FUNDAMENTAL LIMITATIONS ON IMAGE RESTORATION

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Bolt Beranek and Newman, Incorporated

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FUNDAMENTAL LIMITATIONS ON IMAGE RESTORATION

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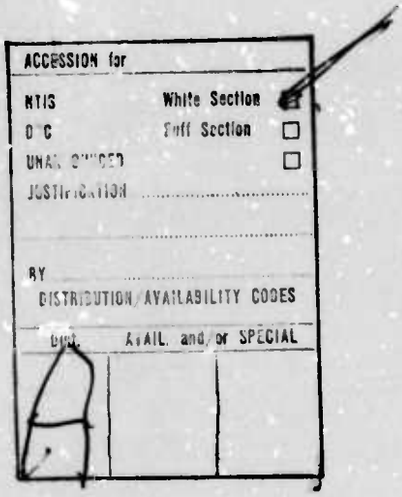

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FUNDAMENTAL LIMITATIONS ON IMAGE RESTORATION

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The research supported under this contract dealt with four problem areas pertinent to optical propagation through the turbulent atmosphere and to the problem of object restoration. The are:

1. The object restoration problem as an improperly posed inverse problem: a solution via singular value decomposition.
2. Sums of independent lognormally distributed random variables.
3. Realizability conditions on the covariance of the wavefront aberration

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function.

4. Statistics of the transfer function; temporal and random amplitude effects.

Chapter I of this report is devoted to a detailed discussion of the object restoration problem. A solution is effected via the method of singular value decomposition and a representative problem is solved and compared with a previous solution via Tichonov regularization. Two situations are considered, the case where the image is noisy but the point spread function is accurately known, and the case where both image and point spread function are noisy. Items 2 - 4 have been discussed and the final versions written up in RADC-TR-74-[illegible]7, October 1974.

SUMMARY

The objective of this program is the analysis of the fundamental limitations placed upon object restoration by the presence of noise and unavoidable atmospheric and optical system degradations. Our approach is to treat individually and in concert the several factors which limit image formation (direct problem) and further limit image restoration (inverse problem), to understand their interrelation and order of importance. Within this overall framework, we have treated various problem areas comprising the chapters of this report: the object restoration problem viewed as an improperly posed inverse problem, sums of random variables individually governed by lognormal probability density functions, and moment behavior of the transfer function random process, including temporal effects and those due to random amplitude errors acting in conjunction with random phase errors, realizability conditions on the covariance of the wavefront aberration function.

Chapter I is devoted to a detailed discussion of the object restoration problem. The purpose of this chapter is two-fold. First to discuss the peculiar mathematical behavior of the object restoration problem; second to discuss computational techniques designed to handle such problems. In particular the method of singular value decomposition is employed to effect a numerically stable inversion. The solution obtained via singular value

decomposition is compared to the solution obtained previously by Barakat and Blackman via Tichonov regularization when the image is a bar target and the optical system is an aberration-free slit aperture. In addition to the image being noisy, calculations are also displayed for the important case where both image and point spread function are noisy (5% noise in both).

Research conducted during the first part of the funding period was basically completed at that time, and the work was written up in final form, we refer the reader to report: RADC-TR-74-277, October 1974 for full details. The research was divided into three topics:

1. Sums of independent lognormally distributed random variables.
2. Realizability conditions on the covariance of the wave-front aberration function.
3. Statistics of the transfer function: temporal and random amplitude effects.

The lognormal probability density permeates much of the current literature on optical propagation through the turbulent atmosphere and has been the source of a good deal of controversy in various contexts of the general problem. Sums of lognormal random variables arise through aperture averaging and other related

processes. Because the lognormal density assumes central importance in many atmospheric propagation problems, certain aspects of the behavior of (sums of) lognormally distributed variables were considered. Particular results include the derivation of an expression for the characteristic function of the lognormal probability density, which is then used to calculate densities for sums of lognormally distributed random variables; a simplified proof that the lognormal probability density is *not* determined by its moments even thougn they exist; an investigation of the observed "permanence" of the lognormal density in terms of its asymptotic behavior, leading to a soundly based explanation of the phenomenon. Realizability conditions on the covariance of the wavefront aberration function were also briefly considered; the class of admissible covariance functions is restricted to those representing (isotropic) random surfaces. Investigations of the statistics of the transfer function random process, predominantly restricted in our prior work to errors of phase, have been extended to include the statistical dependence of the transfer function process on time (and concurrently on integration time), as well as on jointly present random amplitude and phase errors. Results show that the expected value of the time dependent transfer function *does not* depend on integration time, but that the general higher moment behavior *is* collection time dependent. Random amplitude effects are considered within the

usual lognormal and normal assumptions regarding the respective amplitude and phase probability densities, but without the added restriction of independence between the amplitude and phase random processes. In terms of the expected transfer function, a product of four terms arises: one standard deterministic part, one part due to phase errors acting alone, one part due solely to amplitude perturbations, and a cross term which is of unit modulus and depends for its existence on the odd part of the cross correlation between amplitude and phase.

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CHAPTER I

THE OBJECT RESTORATION PROBLEM AS AN IMPROPERLY POSED INVERSE PROBLEM: A SOLUTION VIA SINGULAR VALUE DECOMPOSITION

1. INTRODUCTION

In producing an image of a remote object through an intervening turbulent atmosphere, we are faced with the reality that the image is an imperfect and often inadequate representation of the object. In order to obtain more information one natural approach is post-detection compensation, whereby one attempts to obtain the object given the image and the relevant diffraction characteristics (i.e., point spread function or transfer function) of the optical system. We term this the object reconstruction problem. Unfortunately the object reconstruction problem is one of a class of *inverse* problems that are extremely sensitive to noisy data; in fact so much so that naive and *ad hoc* techniques are virtually powerless to effect a viable solution.

This fact has been implicitly recognized by many people and the existence of a current program on real time *predetection* compensation is proof of this realization. Nevertheless, even if post-detection compensation is currently in disfavor, the fault has been more with the proponents of this approach rather than with the approach itself, in that many methods currently advocated and implemented on a computer are not capable of handling the inherently ill-posed nature of the problem.

The purpose of this part of the report is two-fold. First to discuss in the peculiar mathematical behavior of the object

reconstruction problem; second, to discuss computational techniques specifically designed to handle such inverse problems. Thus this part of the report contains didactic material as well as original material.

It is probably safe to say that inverse problems, such as the object reconstruction problem, are among the most difficult problems facing the scientific community today. The problems are difficult enough in themselves and the overly optimistic attitude of the optical reconnaissance community that brute force computation on large memory, high speed computers can circumvent the inherent difficulties has tended to retard progress in this complicated subject. So much for the polemics, now to the technical problems.

In order to proceed we must postulate or derive a relationship between object and image. Unless we possess such a mathematical or physical model we cannot proceed. If the optical system is illuminated by spatially incoherent, quasi-monochromatic illumination, the transfer of radiation from the object plane to the image plane is a linear operation in the illuminance. Thus by virtue of the superposition integral

$$h(x,y) = \int_{-\infty}^{\infty}\int t(x,x';y,y')o(x',y')dx'dy' \qquad (1.1)$$

where:

$h(x,y)$ = distribution of illuminance in the image

$t(x,y)$ = point spread function

$o(x,y)$ = object intensity function.

Equation 1.1 is an integral equation of the first kind for the unknown object o in terms of the measured (and thus noisy) h and known (but possibly noisy) t. We consider only objects of finite extent, i.e.,

$$o(x,y) \equiv 0 \quad \text{for} \quad x,y \notin R \tag{1.2}$$

where R is a finite region not necessarily symmetric about the origin of coordinates. All that we know about o is that it lies in the larger finite region A (i.e., $o(x,y) \subset R \subset A$) where A is a region about the origin of coordinates for which we have measurements of the image. A is not necessarily symmetric, but probably would be in most application. Thus, Eq. 1.1 becomes

$$h(x,y) = \iint_A t(x,x';y,y')o(x',y')dx'dy' \ . \tag{1.3}$$

Note that we are *not* postulating the isoplanatic condition

$$t(x,x';y,y') = t(x-x',y-y') \ . \tag{1.4}$$

There are no restrictions, such as symmetry, placed on the kernel $t(x,y)$ except that it be the point spread function of an optical system.

Our problem is to determine the unknown $o(x,y)$ in terms of the known $h(x,y)$ and $t(x,y)$, if possible.

There are actually two topics that must be considered in an analysis of the measurements of the type described by Eq. 1.1 which we now write in abstract operator form as

$$h = to \ . \tag{1.5}$$

The first problem is the mathematical one of finding an inverse operator t^{-1} such that $t^{-1}t$ is the unit operator. For the purpose of the present discussion such an inverse is assumed to exist. A "solution" of Eq. 1.5 is then

$$o = t^{-1}h \ . \tag{1.6}$$

The solution is not unique. The solution o in this equation is defined only to within an additive component o_1 where o_1 is any solution of the homogeneous equation

$$to_1 = 0 \ . \tag{1.7}$$

A unique solution of Eq. 1.1 is possible only if information sufficient for the determination of o_1 is available. The meaning of Eq. 1.7 is that the measuring system (i.e., optical system) is completely insensitive to those spectral components of the object contained in o_1. However since we are going to deal only with finite size objects, Eq. 1.2, then $o_1 \equiv 0$.

The second problem in the analysis of Eq. 1.3 arises from the fact that both h and t are subject to statistical uncertainties. In fact small errors in the measured image h can (and usually are) amplified by a nearly singular operator t^{-1} to such an extent that the "solution" of Eq. 1.3 is completely meaningless.

The second problem arises from the fact that the image is a measured quantity and therefore subject to statistical uncertainties. It will turn out that these statistical errors are extremely important. Primary causes of these image errors are: (1) deterministic degradations – aperture and aberration limits; (2) stochastic degradations such as residual random wavefront errors due to the combined effects of turbulence and pre-detection compensation, Poisson (at best) nature of return and Poisson (or similar) detector noise. It is tempting to consider the errors as arising from additive noise so that h(x,y) becomes

$$h_1(x,y) + n(x,y) \tag{1.8}$$

where n(x,y) is the space variant noise and $h_1(x,y)$ the noiseless image, but such as decomposition is simply not possible. The reason is that multiplicative noise also enters and to attempt to separate it out becomes almost impossible. It is better to consider the image h(x,y) as subject to noise and not attempt to specify the noise. Also the point spread function t(x,y) is

generally not known very accurately. Further t(x,y), especially for the types of problems with which we are concerned, can even be a random function; see Barakat (1971), Barakat and Blackman (1973), Barakat and Blackman (1974).

Given this background information let us see why the inverse operator t^{-1} is so badly behaved. The mathematical relation between h and o given by Eq. 1.3 (or equivalently Eq. 1.5) is basically a smoothing operation if we view o as given and seek to calculate h. If the optical instrument were perfect, then assuming the isoplanatic condition, we would have

$$t(x,x';y,y') = \delta(x-x')\delta(y-y') \ . \tag{1.9}$$

Thus Eq. 1.3 would become

$$h(x,y) = \iint_A \delta(x-x')\delta(y-y')o(x',y')dx'dy' \tag{1.10}$$

or

$$\begin{aligned} h(x,y) &= o(x,y) \ , \quad x,y \in A \\ &= 0 \ , \quad x,y \notin A \ . \end{aligned} \tag{1.11}$$

Of course, any point spread function satisfying Eq. 1.10 is not physically realizable. At best one must settle for a broadened response, which has a maximum in the vicinity of $x = x'$, $y = y'$ and tails off to zero as $x - x'$, $y-y' \to \pm\infty$. Such a point spread function necessarily smooths the object somewhat causing a loss

of information in the measured image h. This can be seen, for example, more normally via the Riemann-Lebesque theorem, which states that for integrable t

$$\lim_{\alpha\,\beta \to \infty} \iint_A t(x,x';y,y')\, e^{i\alpha x'}\, e^{i\beta y'}\, dx'dy' = 0 \quad . \quad (1.12)$$

Thus an arbitrarily high frequency component of o(x,y) has a small effect on h(x,y). This is just another way of saying that the optical system has a bandlimited point spread function.

Our problem is to recover this information and thereby determine the unknown object o(x,y); we term this inversion problem the object restoration problem. This is a classical ill-posed problem in the sense of Hadamard. According to Hadamard, a problem is well posed if the following three conditions are satisfied:

(a) solution exists

(b) solution is unique

(c) solution depends continuously on the input data.

We need not concern ourselves with conditions a and b as they are satisfied in the object restoration problem. Barakat (unpublished) has given formal proofs of conditions a and b. However condition c is not satisfied as evidenced by the Riemann-Lebesque theorem, Eq. 1.12; object restoration problems do not satisfy condition c since a small change in the image data h can

correspond to an *arbitrarily* large change in the solution c. Thus the inverse operator t^{-1} does not have a bounded inverse. This numerical instability is inherent in the very nature of the problem.

Our basic problem is to stabilize the solution against the numerical instabilities of such ill-posed problems.

2. ERROR ANALYSIS AND CONDITION NUMBER

Given the preliminary information of the previous sections, we now undertake to discuss in some detail the technical problems which must be faced in attempting object reconstruction on a *rational* basis.

To begin with, we might as well face the fact that the continuous version of the problem is basically useless since the image is measured at discrete points. Therefore we must replace the continuous version by a discrete version. The double integral is replaced by a quadrature formula (the actual one employed is of no great concern) and the continuous variables by discrete mesh points. If the weights for the quadrature formula are denoted by H_k and the quadrature points by x_k, etc., we obtain

$$h(x_i,y_j) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} H_k H_\ell t(x_i,x_k;y_j,y_\ell)\, o(x_k,x_\ell) \quad , \quad (2.1)$$

where $i,j=1,\cdots,m$. If we employ lexicographic ordering we can write this more concisely in matrix notation as

$$\hat{h} = \hat{t}\hat{o} \quad , \quad (2.2)$$

The matrix $\hat{t}$ is $m \times n$ (m rows, n columns) and in general $m \geq n$; $\hat{h}$ and $\hat{o}$ are in this concise rotation column vectors of size m and n respectively.

The validity of this discretization depends on two approximations. First, the replacement of an infinite dimensional function space by a finite dimensional vector space. Second, the replacement of an integral operator by a finite matrix. It will be assumed that both approximations are adequate and attention will be denoted to solving the resultant system of linear equations, Eq. 2.2.

The matrices resulting from ill-posed problems, such as Eq. 1.3, are inherently ill-conditioned irrespective of the particular discretization (quadrature) scheme employed, so long as the quadrature scheme is reasonably faithful in approximating the integral.

Let us consider the sensitivity of the solution of our canonical set of equations, Eq. 2.2, to small variations in $\hat{t}$ and $\hat{o}$ where m = n so that $\hat{t}$ is square. We write

$$\hat{t}(\hat{o}+\delta\hat{o}) = \hat{n} + \delta\hat{n} \quad , \tag{2.3}$$

where $\delta\hat{o}$, $\delta\hat{n}$ are perturbations of the object and image. Then

$$\hat{t}\delta\hat{o} = \delta\hat{n} \quad , \tag{2.4}$$

or

$$\delta\hat{o} = \hat{t}^{-1}\,\delta\hat{n} \quad . \tag{2.5}$$

It is convenient to have a single number which gives an overall assessment of the "size" of a matrix and plays the same role as the modulus in the case of a complex number. For this purpose the Euclidean norm is introduced

$$||\hat{t}|| \equiv \left(\sum_{}^{n} \sum_{}^{n} t_{ij}^{2}\right)^{\frac{1}{2}} \quad . \tag{2.6}$$

By Schwartz's inequality

$$||\delta\hat{o}|| = ||\hat{t}^{-1}\ \delta\hat{h}|| \leq ||\hat{t}^{-1}||\ ||\delta\hat{h}|| \quad , \tag{2.7}$$

we have

$$\frac{||\delta\hat{o}||}{||\delta\hat{h}||} \leq ||\hat{t}^{-1}|| \quad . \tag{2.8}$$

For the relative change in the image, $||\delta\hat{o}||/||\hat{o}||$; we take the norm of Eq. 2.2 and employ Schwartz's inequality, with the final result being

$$||\hat{o}|| \geq ||\hat{h}||\ ||\hat{t}||^{-1} \quad . \tag{2.9}$$

Combining this result with Eq. 2.3, yields

$$\frac{||\delta\hat{o}||}{||\hat{o}||} \leq \frac{||\hat{t}^{-1}||\ ||\delta\hat{h}||}{||\hat{t}||^{-1}\ ||\hat{h}||} = ||\hat{t}||\ ||\hat{t}^{-1}||\ \frac{||\delta\hat{h}||}{||\hat{h}||} \quad . \tag{2.10}$$

This equation shows that the relative change in the object due to relative change in the observed image depends upon the quantity $||\hat{t}||\ ||\hat{t}^{-1}||$. We will term this quantity a *condition number*, $k(\hat{t})$, to indicate how the change of the object solution depends on the change in the input.

Since we have already indicated that the object restoration problem is improperly posed, we must expect that the condition number $k(\hat{t})$ will be very large. For example if $k(\hat{t}) = 10^6$ (a common value in object restoration problems), a perturbation of 2^{-12} in the elements of $\hat{t}$ can change the computed solution

$$\hat{o} = \hat{t}^{-1}\,\hat{h} \quad , \tag{2.11}$$

by a factor of $10^6\ 2^{-12} = (5/2)^6$, that is even the leading digit is wildly inaccurate! A theoretical upper bound for the relative error in the computed solution is given by

$$\frac{||\delta\hat{o}||}{||\hat{o}||} \leq \frac{k(\hat{t})}{1-k(\hat{t})\dfrac{||\delta\hat{t}||}{||\hat{t}||}} \left[\frac{||\delta\hat{h}||}{||\hat{h}||} + \frac{||\delta\hat{t}||}{||\hat{t}||}\right] \quad , \tag{2.12}$$

provided

$$||\delta\hat{t}|| \leq \frac{1}{||\hat{t}^{-1}||} \quad . \tag{2.13}$$

This theorem gives the upper bound of the variation of $\hat{o}$ due to perturbation in $\hat{t}$ and $\hat{h}$. Thus ill-conditioning is associated with a large value for $k(\hat{t})$, since in this case the bound on the error in $\hat{o}$ is very large. This behavior corresponds to the unboundedness of the inverse of the integral operator from which $\hat{t}$ is obtained.

This discouraging state of affairs is characteristic of inversion problems and we must accept the fact that naive attempts at the solution of the problem such as direct inversion, Eq. 2.11

will not succeed. Unfortunately this fundamental fact has not been realized by most previous investigators who have placed undue reliance on machine computation under the assumption that a larger matrix $\hat{t}$ will lead to better results.

Since direct inversion will not work, it is tempting to consider least squares as an inversion technique.

3. LEAST SQUARES INVERSION

If m>n, then the natural method of solution that parallels direct inversion (when the matrix $\hat{t}$ is square) is to minimize

$$||\hat{t}\hat{o} - \hat{h}||^2 \tag{3.1}$$

This is equivalent to solving the linear system

$$\hat{t}^+\hat{h} = \hat{t}^+\hat{t}\hat{o} \tag{3.2}$$

where $\hat{t}^+$ denotes the transpose of $\hat{t}$. The matrix of coefficients $\hat{t}^+\hat{t}$ is now symmetric. The $\hat{o}$ which satisfies Eq. 3.2 is called the least squares solution and is given by

$$\hat{o} = (\hat{t}^+\hat{t})^{-1}\hat{t}^+\hat{h} \tag{3.3}$$

However, in using Eq. 3.3 it is very possible that $\hat{t}^+\hat{t}$ may be even more ill-conditioned than $\hat{t}$ itself! Thus an attempt to invert the square matrix $(\hat{t}^+\hat{t})$ will produce meaningless results similar to those obtained by direct inversion. The reason for this paradoxical state of affairs is bound up with the fact that $\hat{t}^+\hat{t}$ is usually rank deficient even though it is formally over-determined in that there are more columns than rows. That is to say, the rank should be determined during the course of computing; unfortunately information about rank deficiency cannot be obtained from triangular factorization as employed in least square calculations.

Now one way to build up the rank deficiency is to augment the data provided by the optical system (u, the image) with additional *a priori* knowledge of the nature of the physical problem in order to make the computed solution at least physically meaningful. Such constraints usually take the form of inequalities imposed on the solution. These constraints can be either implicit or explicit. The task is to select from the infinite number of possible solutions which satisfy the observational data within experimental error, the one which best satisfies some set of implicit or explicit constraints. We begin by discussing the Tichonov regularization algorithm which employs implicit constraints and which was employed by Barakat and Blackman (1973) in their studies on object restoration.

4. TICHONOV REGULARIZATION ALGORITHM

Most natural objects are smooth. This vague intuitive concept has given rise to a family of inversion algorithms which require the solution to be smooth in some prescribed sense. These algorithms go under the general name of regularization and are associated with the mathematician Tichonov.

The basic idea behind regularization is the replacement of the least squares minimization, Eq. 3.1, by minimization of the functional

$$M^{\alpha} = ||\hat{t}\hat{o}-\hat{h}||^2 + \alpha||\hat{o}'||^2 \qquad (4.1)$$

where α, termed the regularization parameter, is a small non-negative number. It can be shown that the solution via this approach is numerical stable in the norm sense. The functional M^{α} is minimized by the usual machinery of the calculus of variations by setting the first variation equal to zero. We omit the details since they are available in Barakat and Blackman (1973). The final result is that

$$\hat{o} = (\hat{t}^{+}\hat{t} + \alpha\hat{\Delta}^{+}\hat{\Delta})^{-1}\ \hat{t}^{+}\hat{h} \qquad (4.2)$$

where $\hat{\Delta}$ is the matrix of central difference operators. If α is large [i.e., $\alpha \sim o(1)$], then the regularization term tends to swamp the first term and the object solution vector is much too smooth. On the other hand, if $\alpha = o$ then the equation reduces to

$$\hat{o} = (\hat{t}^{+}\hat{t})^{-1}\ \hat{t}^{+}\hat{h} \tag{4.3}$$

which is numerically unstable, being the usual least squares solution already discussed. The choice of α depends on the shape of the object being reconstructed and on the noise level in the image.

Barakat and Blackman (1973) have utilized the Tichonov regularization algorithm in their object reconstruction studies. In the next section we will discuss the use of singular value decomposition for object reconstruction and in order to compare both methods, we first summarize some of the Barakat-Blackman work. Their numerical calculations (not the theory) is confined to that of a one-dimensional unit pulse of half width x_0 imaged by an aberration-free slit aperture. Thus the object and the point spread function are given respectively by

$$\begin{aligned} o(x) &= 0\ , \quad -\infty < x < -x_0 \\ &= 1\ , \quad -x_0 < x < x_0 \\ &= 0\ , \quad x_0 < x < \infty \end{aligned} \tag{4.4}$$

and

$$t(x) = \frac{1}{\pi}\left(\frac{\sin x}{x}\right)^2 \tag{4.5}$$

with the normalization constant π^{-1} determined by the constraint

$$\int_{-\infty}^{\infty} t(x)dx = 1 \ . \tag{4.6}$$

We now assume that the isoplanatic condition holds so that Eq. 1.3 reads

$$h(x) = \int_A t(x-x')o(x')dx' \tag{4.7}$$

where A is the integral ($-10 \leq x \leq 10$). All we know is that $o(x)$ lies within this interval as discussed in Sec. 1.

To compute the image of the pulse, we note that

$$h(x) = \frac{1}{\pi} \int_{x-x_0}^{x+x_0} \left(\frac{\sin x'}{x'}\right) dx' \ , \quad x_0 < 10 \ . \tag{4.8}$$

This can be evaluated explicitly in terms of the sine integral

$$Si(z) = \int_0^z \frac{\sin x}{x} \, dx \tag{4.9}$$

so that

$$h(x) = \frac{1}{\pi} \left\{ Si[2(x+x_0)] - Si[2(x-x_0)] \right.$$
$$\left. - \frac{\sin^2(x+x_0)}{(x+x_0)} + \frac{\sin^2(x-x_0)}{(x-x_0)} \right\} . \tag{4.10}$$

The image $h(x)$ for $x_0 = 4$ is shown in Figs. 1 and 2 (see the

dotted lines), while the object o(x) given by Eq. 4.4 is the heavy solid line.

Barakat and Blackman inverted Eq. 4.7 via the Tichonov algorithm for two cases: one, the academic noise free case shown in Fig. 1; two, the *noisy* case (5% noise in image) shown in Fig. 2. The necessary analytical and computational details are given in their paper along with numerical data on other situations.

Note that in applying the Tichonov algorithm, we have made practically no use of any *a priori* information about the object except that it lies somewhere in the interval $10 \leq x \leq 10$. The reconstructed object, shown in the various figures, has the unrealistic feature of negative illuminance over small intervals. However, considering that we did not demand non-negativity of the reconstructed object the results are very good especially for the noisy case.

5. SINGULAR VALUE DECOMPOSITION

Although the Tichonov regularization algorithm is very powerful, the fact that the resulting object reconstructions can become negative is a mark against the method. Therefore we turn to another inversion algorithm which is even more powerful, the method of singular value decomposition. It should be pointed out that the principal investigator has recently utilized this algorithm to invert photoelectron correlation function data to obtain spectral line shapes (Barakat and Blake, 1975).

The singular value decomposition of the $m \times n$ ($m \geq n$) real matrix $\hat{t}$ is given by the factorization

$$\hat{t} = \hat{U}\hat{D}\hat{V}^{+} \tag{5.1}$$

where $\hat{U}$ is an $m \times m$ orthogonal matrix and $\hat{V}$ is an $n \times n$ orthogonal matrix

$$\hat{U}^{+}\hat{U} = \hat{U}\hat{U}^{+} = \hat{I}_m$$

$$\hat{V}^{+}\hat{V} = \hat{V}\hat{V}^{+} = \hat{I}_n \ . \tag{5.2}$$

Here $\hat{D}$ is an $m \times n$ matrix whose only nonzero elements are on the principal diagonal

$$\hat{D} = \mathrm{diag}(\sigma_1, \sigma_2, \cdots \sigma_n, 0, 0, \cdots, 0) \tag{5.3}$$

where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \tag{5.4}$$

and the remaining (m - n) diagonal elements are zero.

The columns of $\hat{U}$ can be shown to be the orthonormal eigenvectors of $\hat{t}^+\hat{t}$, while the columns of $\hat{V}$ are the orthonormal eigenvectors of $\hat{t}\hat{t}^+$. Finally the singular values of $\hat{t}$, σ_j, are mathematically equal to the non-negative square roots of the symmetric matrices $\hat{t}\hat{t}^+$, $\hat{t}^+\hat{t}$

$$\hat{o} = \mu_j^2 \hat{t}\hat{t}^+\hat{o}$$

$$\hat{o} = \mu_j^2 \hat{t}^+\hat{t}\hat{o} \tag{5.5}$$

so that

$$\sigma_j \equiv +(\mu_j^2)^{\frac{1}{2}} \ . \tag{5.6}$$

However a word of caution; the fact that σ_j and μ_j are related in such a simple manner might tempt one to believe that the problem is perfectly straightforward since one merely solves the eigenvalue problem stated in Eq. 5.5. However, singular values correct to working accuracy for $\hat{t}$ can often be computed when certain small eigenvalues cannot be computed for $\hat{t}^+\hat{t}$ or $\hat{t}\hat{t}^+$. To anyone who has ever done serious computing this fact is not startling; it is caused by the perturbation of an *exact* $\hat{t}^+\hat{t}$ introduced in the multiplication of $\hat{t}^+$ by $\hat{t}$.

The condition number $k(\hat{t})$ of $\hat{t}$ can be expressed in terms of its singular values

$$k(\hat{t}) = \frac{\sigma_{max}}{\sigma_{min}} \tag{5.7}$$

where σ_{max} and σ_{min} are the maximum and minimum singular values of $\hat{t}$. Thus a ill-conditioned matrix is one with a great variation in the magnitude of its singular values. Values of $k(\hat{t}) = o(10^6)$ are common (i.e., have been encountered by the author during the course of the numerical work). To see how this affects the "solution" of our problem $\hat{h} = \hat{t}\hat{o}$, let us substitute Eq. 5.1 into Eq. 2.2. Upon performing the calculations we have

$$\hat{o} = \hat{V}\hat{D}^{+}\hat{U}^{+}\hat{h} \tag{5.8}$$

where

$$\hat{D}^{+} = \text{diag}(\sigma_1^{-1},\cdots\sigma_n^{-1},0,\cdots,0) \ . \tag{5.9}$$

The expansion becomes clearer if the summation is written out explicitly

$$\hat{o} = \sum_j \frac{\hat{u}_j^{+}\hat{h}}{\sigma_j}\,\hat{v}_j \tag{5.10}$$

here $\hat{u}_j, \hat{v}_j$ are the j*th* column vectors of $\hat{U}$ and $\hat{V}$ respectively. The smaller singular values, entering into the denominator tend to magnify greatly any error in the measured image data vector $\hat{h}$

resulting in a spurious solution. To alleviate this state of affairs, we must cut off the expansion before the contamination due to the small singular eigenvalues creeps in. Specifically let us set

$$\sigma_j^{-1} = 0 \quad \text{if} \quad \sigma_j > \varepsilon \tag{5.11}$$

where a reasonable criterion for picking ε is

$$\frac{\varepsilon}{\sigma_0} >> \text{noise} . \tag{5.12}$$

In applying the singular value decomposition to our object restoration problem we employed the programming techniques described in the basic paper by Golub and Reinsch (1970).

In order to demonstrate the power of the singular value decomposition algorithm, we again consider the problem discussed in the previous section. The results of the calculations are summarized in Figs. 3 and 4. In each case, the solution comprises a data set on the interval $|x| \leq 10$. Figure 3 is to be compared with Fig. 1, and Fig. 4 with Fig. 2. Two facts are pertinent. One, the singular value reconstruction has only very small negative intensities; two, the slopes of the singular value reconstruction are much higher than those of the Tichonov algorithm even for the noisy case.

6. NOISY POINT SPREAD FUNCTIONS

Thus for the integral operator t (or equivalently the matrix $\hat{t}$) has been assumed to be subject to no uncertainties. However, as we have already pointed out, the point spread function in many situations is itself subject to noise and it is important to assess the influence of small errors on $\hat{t}$ on the solution $\hat{o}$. Ideally one would like to know under what conditions the solution $\hat{o}$ produced by a given $\hat{t}$ is a continuous function of the matrix elements of $\hat{t}$, as well as a quantative measure of the possible effects of matrix errors.

Some aspects of this difficult problem have already been investigated by Barakat and Blackman (1973). They studied the direct problem

$$h = t\, o \tag{6.1}$$

where t is a random operator for the edge spread function and calculated the expected value of the edge spread function.

In view of this fact, it was felt that some calculations should be made of the influence of a *noisy* $\hat{t}$ on object reconstruction. The author made an attempt to quantify the influence of noisy $\hat{t}$ on $\hat{o}$, but was not very successful since the bounds on $\hat{o}$ were only weakly related to $\delta\hat{t}$.

Therefore it was decided that some numerical calculations involving both a noisy $\hat{t}$ and $\hat{h}$ would be useful. As before we confine ourselves to the situation described by Eqs. 4.4 and 4.5, singular value decomposition was employed. Both the point spread function and the image were subjected to 5% noise. The results of a typical object reconstruction are illustrated in Fig. 5. The reconstructed object is now slightly asymmetric due to the fact that the noisy point spread function is no longer symmetric. Even with the added burden of a noisy t, it would appear that the reconstructed object via singular value decomposition is "better" than the reconstructed object via Tichonov regularization; compared Figs. 5 and 2.

Thus is would appear that singular value decomposition offers a viable approach to object reconstruction when both point spread function and image are mildly corrupted by noise. The fact that the reconstructed object has a small amount of negative illuminance does not appear to be a serious problem.

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FIGURE LEGENDS

Fig. 1. Object reconstruction by 21 point Tichonov regularization: ——— original object, – – – image, · reconstructed object.

Fig. 2. Object reconstruction by 21 point Tichonov regularization: ——— original object, – – – image (5% noise added), · reconstructed object.

Fig. 3. Object reconstruction by singular value decomposition: heavy solid line, object; regular solid line, reconstructed object; dotted line, image.

Fig. 4. Object reconstruction by singular value decomposition; heavy solid line, object with 5% noise added; regular solid line, reconstructed object; dotted line, image.

Fig. 5. Object reconstruction by singular value decomposition when both point spread function and image are corrupted by 5% noise: heavy solid line, object; regular solid line, reconstructed object; dotted line, image.

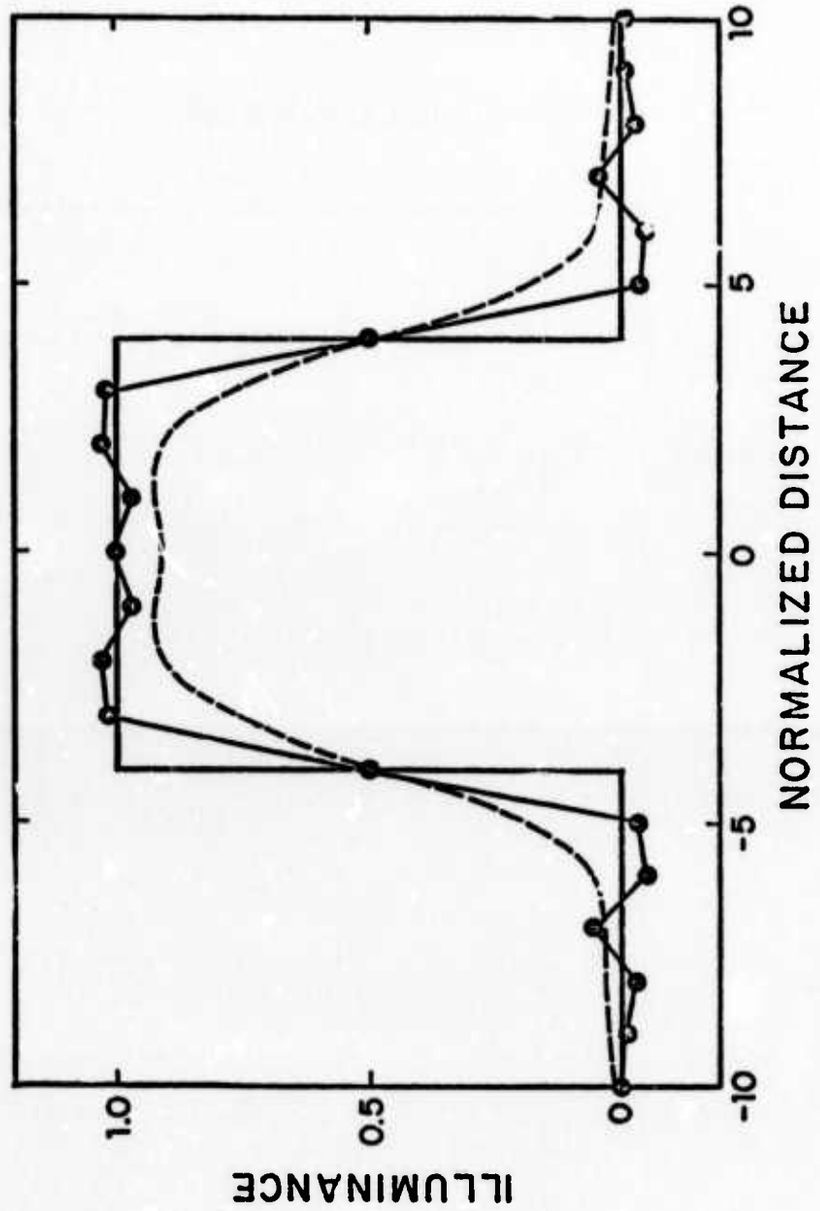

Figure 1

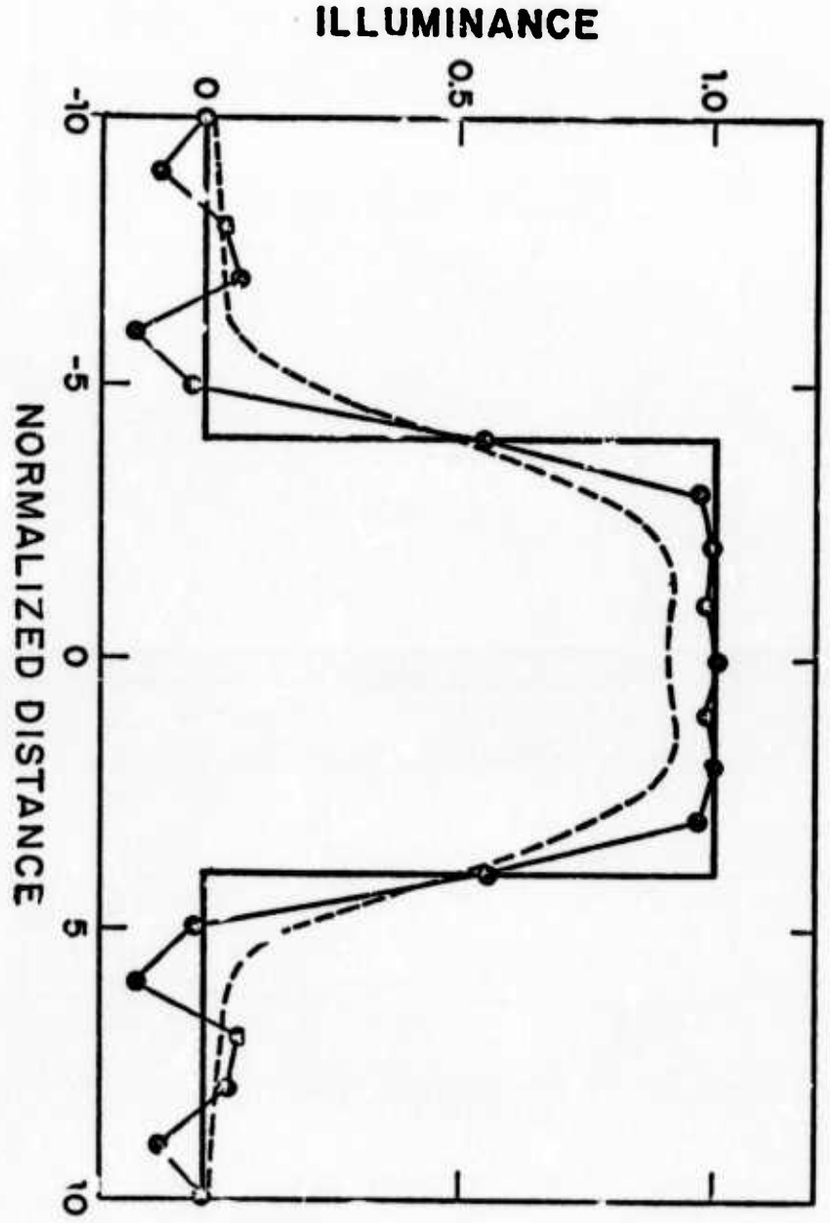

Figure 2

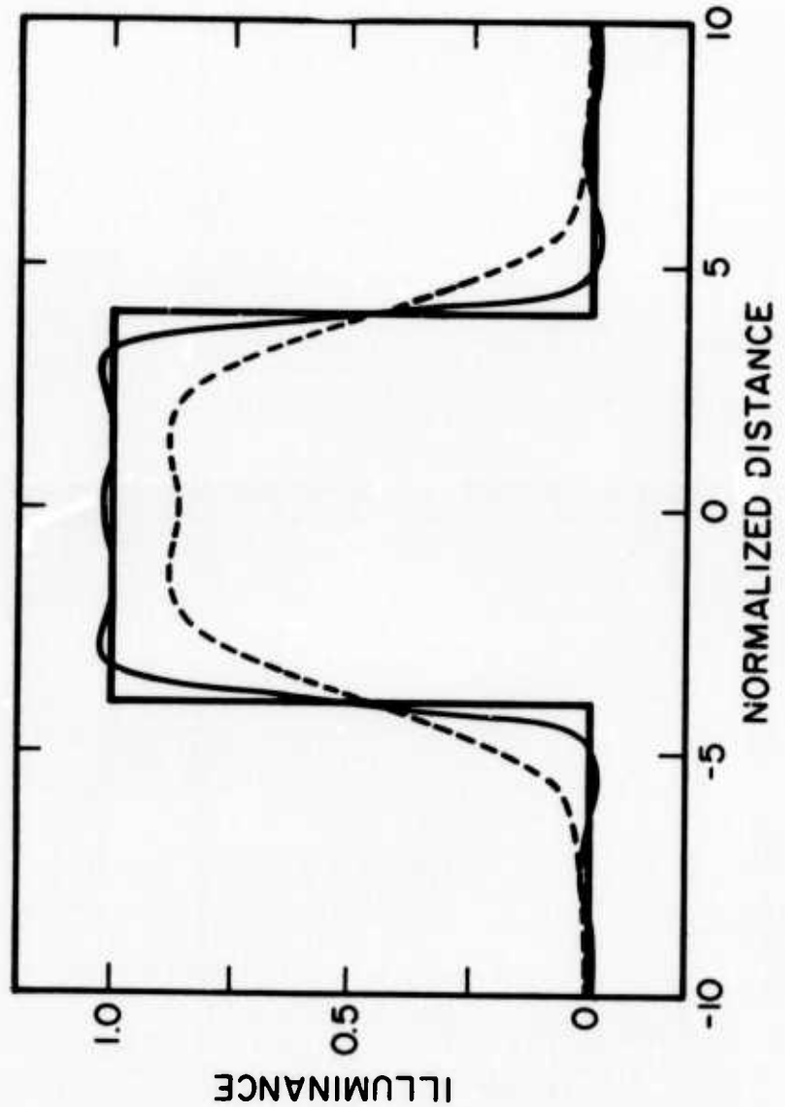

Figure 3

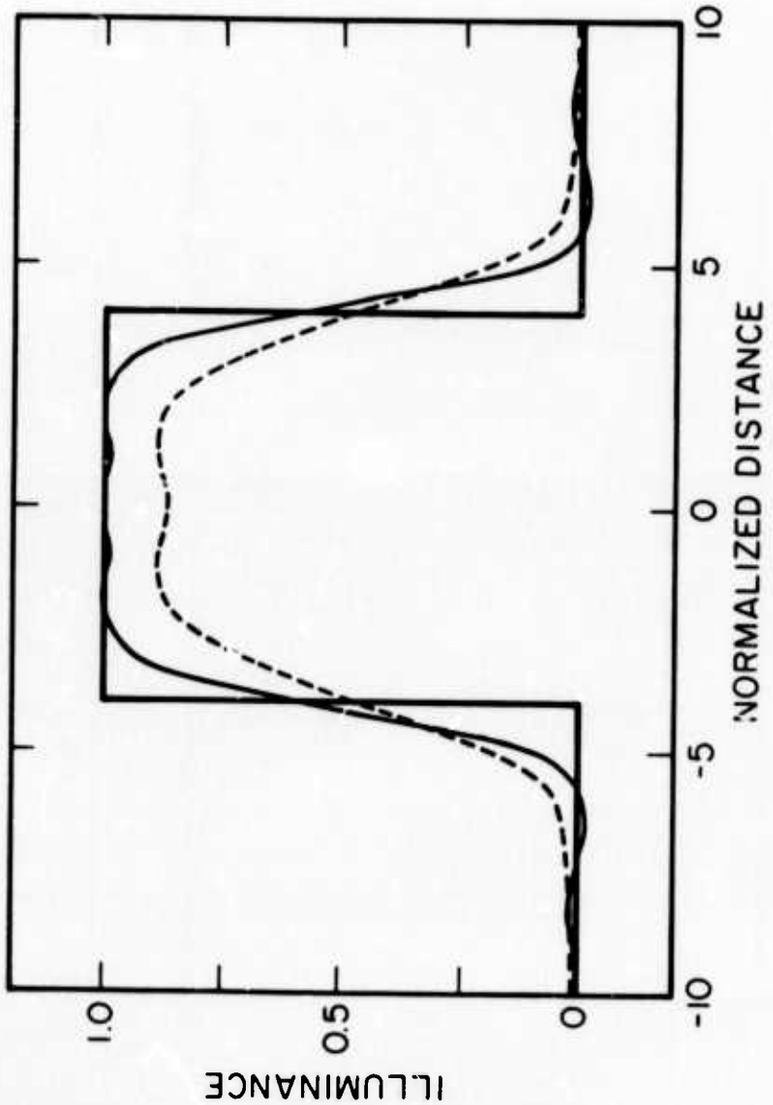

Figure 4

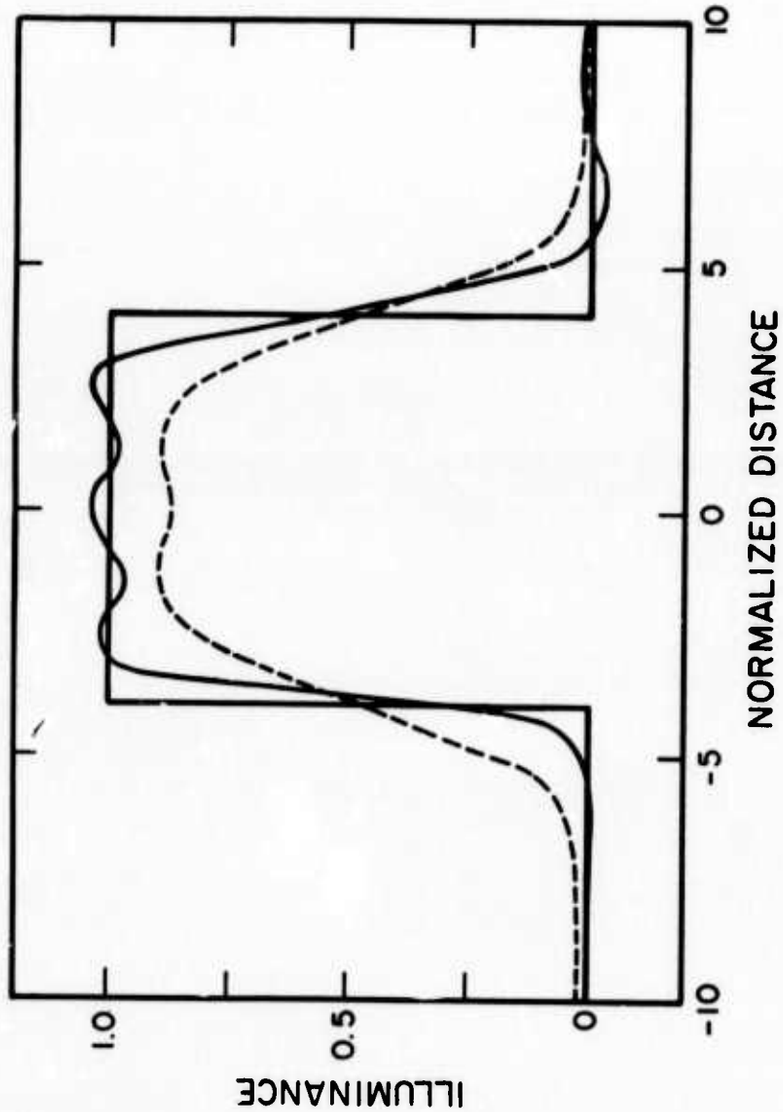

Figure 5

APPENDIX A

The research funded during the beginnings of this contract are written up in full detail in our previous report: RADC-TR-74-277, October 1974. Since this work was basically completed at that time, we refer the reader to that report for full details.

The research was divided into three topics, they are:

1. Sums of Independent Lognormally Distributed Random Variables.
2. Realizability Conditions on the Covariance of the Wavefront Aberration Function.
3. Statistics of the Transfer Function: Temporal and Random Amplitude Effects.

APPENDIX B

The following errata were noted in the interim report:

On p. 3, Eq. (2), the denominator should contain an additional factor of 2 inside the square root.

On p. 4, last paragraph, <u>for</u> $x_k/x_o = y$, <u>read</u> $x_k/x_o = e^y$.

On p. 5, Eq. (10), an additional factor, $e^{i\tau y}$, is required in the integrand.

On p. 8, top line, <u>for</u> On the..., <u>read</u> One of the...

On p. 9, Eq. (24), <u>for</u> $\dfrac{\mu_4 - 3}{\mu_2^{\,2}}$, <u>read</u> $(\mu_4/\mu_2^{\,2}) - 3$,

and subtract 3 from the quotient on the right-hand side of the first line; the second line is correct as is.

On p. 12, first of Eqs. (27), denominator should contain $N^{\frac{1}{2}}$.

On p. 13, Eq. (32), the factor multiplying t^4 in the expansion of the logarithm should have a numerator: $\mu_4 - 3\sigma^4$, and the remainder is $O(N^{-3/2})$, not $O(N^{-5/2})$.

On p. 13, Eq. (33), <u>for</u> $(s < 1)$, <u>read</u> $(|s| < 1)$.

On p. 14, Eq. (34), the fourth term in brackets goes as t^6, not t^4.

On p. 14, Eq. (37), the second term in brackets should have a <u>plus</u> sign in front.

On p. 15, line 3, <u>for</u> tensive, <u>read</u> tensively.

On p. 15, second paragraph, line 8, read "the degree of approximation is governed by the term proportional to the coefficient of excess..." (The coefficient of excess is, by definition, independent of N.)

On p. 16, Eqs. (40) and (41) are of the wrong sign. (See comments above on Eq. (37).) The sentence beginning, "The reason for the negative sign..." should therefore be deleted. It is intuitively obvious that the skewness of a lognormal-like distribution must be positive. (In the penultimate sentence in that paragraph, for recultant, read resultant.)

On p. 18, lines 8-9, read need an algorithm which retains....

On pp. 23, 24, the standard deviation is 0.25, not the variance.

On. p. 41, Eq. (21), the z's on the right hand side of equation should be lower case.

On p. 43, Eq. (24), in the third term on the right hand side of the equation replace τ_r by σ_ψ; in the fourth term replace $r_{\psi w}$ by $r_{w\psi}$.