ERROR STATISTICS FOR NORMAL RANDOM VARIABLES

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Newport, Rhode Island

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PREFACE

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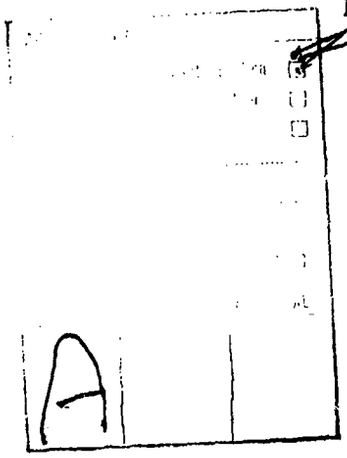

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Four commonly used error statistics (the root mean square error, the geometric mean error, the mean radial error, and the spherical (or circular) error probable) are surveyed and compared in two and three dimensions. Exact or approximate closed-form expressions are given for these statistics for zero-mean, independent, normal random variables. A change of variables reduces the case of correlated variables to the independent case. The expression for the mean radial error

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20. Abstract (cont'd)

in three dimensions and the approximations of the spherical error probable are new. The dependence of the statistics on the variances of the errors is presented graphically.

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INTRODUCTION

The performance of systems involving target positioning, weapon delivery, scoring, etc., is influenced by a variety of random disturbances or errors. Hence, a complete description of system performance requires knowledge of the probability distribution of system errors. Often, a single statistic, derived from these distributions, is used to quantify the results of the tested systems. And, because different statistics are used for different applications, it is difficult (sometimes impossible) to meaningfully assess the relative performance of individual systems. Similarly, it is not always simple to convert from one statistic to another unless the data are available or assumptions are made about the underlying distributions.

The use of the normal probability distribution for describing errors and related phenomena is almost universal. Its applicability in describing errors can be deduced from basic assumptions,[1] or its use can be justified by the central limit theorem.

An n-dimensional normal random vector, whose components represent errors or otherwise, can be completely described statistically by specifying its mean vector and covariance matrix. However, there are many instances where it is necessary to represent the distribution or to describe system performance with a single statistic or number. Four statistics commonly used for this purpose are the root mean square error (RMSE), the mean radial error (MRE), the geometric mean error (GME), and the spherical or circular error probable (SEP or CEP). Each of the above statistics has been individually treated in the literature, but often only for special cases, such as equal variances along the component axes.

This report examines the relationship between these four error statistics for the case where the errors obey a normal or Gaussian probability law. The four statistics are surveyed and compared for zero-mean, normal random variables in two and three dimensions. Exact expressions are given for RMSE, MRE, and GME. Approximate expressions are presented for SEP and CEP. The formulas apply for independent random variables. However, it is shown how these equations can also be used to calculate the statistics for variables having an arbitrary covariance matrix. The results are presented in a form that facilitates the evaluation of the statistics.

STATISTICS FOR INDEPENDENT RANDOM VARIABLES

The joint probability density function for n real, zero-mean, normal random variables $x_1, x_2, \ldots, x_n$ has the form

$$f_X(x_1, x_2, \ldots, x_n) = (2\pi)^{-n/2} |K|^{-1/2} \exp\left(-1/2\, x^T K^{-1} x\right), \quad (1)$$

where x is a column vector, and $|K|$ is the determinant of the $n \times n$ covariance matrix $K = (k_{ij})$, with $k_{ij} = E(x_i x_j)$, the expected value of the product $x_i x_j$. For independent random variables, $k_{ij} = 0$ unless $i = j$. In any case, $k_{ii} = \sigma_i^2$, the variance of x_i. The surfaces of constant probability are found by setting the quadratic form $-1/2\, x^T K^{-1} x$ equal to a constant. These surfaces are ellipses in two dimensions and ellipsoids in three dimensions. Since the variables are uncorrelated with zero means, the density function for the n variables is completely determined by the standard deviations (sigmas) $\sigma_1, \sigma_2, \ldots, \sigma_n$. It will be assumed, without loss of generality, that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n$.

ROOT MEAN SQUARE ERROR (RMSE)

The RMSE for n dimensions is defined as

$$\text{RMSE} = \left(\sum_{i=1}^{n} k_{ii}\right)^{1/2} = \left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2}. \quad (2)$$

This statistic is trivial to compute and is widely used because of its simplicity. It differs from the other three in that the definition itself provides the desired closed-form expression in terms of the variances of the x_i.

MEAN RADIAL ERROR (MRE)

For n variables, the MRE is defined as the expected value of the radial error

$$\left(x_1^2 + x_2^2 + \ldots x_n^2\right)^{1/2}.$$

That is,

$$\text{MRE} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(x_1^2 + x_2^2 + \ldots x_n^2\right)^{1/2} f_X(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n. \quad (3)$$

Or, in shorter notation,

$$\text{MRE} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \|\zeta\| \, f_X(\zeta) \, d\zeta . \tag{4}$$

For $n = 2$, the moments of the radial error are known,[2] with the first moment being

$$\text{MRE} = (2/\pi)^{1/2} \, \sigma_1 \, E\left(1 - \sigma_2^2/\sigma_1^2\right), \tag{5}$$

where $E(m)$ is the complete elliptic integral of the second kind[3]

$$E(m) = \int_0^{\pi/2} \left(1 - m \sin^2 \theta\right)^{1/2} d\theta. \tag{6}$$

Tables for $E(m)$ can be used to evaluate equation (5), and approximations are also available[3] if computer programming is desired.

Expressions for the MRE are not available for $n \geq 3$, except for the special case where $\sigma_1 = \sigma_2 = \ldots \sigma_n$. In this case, both the mean and variance of the radial error can be found,[4] together with the distribution function. The distribution function of the radial error is available and will be discussed later for the case of unequal variances.

A formula for the qth moment of the radial error in n dimensions has recently been developed by D. Childs, where q is any non-negative number, not necessarily an integer. The case $q = 1$ and $n = 3$ gives the radial error in three dimensions. Since that work is not yet published, the proof for the special case is included here. By definition

$$\text{MRE} = (abc)^{1/2} \, (\pi)^{-3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$$

$$\times \exp\left(-ax_1^2 - bx_2^2 - cx_3^2\right) dx_1 \, dx_2 \, dx_3 , \tag{7}$$

where $a = 1/\left(2\sigma_3^2\right)$, $b = 1/\left(2\sigma_2^2\right)$, and $c = 1/\left(2\sigma_1^2\right)$. Hence, $a \geq b \geq c$.

Let

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta.$$

The Jacobian for the transformation is $r^2 \sin\theta$, and

$$\text{MRE} = (abc)^{1/2} (\pi)^{-3/2} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} r^3 \exp(-r^2 \beta) \sin\theta \, dr \, d\varphi \, d\theta, \qquad (8)$$

where $\beta = a \cos^2\varphi \sin^2\theta + b \sin^2\varphi \sin^2\theta + c \cos^2\theta.$

Integration over r gives:[5]

$$\text{MRE} = (abc)^{1/2} (\pi)^{-3/2} \int_0^{\pi} \int_0^{2\pi} \sin\theta / (2\beta^2) \, d\varphi \, d\theta. \qquad (9)$$

Since $\beta = b \sin^2\theta + c \cos^2\theta + (a - b) \sin^2\theta \cos^2\varphi$, the integral over φ becomes

$$I = \int_0^{2\pi} 1/(2\beta^2) \, d\varphi = 4 \int_0^{\pi/2} 1/2 \, (\lambda + \mu \cos^2\varphi)^{-2} \, d\varphi,$$

where $\lambda = b \sin^2\theta + c \cos^2\theta$, and $\mu = (a - b) \sin^2\theta$. In deriving expressions for the MRE in three dimensions, four separate cases are treated. If $a = b = c$, then $I = \pi/c^2$, and it follows from equation (9) that

$$\text{MRE} = 2 (2/\pi)^{1/2} \sigma_1 \quad \text{for } \sigma_1 = \sigma_2 = \sigma_3; \qquad (10)$$

otherwise,

$$I = 2 \int_0^{\pi/2} (\lambda + \mu/2 + \mu/2 \cos 2\varphi)^{-2} \, d\varphi = \int_0^{\pi} (\lambda + \mu/2 + \mu/2 \cos\psi)^{-2} \, d\psi,$$

and

$$I = \pi (\lambda + \mu/2) \left[(\lambda + \mu/2)^2 - (\mu/2)^2 \right]^{-3/2}. \qquad (11)$$

Replacing (11) back in equation (9), expanding μ and λ, and letting $x = \cos\theta$, gives

$$\text{MRE} = (abc)^{1/2}\,(\pi)^{1/2}\int_{-1}^{1}\left[1/2(a+b) - \left(1/2(a+b) - c\right)x^2\right]$$

$$\times\left(a - (a-c)x^2\right)^{-3/2}\left(b - (b-c)x^2\right)^{-3/2}\,dx\,. \tag{12}$$

When $a > b = c$, it follows that

$$\text{MRE} = (2/\pi)^{1/2}\left[\sigma_3 + \sigma_1\left(1 - \sigma_3^2/\sigma_1^2\right)^{-1/2}\sin^{-1}\left(1 - \sigma_3^2/\sigma_1^2\right)^{1/2}\right] \tag{13}$$

$$\text{for } \sigma_1 = \sigma_2 > \sigma_3\,.$$

Equation (13) was written in the above form to illustrate that the ratio of the MRE to σ_1, the largest sigma, depends only on the ratios σ_2/σ_1 and σ_3/σ_1. This property is enjoyed by all the other statistics discussed here, and the equations will be written in a form that displays this fact. For the case $a = b > c$, the result is

$$\text{MRE} = (2/\pi)^{1/2}\left(\sigma_1 + \sigma_3^2/(2\delta\sigma_1)\,\ln\left[(1+\delta)/(1-\delta)\right]\right) \tag{14}$$

$$\text{for } \sigma_1 > \sigma_2 = \sigma_3,$$

where $\delta = \left(1 - \sigma_3^2/\sigma_1^2\right)^{1/2}$.

For the case $a > b > c$, equation (12) can be written in the form

$$\text{MRE} = 2(abc/\pi)^{1/2}\left[(a-c)(b-c)\right]^{-3/2}\left[\tau^2\int_0^1\left(\alpha^2 - x^2\right)^{-1/2}\left(\beta^2 - x^2\right)^{-3/2}\,dx\right.$$

$$\left.+\left(\sigma^2 - \tau^2\alpha^2\right)\int_0^1\left(\alpha^2 - x^2\right)^{-3/2}\left(\beta^2 - x^2\right)^{-3/2}\,dx\right]\,,$$

where $\alpha^2 = b/(b-c)$, $\beta^2 = a/(a-c)$, $\sigma^2 = (a+b)/2$, and $\tau^2 = (a+b)/2 - c$.

Evaluating the integrals[6] gives

$$\text{MRE} = (2/\pi)^{1/2}\left[(\sigma_2\sigma_3)/\sigma_1 + (\sigma_3^2/\sigma_1)\,(1-\sigma_3^2/\sigma_1^2)^{-1/2}\,F(\eta,t) + \sigma_1\,(1-\sigma_3^2/\sigma_1^2)^{1/2}\,E(\eta,t)\right] \quad \text{for } \sigma_1>\sigma_2>\sigma_3\,, \tag{15}$$

where $\eta = \sin^{-1}\left(1-\sigma_3^2/\sigma_1^2\right)^{1/2}$,

$$t = \left[\left(1-\sigma_2^2/\sigma_1^2\right)/\left(1-\sigma_3^2/\sigma_1^2\right)\right]^{1/2},$$

and where $F(\eta, t)$ and $E(\eta, t)$ are elliptic integrals of the first and second kind, respectively:

$$F(\eta,t) = \int_0^{\eta}\left(1-t^2\sin^2\varphi\right)^{-1/2}\,d\varphi$$

$$E(\eta,t) = \int_0^{\eta}\left(1-t^2\sin^2\varphi\right)^{1/2}\,d\varphi\,.$$

Equation (15) must be evaluated with the aid of tables since no approximations are available for E and F. All of the other expressions given can easily be put in algorithmic form.

GEOMETRIC MEAN ERROR (GME)

For n = 2, the GME is defined as the radius of a circle having the same area as the 50-percent ellipse, where the 50-percent ellipse is defined by the equation

$$\iint_{x_1^2/\sigma_1^2 + x_2^2/\sigma_2^2 \le c^2} f_X(x_1,x_2)\,dx_1\,dx_2 = 0.5\,. \tag{16}$$

The substitution $x_1 = \sigma_1\, r\cos\omega$, $x_2 = \sigma_2\, r\sin\omega$ leads to the well known result

$$1 - e^{-c^2/2} = 0.5\,. \tag{17}$$

Hence, $c = 1.177$; and, since the 50-percent ellipse $x_1^2/\sigma_1^2 + x_2^2/\sigma_2^2 = c^2$ has an area of $\pi c^2 \sigma_1 \sigma_2$, one gets

$$\text{GME} = 1.177 (\sigma_1 \sigma_2)^{1/2} . \tag{18}$$

For $n = 3$, the GME is the radius of a sphere having the same volume as the 50-percent ellipsoid. The analog of equation (17) is

$$(2/\pi)^{1/2} \int_0^c r^2 \exp\left(-r^2/2\right) dr = 0.5 .$$

Integration by parts gives the transcendental equation

$$H\left(c/\sqrt{2}\right) - c/\sqrt{2}\, H\left(c/\sqrt{2}\right) = 0.5 , \tag{19}$$

where H and H' are the error function and its derivative:[7]

$$H(x) = 2/\pi \int_0^x \exp\left(-\alpha^2\right) d\alpha ,$$

$$H'(x) = 2/\sqrt{\pi} \exp\left(-x^2\right) .$$

Using the tables gives $c = 1.54$, and

$$\text{GME} = c(\sigma_1 \sigma_2 \sigma_3)^{1/3} = 1.54 (\sigma_1 \sigma_2 \sigma_3)^{1/3} .$$

CIRCULAR AND SPHERICAL ERROR PROBABLE (CEP, SEP)

For $n = 2$, the CEP is defined as the radius of a circle, centered at the origin, within which the random vector (x_1, x_2) lies with probability 0.5. The SEP is similarly defined for a sphere when $n = 3$.

Exact expressions are readily available for CEP and SEP, but they are not in closed form; that is, they are usually integral expressions requiring numerical techniques in their evaluation.[8,9] Some analytical approximations are available for these quantities; but they are generally not accurate for all values of the σ_i. These results often appear as special cases of more general problems regarding the radii of circles or spheres including a certain percent of the normal probability distribution,[10,11] or, equivalently, the

distribution of the radial error. These in turn are included in the problem of calculating the distribution of quadratic forms,[12]

$$Q_k = \sum_{i=1}^{k} a_i x_i^2 ,$$

where the x_i are normal and independent with mean zero and variance one. Many of these problems can also be equated to that of integrating a circular normal distribution over certain regions in space.[13] (The cited references represent only a sampling of the extensive literature available on these topics.)

Values for CEP and SEP have also been tabulated as special cases of the distribution of quadratic forms.[14,15] Here, simple rational expressions are presented that approximate CEP and SEP with accuracies within a few percent for all values of σ_i, $i = 1, 2$ or $i = 1, 2, 3$. The approximations are not obtained analytically, but by fitting equations to existing curves. They are more uniformly accurate than existing approximations and more convenient than tables.

For $n = 2$, $\sigma_1 \geq \sigma_2$, and it is known[16] that CEP/σ_1 is approximately linear in σ_2/σ_1, for $0.3 \leq \sigma_2/\sigma_1 \leq 1$. Also, a quadratic fits the curve well on $[0, 0.3]$. Since $SEP = CEP$ when $\sigma_3 = 0$, it was conjectured that such an approximation would be suitable for all values of σ_3/σ_1. That is, it was assumed that

$$SEP/\sigma_1 \approx \begin{cases} c_1 + c_2\left(\sigma_2/\sigma_1\right) & \text{for } 0.3 \leq \sigma_2/\sigma_1 \leq 1 , \\ c_3 + c_4\left(\sigma_2/\sigma_1\right) + c_5\left(\sigma_2/\sigma_1\right)^2 & \text{for } 0 \leq \sigma_2/\sigma_1 \leq 0.3 , \end{cases}$$

where the coefficients c_i depend on σ_3/σ_1. Using existing curves[8] for SEP/σ_1 versus σ_2/σ_1, with σ_3/σ_1 being a parameter, approximate values for the c_i were found for $\sigma_3/\sigma_1 = 0, 0.1, 0.2, \ldots, 1.0$, and the c_i were plotted versus σ_3/σ_1. A linear function in σ_3/σ_1 was fit through the values of c_2 on $[0,1]$, and a piecewise linear function was fit to c_1 on $[0, 0.3]$ and $[0.3, 1]$. The c_4 and c_5 were nearly constant on $[0, 0.3]$, and c_3 was fit with a quadratic. The resulting approximations are

$$SEP \approx 0.670\sigma_1 - 0.015\sigma_2 - 0.066\sigma_3 + 0.888\sigma_2^2/\sigma_1 + 1.11\sigma_3^2/\sigma_1 \qquad (20)$$

$$\text{for } \sigma_2/\sigma_1 \leq 0.3,\ \sigma_3/\sigma_1 \leq 0.3;$$

$$SEP \approx 0.558\sigma_1 + 0.622\sigma_2 + 0.283\sigma_3 - 0.165\sigma_2\sigma_3/\sigma_1 \qquad (21)$$

$$\text{for } \sigma_2/\sigma_1 \geq 0.3,\ \sigma_3/\sigma_1 \leq 0.3;$$

$$SEP \approx 0.462\sigma_1 + 0.622\sigma_2 + 0.621\sigma_3 - 0.165\sigma_2\,\sigma_3/\sigma_1 \qquad (22)$$

$$\text{for } \sigma_2/\sigma_1 \geq 0.3,\ \sigma_3/\sigma_1 \geq 0.3\ .$$

When $\sigma_3 = 0$, equations (20) and (21) give

$$CEP \approx 0.670\sigma_1 - 0.015\sigma_2 + 0.888\sigma_2^2/\sigma_1 \quad \text{for } \sigma_2/\sigma_1 \leq 0.3\ ; \qquad (23)$$

$$CEP \approx 0.558\sigma_1 + 0.622\sigma_2 \quad \text{for } \sigma_2/\sigma_1 \geq 0.3\ . \qquad (24)$$

These approximations were evaluated and compared to the curves for values of σ_2/σ_1 and σ_3/σ_1 between 0 and 1 in 0.1 increments. The agreement was within 2 percent for all points tested, and the accuracy to which the curves[8] could be read was about 1 percent.

The best of the analytically derived approximations to SEP is probably that of Grubbs,[17] namely

$$SEP \approx \left(\sigma^2\,(1-v/9)^3\right)^{1/2}\ ,$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$

$$v = 2\left(\sigma_1^4 + \sigma_2^4 + \sigma_3^4\right)/\sigma^4\ .$$

This is somewhat simpler than the expressions in equations (20), (21), and (22), and there is agreement within a few percent. The advantage of the form discussed here is that improvements could be made by reducing the error in reading the curves and by using more refined curve fitting techniques. Significantly higher accuracy should be attainable without changing the form of the equations.

STATISTICS FOR CORRELATED VARIABLES

The formulas in the preceding section apply to independent random variables, but a change of variables allows one to use those results for the general case where K is not diagonal.

Since any covariance matrix K is real and symmetric, there exists a real orthogonal matrix P such that

$$PKP^T = PKP^{-1} = D\,, \tag{25}$$

where D is a diagonal matrix. The entries in D are its characteristic values, which are just the characteristic values of K since equation (25) defines a similarity transformation. Making the transformation

$$y = Px, \tag{26}$$

the joint density for the $y_1, y_2, \ldots, y_n$ is given by

$$f_Y\left(y_1, y_2, \ldots, y_n\right) = f_X\left(P^{-1}y\right)\left|J\right|\,, \tag{27}$$

where J is the Jacobian of the transformation P. Since P is orthogonal, $|J| = 1$, and the joint density for the y_i becomes

$$f_Y\left(y_1, y_2, \ldots, y_n\right) = (2\pi)^{-n/2}\left|D\right|^{-1} \exp\left(-1/2\; y^T\, D^{-1} y\right). \tag{28}$$

Hence, the y_i are jointly normal with the diagonal covariance matrix $PKP^{-1} = D$; that is, they are independent with variances equal to the characteristic values of K.

For the two-dimensional case, the variances d_{11} and d_{22} of y_1 and y_2 are the solutions to the quadratic characteristic equation for K; namely

$$d_{nn} = 1/2\left[k_{11} + k_{22} + (-1)^n\left(k_{11}^2 + k_{22}^2 - 2k_{11}\, k_{22} + 4k_{12}^2\right)^{1/2}\right], \tag{29}$$

$$n = 1, 2\,.$$

For the three-dimensional case, the characteristic equation is a cubic

$$d^3 + a_1 d^2 + a_2 d + a_3 = 0\,, \tag{30}$$

where $a_1 = -(k_{11} + k_{22} + k_{33})$,

$$a_2 = k_{11}\,k_{22} + k_{11}\,k_{33} + k_{22}\,k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2 ,$$

$$a_3 = -k_{11}\,k_{22}\,k_{33} + k_{11}\,k_{23}^2 + k_{22}\,k_{13}^2 + k_{33}\,k_{12}^2 - 2k_{12}\,k_{13}\,k_{23} ;$$

let

$$H = (a_2)/3 - (a_1^2)/9 ,$$

$$G = a_3 - (a_1\,a_2)/3 + (2a_1^3)/27 ,$$

and

$$E = G^2 + 4H^3 .$$

The roots to equation (29) must be real and positive. It follows that either $E = 0$ and

$$\begin{aligned} d_{11} = d_{22} &= \left|G/2\right|^{1/3} \operatorname{sgn}(G) - (a_1)/3, \\ d_{33} &= -2\left|G/2\right|^{1/3} \operatorname{sgn}(G) - (a_1)/3; \end{aligned} \tag{31}$$

or $E < 0$ and

$$d_{nn} = 2\,(-H)^{1/2} \cos\left[\frac{2\pi(n-1) + \theta}{3}\right] - (a_1)/3, \quad n = 1, 2, 3 , \tag{32}$$

where

$$\theta = \cos^{-1}\left[\frac{-G}{2(-H^3)^{1/2}}\right] . \tag{33}$$

This is not the most computationally efficient method of solving the cubic; but it is easily programmable and it is exact.

Hence, given an arbitrary covariance matrix K, either 2 x 2 or 3 x 3, its characteristic values can be obtained using equation (29) or equations (31), (32), and (33). Denoting these values by σ_1^2 , σ_2^2 (or σ_i^2 , i = 1, 2, 3), with $\sigma_1 \geq \sigma_2$ (or $\sigma_1 \geq \sigma_2 \geq \sigma_3$), the four statistics can be calculated with the equations given in the preceding section. To justify this procedure, it must be shown that each of the four statistics is invariant under the orthogonal transformation of axes defined by the matrix P.

The square of the RMSE is the trace of the matrix K and, hence, the sum of the characteristic values, which is invariant because K and D are similar matrices by equation (25).

For the other three statistics, only the case $n = 2$ will be discussed, since the arguments are essentially the same for $n = 3$.

The 50-percent probability ellipse has the equation

$$x^T K^{-1} x = c \tag{34}$$

where $c > 0$ is that unique number that satisfies the equation

$$\iint_{x^T K^{-1} x \le c} f_X(x)\, dx = 0.5 . \tag{35}$$

Making the formal change of variables $y = Px$, and recalling that $|J| = 1$ since P is orthogonal, gives

$$\iint_{y^T D^{-1} y \le c} f_X(P^{-1}y)\, dy = 0.5 ; \tag{36}$$

and it follows from (27) that

$$\iint_{y^T D^{-1} y \le c} f_Y(y)\, dy = 0.5 . \tag{37}$$

Hence, the same value of c defines the 50-percent ellipse for the y_i, given by

$$y^T D^{-1} y = c . \tag{38}$$

Let A(x) and A(y) denote the areas of the two 50-percent ellipses, with GME(x) and GME(y) being the geometric mean errors for the x and y vectors. Then, the transformation $y = Px$ gives

$$A(x) = \iint_{x^T K^{-1} x \le c} dx = \iint_{y^T D^{-1} y \le c} dy = A(y) .$$

It follows that

$$GME(x) = \left(A(x)/\pi\right)^{1/2} = \left(A(y)/\pi\right)^{1/2} = GME(y)$$

Similarly, for the MRE, let $\|x\| = (x_1^2 + x_2^2)^{1/2}$ and $\|y\| = (y_1^2 + y_2^2)^{1/2}$. Since orthogonal transformations preserve length, the transformation $y = Px$ yields

$$MRE = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|x\| f_X(x)\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\|P^{-1}y\right\| f_X(P^{-1}y)\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|y\| f_Y(y)\, dy = MRE(y) .$$

Finally, the CEP(x) is defined by the equation

$$\iint_{\|x\| \le CEP(x)} f_X(x)\, dx = 0.5 ;$$

letting $y = Px$ gives

$$0.5 = \iint_{\left\|P^{-1}y\right\| \le CEP(x)} f_X(P^{-1}y)\, dy = \iint_{\|y\| \le CEP(x)} f_Y(y)\, dy ,$$

which is the defining equation for CEP(y). Hence, CEP(x) = CEP(y).

COMPARISON OF STATISTICS

The interesting property enjoyed by each of these statistics is that the ratio of the statistic to σ_1 depends solely on the ratios σ_2/σ_1 and σ_3/σ_1, and not on the absolute values of the σ_i. It is therefore convenient to display results in terms of these ratios.

Figures 1 through 5 give the statistics divided by σ_1 as a function of σ_2/σ_1, for $\sigma_3/\sigma_1 = 0$, 0.25, 0.5, 0.75, and 1.0, respectively. In all cases considered

$$\text{RMSE} \geq \text{MRE} \geq \text{SEP} \geq \text{GME} .$$

It is also evident from these curves that the RMSE, MRE, and SEP all have the same qualitative behavior.

The RMSE, MRE, and SEP are increasing functions of σ_2/σ_1 and σ_3/σ_1, as one would expect. This also appears to be the case for the GME, except for the case when one of the sigmas equals zero, as shown in figure 6. This anomaly results because different definitions for the GME are required for the cases $n = 2$ and $n = 3$. The definition for $n = 3$ would result in GME = 0 whenever one of the sigmas equals zero, because the 50-percent ellipsoid would reduce to an ellipse, having area but no volume. Also, the GME will approach zero as one of the sigmas approaches zero.

The other three statistics, therefore, have a continuity-type property not held by the GME; namely, as one of the sigmas approaches zero, the three-dimensional statistic approaches that for the two-dimensional case.

At first, one might conclude that the GME is not a good representative error statistic, since an error vector could have two large components and one small one, with a small resulting GME. This argument is not necessarily valid, however, since one might be satisfied with an error vector if at least one component is small. In such cases, the GME would be a more suitable criterion than the other three statistics, all of which tend to be large if at least one component is large. These latter statistics appear more suitable for the most common case, where the system error is defined (for $n = 3$) as

$$E = \left(y_1^2 + y_2^2 + y_3^2\right)^{1/2} ,$$

where y_i is the error in the ith coordinate. In this context, the names given to the statistics RMSE, MRE, and SEP are quite natural.

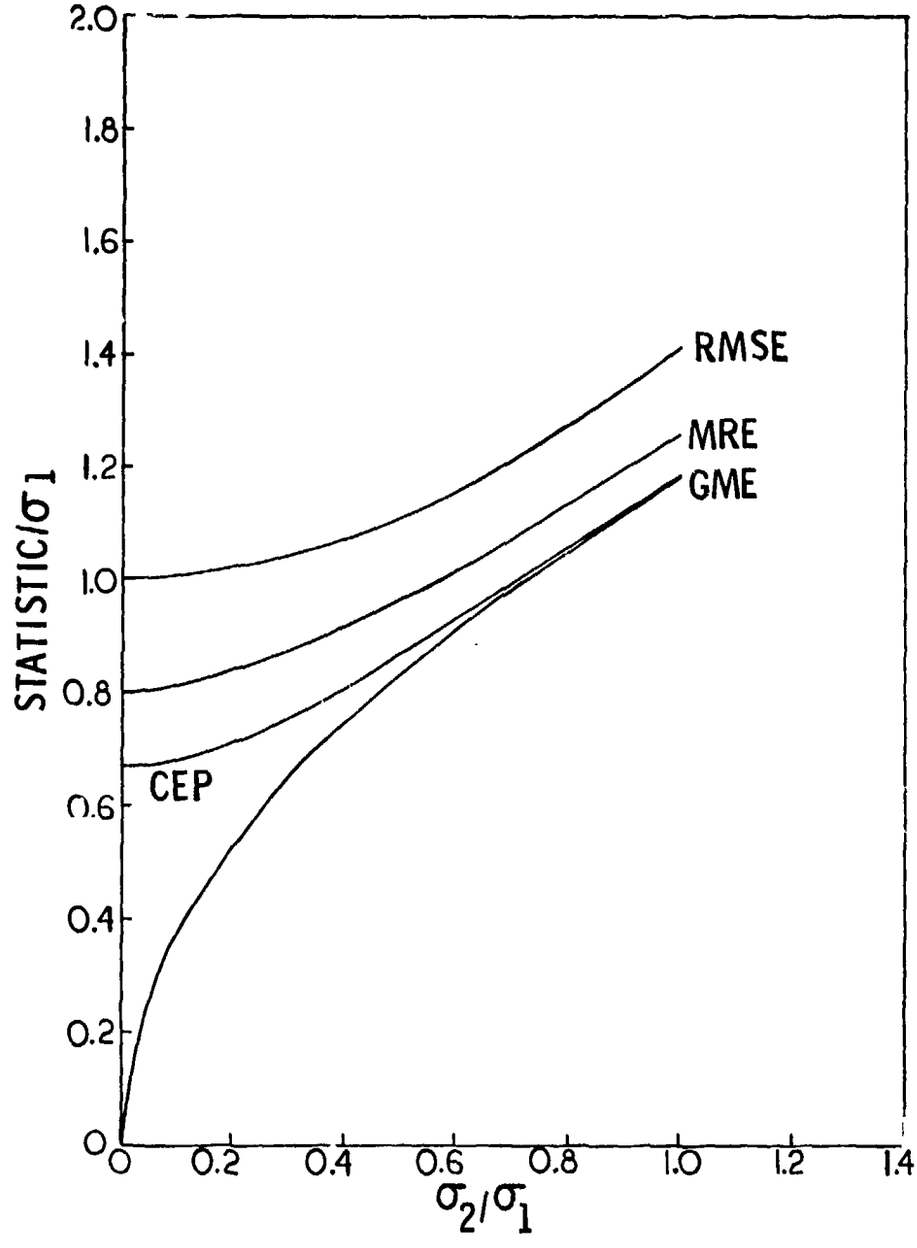

Figure 1. Statistic Normalized by σ_1 for $\sigma_3/\sigma_1 = 0$

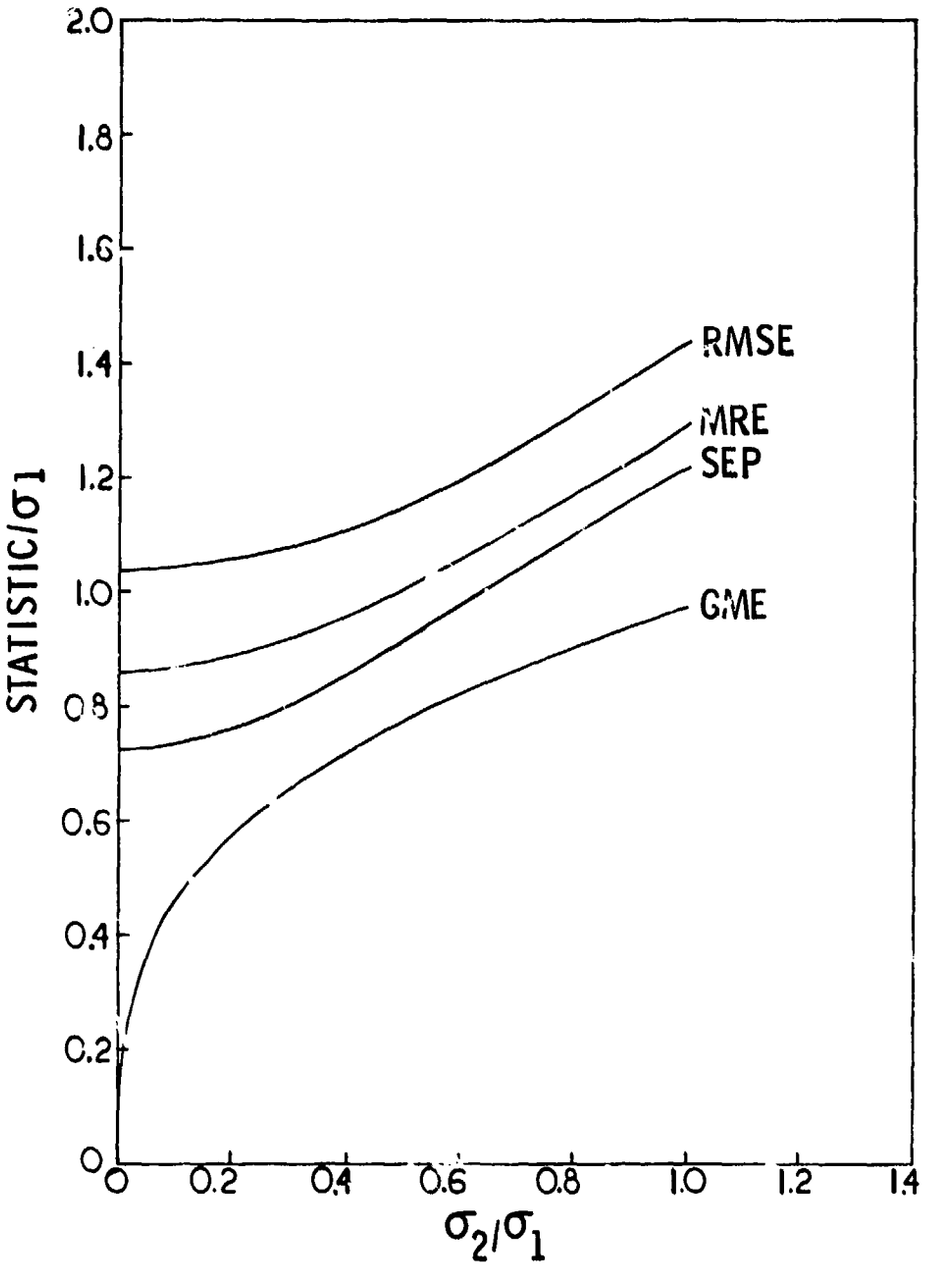

Figure 2. Statistic Normalized by σ_1 for $\sigma_3/\sigma_1 = 0.25$

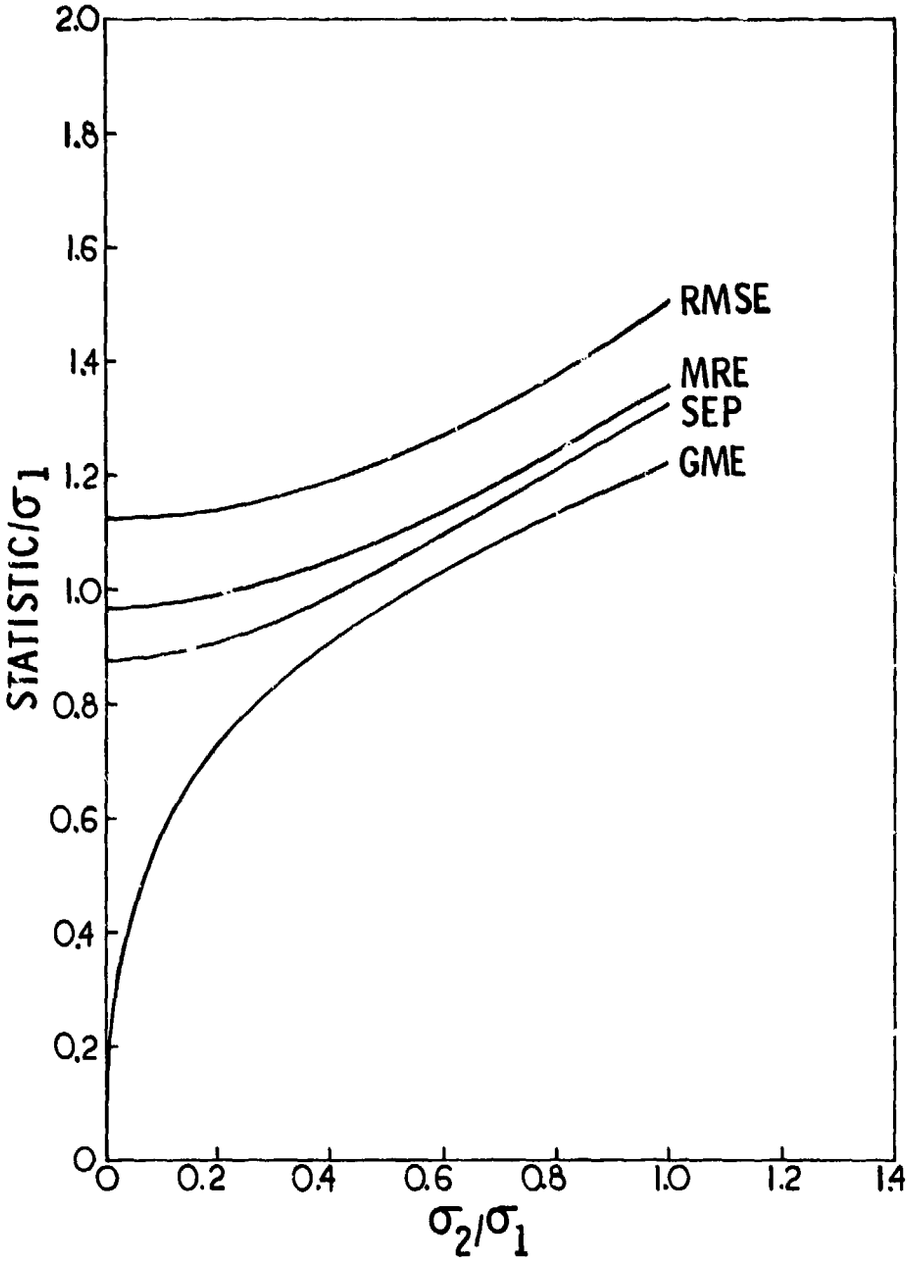

Figure 3. Statistic Normalized by σ_1 for $\sigma_3/\sigma_1 = 0.5$

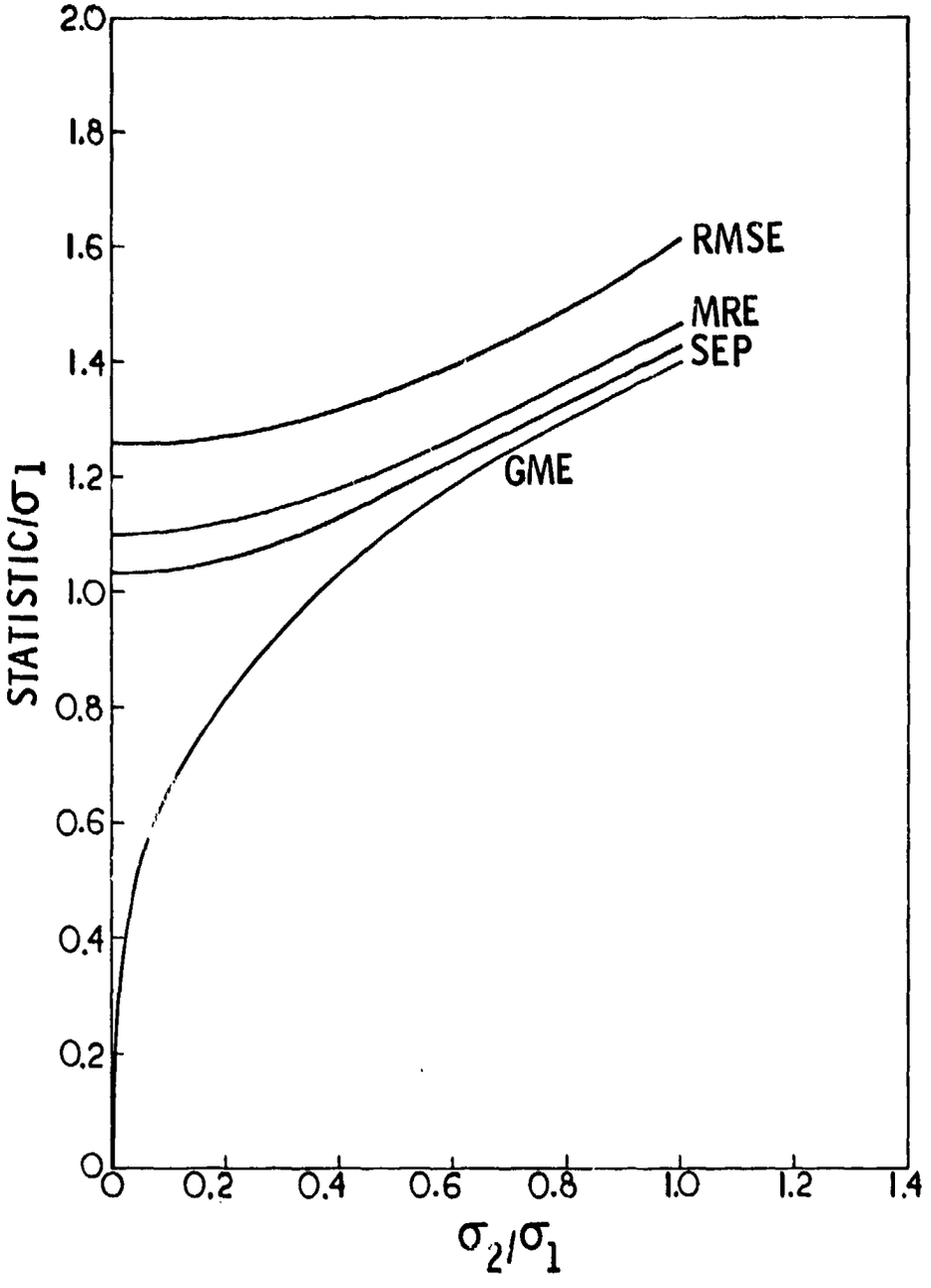

Figure 4. Statistic Normalized by σ_1 for $\sigma_3/\sigma_1 = 0.75$

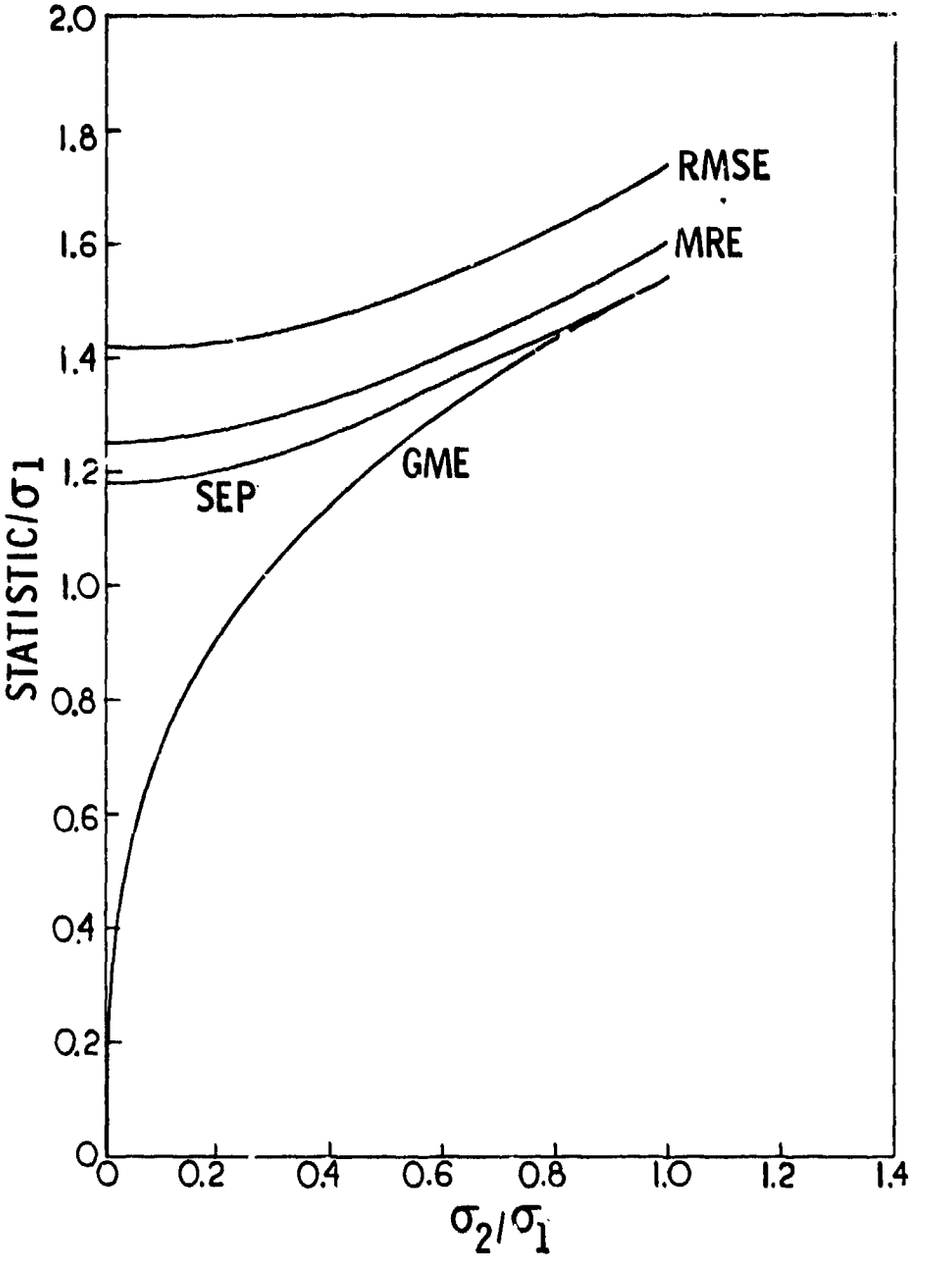

Figure 5. Statistic Normalized by σ_1 for $\sigma_3/\sigma_1 = 1.0$

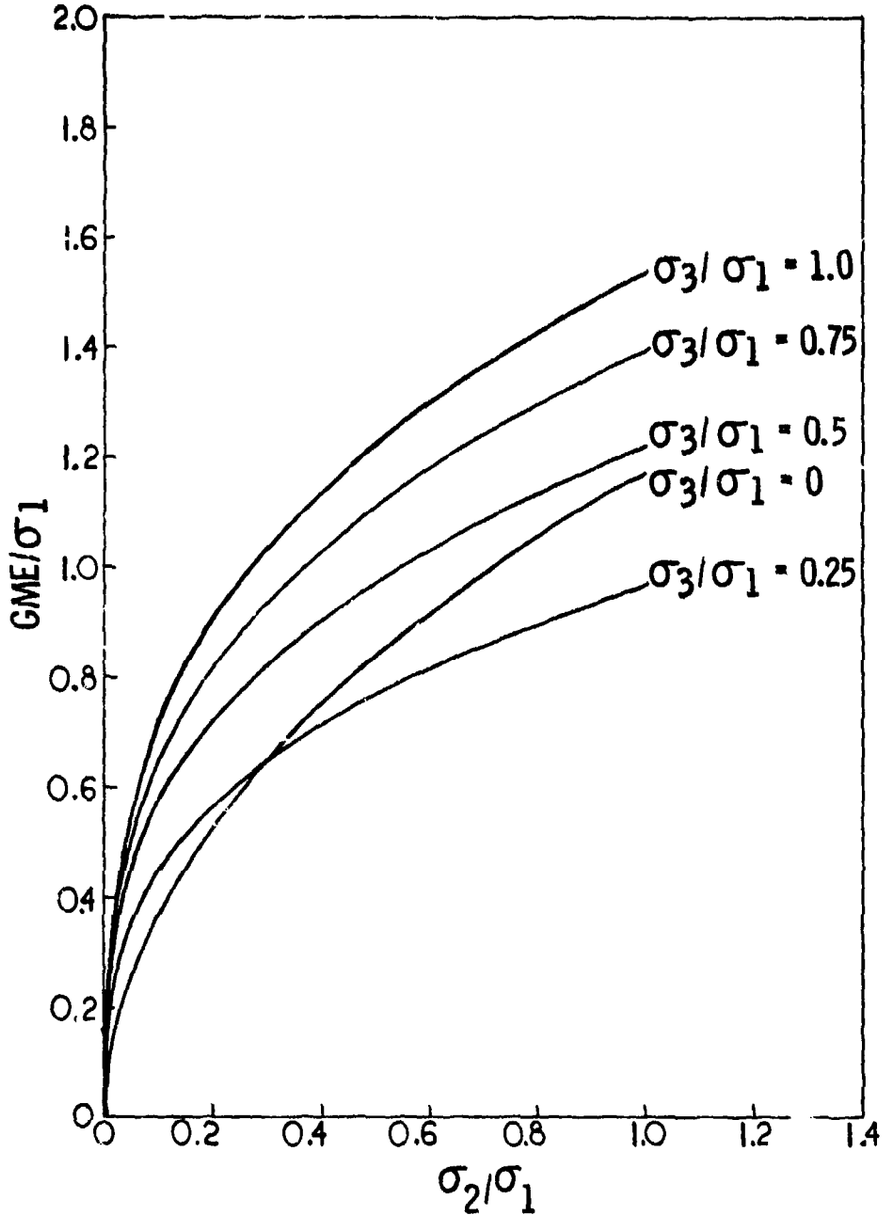

Figure 6. Geometric Mean Error

SUMMARY

Explicit expressions have been given for four error statistics frequently used as performance criteria for systems with random inputs. Some of the expressions are new. The comparative results enable relative statements to be made concerning system performance when measured by various statistics.

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