = 1, j = 2 ,
   (periodic boundary conditions)

   Finally we would like to point out that in the term $\tilde{f}_n$ in (10.7b) one can replace $B_{ch}f_n$ by Bf, without changing the order of the error. This is simply because Lf is AP, hence

(10.20) $\| (B-B_h)f \|_\infty = O(h^j)$.

11. Linear Perturbations

We now consider the perturbed system

$$(11.1) \qquad \dot{y} = [B-\delta]y + D(t)y + f(t)$$

Here $D(t)$ is a bounded operator for each t and is almost periodic as an operator function. We also require

$$(11.2) \qquad \| D \|_\infty \leq \varepsilon_0$$

where ε_0 is to be specified.

The first step in finding AP solutions to (11.1) is to solve the equation

$$(11.3) \qquad y = C(Dy) + C(f) \ .$$

This equation can be solved uniquely by the contracting mapping principle provided $\mu = \varepsilon_0 \bar{K} < 1$, where $\bar{K}$ is the norm of C (see (7.18)). The contracting mapping principle also shows that the assignment of y from f is a bounded operator and in fact

$$(11.4) \qquad \| y \|_\infty \leq \frac{\bar{K}}{1-\mu} \| f \|_\infty \equiv K_1 \| f \|_\infty$$

where K_1 depends only on ε_0 and is independent of D.

We want to show that y satisfies (11.1), and if we assume that D, δ and f are C^1, the results of Section 7 show that it is sufficient to show that y is C^1. This will follow if we observe that y will be the limit of iterates

y_i defined as

(11.5)
(a) $y_0 \equiv 0$

(b) $y_{i+1} = C(Dy_i) + C(f)$.

We will then have for each i

(11.6) $\dot{y}_{i+1} = C(D\dot{y}_i) + C(-\delta y_i + Dy_i + \dot{f})$

If we then define $\bar{y}$ as the unique solution to

(11.7) $\bar{y} = C(D\bar{y}) + C(-\delta y + Dy + \dot{f})$

we will have

(11.8) $\dot{y}_i \rightarrow \bar{y}.$

To see (11.8) note that $\|\dot{y}_i\|_\infty$ is bounded uniformly in i (from (11.6) and $\mu < 1$) and if g_i is defined as

(11.9) $g_i = \sup_{r \geq i} \|\bar{y} - \dot{y}_r\|$

Then $\{g_i\}$ form a nonincreasing sequence satisfying

(11.10) $g_{i+1} \leq \mu\, g_i + \underset{i \to \infty}{o(1)}$

and (11.10) implies $g_i \rightarrow 0$ which, in turn, implies (11.8). The fact that $\bar{y} = \dot{y}$ follows exactly as in Section 7 (see (7.27) ff). Observe that since y is AP, By is also AP (from (11.1)).

Now consider the operators C_h defined by (8.6). The equation

(11.11) $y_h = C_h(Dy_h) + C_h(f)$

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can be solved exactly as (11.3) can. (For simplicity we suppose that $\|C_h\| = \|C\|$ so that we have the same restriction on ε_0.)

Our first task is to show that $y_h \to y$. This will be a simple consequence of the strong convergence of C_h to C. In fact if $w_h = y - y_h$ we have

(11.12) $w_h = [C-C_h](Dy) + C_h(Dw_h) + [C-C_h](f)$

whence

(11.13) $\|w_h\|_\infty \leq \mu\|w_h\|_\infty + \underset{h\to 0}{o(1)}$

and this certainly implies $\|w_h\| \xrightarrow[h\to 0]{} 0.$

Now of course this would not be useful unless we could prove

(11.14) $\|w_h\|_\infty = O(h^j)$

This estimate can be shown provided we make certain assumptions on the interaction of D and the operator L defined in (8.11). We first observe that w_h satisfies the equation

(11.15) $w_h = C_h(Dw_h) + C_h([B-B_h]y)$

and using the analogue of (11.4) for the equation (11.11), together with the uniform boundedness of the operators C_h, we see

(11.16) $\|w_h\|_\infty = O(\|[B-B_h]y\|_\infty)$,

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and it is sufficient to show

(11.17) $\| [B - B_h] y \|_\infty = O(h^j)$.

To obtain (11.17) we make an assumption on the operator
L. We will call it Assumption I as we will later have to make
the same type of assumption for other operators.

Assumption I: Suppose there exists a sequence of
operators $L_0 = I, L_1, \ldots, L_n = L$ such that

(11.18) (a) L_i are closed

(b) $E = D_{L_0} \supseteq D_{L_1} \supseteq \cdots \supseteq D_{L_n}$

(c) L_i commutes with Z_t for $t \geq 0$ (i.e. $Z_t: D_{L_i} \to D_{L_i}$

and $L_i Z_t e = Z_t L_i e$ for $e \in D_{L_i}$)

(d) If $e \in D_{L_\ell}$ then $D(t)e \in D_{L_\ell}$ and

$$L_\ell D(t)e = \sum_{j=0}^{n} E_j^\ell(t) L_j e$$

where the operators $E_j^\ell(t)$ are bounded and AP,
and $E_\ell^\ell = D$.

Assumption I is clearly motivated by the example where
D(t) is multiplication by some function $d(t,x)$ (x is the
spatial variable).

Now Assumption I implies that if $L_\ell f$ is AP then $L_\ell y$ is AP,
for $\ell = 1, \ldots, n$. First note that if $z = C(f)$ then

(11.19) $L_\ell z = C(L_\ell f)$.

This is a consequence of (11.18a,c) using the same argument
as was used for B (see (7.34) ff). Now if we define $z_{\ell;i} = L_\ell y_i$
where the y_i are defined in (11.5); then it certainly follows

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from (11.18d) that $z_{\ell;i}$ is AP for i and for $\ell = 0,\ldots,n$. We will now show that the sequence $\{z_{\ell;i}\}$ converges as $i \to \infty$. To see this first set $\ell = 1$. We then have

$$(11.20) \quad z_{1;i+1} = C(Dz_{1;i}) + C(E_0^1(t)y_i) + C(f)$$

and $z_{1,i} \to z_1$, where z_1 is the solution to

$$(11.21) \quad z_1 = C(Dz_1) + C(E_0^1 y) + C(f) .$$

In fact $w_{1,i} = z_1 - z_{1,i}$ satisfies the equation

$$(11.21) \quad w_{1,i+1} = C(Dw_{1,i}) + C(E_0^1(y-y_i))$$
and

$$(11.22) \quad \|w_{1,i}\|_\infty = \|z_1-z_{1,i}\|_\infty \xrightarrow[i\to\infty]{} 0$$

follows from the same argument used in proving y is C^1 (see (11.9) ff). Finally by the closure of L_1 we must have $z_1 = L_1 y$. It is clear that a simple induction, using the same proof, will establish that

$$(11.23) \quad L_\ell y_i \xrightarrow[i\to\infty]{} L_\ell y , \qquad \ell = 1,\ldots,n.$$

In particular $L (= L_n)y$ is AP and (11.17), hence (11.14) is valid.

Finally we point out that if we assume a chain of operators, as in Assumption I, for the operator B, we will then have BDy AP. We will also have to make such an assumption for the operator $\tilde{L}$ (see (10.16)), when applying the Lax-Wendroff scheme to (11.1), as will be discussed shortly.

We will now apply to (11.1) the two methods of time discretization that have already been described. We are interested in the existence of AP sequence solutions, or equivalently exponential stability for the homogeneous difference equation, with k and h restricted according to relations (9.4) and (10.2). As one might expect this will be a simple consequence of Theorem 2, provided ε_0 is sufficiently small.

We consider first the linear multistep scheme as discussed in Section 9. Since the linear term C_n of the difference scheme (see (2.11)) is a smooth function of its $\ell+1$ arguments, it follows that if $k + k/h^m$ is sufficiently small (i.e. $\|kB_h\|$ is small) then the difference equation for (11.1) can be written

(11.24) $w_{n+1} = C_n^* w_n + k\ O(\varepsilon_0)w_n + k\ \tilde{f}_n$

where C_n^* is the linear term corresponding to the unperturbed version of (11.1), i.e.

(11.25) $\dot{y} = [B_h - \delta]y + f$

Since we have shown in Section 9 that in a region of the form

(11.26) $k + \dfrac{k}{h^{2m}} = O(1)$

the unperturbed homogeneous difference equation

(11.27) $w_{n+1} = C_n^* w_n$

is exponentially stable, with constants K_1 and α_1 independent

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of k and h (see (9.7) ff) it will follow from Theorem 2
(see the remark following (9.16)) that in the same region
the homogeneous version of (11.24) will be exponentially
stable if ε_0 is small enough. It then follows easily
that the inhomogeneous equation (11.24) will have (L_∞^n, L_∞^n)
(or $(\tilde{A}^n, \tilde{A}^n)$) admissible, with an admissibility bound
independent of k and h.

We can thus obtain AP sequence solutions to (11.24)
and, as shown in Section 9 (see (9.18) ff), we will obtain
convergence of $O(k^p)$ provided $\|y_h^{p+1}\|_\infty$ is bounded in h.
However, as one can trivially see, if D, δ, and f are C^{p+1}
then the same argument used to show y is C^1 (see (11.6) ff)
can be repeated p+1 times and applied to the operators C_h
as well as C to show that y and y_h are C^{p+1} and in fact (11.7)
can be generalized to (compare with (9.18))

$$(11.28) \quad y^i = C(Dy^i) + C([(D-\delta)y]^i - (D - \delta)y^i + f^i) ,$$
$$i = 1,\ldots,p+1,$$

with a similar formula for y_h and C_h. This certainly shows
that $\|y_h^i\|$ will be bounded in h for i = 1,...,p+1 and
thus that we can obtain a complete extension of the theory
of Section 9. We note in passing, that the same proof
which led to (11.13) will show that $y_h^i \to y^i$.

The extension of the Lax-Wendroff scheme will be just
as simple. In fact (under the assumption that D, δ and f
are C^3) the difference equation for (11.1) will have the

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same linear term as the equation for (11.25), (10.6)
except for a perturbation

$$(11.29) \quad k[D_n + \frac{k}{2} ((B_{ch} - \delta_n)D_n + D_n(B_{ch} - \delta_n) + D_n^2 + \dot{D}_n)]$$

as one can easily see by carrying out the expansion of
Section 10 for the equation (11.1). It follows that if
$k/h^m = O(1)$ (so that $k\|B_{ch}\|$ is bounded), Theorem 2 is
applicable, provided k and ε_0 are sufficiently small. One
can then obtain AP sequence solutions and carry out the
convergence argument of Section 10 (see (10.17) ff),
with the only difficulty being the verification that

$$(11.30) \qquad \| (B^2 - B_h^2)y\|_\infty = O(h^j)$$

and this will follow immediately if one assumes a chain as
in Assumption I for the operator $\tilde{L}$ introduced in (10.16).

Appendix.   Proof that (L_∞, L_∞) Admissibility Implies

an Exponential Dichotomy.

Here we would like to give a simplified proof of the
proposition that $(\tilde{A}, \tilde{A})$ admissibility for the inhomogeneous
equation

(A.1) $\dot{y} = Ay + f$

implies an exponential dichotomy for the homogeneous equation

(A.2) $\dot{y} = Ay$

It is assumed of course that $A(t)$ is AP although we will
actually do the L_∞ case. The proof is valid only if the
underlying space E is finite dimensional and we can then
regard A as a matrix and y and f as m vectors, where m is
the dimension of E. The reader is referred to Section 5
where an entirely similar proof for the difference equation
case is given.

The general proof is given in M & S (p. 344, Theorem
103.A), but this proof relies strongly on preceding material.
A finite dimensional proof is given in Coppel (p. 134 ff)
for an equation on $[0, \infty)$. His proof can be extended to an
equation on the whole real axis, but the proof given here
is simpler and more in keeping with the ideas of Massera
and Schäffer.

It is shown in M&S (Theorem 103.A) that $(\tilde{A}, \tilde{A})$ admissi-
bility implies (L_∞, L_∞) admissibility and this will be our

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starting point. We thus assume that for every f in L_∞, (A.1) has a unique solution y in L_∞ and the estimate

(A.3)
$$\|y\|_\infty \leq \bar{K}\|f\|_\infty$$

will hold for a certain constant $\bar{K}$.

Now let S_1 be the subspace of initial data which gives rise to solutions to (A.2) that are bounded for $t \in [0,\infty)$. Let S_2 be the analogous subspace giving rise to solutions that are bounded for $t \in (-\infty,0]$. We must have $S_1 \cap S_2 = \{0\}$ because any nontrivial, bounded solution to (A.2) would violate the uniqueness requirement of our definition of admissibility. Let S_3 be any subspace complementary to $S_1 \oplus S_2$ so that

(A.4)
$$E = S_1 \oplus S_2 \oplus S_3$$

Let P_1, P_2, P_3 be the associated projections, i.e. P_1 is the projection onto S_1 along $S_2 \oplus S_3$, etc. We have

(A.5)
$$I = P_1 + P_2 + P_3$$

$$P_i P_j = 0, \quad i = 1,2,3; \quad j = 1,2,3; \quad i \neq j.$$

We will first show $S_3 = \{0\}$, i.e. $P_3 = 0$. This will follow from the variation of constants formula, just as in the difference equation case. If f has compact support and $y(t)$ is the unique L_∞ solution to (A.1), then since y is a bounded solution to (A.2) for large $|t|$, we must have

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$$(A.6) \quad (a) \quad P_2 y_0 = - \int_0^\infty P_2 Y^{-1}(s) \ f(s) \ ds$$

$$(b) \quad P_3 y_0 = - \int_0^\infty P_3 Y^{-1}(s) \ f(s) \ ds$$

$$(c) \quad P_1 y_0 = \int_{-\infty}^0 P_1 Y^{-1}(s) \ f(s) \ ds$$

$$(d) \quad P_3 y_0 = \int_{-\infty}^0 P_3 Y^{-1}(s) \ f(s) \ ds$$

Here $Y(t)$ is the fundamental solution to (A.2).

Now for any vector $z \in E$, if we set $f = \chi_{[0,1]} Y(t) z$ then (A.6b,d) yield immediately $P_3 z = 0$ and we can conclude $P_3 = 0$, i.e.

$$(A.7) \qquad \begin{aligned} E &= S_1 \oplus S_2 \\ I &= P_1 + P_2 \end{aligned}$$

Using (A.6a,c) we also see that for any f with compact support we have

$$(A.8) \qquad y(t) = \int_{-\infty}^\infty G(t,s) \ f(s) \ ds$$

where

$$(A.9) \qquad G(t,s) = \begin{cases} Y(t) P_1 Y^{-1}(s) \ , & t \geq s \\ -Y(t) P_2 Y^{-1}(s) \ , & s \geq t \end{cases}$$

We will next use (A.3) to show that for every fixed t, $G(t,s)$ is in $L_1(-\infty, \infty)$ as a function of s, and in fact

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$$(A.10) \qquad \int_{-\infty}^{\infty} ds \; \|G(t,s)\| \leq \bar{K}$$

where $\bar{K}$ is given in (A.3). This is a simple consequence of using (A.8) as an operator on $L_{\infty}(I)$, where I is any compact interval to conclude that

$$(A.11) \qquad \int_{I} ds \; \|G(t,s)\| \leq \bar{K}$$

and then letting $I \to (-\infty, \infty)$.

We now will show that (A.10) implies

$$(A.12) \qquad \|G(t,s)\| = O(1) \; .$$

To see this we simply observe that $Y^{-1}(s)$ satisfies the adjoint equation

$$(A.13) \qquad Z' = - Z \, A(s)$$

where we have used "'" to stand for d/ds. Hence fixing t and considering the region $s \leq t$, we have

$$(A.14) \qquad G(t,s)' = - G(t,s) \, A(s)$$

Now since A is bounded it follows that $G'(t,s)$ is in L_1, as a function of s, on the interval $(-\infty, t]$ with an L_1 norm which can be bounded uniformly in t. This of course means that $G(t,s)$ must approach a limit as $s \to -\infty$ and since $G(t,s)$ is in L_1 (as a function of s), this limit can only be zero. We thus have

$$(A.15) \qquad G(t,s) = \int_{-\infty}^{s} dr \; G'(t,r) = O(1) \; , \quad s \leq t,$$

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and a similar argument establishes (A.12) for $s \geq t$.

We must now show that (A.12) can be replaced by an exponentially decaying factor, and to do this we consider perturbations of (A.1) and (A.2); namely for small $\varepsilon > 0$ consider the equations

(A.16)
$$\dot{y} = [A-\varepsilon I]y + f$$

(A.17)
$$\dot{y} = [A-\varepsilon I]y$$

If ε is sufficiently small we will have (L_∞, L_∞) admissible for (A.16). This is a simple consequence of the contracting mapping principle analogous to the argument of Section 4C (see (4.9)ff). The contracting mapping principle will also show that the admissibility bound for (A.16) will be given by (see A.3)

(A.18)
$$\bar{K}_\varepsilon = \frac{\bar{K}}{(1-\varepsilon\bar{K})} \leq K , \qquad |\varepsilon| \leq \varepsilon_0 ,$$

if ε_0 is sufficiently small. Thus the admissibility bound for (A.16) can be taken independent of ε, if ε is small enough.

Now the fundamental solution to (A.18) is

(A.19)
$$Y^\varepsilon(t) = e^{-\varepsilon t} Y(t)$$

and if P_1^ε, P_2^ε denote the projections replacing P_1 and P_2 , while $G^\varepsilon(t,s)$ is the new Green's function, then the analysis which led to (A.12), and which depended only on admissibility, is equally valid here and we can conclude

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(A.20) $\qquad \| G^\varepsilon(t,s) \| = O(1) \; ; \; \| G^\varepsilon(t,\cdot) \|_1 = O(1) \; ,$

where the bound in (A.20) can be taken independent of ε (because of (A.18)).

The estimate (A.20) means in particular that for small ε there is a constant K_2 independent of ε such that if $s \geq t$,

(A.21) $\qquad \| Y(t) P_2 Y^{-1}(s) \| \leq K_2 \, e^{-\varepsilon(s-t)}$

and it is only necessary to show that

(A.22) $\qquad \begin{aligned} P_1^\varepsilon &= P_1 \\ P_2^\varepsilon &= P_2 \end{aligned}$

To prove (A.22) we let S_1^ε and S_2^ε replace S_1 and S_2. It then follows from (A.19) and $\varepsilon > 0$

(A.23) $\qquad \begin{aligned} S_1 &\subseteq S_1^\varepsilon \\ S_2^\varepsilon &\subseteq S_2 \end{aligned}$

and (A.22) will be established if we can show that the ranks of the stable and unstable projections are unchanged for small ε, and this, in turn, will follow from

(A.24) $\qquad \| P_1^\varepsilon - P_1 \| = O(\varepsilon) \; .$

To obtain (A.22) we set $H^\varepsilon(t,s) = G^\varepsilon(t,s) - G(t,s)$. Now for any f with compact support we let y and y_ε be the unique bounded solutions to (A.1) and (A.16) respectively. If $w_\varepsilon = y_\varepsilon - y$ we see that w_ε is the unique bounded solution to

(A.25) $\qquad \dot{w}_\varepsilon = A w_\varepsilon - \varepsilon \, y_\varepsilon \; ,$

and we can conclude from (A.3) together with its analogue
for (A.16) that

(A.26)
$$\|w_\varepsilon\|_\infty = O(\varepsilon)\|f\|_\infty$$

Now w_ε can also be written as

(A.27)
$$w_\varepsilon(t) = \int_{-\infty}^{\infty} ds\ H^\varepsilon(t,s)\ d(s)$$

and using the same argument that led to (A.10) we see that

(A.28)
$$\int_{-\infty}^{\infty} ds\ \|H^\varepsilon(t,s)\| = O(\varepsilon)\ .$$

(Note that the bound in (A.28) will be independent of t,
although we will only use it for $t = 0$.)

Now if $s \leq t$ we see that as a function of s, $H^\varepsilon(t,s)$
satisfies the equation

(A.29)
$$H^{\varepsilon'} = - H^\varepsilon A + \varepsilon\ G^\varepsilon(t,s)\ .$$

This equation, together with (A.20) and (A.28), shows that

(A.30)
$$\int_{-\infty}^{\infty} ds\ \|H^{\varepsilon'}(t,s)\| = O(\varepsilon)$$

and since $H^\varepsilon(t,s)$ is in L_1 as a function of s we can write for
$s \leq t$

(A.31)
$$H^\varepsilon(t,s) = \int_{-\infty}^{t} ds\ H^{\varepsilon'}(t,s) = O(\varepsilon)$$

and if we set $t = 0$, $s = 0_-$, (A.31) yields (A.24), which
proves one part of the requirement of an exponential dichotomy
and the other part follows on replacing $-\varepsilon$ by $+\varepsilon$.

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