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CONSTITUTIVE EQUATIONS OF ROCK WITH SHEAR DILATANCY

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13. ABSTRACT

Coupling between dilatation and shear stress referred to as "dilatancy," has been shown to be a significant feature of rocks. Dilatancy has been identified by Lord Kelvin (1875) and by Reiner (1947) as a second-order effect in the isotropic continuum that arises from the application of the "principle of material indifference."

Relations required for the interpretation of tests results in the light of this theory are derived. It is shown that dilatancy in rocks is not the result of cracking at compressive stresses below the limit of shear failure, but its cause. Shear-induced density-gradients of physically significant magnitude are believed to provide the key to the understanding of important aspects of the behavior of rocks and rock-like solids.

Thus, the order of magnitude of the shear-dilatancy coefficient in granite, derived from experimental records, appears to justify the conclusion that the observed form of the premonitory variation of the V_P/V_S (P-wave velocity/S-wave velocity) ratio is the result of the decrease in V_P caused by dilatancy increase accompanying the build-up of tectonic shear forces preceding an earthquake, followed by increase of V_P due to build-up of confining pressure at impending shear failure by restraint of the rapidly increasing dilatancy. This dilatancy-related model explains the observed shape and magnitude of the premonitory velocity variations that immediately precede shocks of significant magnitude.

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Abstract

The effect of pronounced coupling between volume-dilatation and shear stress, usually referred to as "dilatancy," has been shown by experiments in the laboratory and on the site to be a significant feature of the behavior of rocks, even at high confining pressures. While dilatancy has been theoretically identified by Reiner (1947) as one of the second-order effects in the isotropic elastic continuum that arises from the application of the principle of material indifference, the specific relations required for the actual evaluation of the relevant physical parameters from experiments, and for the subsequent interpretation of experimental results in the light of the second-order theory have, so far, not been established.

These equations are derived, and it is shown that dilatancy in rocks is not, as is widely believed, the result of extension cracking that starts from microfissurs at compressive stresses well below the limit of shear failure and increases sharply as this limit is approached, but its cause. It plays, therefore, a most important role in the deformation and fracture of rocks and thus in related geological phenomena, such as the behavior of rock masses subject to shearing forces at different levels of confining pressure. The presence of shear-induced density-gradients of physically significant magnitude is believed to provide the key to the understanding of the difference in the mechanical behavior of rocks and rock-like solids from that of solids in which this effect is insignificant.

Thus, for instance, the order of magnitude of the elastic shear-dilatancy coefficient in Westerly granite, derived from experimental records on the basis of the second-order theory, appears to justify the conclusion that the observed form of the premonitory variation of the V_P/V_S (P-wave velocity/S-wave velocity) ratio is the result of the decrease in V_P caused by the gradual elastic dilatancy increase accompanying the critical build-up of tectonic shear forces preceding an earthquake, independently of the existence of pore-water flow, followed by an increase of V_P resulting from the build-up of the level of confining pressure in the vicinity of the impending shear failure, due to the restraint imposed on the accelerated post-elastic dilatancy in the potential failure region by the surrounding area. This dilatancy related fracture-mechanics model explains not only the observed general shape of the premonitory velocity variations but also the "overshooting" of the V_P/V_S ratio that immediately precedes shocks of significant magnitude.

id.

Constitutive Equations of Rocks with Shear Dilatancy

1. Introduction

Compression tests of crystalline rocks, at confining pressures up to 10 kbars [1] have clearly demonstrated the strong coupling, in such rocks, between volume-deformation and shear stress that has been referred to as "dilatancy" since Reynolds [2] coined the term and discovered the phenomenon in saturated sand ten years after Lord Kelvin [3], in 1875, had predicted, on theoretical grounds, its existence in any isotropic, "condensed" i.e. solid continuum. While Reiner, in about 1945, has been the first to show for both the isotropic viscous fluid [4] and the isotropic elastic solid [5], that in isotropic media the existence of shear dilatancy is the necessary consequence of what is currently referred to as the principle of material indifference [6], systematic observations of shear dilatancy in both "ductile" and "brittle" compact rocks, as well as in micro-fissured, porous and loose rocks have only been accumlating in recent years, particularly since the existence, in rocks, of dilatancy effects of significant magnitude, even at high confining pressure, has been experimentally demonstrated by Handin [7] and by Brace [8].

In spite of the increasing concern by geologists and geophysicists with the dilatancy of rocks in relation to various geotechnical problems [9], the true nature of the phenomenon is not clearly understood. Thus, in a recent state-of-the-art report on "tectonophysics" in which the geophysical significance of dilatancy in rocks is pointed out [10], such dilatancy is defined as "the increase of volume relative to elastic changes, caused by deformation," and its cause identified either as "sliding along rough intergranular surfaces" (in sandstones and limestones) or as "related to cracking" (granite), although it is recognized that dilatancy starts well below the fracture-stress, when "the rock begins to expand relative to elastic changes." Thus dilatancy in compact rock is currently conceived as being the result of and "traceable to open cracks" [11]; this concept, if true, would in fact, justify the definition of dilatancy as a deviation from elastic behavior, instead of its recognition as an integral and characteristic part of the deformational response to applied forces of all isotropic or quasi-isotropic materials, that results from the fact that the formulation of this response in the form of a relation between two symmetric tensors of second rank in the isotropic compressible continuum presupposes the existence of the second-order coupling between shear stresses and volume-dilatation which causes "dilatancy."

The belief that dilatancy is the result of cracks and can therefore be classified as deviation from elastic behavior or as "related to a pseudo-plastic failure mode" [12] seems to be responsible for attempts to deal with dilatancy (erroneously) within the framework of the theory of plasticity with a pressure-dependent yield condition [13]. While the experimental evidence indicates that close to the failure limit in shear the dilatancy-increase is extremely rapid, unless counteracted by a very high confining pressure [14], it also clearly indicates that dilatancy already sets in far below this limit and shows, within this elastic or pseudo-elastic range, its reversibility on unloading, similar and parallel to the first-order deformation [15]. It is only the magnitude of the dilatancy which is related to deviation from elastic behavior, not its existence.

It will be shown [see Eqs.(3.11)] that the magnitude of dilatancy in the elastic medium is an inverse function of both the shear and the bulk modulus. If it is assumed that the analytic generalization, based on Euler's theory of homogeneous functions [16], of linear elastic relations for unidirectional loading in the nonlinear range in the form of power functions provides a useful approximation, the transition from the elastic to the plastic range can be described by a gradual decrease, as an inverse power of the stress, of the effective (tangent) shear modulus ("variable modulus theory"). Retaining, in the transition range, the basic theorem of material indifference, it can be concluded that the magnitude of the dilatancy in this range will increase the faster the more pronounced the deviation, with increasing shear stress, of the deformational response in shear from the linear (elastic) response. The form of this increase will therefore depend on the form of the shear stress-shear-strain-diagram beyond the elastic range, a conclusion that is clearly supported by the experimental evidence (Fig. 1), and therefore justifies the assertion that dilatancy in rocks is not the result of cracking, but causes it when the actual volume expansion becomes large enough to cause internal disruption as the shear stress approaches its limit.

The difference between different rocks and between rocks and other solid materials with respect to the significance of dilatancy effects is thus not the fact of their existence, but that of their magnitude, form and observability which, in turn, depend on the magnitude of the relevant physical constants and the relations between them. These are the characteristic features by which the deformation of quasi-isotropic rocks differs from that of most metals and polymers, and which produce the observed pronounced coupling of shear and volume dilatation. While this coupling is not absent in other materials, it is insignificant at unidirectional small elastic strains and quite difficult to observe or to measure [17]. In rocks, on the other hand, it can be easily demonstrated by comparing volume-change-pressure relations at different intensities of the applied shear stress, recorded in axial compression tests of rocks, at different values of the confining pressure, or at a constant value of the confining pressure and increasing values of the axial loads (Fig. 2).

The principal reason for the difference in the magnitude of dilatancy in different rocks is the difference in their (inelastic) response to shear and their compressibility. The feature which determines the relative magnitudes of the dilatancy in different materials, in addition to its own parameter, is its inverse dependence on the product $(KG^2) = (K^3n^2) = (G^3n^{-1})$, where $n = G/K$, a conclusion derived from second-order elastic theory (see Eq.3.11). In linear elastic theory the ratio n is directly related to Poisson's ratio ν by the equation $n = [3(1 - 2\nu)/2\ (1 + \nu)]$. As ν changes from the low values $0.1 < \nu < 0.25$ characteristic of rocks, to the high values $0.35 < \nu < 0.45$ characteristic of metals and approaches the range $\nu \rightarrow 0.5$, characteristic of linear polymers and elastomers, the ratio n changes from $1.1 > n > 0.6$ to $0.33 > n > 0.10$ and approaches $n \rightarrow 0$ for materials that are practically incompressible. At the same time the ratio $E/K = 3(1 - 2\nu)$ changes from $2.4 > E/K > 1.5$ to $0.9 > E/K > 0.3$ and approaches zero as $\nu \rightarrow 0.5$. On the other hand the ratio $E/G = 2(1 + \nu)$ changes only moderately from 2.2 to 3.0 as ν varies over the whole range $0.1 < \nu < 0.5$.

Hence the fact that the elastic modulus E of igneous rocks at zero load is roughly one order of magnitude lower than that of strong technical metals (for granite $E = 2 - 6\text{x}10^2$ kbars, for steel and tungsten $E = 2.10^3$ and $3.5\text{x}10^3$ kbars, respectively) in conjunction with the low ratio of K/E for rocks (0.4 - 0.7) in comparison to metals (1.1 to 3.3), produces a difference of at least one order of magnitude between the bulk-modulus of rocks and of metals. Since, at the same time, the square ratio n^2 increases by roughly one order of magnitude, the product (K^3n^2) is about two orders of magnitude smaller for rocks than for metals and the dilatancy, therefore, by that order of magnitude larger, assuming the second-order parameter itself to remain of the same order of magnitude. In the practically incompressible elastomers, dilatancy is essentially suppressed by the very high values of the bulk-modulus.

Even in rock the observed, shear-induced volume-dilatation is small in absolute terms, particularly within the elastic range. It attains, however, the order of magnitude of the pressure-induced volume compaction as the shear stresses approach the yield limit, and counteracts this compaction even at high levels of the confining pressure: Brace and Byerlee [18] have found that at confining pressures between 4 and 8 kbars and axial pressures of 17 and 22 kbars, respectively, granite specimens expanded between 1.4 and 0.5 percent against the rather high confining pressures. In these confined compression tests the difference between the measured volume change and the (elastic) volume compaction, based on linear extrapolation with the initial bulk-modulus, becomes significant at stress-differences between 0.35 and 0.6 of the stress-difference at fracture and, immediately preceding fracture, attains values that vary between 0.4 and 2.4 times the (linearly extrapolated) volume compaction at this level, the largest value being associated with zero confining pressure.

In weaker rocks and at lower confining pressures the volume dilatation preceding fracture is larger than in strong rocks. However, in anticipating the differences between different rocks the joint effect of the difference in the parameters on the product (K^3n^2) must be considered. Thus for Cedar City granite at a confining pressure of 2.0 kbars and axial compression of about 10 kbars at failure the observed volume dilatation preceding failure attained 2.0 percent, which is somewhat larger than the (extrapolated) elastic volume compaction, resulting in a small absolute volume expansion at failure against the confining pressure. For sandstone at a confining pressure of 0.35 and axial compression at failure of about 7 kbars, the volume dilatation is of a similar order of magnitude, also with a resultant small absolute volume dilatation at failure against the (small) confining pressure [14]. Concrete under biaxial stresses shows behavior which is similar [19] and so does sand, though there is a significant difference between loose and dense sand: because of the increased compressibility, the magnitude of the volume compaction of the former may exceed the shear-induced quasi-elastic volume expansion before failure, while the response of compacted sands tends to approach that of rocks [20].

It is customary in continuum mechanics to classify dilatancy as a "second order effect", resulting from terms in the constitutive relations that are of second order in the strains. These are, therefore, usually neglected in the classical (infinitesimal strain) theories of elastic and elastic-plastic media, although their inclusion produces physically striking differences in the response to shear of such media and though their magnitude may remain small and require special experimental configurations for their observation: thus, while in the classical theory shearing of a block is produced by purely tangential forces acting on the faces parallel to the plane of shearing, in the general theory additional normal forces on all faces of the block, proportional to the square of the shear angle, must act to produce the same configuration. In elastic media one part of these forces will produce a volume-dilatation, sometimes referred to as Kelvin effect [3], while the remaining normal forces produce elongations or contractions normal to the directions of the shear planes (Poynting effect)[21]; since both effects are proportional to the square of the shear angle, demonstration of their existence requires observational procedures of exceptionally high accuracy for the simultaneous measurement of shear and second-order deformations or forces. In reversed cyclic shear deformation of plastic-work-hardening metals the dependence of the second-order deformation on the square of the applied shear strain results in its one-directional accumulation under shear-strain reversal, producing substantial permanent changes in dimensions and geometry, as observed in the elongation of metal tubes subject to axially unrestrained reversed cyclic torsion (Ronay effect) [22], in the tube formation by reversed rolling of cylindrical bars of kaolinite clays [23] and in cold-rolled metal bars demonstrating the initiation of "tube-formation" by non uniform second order axial contraction (Fig. 3), a phenomenon that is being implicitly utilized in certain processes of metal tube fabrication.

The classification of the above phenomena as "second-order effects" is, however, mathematical not physical; it reflects the mathematical frame selected for their description and not their physical significance. Second-order effects in the constitutive equations of continuous, homogeneous isotropic solid or pseudo-solid media are considered to be those that arise from the deviation of the form of these equations from the tensorial linearity between the significant mechanical variables (stress, strain and their derivatives) on which the continuum theories of the classical media (elastic, viscous, elastic-plastic, visco-elastic) are founded. They are not to be confused with non-linearity of the observed relations between mechanical variables, since such non-linearity may simply reflect the form of the (invariant) parameters of the tensorially linear coordination of these variables, or be the result of the selected measure of the (finite) deformation, without implying the existence of the basic differences of the mechanical response of the material from that associated with classical, tensorially linear constitutive equations. The only source of second-order phenomena is the appearance of second-order non-linearity in the tensorial relations between the mechanical variables. However, their actual physical manifestation requires the creation of a specific experimental or observational framework that facilitates rather than obviates their measurement. Thus the observation of the Poynting effect in elastic metal wires or in thin-walled tubes requires the setting up of experimental conditions permitting free axial extension of the twisted specimen, and of measuring devices for the simultaneous measurement of (second-order) axial elongation. Similarly, the use of a standard experimental device, for instance of a torsion fatigue or creep machine with fixed grips, not only prevents the accumulation of irreversible (second-order) elongation but interferes with the first order (shear) response of the material by the development of axial forces through the existence of constraints. Thus in torsion-creep tests of a highly filled elastomer these axial forces, by interacting with the material response in shear, produced a difference in this response between torsion creep tests with free axial extension and those with fixed length of the specimen [24]. In torsion fatigue tests of workhardening metal tubes the difference between the existence or non-existence of axial constraints produced significant differences not only in fatigue life, but also in the mode of the fatigue-failure [25].

While terms of second order in (small) strain are, by definition, one order of magnitude smaller than the first order terms, this fact does not determine their physical significance, because they are not corrections of the linear terms, with which they might be compared in magnitude in order to assess their significance, but are the source of phenomena that do not exist in "linear" media, and the magnitude of which is therefore independent of that of the first order terms. This fact removes the basis for comparison and therefore any possible justification for their automatic neglect as "terms of higher order". Moreover, the coefficients associated with those terms in the constitutive equations may compensate for their mathematically higher order, so as to reflect their observed physical significance.

2. The Form of Constitutive Equations of Isotropic Media.

Constitutive equations of materials that are isotropic must be invariant under changes of the frame of reference. Since they describe relations between mechanical variables that are second-rank symetric tensors (stress, strain and their time derivatives) their form must be that of isotropic tensor functions of the significant variables. In the case of one variable $\underset{\sim}{D} = f(\underset{\sim}{A})$ the tensor function is isotropic if and only if it has a representation of the form

$$\underset{\sim}{D} = f(\underset{\sim}{A}) = \alpha_o \underset{\sim}{1} + \alpha_1 \underset{\sim}{A} + \alpha_2 \underset{\sim}{A}^2 + \ldots \alpha_n \underset{\sim}{A}^n \tag{2.1}$$

where the coefficients α_k are invariants of $\underset{\sim}{A}$ and can therefore be expressed as functions of the principal invariants of $\underset{\sim}{A}$.

Any reasonably well behaved constitutive equation can be represented as a polynomial isotropic tensor function $\underset{\sim}{D} = f(\underset{\sim}{A})$ in which the components of $\underset{\sim}{D}$ are polynomials in the components of $\underset{\sim}{A}$, so that the coefficients α_k in (2.1) may be expressed as polynomials in the n principal invariants $I_A^{(k)}$ of $\underset{\sim}{A}$; these are the coefficients of the characteristic equation of the tensor-matrix [A]

$$a^n - I_A^{(1)} a^{n-1} + \ldots + (-1) I_A^{(n)} = 0 \tag{2.2}$$

the n roots a_k of which represent the proper numbers (principal values) of $\underset{\sim}{A}$ that are real for a real tensor, for which a non-singular coordinate transformation diagonalizes the matrix [A]. The Hamilton-Cayley theorem, which states that a symetric matrix satisfies its own characteristic equation, transforms Eq.(2.2) into

$$\underset{\sim}{A}^n - I_A^{(1)} \underset{\sim}{A}^{n-1} + \ldots (-1)^n I_A^{(n)} \cdot \underset{\sim}{1} = 0 \tag{2.3}$$

which makes it possible to express the n^{th} power $\underset{\sim}{A}^n$ (and all higher powers) as a linear combination of 1, $\underset{\sim}{A}$, ... $\underset{\sim}{A}^{n-1}$ with scalar coefficients that are polynomials in the principal invariants $I_A^{(k)}$. Eq.(2.1) reduces therefore to

$$\underset{\sim}{D} = f(\underset{\sim}{A}) = \phi_o \underset{\sim}{1} + \phi_1 \underset{\sim}{A} + \phi_2 \underset{\sim}{A}^2 + \ldots \phi_{n-1} \underset{\sim}{A}^{n-1} \tag{2.4}$$

where the coefficients ϕ_k are again polynomials in the principal invariants of $\underset{\sim}{A}$. For the case of principal interest in continuum mechanics which is n = 3, Eq.(2.4) reduces further to

$$D = \phi_o . 1 + \phi_1 A + \phi_2 A^2 \tag{2.5}$$

with $\phi_k = \phi_k(I_A, II_A, III_A)$, where $I_A = I_A^{(1)}$, $II_A = I_A^{(2)}$ and $III_A = I_A^{(3)}$ are the three principal invariants of $\underset{\sim}{A}$.

While Eq.(2.4) as an algebraic theorem is not new [26], Eq.(2.5) has first been applied by Reiner to a viscous fluid [4] and, subsequently, to an elastic solid [5] in two classical papers demonstrating the independence of the magnitudes of the second-order effects (normal stresses in shear flow, Poynting effect in elastic torsion) of those of the first order shear flow or elastic twist, as well as of the invariance of the form of Eq.(2.5) under a change of the measure of strain, which alone confirms the classification of the phenomena related to the quadratic term of this equation as truly of "second order" [27]. Otherwise the arbitrary selection of a measure of strain could produce a second-order effect that would fail to be of second order in a different measure; its magnitude would therefore result from this selection, so that the classification of its order would have no physical significance.

For a polynomial isotropic tensor function of two variables $\underset{\sim}{D} = f(\underset{\sim}{A}, \underset{\sim}{B})$ Rivlin [28] and Rivlin and Ericksen [29] have developed, from a generalization of the Hamilton-Cayley theorem, the explicit representation

$$
\begin{aligned}
\mathbf{D} = {} & \psi_0\mathbf{1} + \psi_1\mathbf{A} + \psi_2\mathbf{B} + \psi_3\mathbf{A}^2 + \psi_4\mathbf{B}^2 + \\
& + \psi_5(\mathbf{A}\mathbf{B}+\mathbf{B}\mathbf{A}) + \psi_6(\mathbf{A}^2\mathbf{B}+\mathbf{B}\mathbf{A}^2) + \\
& \psi_7(\mathbf{A}\mathbf{B}^2+\mathbf{B}^2\mathbf{A}) + \psi_8(\mathbf{A}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{A}_2^2)
\end{aligned}
\tag{2.6}
$$

where ψ_k are polynomials in the ten basic invariants

$$
\begin{aligned}
& \mathrm{tr}\mathbf{A},\ \mathrm{tr}\mathbf{A}^2,\ \mathrm{tr}\mathbf{A}^3,\ \mathrm{tr}\mathbf{B},\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{B}^3 \\
& \mathrm{tr}(\mathbf{A}\mathbf{B}),\ \mathrm{tr}(\mathbf{A}\mathbf{B}^2),\ \mathrm{tr}(\mathbf{B}\mathbf{A}^2),\ \mathrm{tr}(\mathbf{A}^2\mathbf{B}^2)
\end{aligned}
\tag{2.7}
$$

where $\mathrm{tr}\mathbf{A} = I_A$, $\mathrm{tr}\mathbf{A}^2 = I_A^2 - 2II_A$, and $\mathrm{tr}\mathbf{A}^3 = I_A^3 - 3I_AII_A + 3III_A$, so that polynomials in the traces of one tensor variable $\mathbf{A}$ are also polynomials in its principal invariants.

It has been demonstrated by the evaluation of the results of torsion tests of visco-elastic polymers [30] with the aid of Eq.(2.7) that not more than the five coefficients $\psi_1 \ldots \psi_5$ could be reliably determined, the remaining coefficients being associated with effects the observation of which was beyond the (reasonably high) accuracy of the experimental device (Weissenberg rheogoniometer). It appears therefore that for a polynomial isotropic tensor function of two variables $\mathbf{D} = f(\mathbf{A},\mathbf{B})$ a reduced form of equation (2.6) containing terms of not higher than second order.

$$
\mathbf{D} = \psi_0\mathbf{1} + \psi_1\mathbf{A} + \psi_2\mathbf{B} + \psi_3\mathbf{A}^2 + \psi_4\mathbf{B}^2 + \psi_5(\mathbf{A}\mathbf{B}+\mathbf{B}\mathbf{A}) \tag{2.8}
$$

represents a physically adequate explicit representation for use in continuum mechanics, the ψ_k being polynomials in the first seven of the basic ten invariants listed in (2.7).

Eqs.(2.5) and, to a close approximation, (2.8) are the complete forms of explicit representation of isotropic polynomial tensor functions of one and of two variables, respectively, in three-dimensional space. The linearized forms

$$\underset{\sim}{D}' = \phi_0 \underset{\sim}{1} + \phi_1 \underset{\sim}{A} \qquad (2.5a)$$

and

$$\underset{\sim}{D}' = \psi_0 \underset{\sim}{1} + \psi_1 \underset{\sim}{A} + \psi_2 \underset{\sim}{B} \qquad (2.8a)$$

used in the formulation of the constitutive equations of the classical theories of elastic, viscous, plastic, visco-elastic, elastic-plastic and other combined linear media are therefore incomplete forms, the use of which can only be justified by reference to experiment or to analytical convenience, particularly with respect to the difficulties of inversion of the complete tensor polynomials.

Constitutive equations of the polynomial form of Eqs.(2.5) and (2.8) under certain conditions admit "potentials", that are continuously differentiable polynomial invariants of the symetric tensor variables $\underset{\sim}{A}$, and $\underset{\sim}{A}$ and $\underset{\sim}{B}$, respectively. They can therefore be expressed as scalar polynomials in the principal or basic invariants of the tensor variables; their gradients with respect to one of the variables have the form of the tensor functions (2.5) or (2.8). These can therefore be derived by differentiation from the respective "potentials", provided they exist, as in the case of the elastic energy density or the viscous dissipation function.

It is, however, possible to develop explicit forms of Eqs.(2.5) and (2.8) directly, without introducing the problem of the existence or admissibility of a potential, by expanding the coefficients ϕ_k and ψ_k into power series of the principal or basic invariants. Thus with

$$\phi_k = a_{ok} + a_{1k}I_A + (a_{2k} + a_{3k}I_A)I_A^2 + (a_{4k} + a_{5k}I_A)II_A + + a_{6k}III_A \tag{2.9}$$

and limitation to terms of third order in the variable, Eq.(2.5) is transformed into

$$D = (a_{1o}I_A + a_{2o}I_A^2 + a_{3o}I_A^3 + a_{4o}II_A + a_{5o}I_AII_A + a_{6o}III_A)\underset{\sim}{1} + (a_{o1} + a_{11}I_A + a_{21}I_A^2 + a_{41}II_A)\underset{\sim}{A} + (a_{o2} + a_{12}I_A)\underset{\sim}{A}^2 \tag{2.10}$$

and Eq.(2.5a) into

$$D' = a_{1o}I_A\underset{\sim}{1} + a_{o1}\underset{\sim}{A} \tag{2.10a}$$

Since

$$D''' = (a_{3o}I_A^3 + a_{5o}I_AII_A + a_{6o}III_A)\underset{\sim}{1} + (a_{21}I_A^2 + a_{41}II_A)\underset{\sim}{A}$$

are third-order corrections to the two linear terms of Eq.(2.10a), the second order terms of Eq.(2.10) are

$$D'' = (a_{2o}I_A^2 + a_{4o}II_A)\underset{\sim}{1} + a_{11}I_A\underset{\sim}{A} + a_{o2}\underset{\sim}{A}^2 \tag{2.11}$$

so that $D = D'+D''$ represents a simplified form of Eq.(2.10) in which the third-order corrections of the linear terms have been omitted as irrelevant; although their inclusion obviously produces deviations from the linearity implied by Eq.(2.10a), they are however unrelated to the second-order effects defined by Eq.(2.11).

The basic feature of the second-order terms is their independence of the change of sign (direction) of the variable $\underset{\sim}{A}$, while the first order terms and their third-order corrections show the same dependence on the change of sign of $\underset{\sim}{A}$. Thus, while both D' and D''' change sign with a change of sign of $\underset{\sim}{A}$, D'' does not.

With ψ_o to ψ_4 in Eq.(2.8) being of the same form as Eq.(2.9) the explicit polynomial form of Eq.(2.8) consists of the sum of two relations of the form (2.10) with respect to the tensor variables $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and the interaction term $\psi_5(\underset{\sim}{A}\underset{\sim}{B} + \underset{\sim}{B}\underset{\sim}{A})$ where ψ_5 is a polynomial in $tr(\underset{\sim}{A}\underset{\sim}{B})$ which, if this term is to be limited to second order in either of the variables, can consist only of the first two terms $\psi_5 = c_{o5}+c_{15}tr(\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A})$. Hence

$$\begin{aligned} D = {} & [(a_{1o}I_A + a_{2o}I_A^2 + a_{3o}I_A^3 + a_{4o}II_A + a_{5o}I_AII_A + a_{6o}III_A)\underset{\sim}{1} + \\ & + (b_{1o}I_B + b_{2o}I_B^2 + b_{3o}I_B^3 + b_{4o}II_B + b_{5o}I_BII_B + b_{6o}III_B)]\underset{\sim}{1} \\ & + (a_{o1} + a_{11}I_A + a_{21}I_A^2 + a_{41}II_A).\underset{\sim}{A} + (b_{o1} + b_{11}I_B + b_{21}I_B^2 + \\ & + b_{41}II_B).\underset{\sim}{B} + (a_{o2} + a_{12}I_A)\underset{\sim}{A}^2 + (b_{o2} + b_{12}I_B)\underset{\sim}{B}^2 + c_{o5}(\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A}) + \\ & + c_{15}tr(\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A}).(\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A}) \end{aligned} \tag{2.12}$$

with the simplified linearized form containing four constants

$$D' = (a_{1o}I_A + b_{1o}I_B)\underset{\sim}{1} + a_{o1}\underset{\sim}{A} + b_{o1}\underset{\sim}{B} \tag{2.12a}$$

and the second-order terms

$$\begin{aligned} D'' = {} & (a_{2o}I_A^2 + a_{4o}II_A + b_{2o}I_B^2 + b_{4o}II_B)\underset{\sim}{1} + a_{11}I_A\underset{\sim}{A} + b_{11}I_B\underset{\sim}{B} + \\ & + a_{o2}\underset{\sim}{A}^2 + b_{o2}\underset{\sim}{B}^2 + c_{o5}(\underset{\sim}{A}\underset{\sim}{B}+\underset{\sim}{B}\underset{\sim}{A}) \end{aligned} \tag{2.13}$$

The remaining terms of Eq.(2.12), except for the last, are third-order corrections to the linear Eq.(2.12a), while the last term represents a fourth-order correction to the single second-order interaction-term of Eq.(2.13).

While the number of physical constants required for the characterization of the second order terms according to Eqs.(2.11) and (2.13) seems prohibitively large from the point of view of their experimental determination (6 for Eq.2.11 and 13 for Eq.2.13), this number is significantly reduced in the case of the very simple experiments designed specifically for this purpose: (a) uniform triaxial tension or compression; (b) pure (or simple) shear. The form to which these equations reduce in the case of such experimental conditions will now be derived for simple assumptions of material response.

3. Dilatancy in the Isotropic Elastic Medium.

The isotropic elastic medium is defined by the assumption of the existence of a time-independent, reversible relation between stress and strain of the alternative forms $\sigma_{ij} = f(\varepsilon_{ij})$, or $\varepsilon_{ij} = g(\sigma_{ij})$. Comparison of Cauchy's linear elastic constitutive equation

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{3.1}$$

with Eq.(2.10a) produces the identities $\underset{\sim}{D}' = \sigma_{ij}$, $\underset{\sim}{A} = \varepsilon_{ij}$, $a_{10} = \lambda$ and $a_{01} = 2\mu$. For hydrostatic pressure

$$p = \frac{1}{3}\sigma_{ii} = (\lambda + \frac{2}{3}\mu)\varepsilon_{kk} = K\varepsilon_{kk} \tag{3.1a}$$

where $\lambda + \frac{2}{3}\mu = K$ is the bulk modulus. Substracting Eq.(3.1a) from

Eq.(3.1).

$$\sigma_{ij} - p\delta_{ij} = s_{ij} = 2\mu(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}) = 2Ge_{ij} \tag{3.1b}$$

the relation between the stress-and-strain-deviations governed by the shear modulus $\mu = G$ is obtained.

For a tri-axial uniform state of stress with $\sigma_1 = \lambda\varepsilon_{kk} + 2G\varepsilon_1$ and $\sigma_2 = \sigma_3 = \lambda\varepsilon_{kk} + 2G\varepsilon_2$, the equation $(\sigma_1 - \sigma_2) = 2G(\varepsilon_1 - \varepsilon_2)$ can be transformed into

$$(\sigma_1 - \sigma_2) = 2G\varepsilon_{kk} - 6G\varepsilon_2 \tag{3.2}$$

a linear relation between the stress-difference $(\sigma_1 - \sigma_2)$ and the volume-change ε_{kk}, which depends on the lateral constraint represented by the strain-component ε_2. Introducing the dependence of the constraint on the lateral stress σ_2, the alternative linear relation between the longitudinal stress σ_1 and the volume change ε_{kk} is obtained

$$\sigma_1 = (\lambda + 2G)\varepsilon_{kk} - 4G\varepsilon_2 \tag{3.3}$$

For full constraint $\varepsilon_2 = 0$ and $\sigma_2 = \sigma_3 = \lambda\varepsilon_{kk} = \lambda\varepsilon_1$ and therefore $(\sigma_1 - \sigma_2) = 2G\varepsilon_{kk} = 2G\varepsilon_1$ and $\sigma_1 = (\lambda + 2G)\varepsilon_1$ while, for no constraint $\sigma_2 = 0$, $2G\varepsilon_2 = -\lambda\varepsilon_{kk}$ and $2G(\varepsilon_1 - \varepsilon_2) = (2G + 3\lambda)\varepsilon_{kk} = 3K\varepsilon_{kk}$. Hence for full constraint $\varepsilon_1 = \sigma_1/(\lambda + 2G)$, while for zero constraint $\varepsilon_1 = (\sigma_1 - \lambda\varepsilon_{kk})/2G$.

In order to evaluate the second-order effects in this medium under the specified conditions of uniform state of stress and strain the following relations must be considered

$$\mathrm{tr}\underset{\sim}{A}^2 = I_A^2 - 2II_A \text{ or } 2II_A = I_A^2 - \mathrm{tr}\underset{\sim}{A}^2$$

as well as

$$I_A^2 - 3II_A = \tfrac{1}{2}(3\mathrm{tr}\underset{\sim}{A}^2 - I_A^2) = (a_1 - a_2)^2 \tag{3.4}$$

where a_1 and $a_2 = a_3$ are the principal values of $\underset{\sim}{A}$. Therefore

$$3II_A = I_A^2 - (a_1 - a_2)^2 \tag{3.4a}$$

In order to evaluate the third-order corrections to the linear Eq.(3.1) the following relations must be considered

$$\mathrm{tr}\underset{\sim}{A}^3 = I_A^3 - 3I_A II_A + 3III_A \tag{3.5}$$

or

$$III_A = \frac{1}{6}I_A^3 - \frac{1}{2}I_A \mathrm{tr}\underset{\sim}{A}^2 + \frac{1}{3}\mathrm{tr}\underset{\sim}{A}^3$$

and therefore, considering Eq.(3.4a)

$$3III_A = \mathrm{tr}\underset{\sim}{A}^3 - I_A(a_1 - a_2)^2 \tag{3.5a}$$

Moreover

$$I_A II_A = \frac{1}{3}I_A^3 - \frac{1}{3}I_A(a_1 - a_2)^2 = \frac{1}{3}I_A[I_A^2 - (a_1 - a_2)^2]$$

Introducing the above relations into the expressions for the third-order corrections and the second-order effects, the following expressions, valid under the assumption $a_2 = a_3$, are obtained

$$\begin{aligned} D''' = {} & [(a_{3o} + \tfrac{1}{3}a_{5o})I_A^3 - \tfrac{1}{3}a_{5o}I_A(a_1 - a_2)^2]\cdot\underset{\sim}{1} + \\ & + [(a_{21} + \tfrac{1}{3}a_{41})I_A^2 - \tfrac{1}{3}a_{41}(a_1 - a_2)^2]\underset{\sim}{A} \end{aligned} \tag{3.6}$$

and

$$D'' = [(a_{2o} + \tfrac{1}{3}a_{4o})I_A^2 - \tfrac{1}{3}a_{4o}(a_1 - a_2)^2]\cdot\underset{\sim}{1} + a_{11}I_A\cdot\underset{\sim}{A} + a_{o2}\underset{\sim}{A}^2$$

Using the above expressions, the following constitutive equation of the isotropic elastic medium is obtained from Eq.(2.10)

$$\begin{aligned} \sigma_{ij} = {} & [\lambda\varepsilon_{kk} + \alpha_{2o}\varepsilon_{kk}^2 + \alpha_{3o}\varepsilon_{kk}^3 - \alpha_{4o}(\varepsilon_1 - \varepsilon_2)^2]\delta_{ij} + \\ & + [2G + a_{11}\varepsilon_{kk} + \alpha_{21}\varepsilon_{kk}^2 - \alpha_{41}(\varepsilon_1 - \varepsilon_2)^2]\varepsilon_{ij} + a_{o2}\varepsilon_{ik}\varepsilon_{kj} \end{aligned} \tag{3.7}$$

where $\alpha_{2o} = a_{2o} + \frac{1}{3}a_{4o}$, $\alpha_{3o} = a_{3o} + \frac{1}{3}a_{5o}$; $\alpha_{4o} = \frac{1}{3}a_{4o}$; $\alpha_{21} = a_{21} + \frac{1}{3}a_{41}$; $\alpha_{41} = \frac{1}{3}a_{41}$. The term $a_{5o}I_A(a_1 - a_2)^2$, which is a correction to a second-order and not to the first-order term, has been neglected on the basis of the assumption that $a_{5o}\varepsilon_{kk} << a_{4o}$ if a_{5o} and a_{4o} do not differ by more than one order of magnitude.

Eq.(3.7) is transformed by i = j into the pressure-volume change relation

$$p = K\varepsilon_{kk} + \alpha_2\varepsilon_{kk}^2 + \alpha_3\varepsilon_{kk}^3 - \alpha_4(\varepsilon_1 - \varepsilon_2)^2 \tag{3.7a}$$

where

$$\alpha_2 = \alpha_{2o} + \frac{1}{3}a_{11} + \frac{1}{9}a_{o2};\ \alpha_3 = \alpha_{3o} + \frac{1}{3}\alpha_{21};\ \alpha_4 = \alpha_{4o} - \frac{2}{9}a_{o2}$$

and the third-order correction $\alpha_{41}\varepsilon_{kk}(\varepsilon_1 - \varepsilon_2)^2$ to the last (second order) term has been neglected on the basis of the assumption $\alpha_{41}\varepsilon_{kk} << \alpha_4$. Subtraction of Eq.(3.7a) from Eq.(3.7) produces the relation between the deviations

$$\sigma_{ij} - p\delta_{ij} = s_{ij} = 2Ge_{ij} + (a_{11} + \alpha_{21}\varepsilon_{kk})\varepsilon_{kk}e_{ij} + a_{o2}e_{ik}e_{kj}$$

where (3.7b)

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \text{ and } e_{ik}e_{kj} = \varepsilon_{ik}\varepsilon_{kj} - \frac{1}{3}\mathrm{tr}\,\underset{\sim}{\varepsilon}^2\delta_{ij}$$

Thus, according to Eq.(3.7a), the hydrostatic pressure is not only a non-linear (third-order) function of the volume change, but depends also on the square of the maximum shear $(\varepsilon_1 - \varepsilon_2)^2$. The stress-deviation, on the other hand, is a non-linear (second-order) function of the strain-deviation as well as of the volume change. Of the four new constants α_2, α_4, a_{11} and a_{o2} that determine the physical significance of the second-order terms, the constants α_4 and a_{o2} are associated with the two principal second order effects, the coupling of hydrostatic pressure with shear and the "normal-stress effect", while α_2 and a_{11} might be significant only for highly compressible materials that are subject to very high pressures, as they are both coupled with ε_{kk}. The constants may be positive or negative, with the right sign for any specific material to be determined on the basis of experiment.

For the tri-axial uniform state of strain ε_1, $\varepsilon_2 = \varepsilon_3$ and σ_1, $\sigma_2 = \sigma_3$ Eq.(3.7b) reduces to

$$s_1 = 2Ge_1 + \frac{2}{3}a_{o2}(\varepsilon_1^2 - \varepsilon_2^2)$$

and (3.8)

$$s_2 = 2Ge_2 - \frac{1}{3}a_{o2}(\varepsilon_1^2 - \varepsilon_2^2)$$

if the dependence of the stress-deviation on the volume change is neglected. Hence

$$\sigma_1 - \sigma_2 = s_1 - s_2 = 2G(\varepsilon_1 - \varepsilon_2) + a_{o2}(\varepsilon_1^2 - \varepsilon_2^2) =$$

$$2G(\varepsilon_1 - \varepsilon_2)[1 + a_{o2}(\varepsilon_1 + \varepsilon_2)/2G] \qquad (3.9)$$

Since the second term in the bracket is small in relation to unity, the classical relation $(\varepsilon_1 - \varepsilon_2) = (\sigma_1 - \sigma_2)/2G$ seems to be an adequate approximation which, introduced in Eq.(3.7a) produces a simplified equation for the shear dilatancy

$$K\varepsilon_{kk} + \alpha_2\varepsilon_{kk}^2 + \alpha_3\varepsilon_{kk}^3 = \frac{1}{3}(\sigma_1 - \sigma_2) + \sigma_2 + \alpha_4(\sigma_1 - \sigma_2)^2/4G^2 \qquad (3.10)$$

or disregarding, in first approximation, the non-linear terms in ε_{kk}:

$$\varepsilon_{kk} = \frac{p}{K} + \frac{\alpha_4}{4KG^2}(\sigma_1 - \sigma_2)^2 = \frac{p}{K} + \varepsilon''_{kk} \tag{3.11a}$$

or

$$\varepsilon_{kk} - \frac{\sigma_2}{K} = \frac{1}{3K}(\sigma_1 - \sigma_2) + \frac{\alpha_4}{4KG^2}(\sigma_1 - \sigma_2)^2 \tag{3.11b}$$

where the left side of Eq.(3.11b) represents the volume-change with reference to the initially compressed volume under a confining pressure equal to σ_2; this is the volume change that is actually observed in the course of a uniaxial compression test of cylindrical specimens of rock and concrete under constant confining lateral pressure σ_2.

The left side of Eq.(3.11a) may be positive or negative, since in a tri-axial compression test the volume may decrease or increase depending on whether the shear dilatancy due to the stress-difference ε''_{kk} is smaller or larger than the volume-compaction p/K due to the hydrostatic pressure. Since the square of the stress-difference is always positive, a negative coefficient α_4 must be introduced into Eq.(3.11a). Therefore when $\alpha_4 < 0$ shear dilatancy will always exist, independently of the level of the confining pressure, provided the elastic bulk modulus K is finite, since the assumption of incompressibility is obviously incompatible with the existence of elastic shear-dilatancy. However, whether the recorded volume change with reference to the initial unstrained volume is expansion or compaction necessarily depends on the magnitude of the confining pressure, all other things being equal.

Eqs.(3.11) can be utilized for the determination of the constant of shear dilatancy α_4 by fitting this equation to recorded observations in tests of various rocks, of the (elastic) volume change as a function of the stress difference. The influence of the lateral constraint between the limits of full constraint with $\varepsilon_2 = 0$ and no constraint with $\sigma_2 = 0$ can be assessed by introducing these limits into Eq.(3.11b). For full constraint with $\varepsilon_1 = \frac{1}{2G}(\sigma_1 - \sigma_2) = \varepsilon_{kk}$ Eq.(3.11a) is transformed into a quadratic equation for the stress-difference

$$(\sigma_1 - \sigma_2)^2 - \frac{2G}{\alpha_4}(K - \frac{2}{3}G)(\sigma_1 - \sigma_2) + \frac{4G^2}{\alpha_4}\sigma_2 = 0 \qquad (3.12a)$$

the solution of which provides the magnitude of the confining pressure σ_2 as a function of the applied stress σ_1; while in the case of no constraint Eq.(3.11b) provides the relation between the applied longitudinal stress σ_1 and the volume expansion for the unconfined compression test

$$3K\varepsilon_{kk} \doteq \sigma_1(1 + \frac{3\alpha_4}{4G^2}\sigma_1) \qquad (3.12b)$$

Combination of the above equation with the approximate linear relation $\sigma_1 = 2G(\varepsilon_1 - \varepsilon_2)$ produces the strain-components including the dilatancy term

$$\begin{aligned} 3KG\varepsilon_1 &\doteq (K + \frac{G}{3})\sigma_1 + \frac{\alpha_4}{4G}\sigma_1^2 \\ 3KG\varepsilon_2 &\doteq -\frac{1}{2}(K - \frac{2G}{3})\sigma_1 + \frac{\alpha_4}{4G}\sigma_1^2 \end{aligned} \qquad (3.12c)$$

Its effect is to reduce the longitudinal contraction and to increase the lateral expansion, with apparent stress-dependent increase of the classical value of Poisson's ratio ν, given by the (negative) ratio of the first two terms.

Eqs.(3.2) through (3.11) can be easily modified for conditions of tri-axial uniform state of stress in which $\sigma_2 \neq \sigma_3$ and therefore $\varepsilon_2 \neq \varepsilon_3$, including conditions of one-directional full constraint (σ_1, $\sigma_2 = 0$, $\varepsilon_3 = 0$), so as to obtain the coefficient of shear-dilatancy from the records of tests performed under such conditions [19].

The difficulty of estimating the magnitude of the second-order coefficient of shear dilatancy α_4 in the elastic range arises from the absence of reliable observations of reversible dilatancy within this range, containing observed values of K and G, as well as from the fact that the truly elastic range even of hard rocks is rather limited and that the dilatancy is rather small at the upper limit of this range.

The evaluation of recorded stress-strain diagrams for Westerly granite at constant ratio $\sigma_3/\sigma_1 = 0.085$ of longitudinal stress σ_1 and confining pressure σ_3 suggests values of the elastic constants $G = 3300$ ksi $\sim$ 230 kbars and $K = 5000$ksi $\sim$ 350 kbars (Fig. 4a). With these values the shear-dilatancy parameter $-\alpha_4 = 80.000$ ksi $\sim$ 6650 kbars $\sim$ 16K produces a fairly good fit of Eq.(3.11) to the recorded $\varepsilon''_{kk} - p$ curve. Since the ordinate at the vertical tangent $p = 38.2$ksi attained at the shear stress $\frac{1}{2}(\sigma_1 - \sigma_3) \sim 46$ ksi exceeds the elastic range, the recorded shear-dilatation beyond this limit proceeds much faster and at pressures considerable below those in the elastic range that would result from Eq.(3.11), as indicated in Fig. 4a. The deviation from elastic dilatancy above the limiting pressure at which the $\varepsilon''_{kk} - p$ curve has a vertical tangent (see Eq.5.2) is considerably smaller for the observed data presented in Figs. 4b and 5 in accordance with the less sharp deviation from elasticity of the recorded response in shear. With the elastic coefficients $G = 4500$ ksi $\sim$ 320 kbars and $K = 3750$ ksi $\sim$ 260 kbars for Cedar City granite at confining pressure $\sigma_3 = 30$ ksi

$\sim$ 2.1 kbars, the shear dilatancy parameter $-\alpha_4$ = 195,000 ksi $\sim$ 13.800 kbars $\sim$ 53K (Fig. 4b); observations on a much less compressible Westerly granite of similar shear resistance (G = 300 kbars, K = 500 kbars) can be fitted with a shear-dilatancy parameter $-\alpha_4$ = 3100 kbars $\sim$ 6.K (Fig.5). It appears therefore that the shear dilatancy parameter decreases sharply with increasing bulk-modulus. In fact, on the basis of the evaluation of the very small number of observations represented in Figs. 4 and 5 it appears that the product $K.\sqrt{(-\alpha_4)}$ = const., where the constant for granites is approximately 28 - 30×10^3(kbars)$^{3/2}$.

4. Dilatancy in the Isotropic Strain Hardending Medium

While the developed equations of shear dilatancy (3.11) and (3.12) refer to isotropic elastic media, the strong dependence of the second-order term of ε_{kk} on the elastic shear modulus provides the key for its very rapid increase, as the stress-difference approaches and exceeds the range of elastic and thus, at small strain, of linear behavior in shear. In order to represent the time-independent deformation of engineering materials to shear stresses exceeding this range (loading), it has been found useful to apply Euler's theorem of homogeneous functions [16] and to replace the elastic shear modulus 2G by a stress-dependent shear modulus $2\overline{G} = \frac{2}{3}\frac{\overline{\sigma}}{\overline{\varepsilon}}$ where $\overline{\sigma}$ and $\overline{\varepsilon}$ are the (invariant) intensities of stress $\sigma = (\frac{1}{2}\mathrm{tr}\ \underset{\sim}{s}^2)^{1/2}$ and a strain $\overline{\varepsilon} = \sqrt{\frac{2}{3}}\,(\mathrm{tr}\ \underset{\sim}{e}^2)^{1/2}$; the empirical relation $\varepsilon = f(\overline{\sigma})$ in the form of a power function can be deduced from a uni-axial compression test without lateral constraint. Hence $2\overline{G} = \frac{2}{3}\overline{\sigma}[f(\overline{\sigma})]^{-1} = 2\overline{G}(\overline{\sigma})$ can be obtained by fitting an experimental record with the analytically convenient function $f(\overline{\sigma}) = (\overline{\sigma}/\overline{\sigma}^*)^n$ [31], valid for $\overline{\sigma} \geq \overline{\sigma}^*$, where $\overline{\sigma}^*$ is a reference stress-intensity delimiting the range of deviation from elastic behavior, and $n > 0$ is an empirical coefficient (integer) to be determined from experimental records, provided loading is one-directional. Hence the linear approximation to Eq.(3.9) is replaced by

$$\varepsilon_1 - \varepsilon_2 = \left[\frac{(\sigma_1 - \sigma_2)}{(\sigma_1 - \sigma_2)^*}\right]^n \cdot (\sigma_1 - \sigma_2)/2G \qquad (4.1)$$

valid for $(\sigma_1 - \sigma_2) \geq (\sigma_1 - \sigma_2)^*$, the limiting stress difference beyond which the non-linear (inelastic) behavior becomes significant. For rocks this limit varies roughly between zero and two-thirds of the stress-difference at (shear) fracture. Assuming that only the behavior in (volume-constant) shear deviates from elasticity, while the response to hydrostatic pressure remains elastic, and neglecting, as before, the non-linear terms in ε_{kk}, the introduction of Eq.(4.1) into Eq.(3.11b) produces a useful, approximate shear-dilatancy relation beyond the elastic range in the form

$$\varepsilon_{kk} - \frac{\sigma_2}{K} \doteq \frac{1}{3K}(\sigma_1 - \sigma_2) + \frac{\alpha_4}{4KG^2}\left[\frac{(\sigma_1 - \sigma_2)}{(\sigma_1 - \sigma_2)^*}\right]^{2n} (\sigma_1 - \sigma_2)^2 \qquad (4.2)$$

For uni-axial compression without constraint ($\sigma_2 = 0$), therefore

$$3K\varepsilon_{kk} = \sigma_1 + \frac{3\alpha_4 \sigma_1^{*2}}{4G^2}\left[\frac{\sigma_1}{\sigma_1^*}\right]^{2(n+1)} \qquad (4.3)$$

valid for $\sigma_1 \geq \sigma_1^*$, while for $\sigma_1 < \sigma_1^*$ the shear dilatancy is governed by Eq.(3.12b), the constants K, G and α_4 remaining identical in both equations. The observed rapid increase in the shear dilatancy as the stress-difference (or the uni-axial stress) exceeds the pseudo-elastic limit and approaches the plastic range, with G decreasing asymptotically towards zero and the dilatation asymptotically attaining a horizontal tangent, is a reflection of Eq.(4.2). The nature of the transition from elastic to plastic behavior in shear and the associated increase in dilatancy is clearly reflected in the experimental records:

as n increases with increasingly sharp deviation of the response in shear from elastic ($n = 0$) to plastic behavior ($n \to \infty$), the rate of increase of dilatancy becomes increasingly rapid: the difference between sandstone and granite with low values of n on the one hand, and limestone and shale with high values of n on the other, is clearly reflected in their respective pressure-dilatancy functions (Fig. 1)

Obviously, dilatancy could be approached by deriving relation $\dot{\varepsilon}_{ij} = g(\sigma_{ij})$ using the principle of material indifference and matrix polynomials in stress; the third order "corrections" to the linear terms would produce the parametric non-linearity of the stress-strain-relations, heuristically obtained in Section 4 by introducing a stress-dependent shear modulus. The procedure is formally identical with that used for the development of the linear tensor relation $\sigma_{ij} = f(\varepsilon_{ij})$ in Section 3 from the basic equations established in Section 2 and would lead to an alternative constitutive equation of the isotropic "elastic" medium in which ε_{ij} is expressed as a tensor function in the stresses, with coefficients that are matrix polynomials in the principal stress invariants. From those the counterpart equations to Eqs.(3.11), as well as Eq.(3.9), containing second and third order terms in the stress-difference could be directly derived. Although an advantage of the alternative formulation $\varepsilon_{ij} = g(\sigma_{ij})$ appears to be the possibility it provides for a rigorous approach to the plastic range by the introduction of a yield-condition in stress-space, the lack of uniqueness in the definition of strain as well as the fact that in matrix polynomials stresses of higher power are of higher order of magnitude and can therefore not be neglected, makes the approach impractical.

5. Wave Propagation in Dilated Rock.

Since 1969 when Nersesov et al. [32] and Semenov [33] reported an unusual variation of the ratio $\xi = t_s/t_p$ of the travel times of shear waves and compressional waves in the Garm region of the U.S.S.R., followed by local earthquakes, considerable interest was centered on providing an explanation of this phenomenon, particularly after similar observations preceding the 1971 San Fernando [34] and Blue Mountain [35] earthquakes were recorded. The immediate cause of this interest is the hope that the observed changes of $\xi = V_P/V_S$, the ratio of the velocities of propagation of the P- and S-waves in the vicinity of a (potential) earthquake, the occurrence of which seems to coincide with the termination of the process of change, might provide a premonitory warning signal for the prediction of earthquakes. This hope might be justified if the observed changes, schematically presented in Fig. 6, could be conclusively related to significant earthquake related rock-parameters.

An attempt at developing such a relation has been made by Nur [36] who recognized the interrelation of the observed precursory velocity changes with "dilatancy." However, his concept of dilatancy as a crack-induced phenomenon, in conjunction with the results of his study on comparative wave-velocities in dry and in saturated rocks [37], has led him to propose the operation of a dilatancy-crack-induced pore-water-flow gradient as a model for the explanation of the observed velocity changes. The principal weakness of this model is the a priori requirement of the presence of porewater. A model based on shear-dilatancy is proposed here which avoids the necessity of the presence of pore-water, by providing an alternative mechanism for pressure-build up that is related to rock-behavior alone.

It appears that the recognition of the continuum-mechanical nature of dilatancy provides a simple explanation of the observed precursory velocity changes including their principal features. The model shown in Fig. 7 represents the area surrounding a "joint" between two elastic plates subject to forces that produce uniform compressive stresses σ_1 and σ_3, as a result of which the shear stress $\tau_{max} = (\sigma_1 - \sigma_3)/2$ acts in the direction of the joint which, for reasons of convenience, is assumed to coincide with that of maximum shear strain; the compressive stress σ_2 perpendicular to the surface of the plates is assumed to be the intermediate stress ($\sigma_1 > \sigma_2 \geq \sigma_3$).

The "joint" might be modeled in alternative ways either as an array of shear cracks in a continuous medium (fracture mechanics model), or as an actual boundary with "frictional" resistance, or as a narrow "plastic boundary layer. Movement along the joint of one plate against the other releases the strain-energy in the area surrounding it. Since the amount of energy released per unit length of joint is proportional to the area and inversely proportional to the length of the joint, it is proportional to the distance between joints. If it can be assumed that the time required to build up a critical level of strain-energy in a region surrounding a joint is the longer the larger the distance to the nearest joint along which such energy could be released, the magnitude of the energy released per unit length of the joint is the larger, the longer the build-up time.

On the other hand, the shear-stress intensity along the joint at which the relative motion is triggered, is clearly a "property" of the rock, which can be defined in alternative ways for the different models of the "joint": as a limiting "shear fracture toughness," as a limiting "frictional" resistance or as a "yield limit." In most rocks this limit is attained gradually as the shear stress increases beyond the elastic range, with an increasingly fast decrease of the (tangent) shear modulus and associated accelerated increase of dilatancy, which precedes final shear failure of the rock weakened by dilatancy-induced cracking.

Eq.(3.11b) provides an approximate expression for (small) elastic volume changes produced by an external pressure p and a shear stress $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$

$$K\varepsilon_{kk} \doteq p - \alpha_4(\sigma_1 - \sigma_3)^2/4G^2 = \qquad (5.1)$$

$$= \sigma_3 + \frac{1}{3}(\sigma_1 - \sigma_3) - \alpha_4(\sigma_1 - \sigma_3)^2/4G^2$$

The last term on the right side of this equation represents the shear dilatancy, σ_3 is the confining pressure. The volume expansion due to $(\tau_{max})^2$ is counteracted by the pressure $p = \frac{1}{3}(\sigma_1 + 2\sigma_3)$; this fact determines the sign of α_4. The stress difference at the point of reversal of the ε_{kk} versus $(\sigma_1 - \sigma_3)$ relation is obtained by differentiating Eq.(5.1)

$$(\sigma_1 - \sigma_3)^* = \frac{2}{3}G^2/\alpha_4 \qquad (5.2)$$

provided this reversal still occurs within the elastic range. The stress difference at which the absolute value of ε_{kk} changes into expansion is obtained by solving Eq.(5.1) for the stress difference at $\varepsilon_{kk} = 0$:

$$(\sigma_1 - \sigma_3)_{\varepsilon_{kk}=0} = (\sigma_1 - \sigma_3)^* \left[1 \pm \sqrt{1 + 6\sigma_3/(\sigma_1 - \sigma_3)^*}\right] \qquad (5.3);$$

its elastic value depends on the magnitude of the confining pressure σ_3. For the unconfined compression test $(\sigma_3 = 0)$ therefore

$$(\sigma_1 - \sigma_3)_{\varepsilon_{kk}=0} = 2\,(\sigma_1 - \sigma_3)^* \qquad (5.3a)$$

this limit would be in the elastic range. Since, however, in the absence of a very high confining pressure, the attainment of this limit is, according to the test records, an immediate precursor of post-elastic shear failure, its "elastic" value cannot be reached; in Bridgman's unconfined tests on marble the observed ratio $(\sigma_1 - \sigma_3)_{\varepsilon_{kk}=0}/(\sigma_1 - \sigma_3)^* = 1.25$ (Fig. 8), in Brace's confined tests ($\sigma_3 = 4.1$ kbars) on granite it slightly exceeds 1.08, while its "elastic" value, because of the confining pressure, should be close to 3 (Fig. 5).

The transition into the failure range is preceded by an apparent reduction of the shear modulus [see Section 4], resulting in the sharp increase of dilatancy before failure which, at low confining pressures and because of the low tensile strength of rocks, reduces the failure resistance by volumetric crack-formation. The experimental evidence confirms the fact that the post-elastic rapid dilatancy-increase reflects the form of the stress-strain diagram in shear. This supports the conclusion that dilatancy in loading beyond the elastic range can be dealt with, in sufficiently close approximation, by using Eq.(3.11b) with reduced shear modulus, (Eq. 4.2), and suggest the further most important conclusion that dilatancy in rocks is not the result of cracking but its cause. In fact, the hypothesis that the degree of suppression of dilatancy by confining pressure might be a significant contributory cause of the increase, with increasing confining pressure, of the shear failure strength of rocks, appears to merit experimental investigation. The conclusion that "changes of volume (that) accompany the longitudinal plastic yielding of cylindrical specimens under simple compressive stress ... and may vary in a complicated way over the range of stress ... and be manifested as permanent alterations of density on release of stress" was already reached by Bridgman in 1949, who was the first to undertake their systematic observation in different materials [17]. Being unaware of the theory of shear-dilatancy published by Reiner only the year before [5], he suggested an increase in the number of "dislocations" as a model for an energy-related mechanism which would explain "the new picture presented by these experiments, (which) is that <u>fracture is prepared for</u> ... by the reversible creation, by the stress itself, of alternations in the structure; when these alterations have proceeded to a critical degree, the structure becomes unstable and fracture ensues." Bridgman's observations on marble (Fig. 8) clearly indicate both the reversible and the irreversible part of the shear-dilatancy and its correlation with the primary stress-strain relation, and thus support the suggestion that without a systematic, extremely careful study of the correlation of shear-dilatancy and fracture in rocks and rock-like materials (concrete, etc.), the fracture aspects of rock-mechanics will remain in their present state of materials-testing pragmatism.

In order to relate the observed precursory wave velocity changes with dilatancy effects, the order of magnitude of the shear-dilatancy coefficient α_4 must be established by fitting Eqs.(3.11) to recorded observations of volume-change versus stress-difference obtained in laboratory tests of rocks in the elastic range, since it is in this range that such changes start. For granites the order of magnitude of α_4 has been found to be a (negative) multiple of the bulk modulus of the order of magnitude $-\alpha_4 \sim 5 - 50K$, depending on the ratio G/K^3.

Brillouin [38] and Truesdell [39] have provided approximate linearized solutions for the velocities of the P- and S-waves in a non-linear elastic medium the constitutive equation of which contains all second-order terms. Of these equations only that for the P-wave contains the shear-dilatancy term represented by the parameter α_4; disregarding all other nonlinear terms, an approximate expression for the velocity of P-waves

$$\rho_o V_P^2 \doteq (K + \tfrac{4}{3}G)\ (1 + \varepsilon_{kk}) + \alpha_4(\varepsilon_{kk} - \varepsilon_1) \tag{5.4}$$

is obtained, while the velocity of the S-wave

$$\rho_o V_S^2 \doteq G \tag{5.5}$$

is not affected by dilatancy. Evaluating the maximum effect of shear-dilatancy in the elastic range of granite on the classical velocity ratio $V_P/V_S = (\frac{K}{G} + \frac{4}{3})^{1/2}$, it is assumed that $(\varepsilon_{kk} - \varepsilon_1)$ at the upper limit of the elastic range is of the order of magnitude of about 0.01, the ratio $G/K \sim 0.60$ and $\alpha_4 \sim 24K$. The shear-dilatancy related term in Eq.(5.4) is therefore (-0.24K) while the first term is 1.80K. Hence the classical velocity ratio $V_P/V_S = 1.75$ is reduced by the shear-dilatancy at the elastic limit by about 8 percent to 1.62. This decrease of the velocity ratio is of the order of magnitude of the observed decreases (6 percent for the Central Asian, 10 percent for the San Fernando and up to 13 percent for the New York earthquakes), although Eq.(3.11a) is an approximation, and the shear dilatancy parameter used is that of <u>compact</u> granite in laboratory tests, which is low.

As the building-up of the shear stress intensity exceeds the elastic range, the dilatancy increases rapidly with decreasing (tangent) shear modulus in the most highly stressed regions of potential failure [see Eq.(4.2)]. The restraint exerted on the volume expansion of those regions by the surrounding area produces the build-up of hydrostatic pressure; this is reflected by the increase of the velocity of the P-wave which precedes the earthquake that results from the motion along the joint, triggered by shear-failures along it. That such pressure could be quite high might be inferred even from Eq.(3.11b), but still more from Eq.(4.2). Assuming full constraint (for the sake of simplicity) defined by $\varepsilon_{kk} = 0$, the built-up pressure is proportional to the square of the resulting shear strain. Thus, for a shear strain close to failure of not more than 0.03, the pressure would be at least 5 percent of the bulk modulus, or about 5-25 kbars. The velocity ratio is therefore bound to increase as the shear stress approaches the limit of failure; its final steep rise beyond its initial value reflects the triggering process of the actual shear failure as both the shear-stress-strain diagram and the dilatancy diagram tend to flatten out $(d\tau/d\gamma \to 0, d\varepsilon_{kk}/dp \to 0)$. A close scruting of the Central Asian [33] precursory records suggests, in fact, that not only the principal shock, but almost every strong aftershock is preceded by a steep rise of the velocity ratio.

The intensity of the limiting shear-failure stress, the exceedance of which triggers the relative motion, depends on the interaction of shear resistance and post-elastic dilatancy. Being a physical parameter that characterizes the rock in the focal area, it is hardly to be expected that the <u>magnitude</u> of the precursory velocity changes in a certain area, which only depends on this property of the rock, be related to the extent of the motion, which determines the severity of the earthquake. The magnitude of the decrease of the velocity ratio might be correlated with the shear-stress at which post-elastic dilatancy starts and thus predicted on the basis of Eq.(5.2), which relates it to the two material parameters G and α_4.

The duration of those changes, however, the first period of which (decrease of velocity-ratio) reflects the process of elastic stress-build-up, the second (increase of velocity-ratio) the subsequent process of post-elastic pressure build-up at accelerating shear deformation and post-elastic dilatancy in the vicinity of the joint, is likely to be determined by the rate of build-up of the stresses in the focal area; it may therefore be a function of the magnitude of the area involved. Since this magnitude determines the total elastic strain-energy that can be built-up before failure, and is subsequently released by motion along the joint, the duration of the process of precursory velocity changes does therefore appear to provide an indication of the severity of the subsequent earthquake, a conclusion supported by the observations.

Many more systematic laboratory studies and in-situ measurements of dilatancy in various rocks will be required to quantitatively verify the proposed model in all its details. However, it appears that this model provides strong evidence for the significance, in rock-mechanics and geophysics, of both elastic and post-elastic dilatancy in rocks, as well as for the belief that the study of "premonitory" changes of travel times or velocity-ratios in the light of the proposed shear-dilatancy model could provide a most valuable tool for the prediction of earthquakes.

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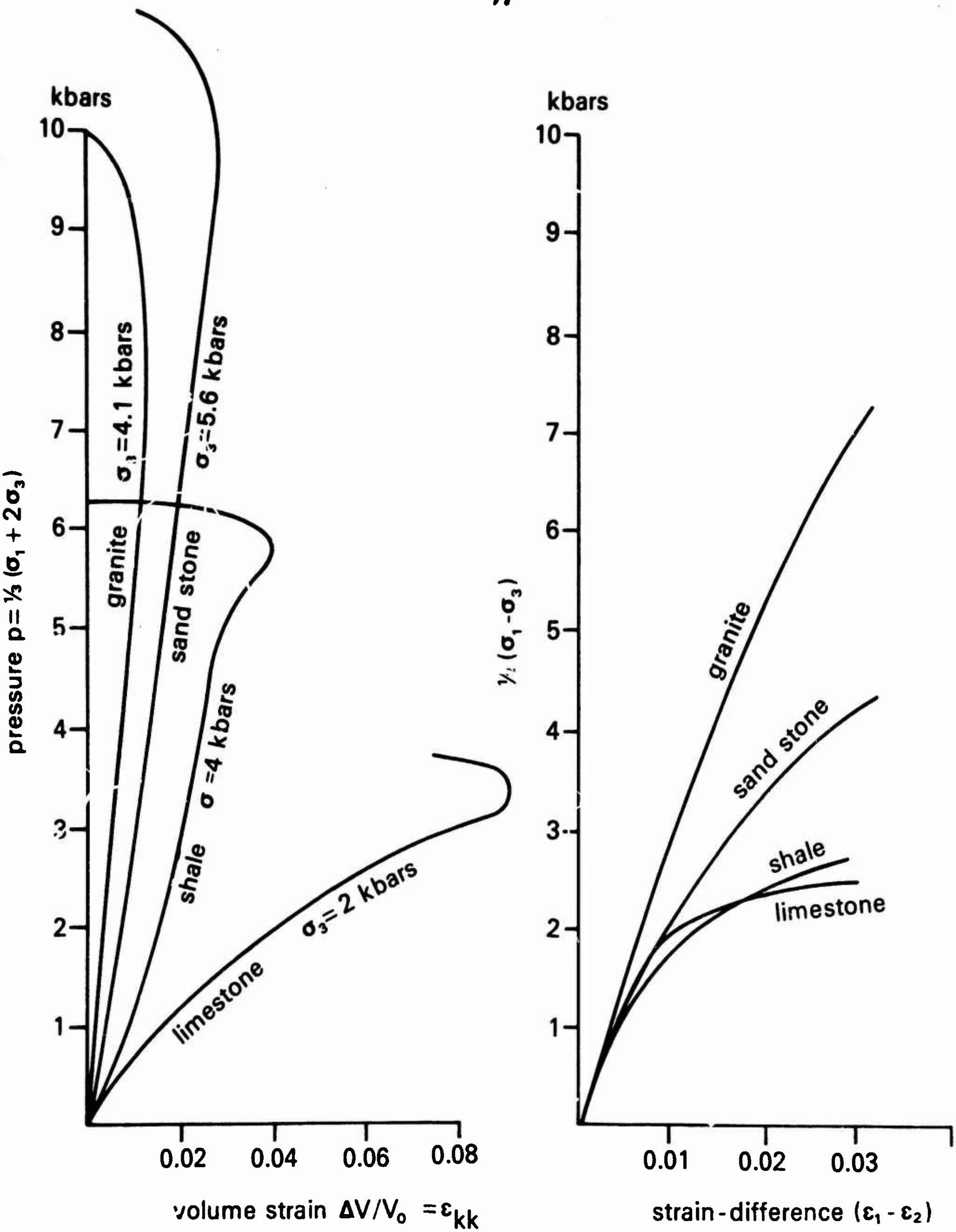

Fig. 1 Shear-stress shear-strain and dilatancy pressure diagrams for various rocks. [1]

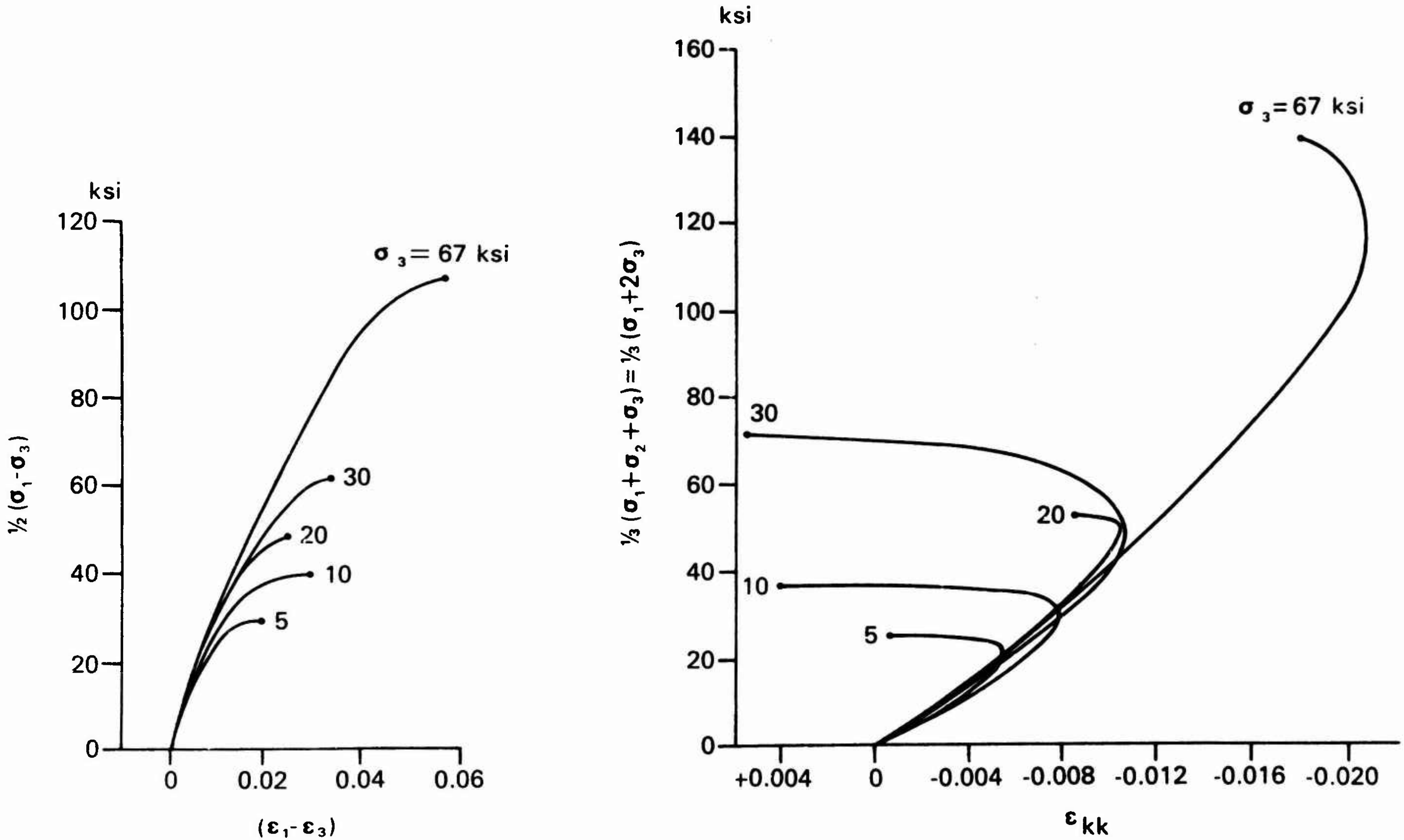

Fig. 2 Shear-stress shear-strain and pressure dilatancy diagram in compression tests on Cedar City granite at different confining pressures σ_3. [14]

Fig. 3 Irreversible normal-stress (Weissenberg-Ronay) effect produced by cold-rolling of Ti alloy bar.

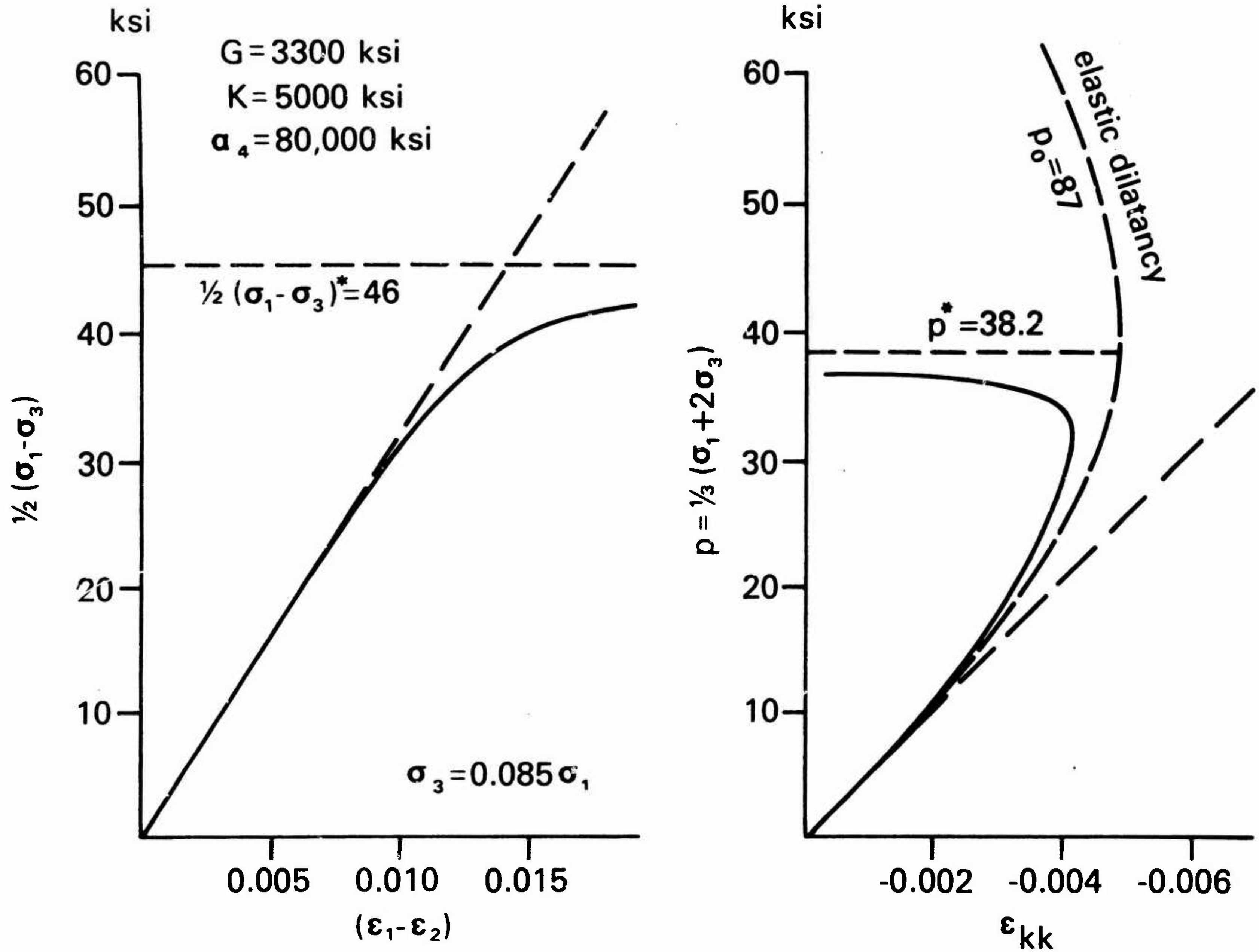

Fig. 4a Fitting of results of compression tests with confining pressure σ_3 for Westerly granite [14] by Eqs.(3.5) to (3.7)

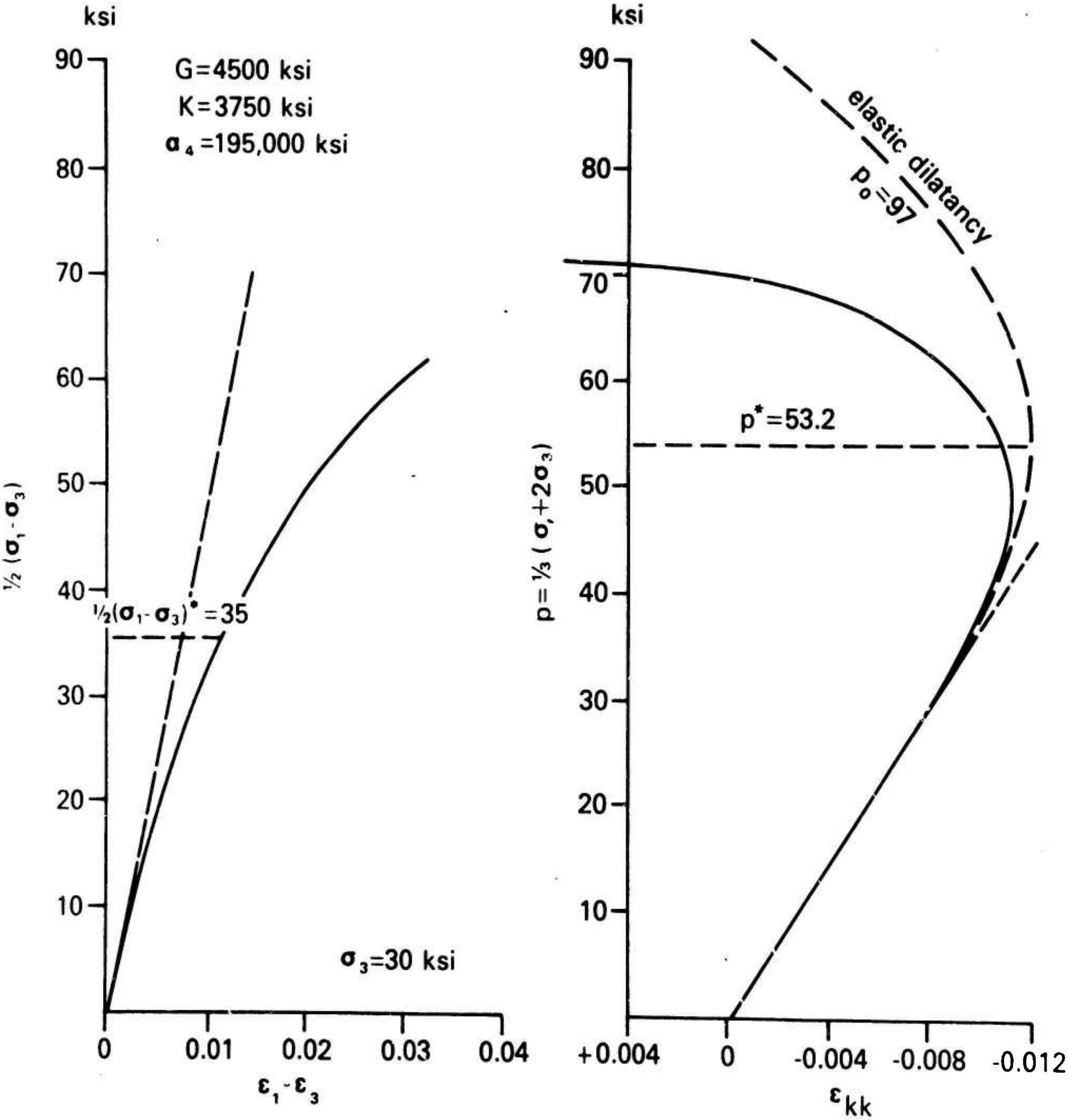

Fig. 4b Fitting or results of compression tests with confining pressure σ_3 for Cedar City granite [14] by Eqs. (3.5) to (3.7).

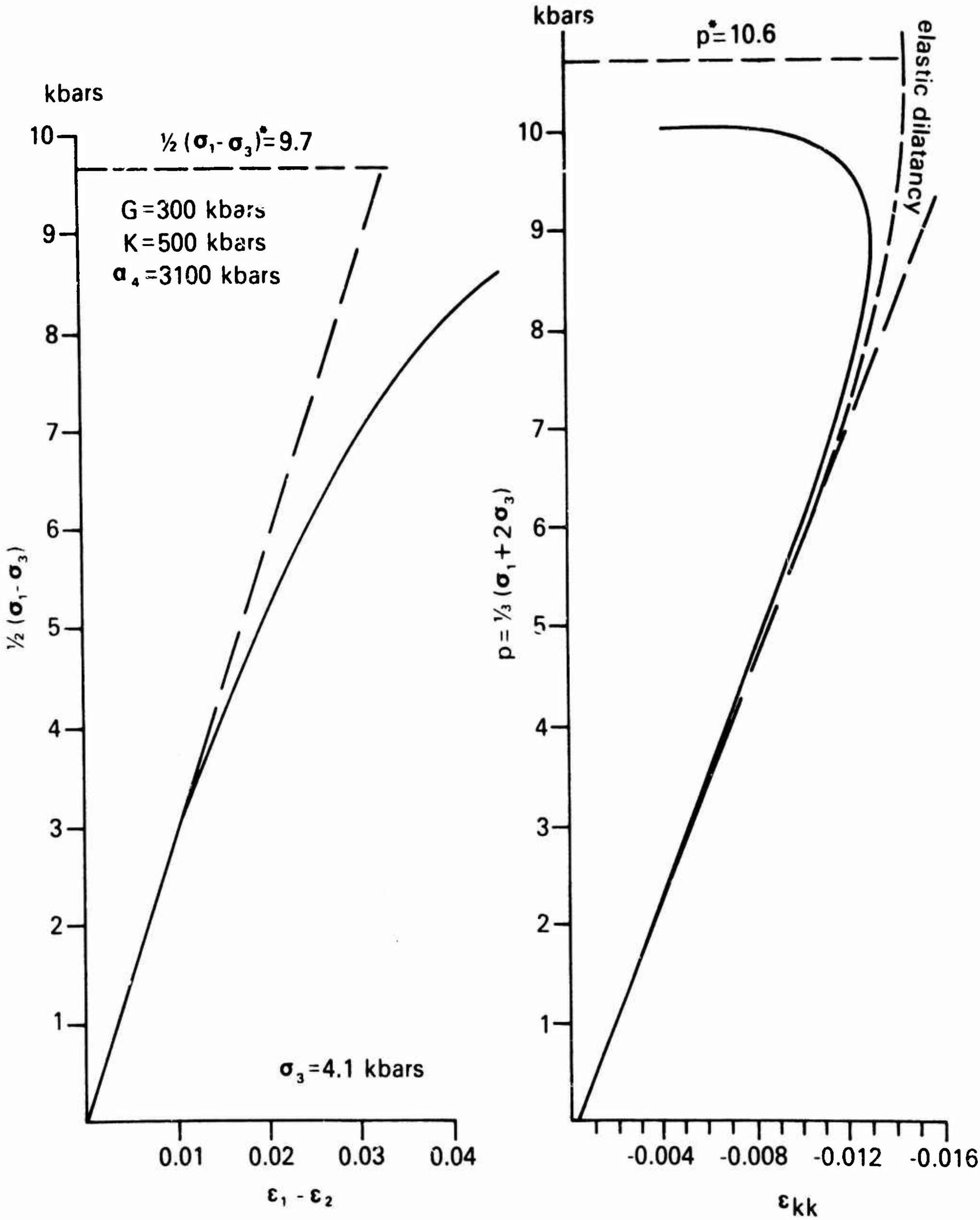

Fig. 5 Fitting of results of compression tests with confining pressure σ_3 for Westerly granite [11] by Eqs.(3.5) to (3.7).

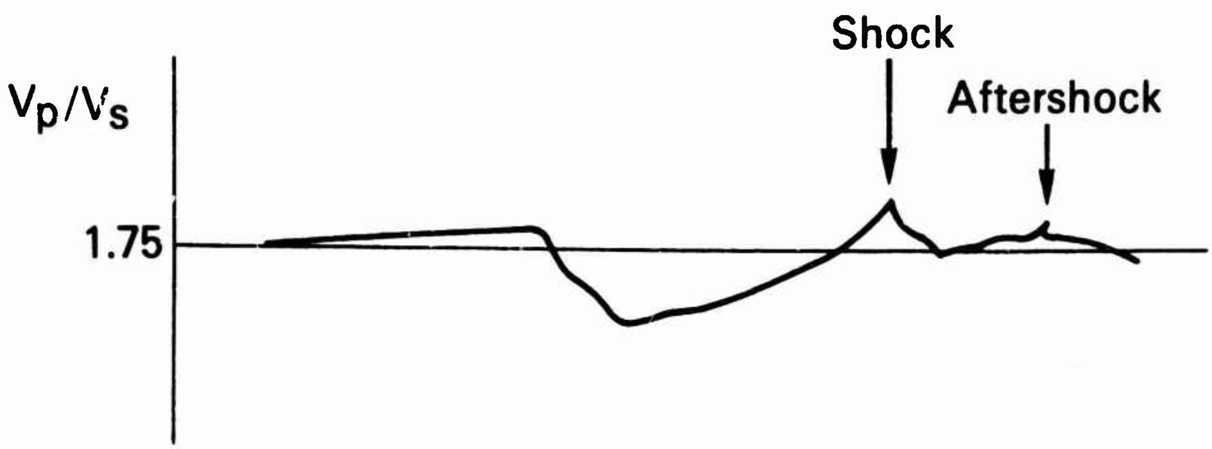

Fig. 6 Schematic form of the observed precursory changes of the velocity-ratio $\xi = V_p/V_S$.

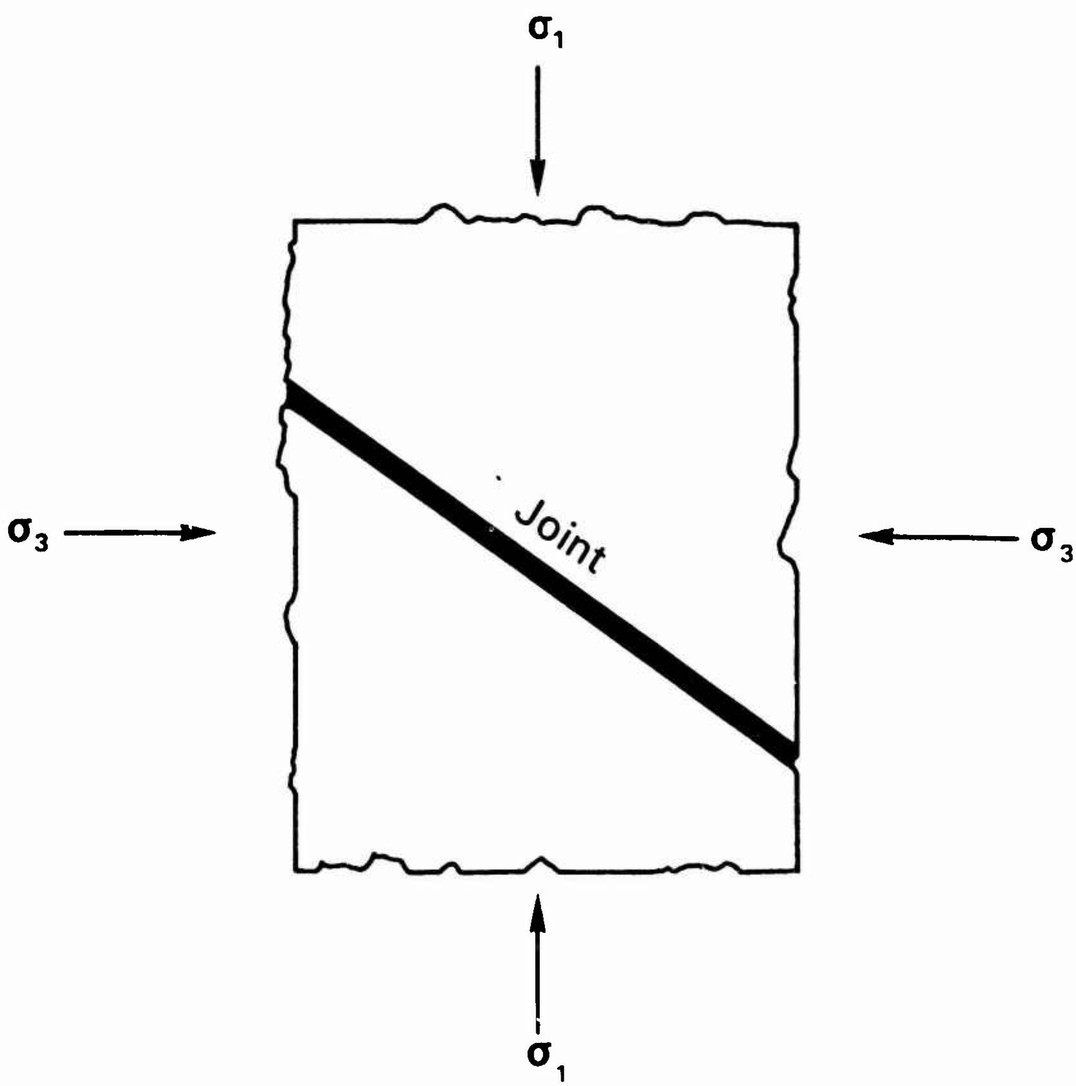

Fig. 7 Model of "joint" along which strain-energy is released.

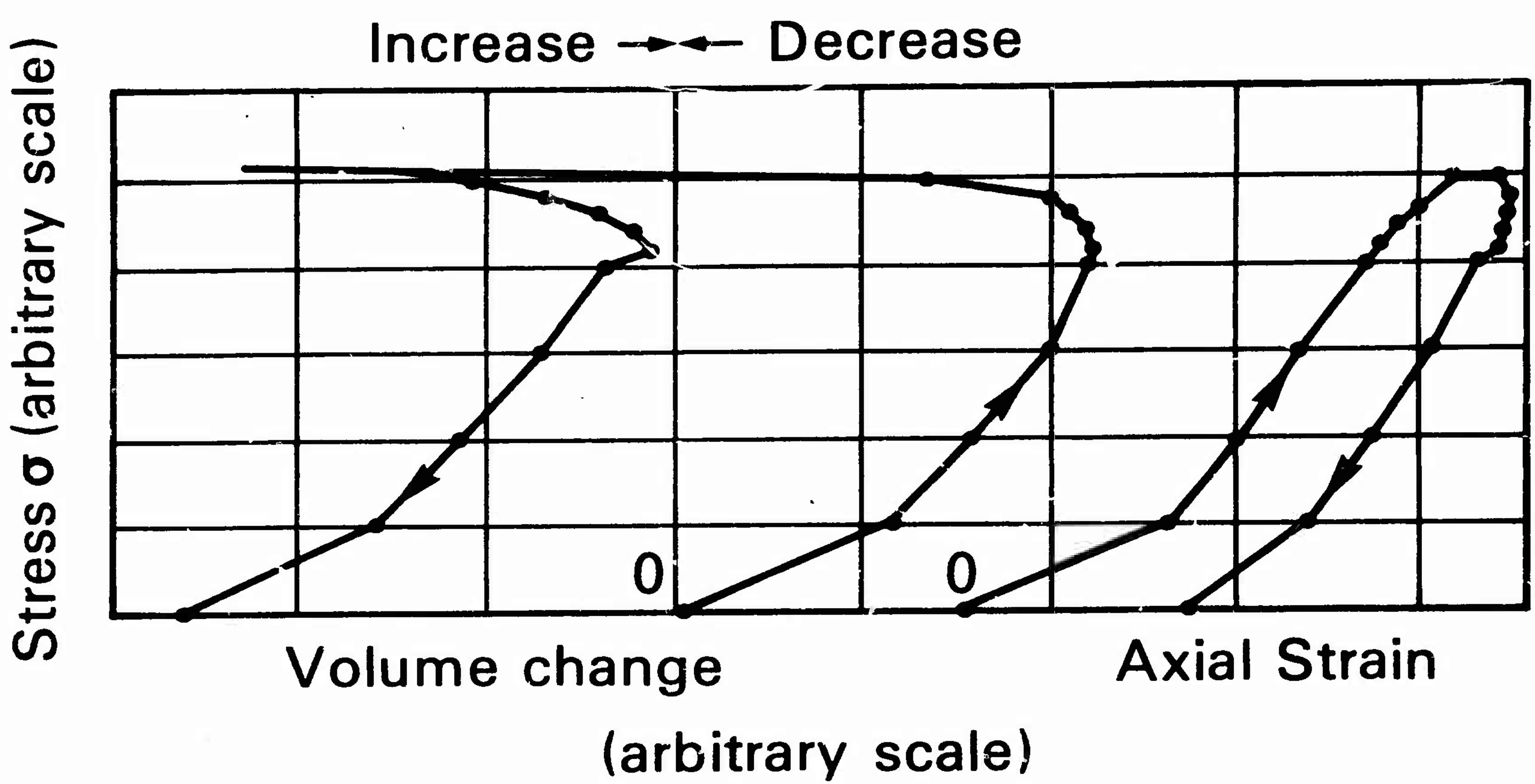

Fig. 8 Change of length and change of volume versus compressive stress in unconfined compression test of marble. [17]