AD-A009 926

MANPOWER PLANNING MODELS. II. CROSS SECTIONAL MODELS

R. C. Grinold, et al

Naval Postgraduate School
Monterey, California

March 1975

NPS55Mt75021

NAVAL POSTGRADUATE SCHOOL

Monterey, California

MANPOWER PLANNING MODELS - II CROSS SECTIONAL MODELS

by

R. C. Grinold

and

K. T. Marshall

March 1975

Prepared for:
Office of Naval Research, Arlington, Virginia 22217

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral Isham Linder
Superintendent

Jack R. Borsting
Provost

This work was supported in part by the Office of Naval Research and the Manpower Planning Division (MPI20) of the Marine Corps.

Prepared by:

Kneale T. Marshall

Kneale T. Marshall

R. C. Grinold (KTM)

R. C. Grinold

Reviewed by:

David A. Schrady

David A. Schrady, Chairman
Department of Operations Research
And Administrative Sciences

Released by:

Robert Fossum

Robert Fossum
Dean of Research

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55Mt75021	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Manpower Planning Models - II Cross Sectional Models		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. C. Grinold K. T. Marshall		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P. O. 3-0092
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE February 1975 13. NUMBER OF PAGES [illegible]
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Manpower, Markov Chain, Planning, Models, Flow

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is the second in a series of reports on Manpower Planning Models. The emphasis in this report is on the formulation, theory, and application of cross-sectional models. Included are sections showing applications to planning and forecasting university faculty movement, a university retirement system, and forecasting student enrollments.

DD FORM 1473 1 JAN 73 EDITION S/N

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

TABLE OF CONTENTS

1.	Introduction	1
2.	Fractional Flow Assumptions	3
3.	Fractional Appointments with Hindsight	5
4.	Fractional Appointments with Foresight	8
5.	Analysis of Fractional Appointment Policies	10
6.	Stationary Appointment Policies	14
7.	A Requirements Model	32
8.	A University Faculty Model	35
9.	Probabilistic Interpretation of the Fractional Flows	45
10.	A Retirement System Model	49
11.	Student Enrollment Forecasting Models	64
12.	A Vacancy Model	81
13.*	More General Fractional Appointment Policies	87
14.*	Evolution of Fixed Size Systems	92
15.	Notes and Comments	101

II. CROSS-SECTIONAL MODELS

1. Introduction.

The general flow model discussed in Report No. I of this series is useful for demonstrating the basic concepts of flow conservation and equilibrium. However, it has very little structure and as a result has a very large number of variables (the flows $f_{ij}(t)$). Without additional constraints it admits many unrealistic flow patterns.

In this report we present some models which essentially describe how a manpower system changes from one set of stock levels $\{s_i(t)\}$ at an accounting point t to another set $\{s_i(t+1)\}$ at the point $t + 1$. The common feature of all these models is that knowledge of historic personnel movement prior to t is not required by the model. The only data requirements will be the cross-sectional structure of the system at a given time; hence the name "cross-sectional models." A strong point in favor of such models is that most organizations keep current files on personnel such that determining the structure of the organization at say month-end, or year end, is easy. In almost all cases with which the authors are familiar, the information system in an organization does not allow for easy tracking of historical data over time.

Section 2 presents the basic fractional flow assumptions. Sections 3-7 develop the theory and properties of cross-sectional models, and section 8 presents an application to University Faculty Planning. Section 9 gives a probabilistic interpretation of the fractional flow assumptions, and sections 10 and 11 give applications using these probabilistic interpretations; section 10 presents a university faculty retirement system, and section 11 some models for student enrollment forecasting. Sections 12-14 discuss more advanced theoretical concepts, and the report ends with notes and comments in section 15.

2. Fractional Flow Assumptions.

In the fractional flow model we assume that the fraction of the stock in class i at time $(t-1)$ that flows to class j at time t is a fixed number q_{ji}, independent of t and $s_i(t-1)$. Thus

$$f_{ij}(t) = q_{ji} s_i(t-1) \tag{1}$$

$$\text{for all } t \text{ and for } i = 1,2,\ldots,N$$
$$j = 0,1,2,\ldots,N.$$

This assumption is often difficult to justify and is only an approximation in a great many cases. The user will have to balance the shortcomings of the assumption against the simplicity and utility of the resulting model for any specific application. In the examples we present in this chapter, we shall discuss the validity and shortcomings of the fractional flow assumption.

The flows and stocks are nonnegative, thus $q_{ji} \geq 0$. Also, if we sum (1) over the index j we obtain from[†] (I.1)

$$s_i(t-1) = \sum_{j=0}^{N} f_{ij}(t) = \sum_{j=0}^{N} q_{ji} s_i(t-1),$$

which implies $\sum_{j=0}^{N} q_{ji} = 1$. The fractions q_{ji} partition the stock of manpower in class i into fractions that flow into each class j.

From the other basic conservation relation (I.1) and (1) above we obtain

$$s_j(t) = \sum_{i=0}^{N} f_{ij}(t) = f_{0j}(t) + \sum_{i=1}^{N} q_{ji} s_i(t-1) \tag{2}$$

$$\text{for } j = 1,2,\ldots,N.$$

† The notation (I.k) refers to equation (k) in report number I in this series.

Let $f_0(t)$ be the vector $[f_{01}(t), f_{02}(t), \ldots, f_{0N}(t)]$ of new appointments during period t. Recall that $s(t) = [s_1(t), s_2(t), \ldots, s_N(t)]$. Finally let Q be the $N \times N$ matrix $[q_{ji}]$ for j and i between 1 and N. In matrix notation equation (2) becomes

$$s(t) = Qs(t-1) + f_0(t). \tag{2}$$

In this chapter we shall treat the N-vectors $s(t)$ and $f_0(t)$ as $(N\times 1)$ matrices which are commonly called column vectors. To avoid possible confusion we shall write out important sets of equations explicitly.

Equation (2) is the basic fractional flow model. Given the stocks at time $t-1$, the new appointments in period t and the matrix Q it is possible to predict the stocks at time t. The model is cross sectional since it uses the cross section data $s(t-1)$ and is independent of all stocks and flows prior to time $t-1$. The first term on the right hand side of (2), namely $Qs(t-1)$, is the <u>legacy</u> left over from appointments made in periods before t. The second term, $f_0(t)$, is the vector of new appointments in t. The sections that follow treat variations of the basic model (2) and present some interesting applications of (2) in a variety of contexts.

3. Fractional Appointments with Hindsight.

Let us define $s_0(t)$ as the number of vacant positions at time t, and

$$\lambda(t) = \sum_{j=0}^{N} s_j(t) \tag{3}$$

as the total number of positions in the system. We write $s(t)$ for the N-vector $[s_1(t), s_2(t), \ldots, s_N(t)]$ and $s^*(t)$ for the $(N+1)$-vector $[s_0(t), s(t)]$; the sum[†] $es(t) = \lambda(t) - s_0(t)$ is the number of individuals filling jobs within the organization at time t.

In this section we present an appointment policy which allows one or more accounting points to pass before a vacancy is filled. The next section presents an appointment policy that anticipates future vacancies.

We can distinguish the vacancies at time t by their source, since $s_0(t) = \sum_{i=0}^{N} f_{i0}(t)$. First, let $f_{00}(t)$ be the number of vacancies at time $t-1$ that are not filled during period t. The other flows are given by (1); thus

$$s_0(t) = f_{00}(t) + \sum_{i=1}^{N} q_{0i} s_i(t-1). \tag{4}$$

The lagged fractional appointment policy is determined by a scalar a_0 and an N vector $a = [a_1, a_2, \ldots, a_N]$. For $j = 1, 2, \ldots, N$, we let a_j be the fraction of vacancies $s_0(t-1)$ observed at time $(t-1)$ that are filled by appointing individuals in class j. We say that a_0 is

---

† The vector e is a row vector of appropriate length with each element equal to 1, which is used to sum the elements of a given vector. Thus $es(t) = \sum_{j=1}^{N} s_j(t)$.

the fraction of the vacancies that remain open during period t. The numbers a_j $j = 0,1,2,\ldots,N$ are independent of t and $s_0(t)$, are nonnegative, and sum to one.

From this definition we see that the appointments, or input flows, are given by

$$f_{0j}(t) = a_j s_0(t-1) \quad j = 0,1,2,\ldots,N.$$

Now for $j = 1,2,\ldots,N$ we define $w_j = q_{0j}$ as the fraction of those in class j at time $(t-1)$ who withdraw from the system during period t. Finally let P^* be the $(N+1) \times (N+1)$ matrix

$$(5) \qquad P^* = \left[\begin{array}{c|c} a_0 & w \\ \hline & \\ a & Q \\ & \end{array}\right],$$

where Q is the $N \times N$ matrix in (2).

P^* is a stochastic matrix; each element is nonnegative and the column sums are equal to one. The status of the manpower system at time t is given by the $N + 1$ vector $s^*(t) = [s_0(t), s(t)]$ where $s(t) = [s_1(t), s_2(t), \ldots, s_N(t)]$. From our definitions the lagged constant size model is

$$(6) \qquad s^*(t) = P^* s^*(t-1).$$

Example 1: Consider the example of university faculty with $N = 3$ classes, 1 - nontenured, 2 - tenured, 3 - retired. Let the time period be one year and assume that in one period 25% of the nontenured faculty become tenured, 25% stay nontenured and the remainder leave. Assume that 80% of the tenured faculty stay tenured, 10% leave, and 10% retire. Assume that 80% of the retired remain retired and 20% die (leave the system). Let us assume that all new hirings are into the non-tenured ranks. Then

$$P^* = \begin{bmatrix} 0 & .5 & .1 & .2 \\ 1 & .25 & 0 & 0 \\ 0 & .25 & .8 & 0 \\ 0 & 0 & .1 & .8 \end{bmatrix} .$$

<u>Problem 1</u>. Find the stock levels after one year if the current levels are:

	(a)	(b)
vacancies	800	828
nontenured	1000	1103
tenured	2000	1379
retired	200	690
<u>Total Positions</u>	4000	4000

Notice that $s^*(t)$ sums to $\lambda(t)$ for all t. since

$$\lambda(t) = es^*(t) = eP^*s^*(t-1) = es^*(t-1) = \lambda(t-1).$$

Thus the system remains of constant size and the $N + 1$ vector $\frac{s^*(t)}{\lambda(t)}$ is nonnegative and sums to one. The notation and form of (6) suggests an analogy to Markov chain theory. Indeed (2) and (6) are sometimes called Markov models. It is both desirable and natural that results of Markov chain theory be used wherever applicable, but the reader should keep in mind that we are discussing a deterministic model and we carefully avoid reference to probabilities. As we shall mention later, too deep an analogy to the stochastic behavior of Markov chains can be quite misleading.

4. Fractional Appointments with Foresight.

It is possible to construct a fractional appointment policy which anticipates the vacancies that will occur in period t, and which hires enough replacements to fill the vacated positions. With this policy $s_0(t) = 0$ for all t.

If $s(t-1)$ is the manpower stock at time $t-1$, then $\sum_{i=1}^{N} w_i s_i(t-1)$ vacancies will be created in period t. Of all these vacancies a fraction a_j will be filled by appointing new individuals into class j. There are no vacancies left unfilled, thus $a_0 = 0$ and $\sum_{j=1}^{N} a_j = 1$. The flow of new appointments in class j in period t is thus

(7) $$f_{0j}(t) = a_j \sum_{i=1}^{N} w_i s_i(t-1)$$

and using

(8) $$s_j(t) = \sum_{i=1}^{N} (q_{ji}+a_j w_i) s_i(t-1)$$

for $j = 1,2,\ldots,N$.

In matrix notation, let $a \cdot w$ be the $N \times N$ matrix with elements $a_j w_i$. This is the same as a matrix product if a is considered as a $(N\times 1)$ matrix (commonly called a column vector) and w is a $(1\times N)$ Matrix (a row vector). With this convention let $P = Q + a \cdot w$. Then

(9) $$s(t) = Ps(t-1) .$$

Notice that P is a stochastic matrix (all its columns sum to 1). First, since w and a are nonnegative, $P_{ji} = q_{ji} + a_j w_i \geq 0$. In addition

$$\sum_{j=1}^{N} P_{ji} = \sum_{j=1}^{N} q_{ji} + w_i \sum_{j=1}^{N} a_j = 1,$$

since $\sum_{j=1}^{N} a_j = 1$ and $w_i = q_{0i} = 1 - \sum_{j=1}^{N} q_{ji}$. Thus (9) has the same mathematical structure as (6). However, we have one less equation, a different type of hiring policy, and a stochastic matrix in which the effects of changes in a_j or w_i are not readily apparent.

Example 2: (Continue Example 1). Suppose we use the same fractional hiring policy a, but that we anticipate vacancies. The P matrix derived from the P^* matrix in Example 1 is

$$P = \begin{bmatrix} .75 & .1 & .2 \\ .25 & .8 & 0 \\ 0 & .1 & .8 \end{bmatrix}.$$

Since state 0 (outside the system) does not explicitly appear in this model it appears that certain flows take place which are not natural. For example 20% of those in retirement appear to return to the non-tenured ranks. This flow is of course due to new hiring.

Problem 2: Find the stock levels after one year if the current levels are:

	(a)	(b)
nontenured	1800	1391
tenured	2000	1739
retired	200	870
Total Positions	4000	4000

□

5. Analysis of Fractional Appointment Policies.

Both the hindsight model (6) and the foresight model (9) lead to algebraic equations similar to the transition equations of a finite state Markov chain. This section will examine how the results of regular Markov chain theory (Kemeny and Snell Chapter IV) can be used in our model. We shall concentrate on the foresight model (9) the results and algebra are identical for the hindsight model (6).

The reader who pursues parts (b) of problems 1 and 2 will notice the interesting fact that an equilibrium has been reached. For some value of $s(0)$ we obtain $s(1) = s(0)$ and therefore, $s(t) = s(0)$ for all t. This equilibrium can be explained using Markov chain theory.

If $s(0)$ gives the initial stock levels then

$$s(1) = Ps(0),$$

$$s(2) = Ps(1) = P(Ps(0)) = P^2 s(0),$$

and in general

(10) $$s(t) = P^t s(0).$$

Under reasonable assumptions on the matrix P equation (10) has an interesting structure for large t. We do not wish to go into the technical details of these assumptions since they involve concepts used in Markov chain theory and have little pertinence to our manpower flow models. In all the examples discussed in this book the assumptions hold. They lead to the fact that for some t large enough P^t has all positive elements, and that

(11) $$P^t \to V \quad \text{as} \quad t \to \infty, \quad \text{where}$$

(i) every column of the matrix V is the same, say $[v_1, v_2, \ldots, v_N]$, and thus

$$V = \begin{bmatrix} v_1 & \cdots & v_1 \\ v_2 & & v_2 \\ \vdots & & \vdots \\ v_N & & v_N \end{bmatrix},$$

(ii) the vector $v = [v_1, v_2, \ldots v_N]$ satisfies

$$v = Pv$$

$$ev = 1$$

$$v_i > 0 \text{ for every } i = 1,2,\ldots,N.$$

For large t we have, by (10), that $s(t) \cong Vs(0)$. From (ii) we see that $Vs(0) = (v_1, v_2, \ldots v_N)\lambda(0)$, where $\lambda(0)$ is the system size at time zero. If the numbers $(v_1, \ldots, v_N)$ could be determined, they would tell us what the distribution of people among classes would be after some time periods had elapsed. Although this is a limiting result as $t \to \infty$ the distribution v is often obtained approximately in only a few time periods.

Example 3: Using the distribution of people in part (a) of Problem 1 as $s^*(0)$, and using P^* from example 1 the stock levels at various times t are:

Time	0	1	2	4	8	∞
Vacancies	800	740	782	791	814	828
Nontenured	1000	1050	1003	1028	1075	1103
Tenured	2000	1850	1742	1574	1422	1379
Retired	200	360	473	607	689	690

Problem 3: Perform the calculations in Example 3 using part (a) of Problem 2 as $s(0)$ and P from Example 2. $\square$

The reader who solves Problem 3 will realize how tedious the calculation of $s(t)$ is using (10), especially for t quite large. If we let s be the limiting vector of $s(t)$ as t becomes large, then by using (11) it must be that s satisfies

$$s = Ps. \tag{12}$$

If λ is the total system size, then

$$\lambda = \sum_{j=1}^{N} s_j . \tag{13}$$

Equations (12) and (13) comprise $(N+1)$ equations in the N unknown stock levels s. It is easy to show that the equations in (12) are linearly dependent. Let e be a vector with all elements equal to one (recall that whenever we use e we shall assume its dimension is compatible in the equation in which it appears). Then (12) and (13) can be written as

$$\begin{aligned}(I-P)s &= 0\\ es &= \lambda .\end{aligned} \tag{14}$$

If one of the first N equations is ignored, the remaining N can be solved uniquely for the steady-state stocks s. If we drop the first equation in (14) we obtain N equations in N unknowns.

$$\begin{bmatrix} -p_{21} & (1-p_{22}) & & & -p_{2N} \\ -p_{31} & -p_{32} & (1-p_{33}) & \cdots & -p_{3N} \\ & \cdot & \cdot & & \cdot \\ & \cdot & \cdot & & \cdot \\ & \cdot & \cdot & & \cdot \\ -p_{N1} & -p_{N2} & -p_{N3} & \cdots & (1-p_{NN}) \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \lambda \end{pmatrix} . \tag{15}$$

Equation (15) shows how the steady state vector of stocks depends on the appointment policy.

Problem 4: Write down the equivalent of equation (15) for the hindsight model (6).

Problem 5: Continuation of Problem 3, with $\lambda = 4000$, find the steady state stock levels.

Problem 6: Contrast the difficulty involved in recalculating steady state stock levels in both the hindsight and foresight models when: (i) the size λ is changed, (ii) the appointment policy is changed. □

The cross-sectional models discussed to this point assume a system of constant size. If vacancies are considered as a separate state, equation (6) can be used for forecasting future stock levels in the time periods immediately ahead, given the current stock levels and the matrix P^*. If the long-run effects of a matrix P are required, then the analog of (15) should be used. If vacancies are assumed to be filled quickly it may be more appropriate to use an N state model. Short-range forecasts can be made with (9) and long-range forecasts with (15).

6. Stationary Appointment Policies.

Returning to equation (2) of the basic fractional flow model, the stock level in class j at t is given by

$$s_j(t) = \sum_{i=1}^{N} q_{ji} s_i(t-1) + f_{0j}(t), \quad i = 1,2,\ldots,N,$$

or in matrix form $\quad s(t) = Qs(t-1) + f_0(t).$

Thus given an initial stock vector $s(0)$, and new input vectors $f_0(1), f_0(2), \ldots, f_0(t)$, the stock levels $s(1), s(2), \ldots, s(t)$ are easily calculated.

In the previous section $f_0(t)$ was chosen so that the total number of positions in the system remained constant. In this section we consider a different form of $f_0(t)$ which allows for growth or decay of the system.

a. Geometric Growth.

Let $f_0(0) = f$, the input vector in period 0, and let θ be some positive number. Now let the input vector in period t be

$$f_0(t) = \theta^t f, \quad t \geq 0. \tag{16}$$

If $\theta > 1$ the new input grows geometrically, if $\theta < 1$ it decays geometrically, and if $\theta = 1$ the input is constant in each time period. Substituting (16) into (2) gives

$$s(t) = Qs(t-1) + \theta^t f, \quad t \geq 1.$$

The question of interest here is, how do the stock levels behave over time for various values of θ? Given a starting stock level $s(0)$ repeated application of this equation for increasing t gives

$$
\begin{aligned}
s(1) &= Qs(0) + \theta f, \\
s(2) &= Qs(1) + \theta^2 f, \\
&= Q^2 s(0) + Q\theta f + \theta^2 f,
\end{aligned}
$$

and in general

$$
(17) \qquad s(t) = Q^t s(0) + \left(\sum_{j=0}^{t-1} \theta^{t-j} Q^j \right) f.
$$

In order to investigate the behavior of the stock levels $s(t)$ as t increases it is necessary to rely on some of the results of linear algebra, and in particular some results from the theory of nonnegative matrices. It is not our intent to reproduce this theory here, but rather to use it as it applies to our manpower problem. The interested reader should consult the references at the end of the report for details.

Let us assume that it is possible to leave the system eventually from any class i. This is not to say that we must leave directly from i, but only that if one is ever in a class i one can eventually leave by some route. It is hard to imagine an organization where this is not true! Define a matrix R as the matrix Q with each element divided by θ, and write $R = Q/\theta$. Then the theory tells us that there is a number ρ greater than zero and less than 1 such that if θ is greater than ρ the elements of the matrix R^t each go to zero as t increases. We write this as

$$
(18) \qquad R^t \to 0 \quad \text{as} \quad t \to \infty, \quad \text{where}
$$

0 is an $N \times N$ matrix with all elements equal to zero. Also when θ is greater than ρ the inverse of $(I-R)$ exists and is nonnegative, with

$$(I-R)^{-1} = \sum_{t=0}^{\infty} R^{t}.$$

This matrix plays an important role in our models, and we denote it by

$$D(\theta) = (I-R)^{-1}.$$

In particular, for $\theta = 1$ we have $D(1) = D = (I-Q)^{-1}$.

Returning now to our problem of investigating the behavior of $s(t)$, divide (17) by θ^{t} and substitute R for Q/θ. Then

$$\frac{s(t)}{\theta^{t}} = R^{t}s(0) + \left(\sum_{j=0}^{t-1} R^{j}\right)f. \tag{19}$$

It can be shown that $\sum_{j=0}^{t-1} R^{j} = (I-R^{t})D(\theta)$ when $\theta > \rho$. Using this in equation (19) gives

$$\frac{s(t)}{\theta^{t}} = D(\theta)f + R^{t}[s(0) - D(\theta)f]. \tag{20}$$

This equation for the stock levels at time t is in a form which is very useful in determining the behavior of $s(t)$ as t increases. Its behavior will depend on the magnitude of θ and we consider various cases.

The first case considered is when θ is greater than 1. In this case $f_0(t)$ increases geometrically without bounds so that the organization keeps growing. The first term on the right hand side of (20) is constant whereas the second term varies as R^{t}. But if $\theta > 1$ then $\theta > \rho$ since $\rho < 1$; thus $R^{t} \to 0$ as $t \to \infty$. It follows that

$$\frac{s(t)}{\theta^{t}} \to D(\theta)f,$$

and that for large t we can approximate the stock levels by

$$s(t) \approx \theta^t D(\theta) f. \tag{21}$$

This is a simple result from which stock levels can be easily calculated. However, the main purpose of this analysis is not to determine simple methods of computation, but rather to derive simple expressions which give insight into how the system stock levels grow relative to each other as the input grows. Equation (21) tells us that the <u>fractions</u> in each class i stay the same eventually, and thus if $s(t-1)$ gives the stock levels at $t - 1$ then $\theta s(t-1)$ gives them at t. Each stock level is increased by the multiplier θ and therefore, the distribution of total personnel among the classes stays the same.

<u>Example 4</u>: Consider a system with 2 classes with

$$Q = \begin{bmatrix} .4 & .1 \\ .3 & .7 \end{bmatrix}, \quad \theta = 1.05, \quad f = [100,0],$$

This Q assumes 40% of those in class 1 remain in 1 in a time period, 30% move to 2, and 30% leave the system. Of those in class 2 10% move to class 1, 70% remain in class 2 and 20% leave the system. The system starts with an input of 100 into class 1 and none into class 2, and input grows at a rate of 5% per year.

The matrix R is given by

$$R = \begin{bmatrix} .381 & .095 \\ .286 & .667 \end{bmatrix},$$

and

$$D(1.05) = \begin{bmatrix} 1.86 & .532 \\ 1.59 & 3.46 \end{bmatrix}.$$

Thus $D(\theta)f$ is [186,159], and for large t we can write

$$s(t) \approx [.539,.461]345(1.05)^t.$$

This equation can be used to determine $s(t)$ for large t. However, an important use is that it tells us that eventually the system has about 54% of its people in class 1 and 46% of its people in class 2, and these proportions stay constant even though the total number in the system is growing geometrically.

Figure II.1 illustrates this example and the use of the analysis leading to equation (20). The axes represent the numbers in each class and the vectors $s(1)$, $s(2)$, etc. are plotted starting with $s(0) = [0,100]$. They approach a line drawn through the point (539,461) and through the origin. Equation (20) tells us that $s(t)$ eventually approaches this line, independent of the value of $s(0)$.

Problem 7: Using Q, θ and f of the example, plot $s(t)$ starting with $s(0) = [200,0]$.

Returning to the analysis of equation (20) we now consider the case $\theta = 1$. Thus the input in period t, $f(t)$ is simply f for all t. Note that $R = Q$, $R^t \to 0$ since $\rho < 1$, and the stock levels stay bounded and approach the vector $s = Df$, independent of the starting stocks $s(0)$.

Example 5: (continuation of example 4). For this example the system stocks approach 200 in class 1 and 200 in class 2 for a total system size of 400. The values of $s(t)$ are plotted in Figure II.2 starting with $s(0) = [0,100]$.

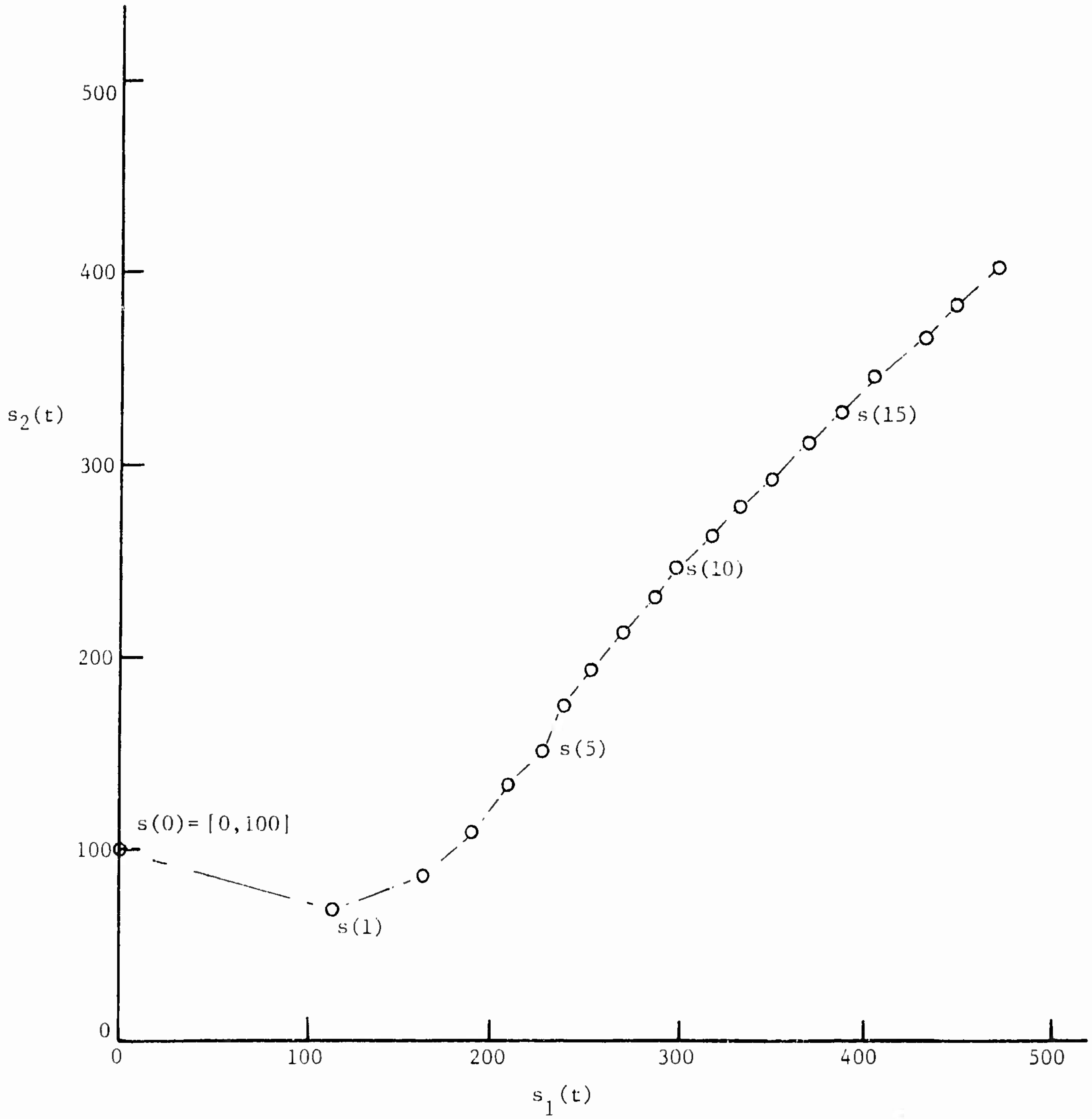

Figure II.1: Plot of s(t) for Example 4, with δ=1.05, s(0)=[0,100]

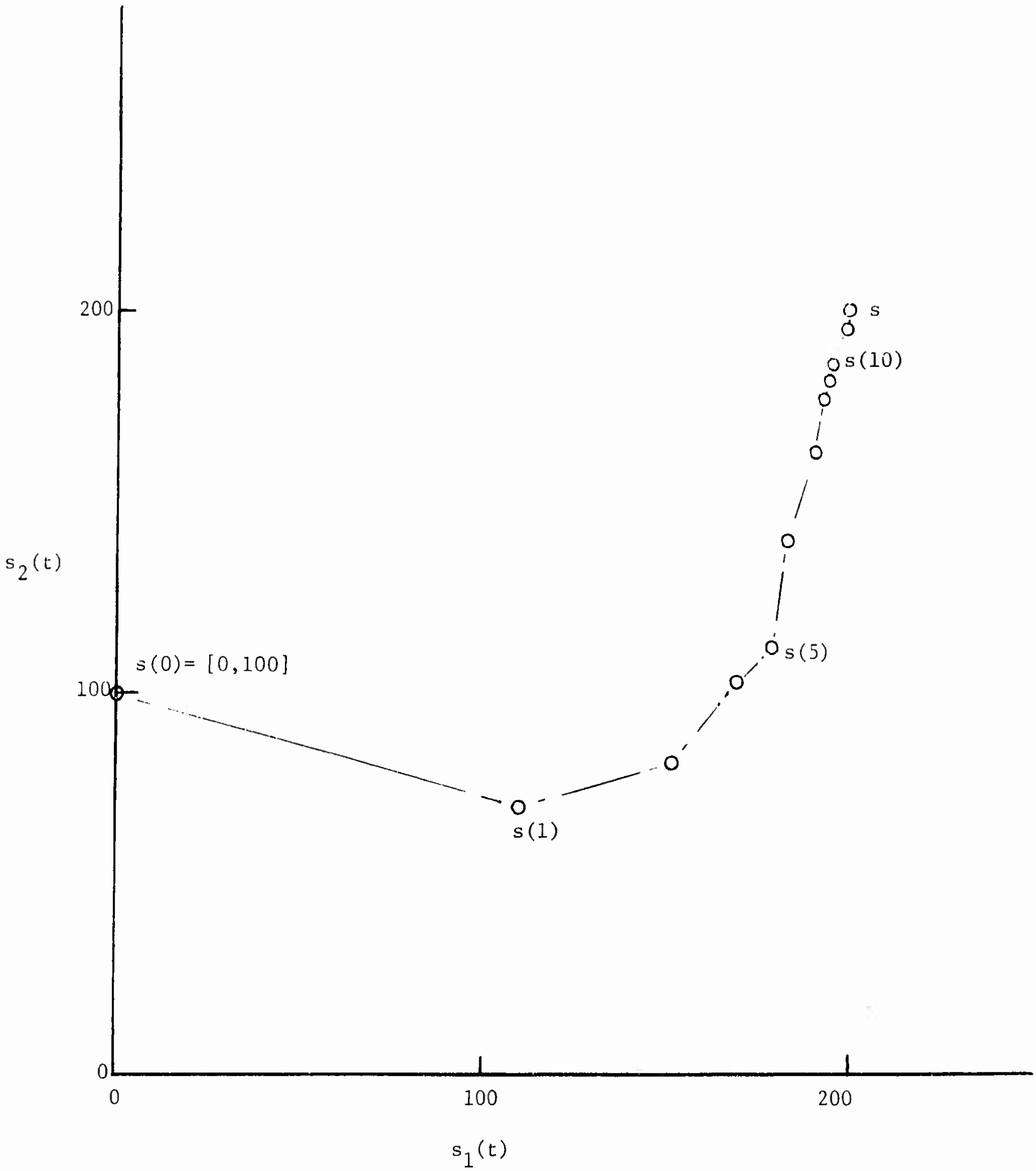

Figure II.2: Plot of s(t) for Example 5 using θ=1, s(0)=[1,100]

Problem 8: Calculate $s(1), s(2), \ldots$ for the case $s(0) = [200,0]$ and plot on figure II.2.

The third case to consider in equation (20) is when $\rho < \theta < 1$. In this case the new input each year decreases geometrically at a rate slower than ρ. In this case R^t still converges to a zero matrix and for large t $s(t)$ is given by (21). Thus $s(t)$ eventually goes to zero, but geometrically at a rate θ and with the stocks in the same proportions as given by $D(\theta)f$. The reader is cautioned that R varies as θ varies, so that decreasing the value of θ from a number above 1 to one between ρ and 1 will change the proportions of the stocks in each class in steady state.

Example 6: (continuation of example 4). It can be shown that ρ for Q in example 1 is .779. Using a θ of 0.9 we find

$$R = \begin{bmatrix} .444 & .111 \\ .333 & .778 \end{bmatrix},$$

$$D(.9) = \begin{bmatrix} 2.57 & 1.29 \\ 3.86 & 6.42 \end{bmatrix},$$

$$D(\theta)f = [257,386] = 643[.4,.6].$$

After 20 periods the stock levels are given approximately by [31,44] which are in the ratio 4 to 6. Figure II.3 shows the plot of $s(t)$ starting with $s(0) = [0,100]$. Notice that the stocks are going to zero along the line through the origin and the point (40,60).

Problem 9: Calculate $s(1), s(2), \ldots$ for example 6 starting with $s(0) = [200,0]$ and plot on figure II.3.

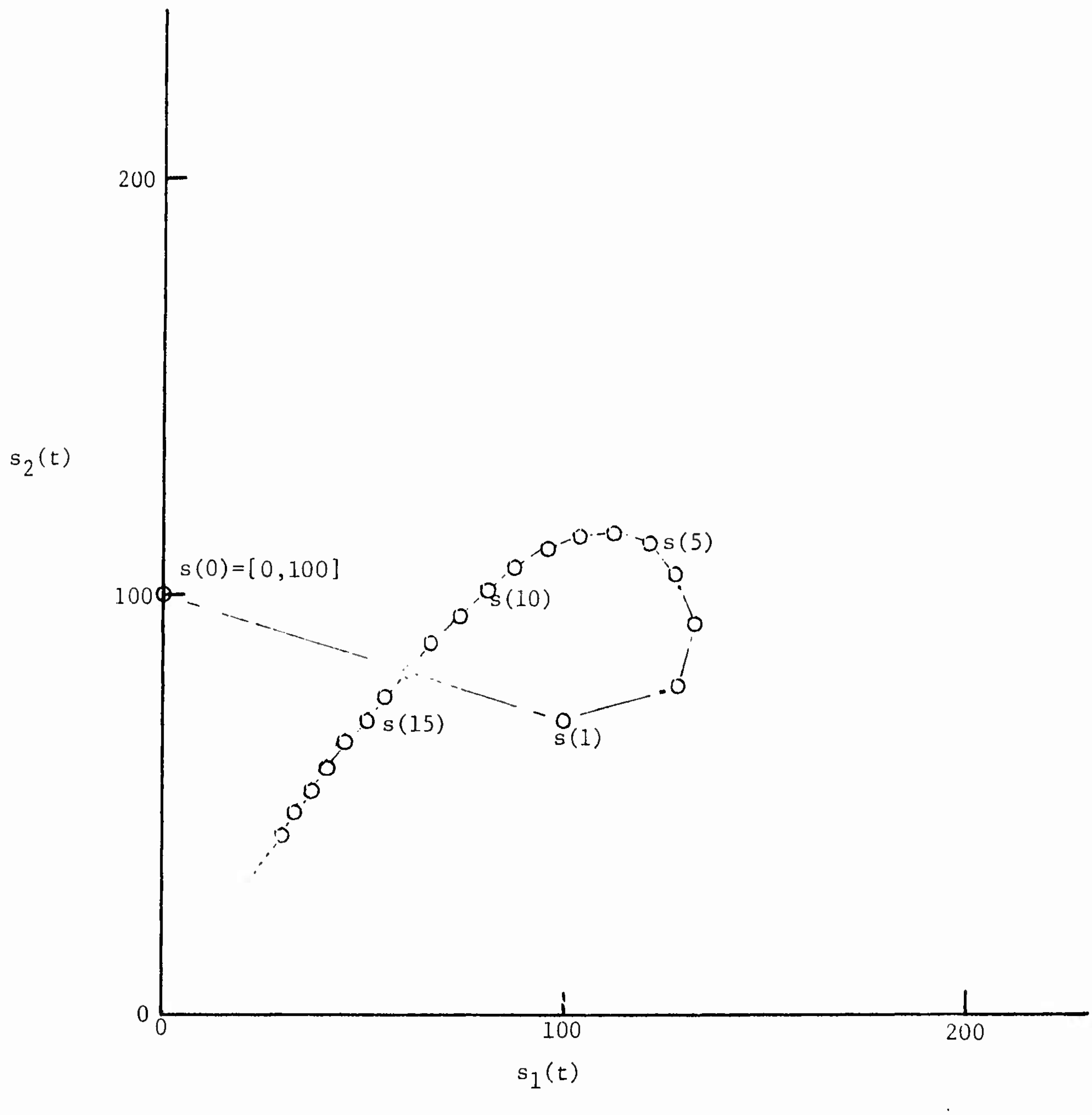

Figure II.3: Plot of s(t) for Example 6 using θ=.9, s(0)=[0,100]

The fourth and final case to consider in equation (20) is when $\theta \le \rho$. For example, suppose there is no new input each year and the system is simply allowed to die out on its own. In this case θ would be zero. The case of firemen on the railroads might be an example. No new ones are added to the system but the number in the system decreases through natural attrition. It should be clear that if θ is too small the system cannot shrink at the rate θ (the example $\theta = 0$ illustrates this). The value ρ is important here, since if $\theta < \rho$ it is ρ which determines the rate at which the organization shrinks. The mathematics referred to earlier support this. If $\theta \le \rho$ then $(I-R)^{-1}$ does not exist and equations (20) and (21) have no meaning. The number ρ is a lower bound on the rate of contraction. This is because attrition or withdrawal from the system depends only on the coefficients q_{ji} and w_i (recall $w_i = 1 - \sum_{j=1}^{N} q_{ji}$) and is independent of the appointment rate.

Returning for a moment to the theory of nonnegative matrices we use the following result. One can find an N-dimensional vector which we call s whose elements are nonnegative and add to one, and which satisfies $\rho s = Qs$.

Suppose this s is used as the initial stock vector $s(0)$ (the fact that its elements add to one is a convenience and is not necessary), and we have no new input so that $f_0(t) = 0$ for all $t \ge 1$. Then from (2)

$$s(t) = Q^t s$$

and from our choice of s

$$s(t) = \rho^t s.$$

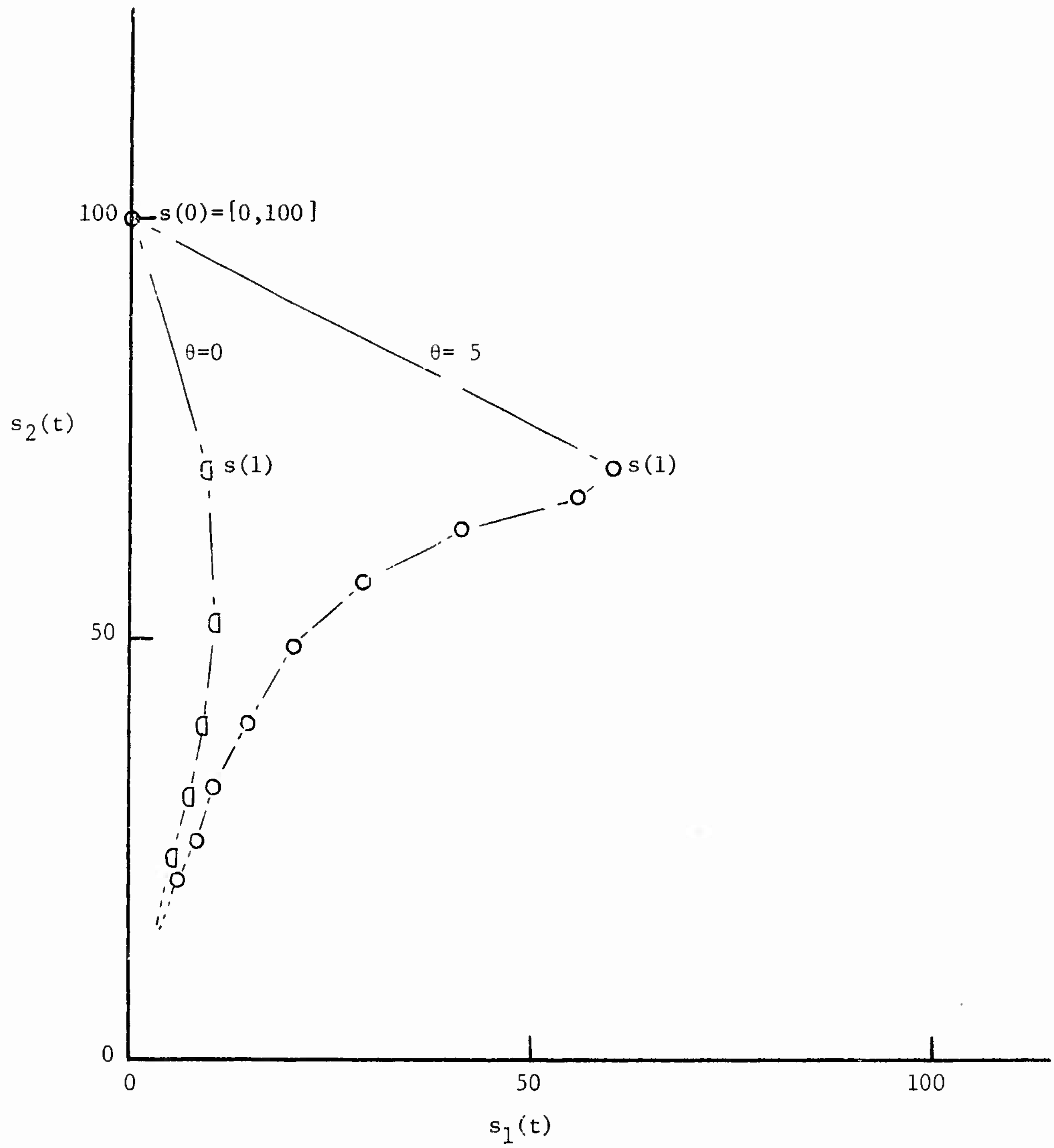

Figure II.4: Plot of s(t) for example 7 using θ=.5 and θ=0, s(0)=[0,100]

Summing the elements on each side of the equation gives

$$es(t) = \rho^t,$$

so if the system starts at time 0 with n people, after t periods it will have $n\rho^t$ people. This shows that ρ determines the speed at which the system dies out. For if we take θ positive but less than ρ, then for large enough t θ^t will be very small compared to ρ^t and any new input will have a negligible effect on the size of the system relative to what is left in the system after natural attrition.

Example 7: (continuation of example 4). For this example $\rho = .779$, and the vector s is found to be [.21,.79]. Thus if the system starts with 100 people, 21 in class 1 and 79 in class 2, and if no new input is added than $s(t) = (.779)^t[21,79]$. If the system starts with $s(0) = [0,100]$ and $f = [100,0]$, $\theta = .5$, the values of $s(1), s(2), \ldots$ are plotted in Figure II.4. The values are also plotted for the case of no input $(\theta=0)$.

Problem 10: Calculate $s(1), s(2), \ldots$ for example 7 starting with $s(0) = [200,0]$ and plot on Figure II.4. Repeat with $\theta = 0$. □

The case $\theta = 0$ is of interest if a system is to undergo a reduction in size. Consider a cutback in a manpower system from a level of, say N_1 people which has been maintained using a given constant appointment policy f_1, to a lower level, say N_2, where this reduction must be brought about only by natural attrition. Let us assume that the distribution between grades is to remain the same when the new level is reached. One way to model such a change in structure is to assume that the vector of numbers to be enlisted in each future period is given by $f_2 = \frac{N_2}{N_1} f_1$, and that a vector $\frac{N_2}{N_1}$ s of people and positions in the system have been singled out. Since es = N_1 by assumption, we have singled out N_2 positions. Attrition from these will be filled by the appointment vector z and we assume the same Q matrix holds. It is easy to show that this input vector maintains the steady state distribution but with N_2 people. The remaining people, given by a vector $\frac{N_1 - N_2}{N_1}$ s, are not replaced as they leave. Thus the behavior of the system in the transition period from the level N_1 to the level N_2 is described by case 4 described above, $\theta = 0$, applied to a steady state system containing $(N_1 - N_2)$ people.

Such a splitting of the system into two groups is simply a convenience which allows us to use the models developed this far. No actual splitting need occur in practice. It is simply a convenient trick which allows us to investigate the transient behavior of the system.

Example 8: (continuation of example 5). Assume we have a system with 400 people, Q matrix as in example 4 and in steady state with f = [100,0]. The steady state stocks are [200,200] (see Figure II.2). Assume the system is to be reduced to a total of 300, with 50% of these in each of the two classes. We can think of a system of 300 which continues as before with a new input vector [75,0] in each period. This will maintain the subsystem at [150,150] in each period. The remaining 100 are taken to be another system, with $f(t) = 0$, $t \geq 0$. Figure II. 5 gives the stocks in each period and shows how the system approaches its new steady state.

b. Arithmetic Growth.

Let f and g be two N-vectors (assumed nonnegative). In this section we assume that the input vector at t is given by

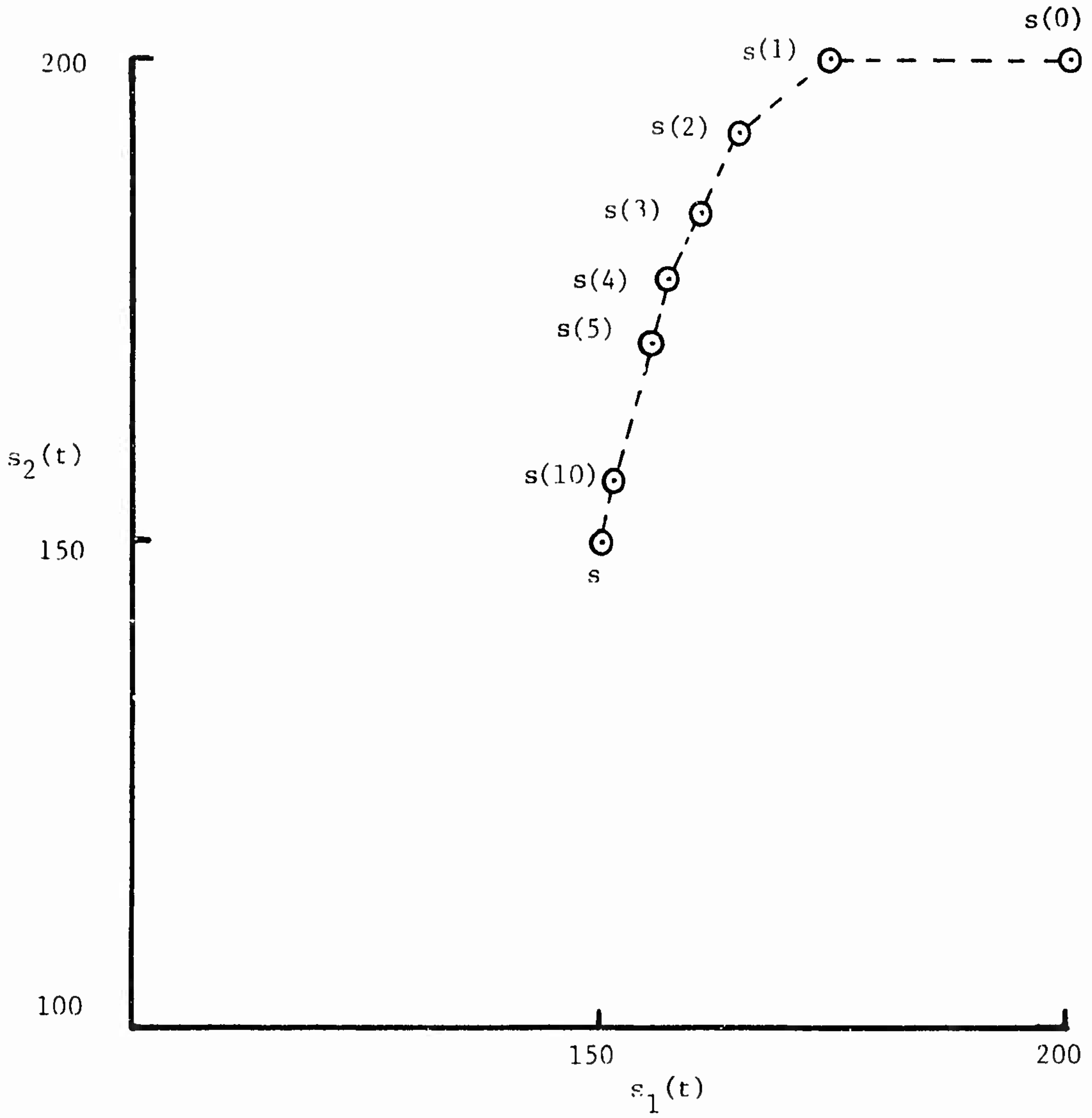

Figure II.5: Plot of S(t) for Example 8, a decreasing organization

$$f(t) = f + gt, \quad t \geq 0. \tag{22}$$

When $f(t)$ satisfies (22) we say we have arithmetic growth.

The stocks are given by

$$s(1) = Qs(0) + f + g,$$

$$s(2) = Qs(1) + f + 2g,$$

$$= Q^2 s(0) + Qf + Qg + f + 2g,$$

and in general

$$s(t) = Q^t s(0) + \sum_{j=0}^{t} jQ^{t-j} g + \sum_{j=0}^{t-1} Q^j f$$

$$= Q^t s(0) + t \sum_{j=0}^{t} Q^j g - \sum_{j=0}^{t} jQ^j g + \sum_{j=0}^{t-1} Q^j f. \tag{23}$$

In order to investigate the long-run behavior of the system we need to know how each of the terms on the right-hand-side of (23) behaves.

The first term vanishes for large t since $Q^t \to 0$, and the last term converges to Df. The third term must be investigated (which is done below), but as we shall see it remains finite. The second term, however, increases linearly in t for large t, since if t is large the sum is approximately Dg.

To return to the third term, this sum can be written (without multiplication by g)

$$\begin{aligned} \sum_{j=0}^{t} jQ^j = \; & Q \\ & + Q^2 + Q^2 \\ & + Q^3 + Q^3 + Q^3 \\ & + \vdots \\ & + Q^t + Q^t + \ldots + Q^t, \end{aligned}$$

and summing each column, using $\sum_{j=0}^{t-1} Q^j = (I-Q^t)D$, gives

$$\sum_{j=0}^{t} jQ^j = Q[(I-Q^t)D - tQ^{t+1}]D.$$

As t becomes large this expression approaches QD^2. Thus from (23) we can say

$$\lim_{t\to\infty} (s(t)-tDg) = Df - QD^2g,$$

or

$$s(t) \approx D(tg+f-QDg).$$

This expression tells us that in the long run the number in the system increases linearly. The number in a given state i will be given by the $i^{\underline{th}}$ element of the vector $[f-Qdg]$ plus t times the $i^{\underline{th}}$ element of the vector Dg. Again the importance of the matrix $D = (I-Q)^{-1}$ is demonstrated.

Problem 11: Using the Q and $s(0)$ of problem 7, let $f = [100,0]$ and $g = [10,10]$. Find Dg and $f - QDg$. Plot the two lines $y_i = (Dg)_i t + (f-QDg)_i$, $i = 1,2$. Determine $s(1), s(2), \ldots, s(20)$, and plot each element on the same paper as the lines.

The following problems are more advanced ones which demonstrate how geometric growth models can be formulated as foresight or hindsight models.

Problem 12: When $\theta = 1$, show $ef = wDf$. What is the interpretation of this formula? Derive and interpret the formula in the general case $\theta > \rho$.

Problem 13: Given $s(0)$, $\theta = 1$, and a fractional appointment policy a, determine (ef), the number of appointments per period so that in the equilibrium system with stocks $s = Qs + (ef)a$ we have $es = es(0)$, i.e., the total in the system in steady state is equal to the current total in the system.

Problem 14: Given $s^*(0) = [s_0(0), s(0)]$, $\theta = 1$, and a fractional appointment policy a, determine (ef), the number of appointments per period so that an equilibrium system with stocks $s = Qs + (ef)a$, $s_0 = ws$, has $s_0 + es = s_0(0) + es(0)$.

Problem 15: Show that if

$$P = Q + \frac{f}{ef} \cdot [w + (\theta-1)e]$$

and $s(t) - Ps(t-1)$, that

(i) $eP = \theta e$

(ii) $es(t) = \theta^t es(0)$,

(iii) $\dfrac{s(t)}{\theta^t} \to \dfrac{Df(es(0))}{(eDf)}$, where $D = (I - Q/\theta)^{-1}$

Interpret the results.

Problem 16: Show that if

$$P^* = \left[\begin{array}{c|c} 0 & [w+(\theta-1)e] \\ \hline \dfrac{\theta f}{ef} & Q \end{array}\right]$$

(i) the columns of P^* each add to θ,

(ii) (ef, Df) solves $s^* = \left(\dfrac{P^*}{\theta}\right) s^*$, where $D = (I-Q/\theta)^{-1}$

Interpret the result if $s^*(t) = P^*s^*(t-1)$.

Problem 17: Show that if

$$P^* = \left[\begin{array}{c|c} (\theta-1) & \theta[w+(\theta-1)e] \\ \hline \frac{\theta f}{ef} & Q \end{array}\right]$$

then

$$(ef,Df) \text{ solves } s^* = (\frac{P^*}{\theta})\ s^* .$$

Interpret the result if $s^*(t) = P^*s(t-1)$ and contrast with problem 16.

Problem 18: It is shown in problems 16 and 17 that for the case of geometric growth an equivalent hindsight model with (N+1) states can be formulated. In both these cases $s^*(t) = P^*s(t-1)$, and $\frac{s^*(t)}{\theta^t} \to k(ef,Df)$, for some constant k. The stocks in any finite period t differ in the two models. Show that in general it is not possible to construct a matrix P^* such that $s^*(t) = \theta^t s^*(0)$ for all t.

7. A Requirements Model.

In this section the fractional flow model is used to determine the sequence of input vectors $f_0(t)$ which are needed to exactly meet a given sequence of required stock levels $s(t)$. In previous sections the input flow vectors $f_0(t)$ were assumed to be given and the behavior of the resulting stock vectors was analyzed. Now we reverse the problem.

Let the current time period have index zero, and assume stock vectors $s(0), s(1), \ldots, s(T)$ are given for some planning horizon T. The vectors $f_0(1), \ldots, f_0(T)$ are to be determined in order to meet these stock levels. From equation (2) we have $f_0(t) = s(t) - Qs(t-1)$, $t = 1,2,\ldots,T$. Let us assume that the only feasible input vectors are those which are nonnegative. That is to say, requirements can be met only with appointments. Forced attrition cannot be used.

The first question to ask is, can the given sequence of stocks $s(1), \ldots, s(T)$ be met with any feasible set of appointments $f_0(1), \ldots, f_0(T)$? The answer is yes if and only if $s(t) \geq Qs(t-1)$ for each $t = 1, \ldots, T$. There are $T \times N$ inequalities which are simple to check. Suppose our requirements are changing geometrically, so that $s(t) = \theta^t s(0)$, $\theta > 0$. Then it is easy to see that only the N inequalities

$$s(0) \geq \frac{Q}{\theta} s(0) \tag{24}$$

need be tested.

From the theory referred to in section 6, if $\theta < \rho$ then (24) has no solution. This simply says that if the requirements die out too quickly, natural attrition is not enough to reduce the legacy sufficiently to meet the requirements. Letting $R = \frac{Q}{\theta}$ as in section 6 we see from (24) that not all starting stock levels $s(0)$ lead to feasible appointments. Let the N-vector x be any solution of the inequalities

$$(25) \qquad \begin{aligned} (I-R)x &\geq 0 \\ x &\geq 0. \end{aligned}$$

Then such an x is a feasible starting stock level if requirements are geometrically changing at rate θ. It is easy to see from (25) that any x satisfying (25) is also a feasible starting stock level for any $\theta' \geq \theta$.

Example 9: Let Q be given as in example 4. Thus

$$Q = \begin{bmatrix} .4 & .1 \\ .3 & .7 \end{bmatrix}.$$

The values of x satisfying (25) are plotted in Figure II.6 for $\theta = 1.05$, 0.9, 0.8, and $\theta = \rho = .779$. Note that the set of feasible starting vectors increases with θ and in each case forms a cone. As θ decreases to ρ, this cone degenerates to a line.

Problem 19: Show that the requirements $s(t) = .9^t s(0)$, with $s(0) = [50,60]$ cannot be met for any t, but the requirements $s(t) = (1.05)^t s(0)$ can be met starting at $[50,60]$. Find the input vectors $f_0(1),\ldots,f_0(5)$ in this case. □

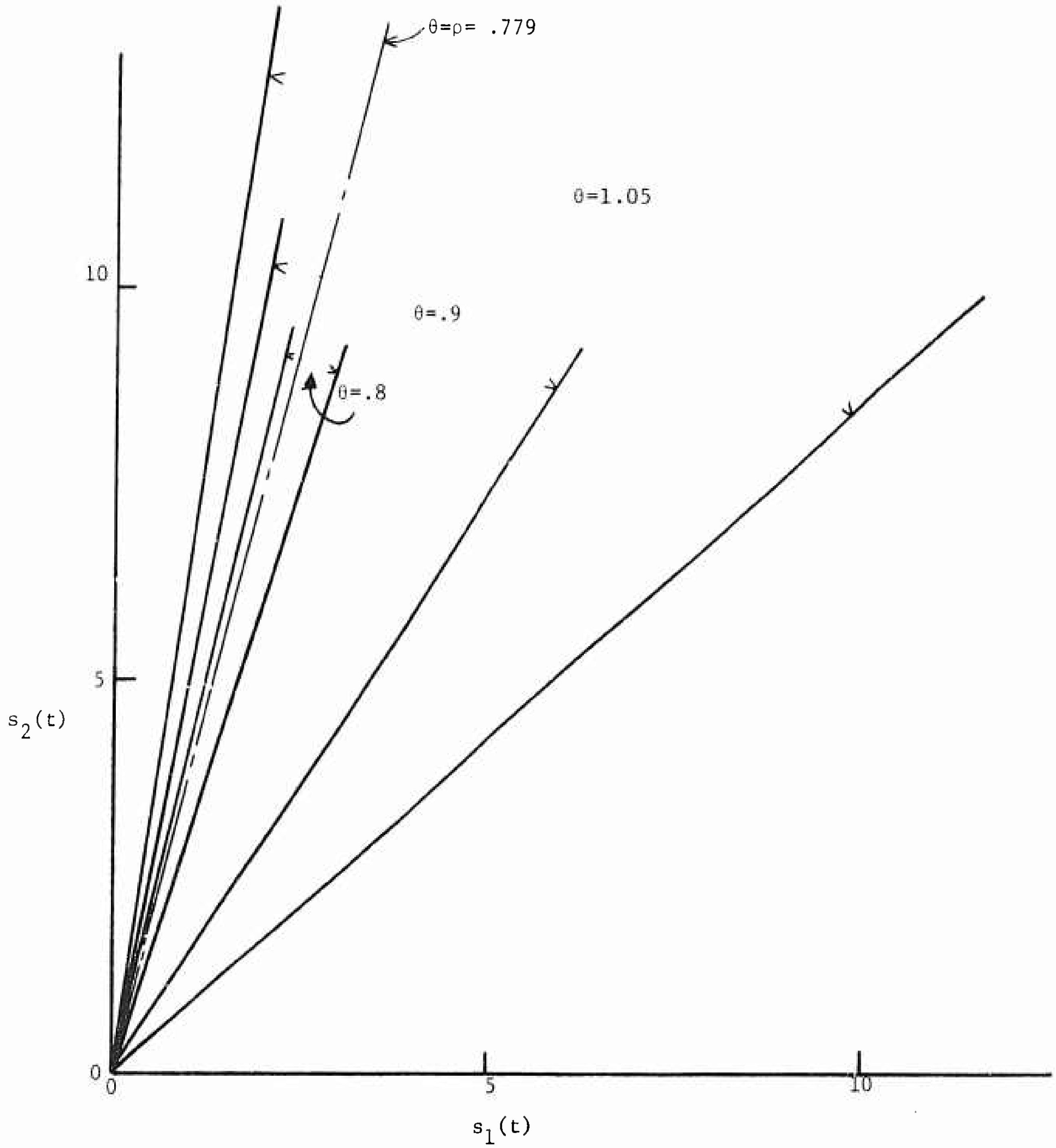

Figure II.6: Sets of Feasible Initial Vectors for various Geometric Growth rates

8. A University Faculty Model.

The examples discussed so far have been of simple fictitious systems with only two or three states. These allowed us to investigate various implications of the cross sectional model without becoming involved in systems with large numbers of classes. In this section we describe a cross-sectional flow model of a University faculty using real data and appointments.

The faculty of a university can be partitioned in many ways. For example, they could be partitioned into classes depending on their academic department, their status, their pay grade, their age or some combination of these. The choice of a classification scheme must reflect the intended use of the model. In the example treated here the basic questions were of rank structure. In the institution in question, namely the campus of the University of California, Berkeley, models were required which would describe movement of faculty between ranks and which could be used to determine the effects of various hiring and promotion policies on rank structure and tenure/non-tenure ratios.

As part of a larger study in University Planning, Branchflower [1970] formulated and analyzed a model of faculty flow of the type discussed in this chapter. The data from his work is summarized in Table II.1, which gives the actual movement of faculty through the thirteen ranks of the College of Engineering at Berkeley in the period 1 July 1960 to 1 July 1968. Since the purpose of this model is to study the distribution of faculty in the active ranks, the "retired" class was considered external to the system.

RANK AND STEP

			(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	Ret. (14)	Res. (15)	Dec. (16)	TOTAL
(1)	Ass't Prof.	Step 1	7	14	1	1											2		25
(2)	Ass't Prof.	Step 2		34	39	5											4		82
(3)	Ass't Prof.	Step 3			39	26	24										4	1	94
(4)	Ass't Prof.	Step 4				13	15	2									3		32
(5)	Assoc. Prof.	Step 1					42	38	9										89
(6)	Assoc. Prof.	Step 2						55	40	1							1		97
(7)	Assoc. Prof.	Step 3							130	70	1						4		205
(8)	Full Prof.	Step 1								135	60	3	1			1	2		202
(9)	Full Prof.	Step 2									120	44	4			3	2	1	174
(10)	Full Prof.	Step 3										141	48	1		2	6	1	199
(11)	Full Prof.	Step 4											95	32	1		2		130
(12)	Full Prof.	Step 5												65	17		2		84
(13)	Full Prof.	O/S													30		2		32
	Appointments		13	33	25	4	6	6	6	4	1	0	0	0	0				98

TABLE II.1 MOVEMENT OF FACULTY FOR THE COLLEGE OF ENGINEERING AT BERKELEY FOR THE PERIOD 1 JULY 1960 TO 1 JULY 1968 ACCUMULATED STATISTICS

The numbers in Table II.1 require some explanation. They are aggregated figures over the eight year period. What Branchflower did was to determine the stock levels as of 1 July in a given year (say 1960), and determine how many of these either stayed in the same rank, moved to some other rank or left the active ranks by 1 July of the following year. Thus, he determined eight matrices with numbers of actual movements in each one. The table given here is the sum of these eight matrices. Thus, in the period 1960 to 1968 a total of 70 faculty in rank Associate Professor step 3 moved to rank Full Professor step 1 in a one-year period. In the eight year period only one person obtained a double promotion from Associate Professor step 3 to Full Professor step 2. Columns 14, 15 and 16 show the total numbers who retired, resigned and died respectively in this period, and the final column gives the row sums. Row 14 gives the total number of new appointments to each rank in the eight year period, with a total of 98 new appointments in the eight year period. The Q matrix for this system is calculated from this aggregated data and is shown in Table II.2. Only the non-zero entries are shown. A characteristic of this system is immediately obvious from this Q matrix. Since no demotions occur and since the ranks have been ordered in increasing order of seniority Q has a lower triangular structure; that is, all q_{ji} above the main diagonal (all elements q_{ji} with $i > j$) are zero. The dominant fractions lie on the main diagonal and the one below it, showing that one either stays in the same rank or moves to the next highest one except for rare double promotions. This structure for Q is found in many systems.

The reader should question the aggregation of eight years of data to determine Q. Why eight? Why not one, three, six, etc.? No attempt

State j \ i	1	2	3	4	5	6	7	8	9	10	11	12	13
1	.28												
2	.56	.41											
3	.04	.47	.44										
4	.04	.07	.37	.38									
5			.14	.52	.47								
6				.04	.43	.57							
7					.10	.41	.64						
8						.01	.34	.67					
9								.30	.69				
10								.01	.25	.71			
11									.02	.24	.73		
12										.01	.25	.77	
13											.01	.20	.97

TABLE II.2: THE Q MATRIX FROM 1960 - 1968 DATA has been made here to do any statistical studies on this data. Such a study would be outside the scope of this book. Our only excuse for using eight years of data is that it was available and gave reasonable numbers in the non-zero cells so that fractions could be calculated. The interested reader can obtain more details on the original data and questions of the stationarity of Q from Branchflower's paper.

Suppose we take 1 July 1968 as our point $t = 0$. The stock levels $s(0)$ are given in row 14 of Table II.3. Thus, on that date the faculty had a total size of 210 people. Let us assume that the faculty is to stay fixed at this size with no vacancies unfilled. Various hiring policies can be tried using our constant size-predictive model with $N = 13$ to determine the long-run effect of these policies.

First we calculate w , the vector of fractional withdrawals from each state from Q in Table II.2. Thus

$$w = (.08,.05,.05,.06,.00,.01,.02,.02,.04,.04,.01,.03,.03).$$

If this is appended as a row to Q the columns will each sum to one. Suppose the hiring policy of interest is one in which all new faculty are hired into Assistant Professor step 1. Then

$$a = [1,0,0,0,0,0,0,0,0,0,0,0,0],$$

and the matrix

$$P = Q + a.w$$

is shown in Table II.3. Using this matrix the steady state vector v of fractions in each state is calculated, using equation (12) and multiplying

State j \ i	1	2	3	4	5	6	7	8	9	10	11	12	13
1	.36	.05	.05	.06		.01	.02	.02	.04	.04	.01	.03	.03
2	.56	.41											
3	.04	.47	.44										
4	.04	.07	.37	.38									
5			.14	.52	.47								
6				.04	.43	.57							
7					.10	.41	.64						
8						.01	.34	.67					
9								.30	.69				
10								.01	.25	.71			
11									.02	.24	.73		
12										.01	.25	.77	
13											.01	.20	.97
s(0) (1968)	1	4	17	16	11	14	20	22	26	19	22	22	16
s(5) (1973)	10	9	7	5	8	12	20	21	21	20	20	22	35
s(10) (1978)	10	9	8	6	8	8	13	17	18	19	19	22	53
Steady state Stocks s	9	9	8	6	8	9	13	13	13	11	11	13	87

TABLE II.3: THE P MATRIX AND STOCK LEVELS AT VARIOUS TIMES.

this by the total faculty size 210 gives the numbers in each rank in the long run. These are shown in row 17 in Table II.3.

The results obtained might come as somewhat of a surprise to a university administrator. In steady state, if the size of the institution stays constant, and if the same retention and promotion structure holds in the future, the faculty will finish up with over 41% in Full Professor rank at overscale grade, 15% Assistant Professors, 14% Associate Professors, and, since in this institution tenure is given to all grades of Associate Professor and above, a faculty with 85% tenured.

Such a result may not be so disturbing if the time it takes to reach this distribution is very long. The distribution in 1968 had 7.6% full professors overscale, 18% assistant professors, 21% associate professors and 82% tenured. Thus the tenure fraction is not changing much but the average grade of faculty is increasing significantly. A calculation of the stocks at five years (1973) and ten years (1978) is shown in Table II.3 in rows 15 and 16 respectively. The steady state distribution and those at times 0, 5 and 10 are shown plotted in Figure II.7. Also for simplicity the percentage in each of the major groups, assistant professor, associate professor, full professor (regular), and full professor overscale are shown in Table II.4.

It is clear from Figure II.7 and Table II.4 that the use of the historic Q matrix to a system of constant size leads to a very large increase in the highest ranks, even though <u>all</u> new appointments are made in the lowest rank. Any other appointment policy would lead to an even more top heavy structure. A look at the historical data (not given here) shows that in periods up to 1968 this particular institution was expanding. We might

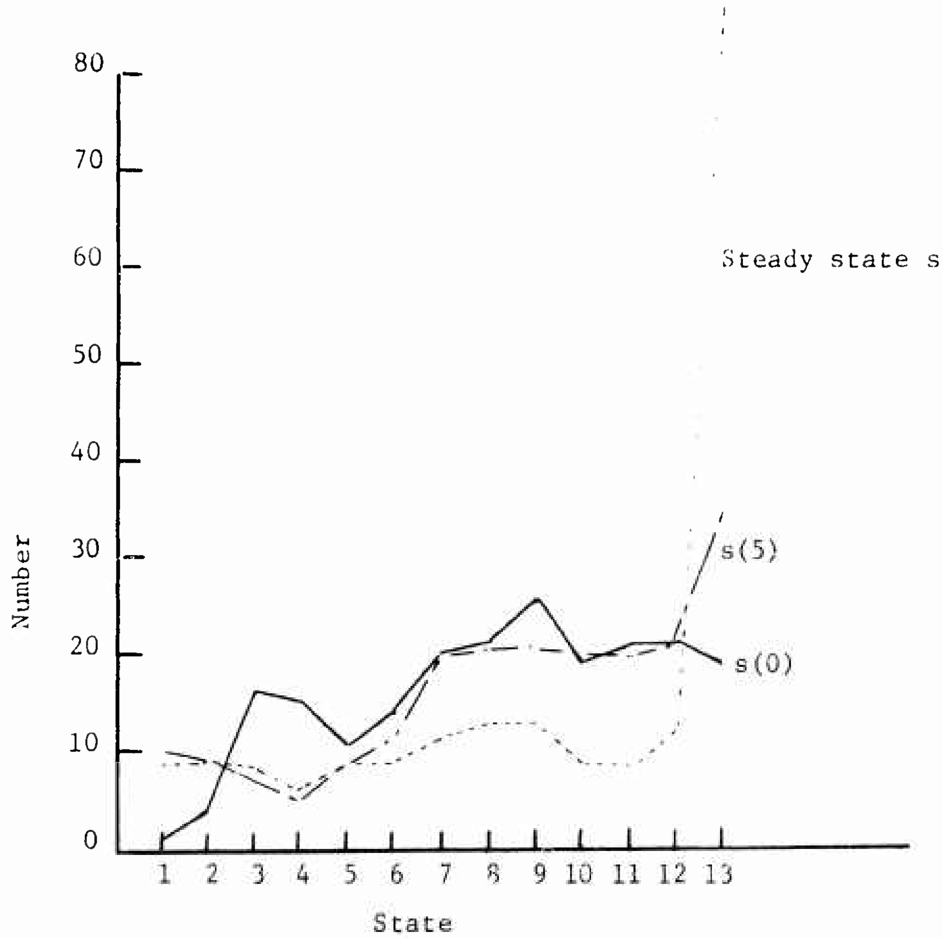

FIGURE II.7: DISTRIBUTIONS BETWEEN RANKS

Period	Assistant	Associate	Full	Overscale
0	18.1	21.4	52.9	7.6
5	14.8	19.0	49.5	16.7
10	15.7	13.8	45.3	25.2
Steady State	15.2	14.3	35.2	41.4

TABLE II.4: PERCENTAGE DISTRIBUTIONS BY MAJOR RANK.

ask what the distribution among ranks would be if the system could continue to expand. Table II.5 gives the steady state percentages in each major rank for growth rates in new input of 5%, 3%, constant input, and a decrease in input size of 1% per year.

It is clear from this table that a small growth rate substantially increases the proportions in the lower ranks. The large "pile-up" in the overscale rank which takes place with no growth is substantially reduced with a 3% growth rate. Notice how the overscale percentage increases if the system starts to decrease in size.

The calculations in Table II.5 were made using equation (21) in the following way. The input vector f had seven people entering in state 1 and no one else in the other states. The matrix $D(\theta) = (I-Q/\theta)^{-1}$ was calculated using Q of Table II.2 and values of θ equal to 1.05, 1.03, 1.00 and .99 for 5%, 3%, 0 and -1% growth. The vector $D(\theta)f$ was found and normalized to sum to 100.

Growth Rate	Assistant Professor	Associate Professor	Full Professor	Overscale Professor
+ 5%	35.4	24.1	29.5	11.0
+ 3%	28.2	21.5	31.9	18.4
0	15.2	14.3	35.2	41.4
- 1%	10.5	10.3	23.9	55.3

TABLE II.5: PERCENTAGE IN EACH MAJOR RANK FOR VARIOUS GROWTH RATES.

9. Probabilistic Interpretation of the Fractional Flows.

In earlier sections q_{ji} denotes the fraction of people in class i at the end of one period who are counted in class j at the end of the following period. Each individual follows his own path through the system and these paths vary greatly from one individual to the next. These individual paths have not been considered to this point. They have purposely been suppressed, in fact, since a prime aim in the choice of our models is to have them contain as little amount of detail as possible in order to answer questions of interest. In later sections it will be necessary on occasion to follow individuals from class to class as they move through the system. Each path can be considered to be in some sense random. By this we mean that if a person is chosen from the system and his path examined, the successive classes of the individual and the times when he enters the classes will not be predictable with certainty. What we can say is that there will be a certain probability that the individual will be in a given class at a given time, or that he entered the class at a given time. It turns out that our earlier model will suffice if the fractions q_{ji} are interpreted correctly.

Consider the path that an individual takes as he moves through the system. Let us suppose he enters in period u and is first counted at time u in class k. Let us further suppose that t periods later he is counted in class i. Where will he be at time $(t+u+1)$ if at $(t+u)$ he is at i and he entered in k at time u? We cannot say with certainty, but let us say that q_{ji} is the probability that an individual who enters in period u in class k and is in class i at $(t+u)$, is in class j at $(t+u+1)$. Notice that we are assuming that this probability is independent of t, u, and k, and is the same for all individuals. This group of strong assumptions says that the probability an individual who is in some class i in a time period moves to class j in the next time period, is independent of the particular time period and all previous history of the individual. In addition we assume that all individuals in the organization behave independently of each other. In the language of probability theory, each individual follows a path which evolves according to the laws of a homogeneous finite state Markov chain.

These are very strong assumptions and in many cases are unrealistic. As we shall see they do lead to the fractional flow model of earlier sections. However, the reader should understand that though these detailed assumptions lead to the earlier model, that they are not required to hold in order to justify the fractional flow model. In mathematical terms they are <u>sufficient</u> to lead to the earlier model but not <u>necessary</u>. Here is a case where a too detailed look at the real system, by trying to describe individual flow patterns, can lead to confusing and unnecessary assumptions. The art of good modelling is to go into only enough detailed structure as is necessary for the particular application. For a retirement model described in section 10 we need a probabilistic interpretation. For a faculty flow problem in section 8 the probabilistic interpretation was not required.

The symbol $s_j(t)$ has been used earlier to indicate the stock level in class j at time t. Now it must be interpreted as the expected stock level in j at t. The upper case letter $S_j(t)$ is the random variable which denotes the (uncertain) stock level, and $E[S_j(t)] = s_j(t)$.

If the stock levels at t are given it is easy to find the expected stock levels at $(t+1)$ in terms of these. If there are $S_i(t)$ people in class i at t, the expected number of these in j at $(t+1)$ will be $S_i(t)q_{ji}$. This holds for all $i = 1,\ldots,N$, and if we add in the new flows into j from outside we have

$$E[S_j(t+1)\ S_1(t),\ldots,S_N(t)] = \sum_{i=1}^{N} q_{ji}S_i(t) + f_{0j}(t+1).$$

Taking expectations of the stocks $S_1(t),\ldots,S_N(t)$ gives

$$s_j(t+1) = \sum_{i=1}^{N} q_{ji}s_i(t) + f_{0j}(t+1), \quad j = 1,2,\ldots,N,$$

or in matrix notation

$$s(t+1) = Qs(t) + f_0(t+1). \tag{26}$$

This is the same as the basic flow equation (2) in the fractional flow model. Thus, in terms of expected values the Markov assumptions lead to the fractional flow model. One could postulate a number of detailed models which would lead to equation(26) in terms of expected values. In many applications the only variables of interest are these expected values and many of the detailed assumptions are unimportant in calculating these. However, if variances and covariances are to be calculated to estimate the effects of uncertainty the reader must be much much more careful in the choice of a model.

Many of the results in earlier sections have a probabilistic interpretation in terms of the Markov model. For example let us look at the matrix $D = (I-Q)^{-1}$. First let us look at the (j,i) the element of the matrix Q^n for some fixed $n > 1$. Call this $q_{ji}^{(n)}$. Then from matrix multiplication we have

$$q_{ji}^{(n)} = \sum_{k=1}^{N} q_{jk}^{(n-1)} q_{ki}.$$

This equation shows us that $q_{ji}^{(n)}$ is the probability that an individual is in class j n periods after being observed in class i. Now since $(I-Q)^{-1}$ can be written as $\sum_{n=0}^{\infty} Q^n$, the (j,i)th element of D can be written

$$d_{ji} = q_{ji} + q_{ji}^{(2)} + q_{ji}^{(3)} + \dots + q_{ji}^{(n)} + \dots, \quad \text{if } i \neq j.$$

(27) and

$$d_{ii} = 1 + q_{ii} + q_{ii}^{(2)} + q_{ii}^{(3)} + \dots + q_{ii}^{(n)} + \dots .$$

Suppose an individual enters the system in class i. How many periods can he be expected to spend in class i? He spends the first one there since he entered in this state (by our accounting assumptions). He spends the nth period in i with probability $q_{ii}^{(n)}$. Thus, the second equation in (27) gives the total expected time an individual spends in i (that is, expected number of periods) if he enters in i. For $j \neq i$ he cannot be in j the first period. Thus, the first equation in (27) holds in this case. The matrix D gives us the expected durations an individual who enters in a given state spends in each of the states.

Problem 20: Show that eD is an N-vector which gives the expected number of periods an individual spends in the system if he enters in a given state. □

10. A Retirement System Model.

The theory of cross-sectional manpower flow is used in this section as the basis of a model of a retirement system. This retirement system model allows a decision maker to investigate various retirement policies. Concepts developed in section 9 are used, and an early retirement scheme is investigated as an illustrative example.

The retirement model relates the manpower flow process to the financial parameters that describe the retirement system. First we calculate the expected present value of the annuity that an individual will receive discounted to the time of retirement. Then we calculate the present value at retirement of all contributions to the individual's retirement fund, and match this with the annuity.

In this section time period t is assumed to be the period in which retirement takes place, and periods are assumed to be of one year duration for convenience. If an individual entered the system in period $(t-k)$ it is assumed his length of service is k (note that it is actually between (k and $k+1$). A person who enters in period t (and is then counted at time t) has length of service zero. We shall say that a person who is in class i and has length of service k is in state (i,k).

Let $a_i(k,t)$ be the annuity paid an individual who retires in period t while in state (i,k), and let $m_j(i,k)$ be the probability that this individual will receive exactly j annuity payments in retirement. Note that $\sum_{j=0}^{\infty} m_j(i,k) = 1$ for all states (i,k) and let μ_{ik} and σ_{ik}^2 be the mean and variance of this distribution respectively. If the value of future annuity payments is discounted to the retirement period t using a discount factor β, and the retired individual receives exactly j payments, then the value of his payments at retirement is

$$a_i(k,t)(1+\beta+\beta^2+\ldots+\beta^{j-1}) = a_i(k,t)\ \frac{(1-\beta^j)}{1-\beta}\ .$$

Now unconditioning on j, the expected value at retirement of all payments to a person who retires from state (i,k) is

$$[1 - \sum_{j=0}^{\infty} \beta^j m_j(i,k)]a_i(k,t)/(1-\beta). \tag{28}$$

If there is any variance in the lifetime of an individual after retirement we would expect this to affect the total of all annuities paid out if the discount factor β is not equal to 1.

Problem 21: Using (28) show that if the discount factor β is 1 (no discounting of future monies) then the total expected value of all payments is simply the value of a single payment times the average number of payments; it is independent of the variance of the distribution $m_j(i;k)$. Explain this result. □

It is realistic to assume that β is less than, but close to, one. By using a Taylor series expansion of (28) about the point $\beta = 1$, and ignoring terms in $(1-\beta)^j$, $j \geq 2$, the total expected present worth at retirement of annuity payments can be shown to be well approximated by

$$[\mu_{ik} - \frac{(1-\beta)}{2}\ (\sigma_{ik}^2+\mu_{ik}^2-\mu_{ik})]a_i(k,t). \tag{29}$$

Let $v_i(k,t)$ be the expected value of the retirement fund in period t of a person who retires from state (i,k). By equating fund size to payments we see that the annuity he can be paid is given by

$$a_i(k,t) = \frac{v_i(k,t)}{\mu_{ik} - \frac{1-\beta}{2}(\sigma_{ik}^2+\mu_{ik}^2-\mu_{ik})} . \tag{30}$$

The quantity $v_i(k,t)$ is of course made up of contributions to the fund during the person's time in the organization. It will depend on his previous salaries, which in turn will depend on the individual's detailed movement between states. It is this quantity $v_i(k,t)$ which we now investigate, and we shall need the cross-sectional flow model with its probabilistic interpretation.

Let $c_i(k,u)$ be the contribution to the retirement fund (employer plus employee) in period u for an individual in state (i,k) in that period. If we consider a sample history of a person and assume that the fund earns interest at rate α, then we can trace the growth of the retirement fund over time.

First we see that $v_i(0,u) = c_i(0,u)$, the contributions for a person in period u who entered in that period in class i. Using conditional probability arguments we find an expression for the expected present value of the fund.

Given that a person is now (period t) in state (j,k) <u>and</u> was in state $(i,k-1)$ in $t-1$, then the expected value of the fund is

$$c_j(k,t) + (1+\alpha)v_i(k-1,t-1). \tag{31}$$

Let $p_{ij}(k)$ be the probability that an individual who is now in state (j,k) was in state $(i,k-1)$ in the previous period.* Note that for

---

*Care must be taken here. It may not be possible to be in some state (j,k), depending on how people enter the system and how promotions are made. In this case we would be conditioning on events which occur with probability zero. This technical difficulty can easily be overcome in a number of ways. For simplification in exposition we assume that all states (j,k) for which $v_j(k,t)$ is defined can be obtained with positive probability.

$k \geq 1$, $\sum_{i=1}^{N} p_{ij}(k) = 1$, and we are assuming these probabilities are independent of the period. We return to this problem later. Now unconditioning (31).

$$v_j(k,t) = c_j(k,t) + (1+\alpha) \sum_{i=1}^{N} v_i(k-1,t-1)p_{ij}(k),$$

or, using vector/matrix notation,

$$v(k,t) = c(k,t) + (1+\alpha)v(k-1,t-1)P(k), \qquad k \geq 1. \tag{32}$$

Here, $v(k,t)$ and $c(k,t)$ are N-component row vectors, and $P(k)$ is an $N \times N$ stochastic matrix (each column sums to 1).

By successive substitution (32) is solved for $v(k,t)$, and we obtain

$$v(k,t) = \sum_{j=0}^{k} (1+\alpha)^j \; c(k-j,t-j) \prod_{i=1}^{j} P(k-i+1), \qquad k \geq 0, \tag{33}$$

where an empty product is taken to be I, the identity matrix, and

$$\prod_{i=1}^{j} P(k-i+1) = P(k-j+1).P(k-j+2).\cdots.P(k), \qquad j \geq 1.$$

Equation (33) gives the expected value of the fund for all states (i,k) in terms of the contribution vectors $c(0,t-k), c(1,t-k+1), \ldots, c(k,t)$, the interest rate α, and the matrices $P(i)$, $i = 1,2,\ldots,k$. It remains to investigate these matrices, which are of course related to the underlying cross-sectional flow model (which we have not used to this point).

Consider an individual who entered the system in period $t - k$ and who is in class j at time t. Let $p_{ij}(t,k)$ be the probability that this individual was in class i at $t - 1$. We wish to discover what assumptions are necessary so that $p_{ij}(t,k)$ will be independent of t, so that $p_{ij}(k)$ is well defined.

Define $Z(t)$ as the person's class at time t, and E as the period of entry, both random variables. Then

$$p_{ij}(t,k) = P[Z(t-1) = i \mid Z(t) = j \, , \, E = t-k].$$

From conditional probability arguments,

$$(34)\quad p_{ij}(t,k) = \frac{P[Z(t) = j \mid Z(t-1) = i, E = t-k]P[Z(t-1) = i \mid E = t-k]}{P[Z(t) = j \mid E = t-k]} .$$

From the Markov property of our fractional flow model

$$(35)\qquad P[Z(t) = j \mid Z(t-1) = i, E = t-k] = q_{ji},$$

independent of k and t. Now define

$$z_i(t,k) = P[Z(t) = i \mid E = t-k].$$

Now $z_i(t,0) = P[\text{a person who enters in } t \text{ does so in class } i]$.

Therefore, if $f_{0i}(t)$ is the flow from outside the organization into class i in period t, and $f_0(t)$ is the N-vector of these flows,

$$(36)\qquad z_i(t,0) = f_{0i}(t)/ef_0(t),$$

since $ef_0(t)$ is the total new input in period t. By a straightforward conditioning argument

$$z_i(t,k) = \sum_{i=1}^{N} q_{ji} z_i(t-1,k-1),$$

or in vector notation

$$(37)\qquad z(t,k) = Qz(t-1,k-1) = Q^k z(t-k,0).$$

From this equation we see that $z(t,k)$ is independent of t if and only if $z(t-k,0)$ is independent of t. For this to be true we can see from (36) that the <u>fraction</u> appointed into any class i should be the same for all periods. This does not say that we have to appoint the same <u>number</u> into i in each period. Indeed this can vary from period to perio ; but the <u>fraction</u> must stay constant.

We now assume that for any period u

$$a_i = f_{0i}(u)/ef_0(u), \qquad i = 1,2,\ldots,N,$$

and let a be the N dimensional column appointment vector. Then from (37)

$$z(t,k) = Q^k a. \tag{38}$$

Using this with (35) in (34) we obtain*

$$p_{ij}(k) = p_{ij}(t,k) = q_{ji}z_i(t-1,k-1)/z_j(t,k). \tag{39}$$

Equations (39) and (38) show that the matrix $P(k)$ depends only on Q and a, and (39) can be written in matrix form.

For any vector $x = (x_1,x_2,\ldots,x_N)$, let $[x]_{DG}$ represent the $N \times N$ matrix whose diagonal elements are the same as the corresponding elements of the vector, and all off-diagonal elements are zero. Thus

$$[x]_{DG} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & x_N \end{bmatrix}.$$

---

*Note that from ou assumptions $z_j(t,k) > 0$ for all j and k.

Also let Q' denote the transpose of Q.

Using (38) and (39) we see that*

$$P(k) = [Q^{k-1}a]_{DG}\, Q'[Q^k a]_{DG}^{-1}, \qquad k = 1,2,\ldots .$$

This representation of $P(k)$ clearly shows its dependence on Q and a. This can now be used in (33) to show that

$$(40) \qquad v(k,t) = \sum_{j=0}^{k} (1+\alpha)^j\, c(k-j,t-j)[Q^{k-j}a]_{DG}[Q']^j[[Q^k a]_{DG}]^{-1}.$$

Equation (40) now gives the expected value of the fund for each state (i,k) in terms of the interest rate α, the contributions $c(0,t-k),\ldots,c(k,t)$, the appointment vector a and the transition matrix Q.

Problem 22. Assume that all new admissions are made into class 1. Thus a is the column vector $[1,0,\ldots,0]$. Use equation (40) to show that for this case

$$v_i(k) = \sum_{j=0}^{k} (1+\alpha)^j \sum_{\ell=1}^{n} c\,(k-j,t-j)q_j^{(j)}q_{\ell 1}^{(k-j)}/q_{i1}^{(k)},$$

and interpret this result. □

In many institutions contributions are related to salary. Let us assume that a fraction δ of an individual's salary is placed in his retirement fund each period, and let $s_i(t)$ be the salary of an individual in class i at t, independent of his length of service k. Let us further assume that salaries have been growing at rate γ per year.

---

*The inverse of $[Q^k a]_{DG}$ exists if all diagonal terms are positive. This is true if all states (j,k) can be held with positive probability.

Then the contribution in period $t - j$ of a person who was then in class i would have been

$$c_i(k-j,t-j) = \frac{\delta s_i(t)}{(1+\gamma)^j} .$$

If $s(t)$ is a row vector of current salaries by class, then (40) becomes

$$(41)\qquad v(k,t) = \delta \sum_{j=0}^{k} \left(\frac{1+\alpha}{1+\gamma}\right)^j s(t)[Q^{k-j}a]_{DG}[Q']^j[[Q^k a]_{DG}]^{-1} .$$

Problem 23. Interpret (41) when the salaries have been increasing at the same rate as the interest rate on the retirement fund. □

Equation (41) can now be used to calculate $v_i(k,t)$ and this used in equation (30) to calculate the annuity $a_i(k,t)$. We illustrate the model in analyzing alternative retirement and appointment schemes for a university faculty.

A university faculty model is formulated in which faculty can be in any one of the following 15 classes:

Class	Description
1	Nontenure
2	Tenure - Age 30 to 34
3	Tenure - Age 35 to 39
4	Tenure - Age 40 to 44
5	Tenure - Age 45 to 49
6	Tenure - Age 50 to 54
7	Tenure - Age 55 to 58, Low salary
8	Tenure - Age 59 to 61, Low salary
9	Tenure - Age 62 to 64, Low salary
10	Tenure - Age 55 to 58, Medium salary
11	Tenure - Age 59 to 61, Medium salary
12	Tenure - Age 62 to 64, Medium salary
13	Tenure - Age 55 to 58, High salary
14	Tenure - Age 59 to 61, High salary
15	Tenure - Age 62 to 64, High salary.

Table II.6 gives the basic fractional flows between the 15 classes of faculty. We call this Q_0. Table II.7 gives the fractional flows after an early retirement scheme has been emplemented. We can this Q_1. Table II.8 contains three different appointment vectors and a vector s of current salaries. The appointment vector a_0 is the one used before early retirement, a_1 is the one used after early retirement, and a_2 is the case where all appointments are made into non-tenure. It was assumed that salaries had been growing at 4% per year and the interest rate on the fund was 6% per year. The fund was incremented with 16% of the salary level each year.

Equation (41) was solved for 3 cases, and the results for classes 9 (tenured low salary, age 62-64) and 15 (tenured high salary, age 62-64) for the expected fund values are given in tables II.9 and II.10 Table II.11 shows the steady state stock levels for cases 1 and 2. These we calculated using the "hindsight" model of section 3 and a total system size of 413 faculty. For case 1 the calculations were made using Q_0 and a_0. For case 2 Q_1 and a_1 were used, and for case 3 Q_0 and a_2 were used. All other parameters were kept the same in the three cases.

Notice that with the implementation of the early retirement scheme the expected fund size is almost unchanged, even though there are large differences in the diagonal elements of Q_0 and Q_1 for classes 7 through 15. However, the steady state distribution of faculty has changed considerably, with a higher percentage of younger faculty. The slight decrease in retirement fund size (about 1%) may be tolerable when viewed in light of the improvement in the distribution of faculty.

Class j \ Class i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.78														
2	.043	.70													
3	.02	.28	.77												
4	.007		.19	.76											
5				.20	.79										
6					.18	.80									
7						.06	.78								
8							.19	.77							
9								.20	.88						
10						.06				.77					
11										.21	.74				
12											.23	.84			
13						.06							.74		
14													.23	.68	
15														.30	.70

Table II.6: Fractional Flows Q_0 in a Faculty Early Retirement Model.

Class j \ Class i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.78														
2	.43	.7													
3	.02	.28	.77												
4	.007		.19	.76											
5				.20	.79										
6					.18	.8									
7						.06	.69								
8							.19	.54							
9								.2	.42						
10						.06				.71					
11										.21	.57				
12											.23	.47			
13						.06							.73		
14													.23	.64	
15														.30	.62

Table II.7: Fractional Flows Q_1 in a Faculty Early Retirement Model

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_0	.8	.028	.044	.044	.036	.01	0	0	0	0	0	0	.01	.028	0
a_1	.83	.024	.037	.037	.03	.008	0	0	0	0	0	0	.008	.024	0
a_2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s	12013	15498	18428	19784	21689	22423	17283	17283	17283	21890	21890	21890	27618	27618	27618

Table II.8: Appointment and Salary Vectors for Faculty Example.

Length of Service (years)	Present Value of Fund		
	Case 1	Case 2	Case 3
20	94,579	92,114	86,787
25	117,895	115,341	111,524
30	143,897	141,085	138,387
35	172,578	169,378	167,584
40	204,029	200,347	199,337

Table II.9: Retirement Fund for Faculty in Class 15 at Retirement.

Length of Service (years)	Present Value of Fund		
	Case 1	Case 2	Case 3
20	77,313	77,569	74,015
25	99,799	99,634	96,203
30	124,306	123,913	120,564
35	151,166	150,648	147,319
40	180,671	180,047	176,705

Table II.10: Retirement Fund for Faculty in Class 9 at Retirement.

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Vacancies
Case 1	110	19	38	39	42	40	11	9	15	10	8	12	10	10	10	30
Case 2	120	21	43	43	46	43	9	3	1	9	4	2	11	9	7	34

Table II.11: Steady State Stocks Without and With Early Retirement.

Case 3 is added for illustration. It shows that if a policy change were made to appoint only non-tenured faculty, this would change the expected value of the retirement fund even if no changes were made in the fractional flows. It is interesting to see that for the cases studied the expected value of the fund decreases if appointments are made only into non-tenure.

Figure II.8 shows a plot of the fund value for classes 9 and 15 for increasing length of service at retirement.

In conclusion, this section has related some financial variables to a cross-sectional manpower flow model and indicated how to calculate some relevant expected values such as the value of a retirement fund. The spirit of the model is more important than the particular formulas and examples presented here. It can be possible with a cross-sectional flow model to study financial questions related to manpower policy. The particular model, classification scheme, and variables under study, will depend on the policies to be investigated and on the questions you wish to answer about the system.

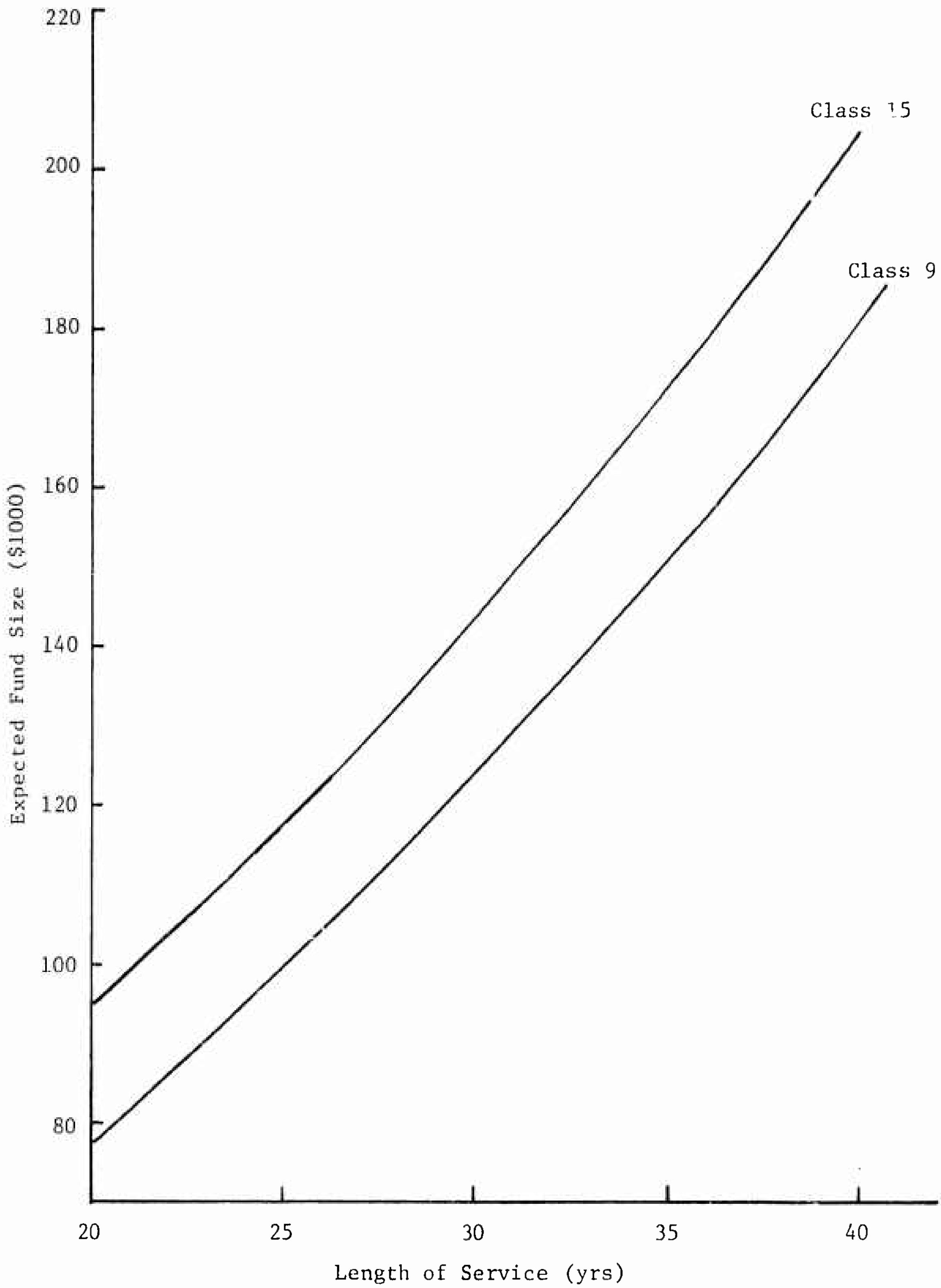

Figure II.8: Expected Retirement Fund Size for Faculty Example.

11. Student Enrollment Forecasting Models.

Of the many areas of application of the fractional flow model, the authors have gained considerable experience from its use in student enrollment forecasting of university undergraduates. Let us consider the classes of students to be freshmen, sophomores, juniors and seniors; thus we have a four-class model. The time periods are taken to be semesters. We shall find that a number of problems arise in the application of a simple fractional flow model. These will be discussed as they arise. The model will then be used to predict student attendance patterns, and will be checked against a set of independent data on such attendance patterns.

The first problem encountered is that the natural accounting and enrollment period in a university student model is either the semester or quarter. For simplicity we use the semester here. The reader should have no trouble in extending the results to a quarter system.

The fractional flows between classes from the fall semester to the spring semester differ considerably from those between classes from the spring semester to fall semester. If little or no new input of students takes place in the spring, and one is only interested in forecasts for the fall, the detailed flows in intermediate semesters could be ignored. However, in the institution we studied (the Berkeley campus of the University of California) significant new input occurs each spring. Table II.12 shows the new input into the four classes from the Fall of 1962 to Fall of 1966. Table II.13 shows the fractional flows Q and Q^* between classes, from Fall to Spring and Spring to Fall respectively.

Semester	(1) Freshmen	(2) Sophomores	(3) Juniors	(4) Seniors	Total
Fall 1962	3525	678	1416	184	5803
Spring 1963	328	187	324	42	881
Fall 1963	3620	738	1569	199	6126
Spring 1964	346	209	408	45	1008
Fall 1964	3427	602	1442	202	5673
Spring 1965	256	180	452	49	937
Fall 1965	2579	390	1042	125	4136
Spring 1966	291	210	476	66	1043
Fall 1966	3053	733	1418	205	5409

Table II.12: New Admissions, University of California, Berkeley.

$$
Q = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array}
\begin{array}{c}
\begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\
\begin{bmatrix} .926 & & & \\ .001 & .857 & & \\ & .031 & .902 & \\ & & .005 & .789 \end{bmatrix}
\end{array},
$$

$$
Q^* = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array}
\begin{array}{c}
\begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\
\begin{bmatrix} .103 & & & \\ .699 & .115 & & \\ & .792 & .158 & \\ & & .749 & .312 \end{bmatrix}
\end{array}.
$$

Table II.13: The Fractional Flows for Fall/Spring and Spring/Fall.

Let time $t = 0$ be the start of a fall semester (end of a spring semester; the summer period is ignored), and let the current enrollments be given by the 4-dimensional vector $s(0)$. Then in the spring enrollments will be

$$s(1) = Qs(0) + f(1),$$

where $f(1)$ is the vector of new admissions in the spring. Now

$$\begin{aligned} s(2) &= Q^*s(1) + f(2) \\ &= Q^*Qs(0) + Q^*f(1) + f(2) \cdot \end{aligned}$$

If we let $Q^*Q = \bar{Q}$, then the enrollments in successive fall quarters are given by

$$s(n) = \bar{Q}s(n-2) + Q^*f(n-1) + f(n), \qquad n \text{ even.} \tag{42}$$

The reader should now see how the basic model (equation 2) can be modified for cyclic situations.

Problem 24. Show that the stocks in successive spring semesters are given by

$$s(n) = \hat{Q}s(n-2) + Qf(n-1) + f(n), \qquad n = 3,5,\ldots,$$

where $\hat{Q} = QQ^*$.

Problem 25. Determine $\bar{Q}$ and $\hat{Q}$ for the data in Table II.13

Problem 26. Determine the forecasting formula for successive fall quarters assuming (1) three quarters operation, (2) year-round operations in 4 quarters. □

A second problem was found when the university data was analyzed. A significant number of students have breaks in their attendance (in addition to the obvious summer periods) between initial entrance and graduation, campus transfer, or drop-out. It seems that students often take a semester off and return at a later date to finish their degree program. However, how is one to distinguish between a student who has left permenently, and one who is on a temporary 'vacation'? Indeed, the student himself may not know which state he is in! Records are kept which distinguish "continuing" from "returning" students once they are re-admitted. It was found that this data could be used to estimate the fractional flows to and from a 'vacation' state. Thus a 5-state model is postulated to more realistically model the students' attendance patterns. For details the reader should consult Marshall, Oliver and Suslow [1970].

Table II.14 shows the fractional flows between classes, where state 5 represents the 'vacation' state. Notice that of those on vacation in a fall semester .632 stay on vacation in the spring; but of those on vacation in the spring only .342 stay on vacation the following fall. This shows that students prefer to return to their studies in the fall each year, a not unexpected observation.

One could now use either the 4-state or 5-state models in equation (42) to forecast student enrollments, and more will be said about this later in this section. At this point we use the model to investigate attandance patterns, and check the results against an independent set of data. The attendance patterns of all students who entered the Berkeley campus for the first time in the fall semester of 1955 were studied, as were those of the similar group who entered for the first time in 1970 (for details see Suslow, et al [1968]). A group with a common

$$Q = \begin{bmatrix} .913 & & & & .040 \\ .001 & .831 & & & .074 \\ & .031 & .852 & & .155 \\ & & .005 & .757 & .099 \\ .012 & .026 & .050 & .032 & .632 \end{bmatrix}$$

$$Q^* = \begin{bmatrix} .067 & & & & .084 \\ .699 & .039 & & & .145 \\ & .792 & .059 & & .234 \\ & & .749 & .207 & .193 \\ .036 & .076 & .100 & .104 & .342 \end{bmatrix}$$

Table II.14: Fractional Flows for the 5-state Model.

characteristic such as entrance date and state at entry is called a 'cohort.'

Table II.15 gives the numbers and fractions of students in both the 1955 and 1960 cohorts who attended a given semester after entrance. The fractions are plotted in Figure II.9.

A striking feature of this data is its stability over time. Attendance on any given semester after entrance varies little from the 1955 to 1960 group. Changes in University probation policy for freshmen and sophomore students appears to explain the small discrepancy in the third and fourth semesters (see Suslow, et al [1968]).

Semester after Entrance	1955 Entering Group			1960 Entering Group		
	Specific semester	Number of students	Fraction attending given semester	Specific semester	Number of students	Fraction attending given semester
1st	F 1955	2,067	.972	F 1960	3,228	.978
2nd	S 1956	1,924	.905	S 1961	3,002	.910
3rd	F 1956	1,585	.756	F 1961	2,331	.706
4th	S 1957	1,455	.684	S 1962	2,104	.637
5th	F 1957	1,260	.593	F 1962	1,891	.573
6th	S 1958	1,194	.562	S 1963	1,799	.545
7th	F 1958	1,114	.524	F 1963	1,753	.531
8th	S 1959	1,058	.498	S 1964	1,690	.512
9th	F 1959	424	.199	F 1964	693	.210
10th	S 1960	276	.130	S 1965	461	.139
11th	F 1960	107	.050	F 1965	182	.055
12th	S 1961	77	.036			
13th	F 1961	37	.017			
14th	S 1962	34	.015			
15th	F 1962	25	.011			
16th	S 1963	15	.007			

Table II.15: Number of Students Attending any Given Semester.

An important point to note is that the behaviour of a cohort appears to be independent of its size, a feature which may prove very useful when major changes from today's admission policies are considered. The cohorts of 2126 and 3290 students have essentially the same attendance characteristics.

Table II.16 gives the numbers and fractions of students in each cohort who attended without interruption at least the given number of semesters. The fractions are plotted as the lower curve in Figure II.10.

This curve has essentially the same shape as that in Figure II.9 with a sharp break point at the eighth semester and an increased tendency for a student to leave the system after one or two years. We again have close agreement between the 1955 and 1960 data.

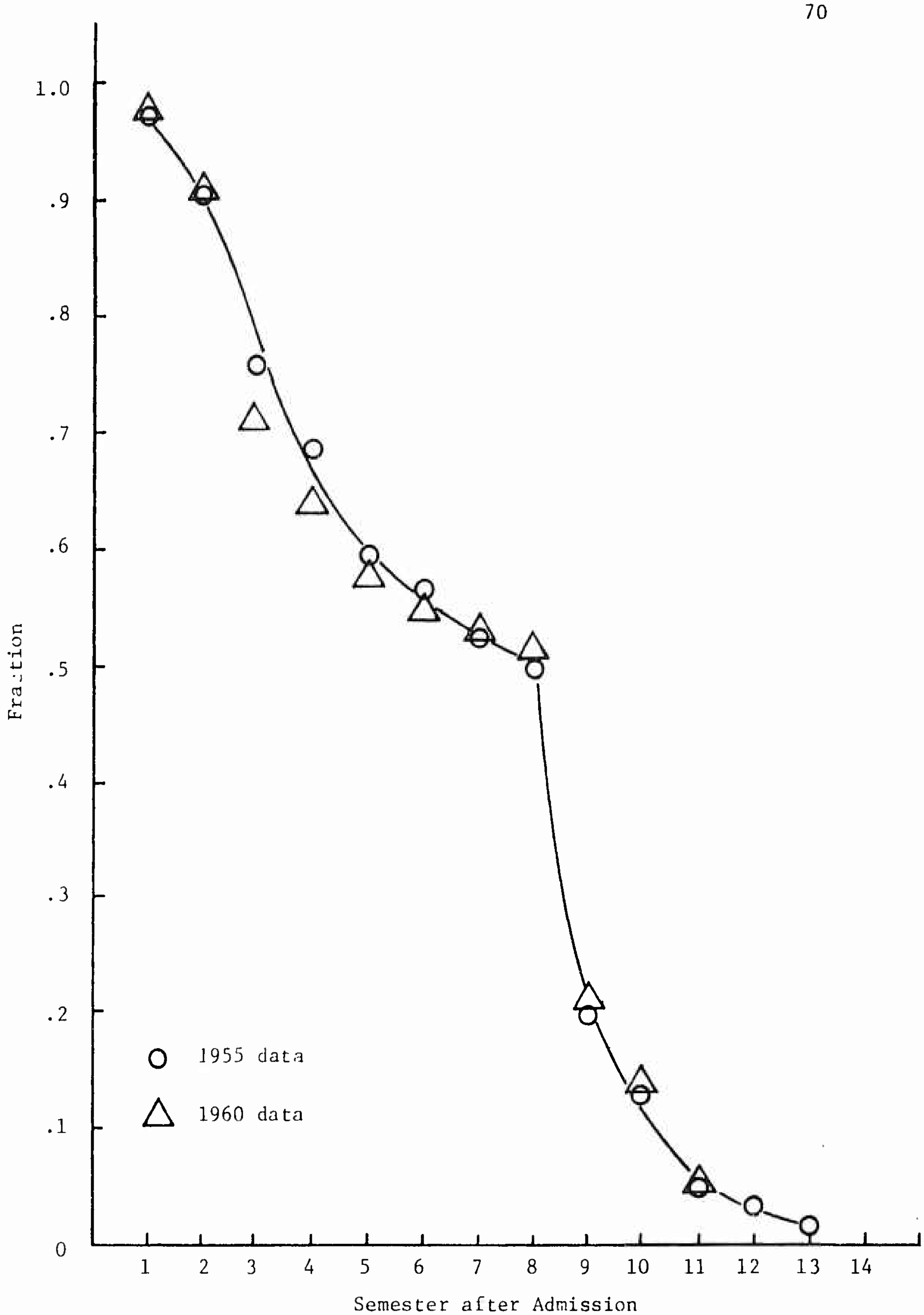

Figure II.9: Fraction Attending given Semester after Admission.

Table II.16 gives the fractions of students from each cohort which spent at least a given number of semesters in the system. This includes semesters in attendance and semesters on vacation.

These results are plotted as the upper curve in Figure II.10. Both curves in Figure II.10 would coincide if students had no vacations but the large difference between the two curves (15% by the 8th semester) shows that a significant proportion of students interrupt their consecutive attendance patterns by vacations.

Returning now to our fractional flow model, we first calculate the fraction of students attending the $n^{\underline{th}}$ semester after entrance in a fall semester as a freshman. For the $1^{\underline{st}}$ semester after entrance (that is, the entering semester) we take it to be 1. For the second semester the first column of Q gives the fractions of these freshmen in each state. Summing over the four attending states gives the fraction in attendance.

Problem 27: Show that for n even, the fraction attending the $n^{\underline{th}}$ semester after attendance is given by summing the first four elements of the first column of $\bar{Q}^{\frac{n}{2}}$, and for n odd, $\hat{Q}^{\frac{n-1}{2}}$.

Table II.18 shows the calculated probability of attendance in each semester for the four state model and this is plotted in Figure II.11. The calculated distribution agrees well with that of the 1955 and 1960 freshmen cohorts. However, since there are no vacation states, attendance, consecutive attendance and elapsed time are the same in this model and we naturally get poor agreement with the cohort data for consecutive attendance and elapsed time.

Number of Consecutive Semesters Completed	1955 Entering Group (2,126 students)		1960 Entering Group (3,298 students)	
	Number of Students	Fraction	Number of Students	Fraction
1	2,067	.972	3,228	.978
2	1,923	.904	2,994	.907
3	1,554	.730	2,301	.697
4	1,373	.645	2,018	.611
5	1,112	.523	1,679	.509
6	1,027	.483	1,554	.471
7	883	.415	1,371	.415
8	819	.385	1,291	.391
9	222	.104	363	.110
10	112	.052	181	.054
11	15	.007	33	.010

Table II.16: Students Completing Each Consecutive Semester With no Interruptions in Attendance

Number of Semesters in the System	1955 Data	1960 Data
1	.976	.984
2	.929	.926
3	.804	.769
4	.752	.718
5	.655	.660
6	.625	.634
7	.565	.589
8	.528	.556
9	.207	.234
10	.129	.155
11	.033	.053
12	.019	.000

Table II.17 : Fraction Completing at Least a Given Number of Semesters.

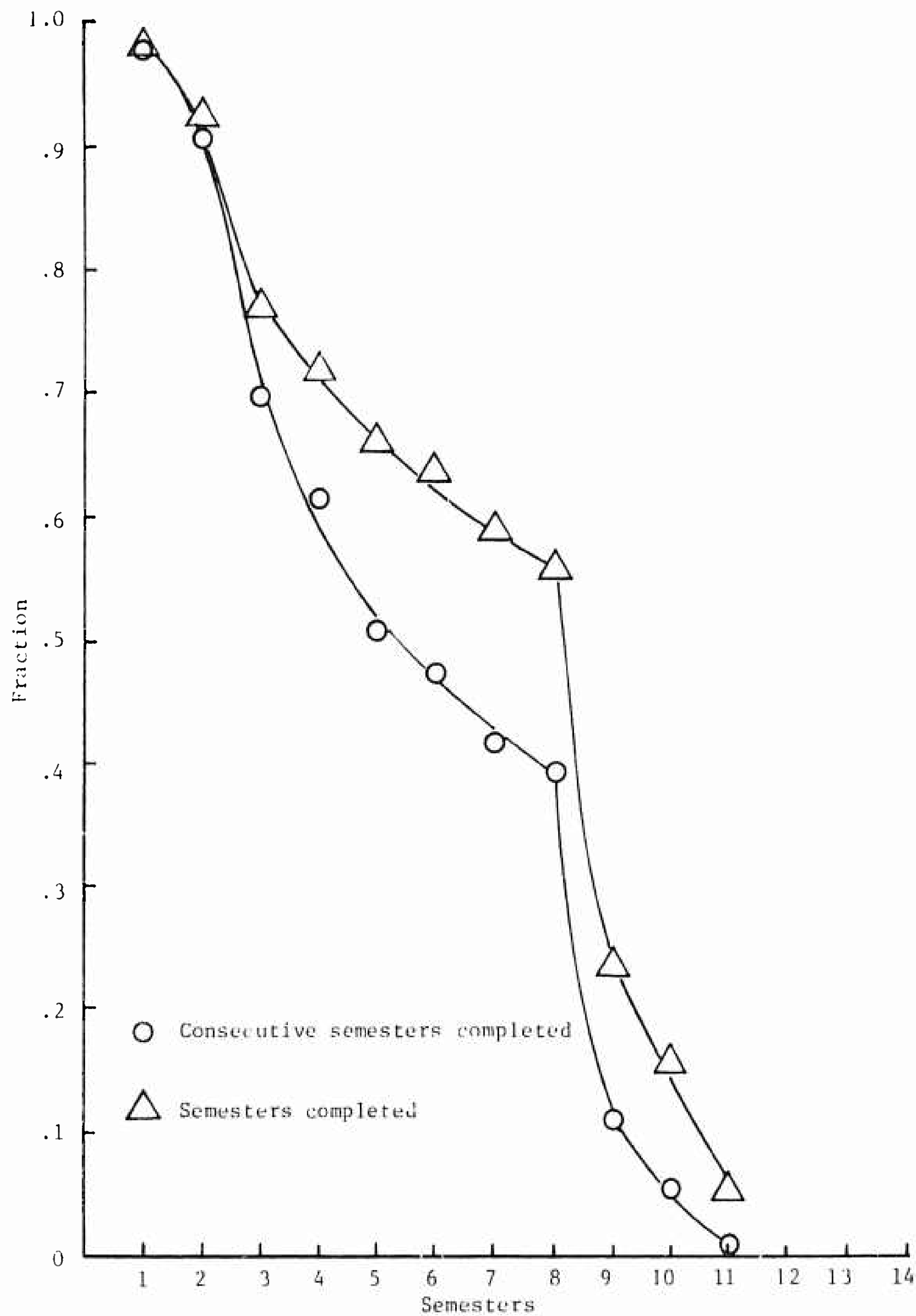

Figure II.10: Fraction with Consecutive Attendance and Elapsed Time (1960 Data).

Semester n	Fraction in Attendance	Fractions from 1970 Data
1	1.000	.978
2	.926	.910
3	.743	.706
4	.564	.637
5	.593	.573
6	.534	.545
7	.475	.531
8	.395	.512
9	.213	.210
10	.173	.139
11	.073	.055
12	.059	-

Table II.18: Fraction in Attendance for the 4 State Model.

Semester n	Attending Semester n	n Consecutive Semesters	At least n Semesters
1	1.000	1.000	1.000
2	.914	.914	.926
3	.709	.700	.746
4	.628	.607	.669
5	.541	.501	.601
6	.483	.428	.546
7	.414	.339	.482
8	.350	.263	.407
9	.178	.088	.232
10	.160	.067	.201
11	.097	.017	.126
12	.089	.013	.111

Table II.19: Calculated Fractions for the 5 State Model.

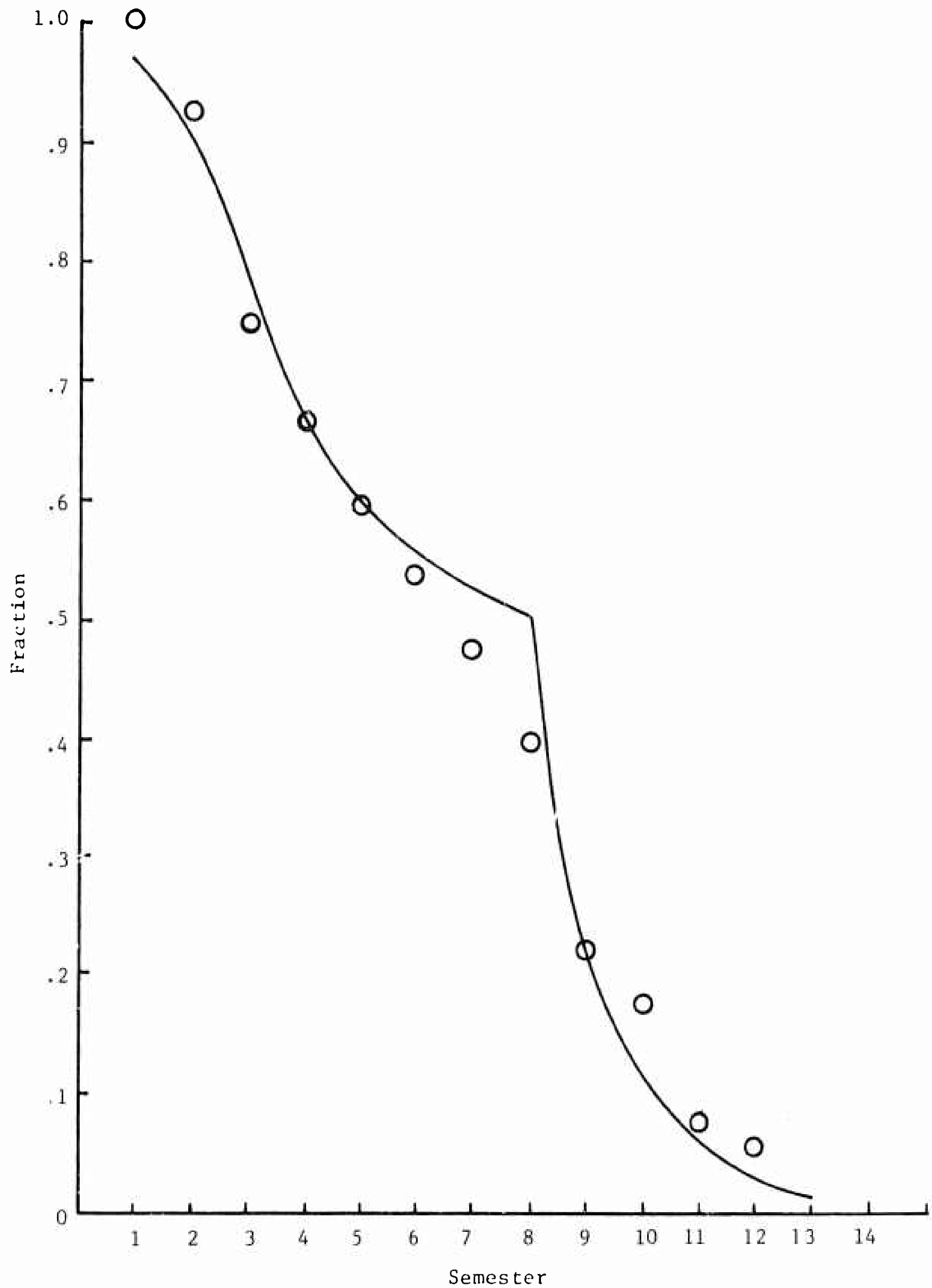

Figure II.11: Fraction in Attendance; 4 Class Model.

Table II.19 gives the calculated distributions of attendance, consecutive attendance and elapsed time for the single vacation state model. The distribution of attendance is plotted in Figure II.12 for comparison with the cohort data.

The results for the fraction in attendance do not agree as closely as those obtained with the four state model. The major discrepancy occurs in semesters seven and eight. Very good agreement is found in the first five semesters. This characteristic is found in the distributions of consecutive attendance and elapsed time also, although in these two distributions the five state model naturally gives an improvement over the four state.

The consistently poor agreement near the break at semester eight requires explanation. Remember that we are comparing fractions calculated from a cross-sectional model, with fractions observed from longitudinal studies on students. The cross-sectional data includes students who enter in all classes, and a feature of the model is the assumption that a student who enters as, say, a junior, behaves in the same way as a student who entered as a freshman when he becomes a junior. Thus the cross-sectional data includes numerous different cohorts superimposed at one time, and it is difficult (if not impossible) to identify to which cohort a student belongs using the available data.

One of the main purposes for our including this example of student forecasting is to motivate the types of models presented and analyzed in report number III. Perhaps by keeping a little more data on an individual in addition to his current class, such as how long he has been in the system and what was his class at entry, his longitudinal behavior patterns can be used as the basis for a model. As we shall see, the longitudinals models tend

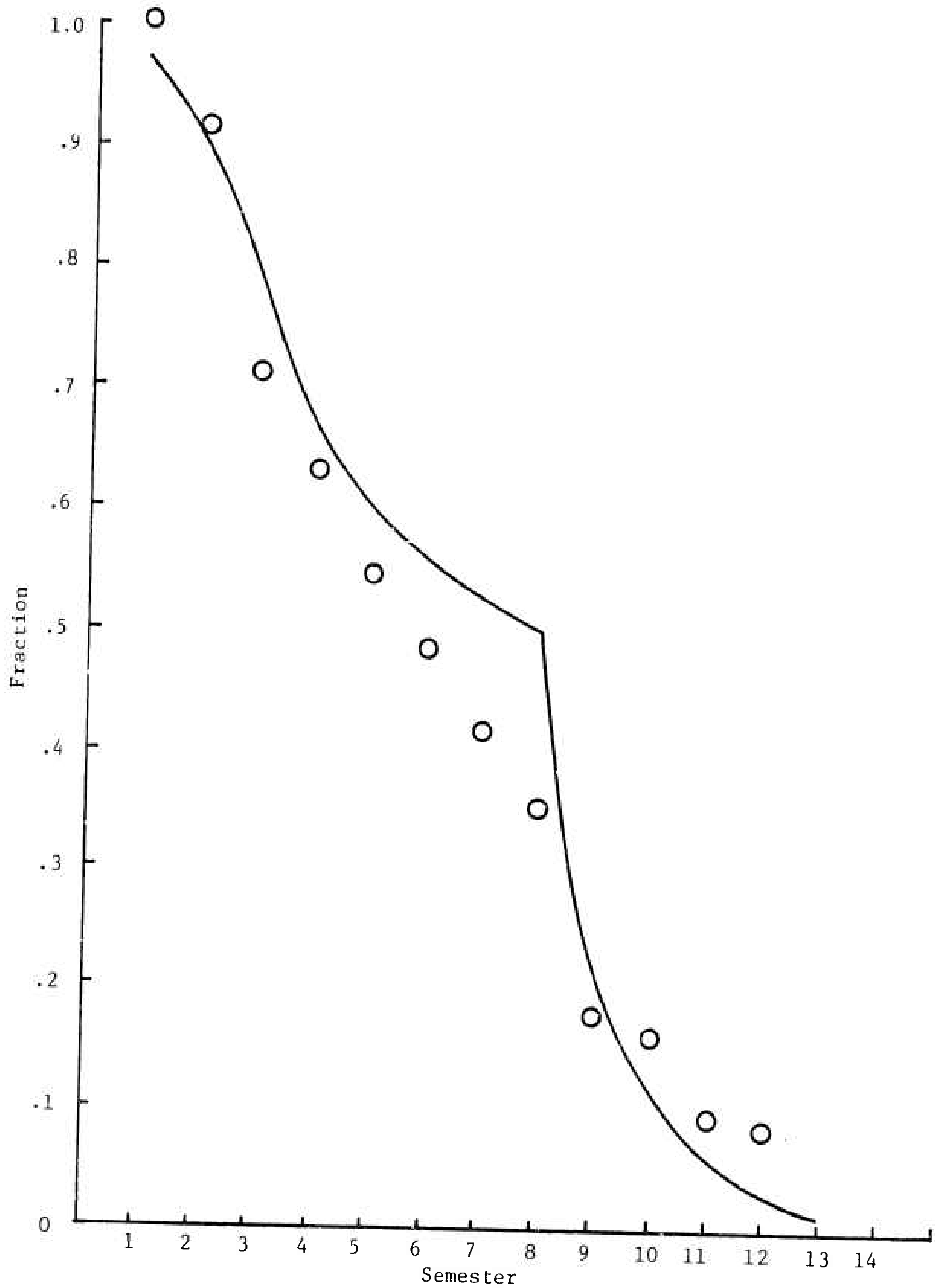

Figure II.12: Fraction in Attendance; 5 Class Model.

to better represent actual movement of individuals through a system, but the price paid is in larger amounts of less frequently available types of data.

To end this section we include enrollment predictions for the years 1962-1966 for the Berkeley campus by Freshmen, Sophomores, Juniors, and Seniors, using the four and five state models. Actual enrollments are given for comparison. In the predictions, new inputs were taken as fixed and known and only the continuing portions were calculated. The results are given in Table II.20.

In order to attempt to measure the suitability of each model in a simple manner two types of error are defined and computed for each model. We first compute the absolute error by taking the absolute value of the difference between the actual enrollment and the predicted value for a given state and year and summing over the states. These are given in Table II.21. Secondly, we find the mean square error for each year group in the usual manner for the four states. These are given in Table II.22.

In terms of absolute error, the five state model appears to be superior to the four state model over short prediction periods. Our results show that for longer periods the four state model gives better results.

In terms of absolute error the five state model seems to be consistently better, although there is little difference in them at five years. If enrollment predictions are required over one or two years in the future, it seems that the five state model is the most desirable one.

		1961	1962	1963	1964	1965	1966
	4 STATE		3930	4042	3865	2986	3388
FRESHMEN	5 STATE		3914	4032	3861	2997	3428
	ACTUAL	3843	3972	4106	4186	3307	3633
	4 STATE		3790	3899	3875	3479	3245
SOPHOMORES	5 STATE		3748	3866	3855	3480	3289
	ACTUAL	3778	3649	3846	3468	3349	3126
	4 STATE		4809	5049	5026	4619	4689
JUNIORS	5 STATE		4743	4927	4930	4529	4660
	ACTUAL	4180	4762	4806	5429	5311	5624
	4 STATE		4311	4859	5053	5220	5078
SENIORS	5 STATE		4289	4772	4899	5080	4949
	ACTUAL	3943	4210	4789	4585	4581	4364
	4 STATE		16840	17849	17819	16304	16400
TOTAL	5 STATE		16694	17597	17545	16086	16326
	ACTUAL		16593	17547	17668	16548	16747

Table II.20: Enrollment Forecasts and Actual Enrollments, 1962-1966.

MODEL	1962	1963	1964	1965	1966
4 STATE	247	302	151	244	347
5 STATE	101	50	123	462	421

Table II.21: Calculated Absolute Prediction Errors.

MODEL	1962	1963	1964	1965	1966
4 STATE	92	133	403	501	603
5 STATE	70	72	388	493	579

Table II.22: Calculated Mean Squared Error.

12. A Vacancy Model.

In previous sections we have considered stocks of manpower in various classes but have not specifically concerned ourselves with positions which people fill. In this section we consider an organization where both positions and people are accounted for.

Let $x_i(t)$ be the number of class i positions available at time t, and let $s_i(t)$ be the number of people in class i. The number of vacancies in class i at time t is given by $v_i(t) = x_i(t) - s_i(t)$.

We assume the flow of positions is governed by the simple equation

$$x(t+1) = x(t) + y(t+1), \tag{43}$$

where $y_i(t+1)$ is the number (perhaps negative) of new class i position that are made available in period $t + 1$. If $y_i(t+1)$ is positive, positions are added. If negative they are removed.

The flow of manpower in period $t + 1$ is determined in large part by the vector of vacancies $v(t)$ that exist at time t.

We assume that all of the $v_i(t)$ vacancies in class i are filled during period t. Moreover, we assume that a fraction $r_{ji} \geq 0$ of these vacancies are filled by individuals from class j for $j = 1,2,\ldots,N$, and $a_i \geq 0$ is the fraction of vacancies filled by new appointments. The fractional flow assumption is

$$\begin{aligned} f_{ji}(t+1) &= r_{ji}v_i(t) \quad \text{for} \quad j \neq i \\ f_{0i}(t+1) &= a_iv_i(t). \end{aligned} \tag{44}$$

Since we assume all vacancies are filled we have

$$\sum_{j=1}^{N} r_{ji} + a_i = 1.$$

The inflow of people into rank i in period t will be

$$v_i(t) = \left(\sum_{j=1}^{N} r_{ji}\right)v_i(t) + a_i v_i(t).$$

The outflow of people from class i to other classes will be:

$$\sum_{j=1}^{N} r_{ij}v_j(t) + f_{i0}(t+1).$$

When these relations are combined we find

$$s_i(t+1) = s_i(t) + v_i(t) - \sum_{j=1}^{N} r_{ij}v_j(t) - f_{i0}(t+1).$$

Let $h_i(t+1) = f_{i0}(t+1)$, the flow out of the system from i in $t + 1$ and R the matrix of elements r_{ij}. Then

(45) $$s(t+1) = x(t) - Rv(t) - h(t+1).$$

Subtracting (45) from (44) we obtain

(46) $$v(t+1) = Rv(t) + h(t+1) + y(t+1).$$

Equation (46) determines the flow of <u>vacancies</u> in the organization. Although it is possible to have $y_i(t+1) \leq 0$, we assume that $v_i(t) \geq 0$ for all i and t. Thus we can obtain from (46) an explicit lower bound on $y(t+1)$, the change in positions in $t + 1$;

$$y(t+1) \geq -(Rv(t)+h(t+1)).$$

Positions cannot be removed faster than this without causing negative elements in the vacancy vector.

The equilibrium solution of the system given by (46) is easily seen to be

(47)
$$v = (I-R)^{-1}h, \quad y = 0$$
$$s = x - v.$$

From (47) we can calculate the equilibrium flows: $f_{ij} = r_{ij}v_j$ for $i \neq j$, and $f_{ii} = s_i + v_i$. Thus the average lifetime an individual spends in class i is given by

$$\ell_i = \frac{1}{1 - f_{ii}/s_i} = \frac{s_i}{v_i}.$$

The model above operates with hindsight. We observe the vacancies at time t, and act during period $(t+1)$ to fill the time t vacancies. These actions in turn create new vacancies at time $t + 1$. An alternate foresight model is possible which eliminates vacancies by planning ahead. Although the assumption of being unable to eliminate vacancies is unrealistic, the model that we obtain is mathematically simpler, and the policies are more forward looking.

Let $z_i(t+1)$ be the number of class i positions filled during period $t + 1$ (recall that period $t + 1$ is the interval $(t,t+1]$). It follows that

(48)
$$z_i(t+1) = \sum_{j\neq i} f_{ji}(t+1).$$

Let r_{ji} for $j = 1,2,\ldots,N$ $(j \neq i)$ be the fraction of these positions that are filled by people from rank j, and let a_i be the fraction that are filled by new appointments. Set $r_{ii} = 0$. We require that all positions are filled. Thus

$$\sum_{j=1}^{N} r_{ji} + a_i = 1.$$

Vacant positions in class i during period $t + 1$ are created by departures $h_i(t+1)$, addition of new positions $y_i(t+1)$, and by flow of individuals in class i to fill other positions. The total number of vacancies created is

$$h_i(t+1) + y_i(t+1) + \sum_{j=1}^{N} r_{ij} z_j(t+1). \tag{49}$$

If all vacancies are filled by time $t + 1$, then (48) and (49) must be equal. In matrix notation we have

$$z(t+1) = h(t+1) + y(t+1) + Rz(t+1). \tag{50}$$

Equation (50) can be solved for z in terms of h and y,

$$z(t+1) = (I-R)^{-1}(h(t+1)+y(t+1)).$$

In the steady state we obtain

$$z = (I-R)^{-1}h, \quad y = 0, \quad s = x. \tag{51}$$

From this we can compute new appointments $f_{0i} = z_i a_i$, and internal flows $f_{ij} = r_{ij} z_j$. Specification of s_i allows us to compute $f_{ii} = s_i - z_i$ and the average lifetime in class i for an individual is

$$\ell_i = \frac{s_i}{z_i}.$$

Problem 28: Show that the equilibrium vacancy model is equivalent to an equilibrium Markov model with

$$q_{ji} = r_{ij}v_j/s_i \quad \text{for} \quad i \neq j,$$

$$q_{ii} = (\ell_i - 1)/\ell_i, \qquad f_i = v_i a_i.$$

Example 10: Consider a six class organization where classes 1, 2, 3 are in the bottom stratum, classes 4 and 5 are in the middle stratum and class 6 the top stratum. Let us assume

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} & .2 & .1 & .15 & .1 & \\ .15 & & .1 & & .2 & \\ .05 & .1 & & .25 & .05 & \\ & & & & .5 & .4 \\ & & & .4 & & .3 \\ & & & & & \end{bmatrix} \end{array}$$

a =	.8	.7	.8	.2	.15	.3
h =	17	9	13	4	5	1
s =	400	470	375	85	90	10 .

Notice from R that one can move up in strata but not down.

Of the vacancies created in class 4, for example, 15% are filled from class 1, 25% from class 3, 40% from class 5 and 20% by new people. Nine people leave the system from class 2 each period, 13 from 3, 17 from 1, etc. The system has 400 people in 1, 470 in 2, 375 in 3, etc.

First we calculate $(I-R)^{-1}$ which is

$$\begin{bmatrix} 1.040 & .220 & .126 & .311 & .310 & .210 \\ .160 & 1.050 & .120 & .183 & .323 & .170 \\ .680 & .166 & 1.020 & .371 & .266 & .228 \\ & & & 1.250 & .025 & .687 \\ & & & .500 & 1.250 & .575 \\ & & & & & 1.000 \end{bmatrix}$$

From equations (47) and (51) we see that the vectors v (steady state vacancies) and z (steady state positions filled) are equal and both are given by the vector

$$(18.1,15.3,18.1,8.8,8.9,1.0).$$

The average lifetime of an individual in each class is given by the vector

$$(22.1,30.7,20.7,9.6,10.2,10.0).$$

13. More General Fractional Appointment Policies.*

In this section we consider the fractional flow model

$$s(t+1) = (Q+A)s(t). \tag{52}$$

We assume $A \geq 0$, and $e(Q+A) = e$. In this model the appointment policy is

$$f(t+1) = As(t), \tag{53}$$

which is a linear function of the current stocks of manpower. By assuming $A \geq 0$, we have insured that $f(t) \geq 0$. Also the system will stay of constant size since

$$es(t+1) = e(Q+A)s(t) = es(t).$$

It is apparent that w must equal eA. We can interpret a_{ij}/w_j as the fraction of departures in class j that are replaced by appointments in class i. This type of appointment policy is clearly more flexible than the policy presented in section 4 where $a_{ij} = a_i w_j$ for all i and j.

The equilibrium version of equation (52) is

$$s = (Q+A)s, \quad f = As. \tag{54}$$

The question arises, is it possible to obtain equilibrium solutions of (54) that are not possible when we restrict a_{ij} to the form $a_i w_j$ with $\sum_{j=1}^{N} a_j = 1$? The answer is no as we now show.

Let s and f be any equilibrium solution of (54); then let

$$a = As/ws, \tag{55}$$

Clearly $$ea = eAs/ws = 1$$

and
$$(Q+a.w)s = Qs + \frac{As}{ws}\, ws = s.$$

However, if we use the fractional appointment policy $a = \frac{As}{ws}$ for $s(0) \neq s$, then the path to equilibrium using this appointment policy will in general differ from the path taken using appointment policy A.

Problem 29: Suppose we require $(Q+A) \geq 0$, and $e(Q+A) = e$; but allow elements of A to be negative. Examine and interpret this model.

Example 11: Consider a university faculty consisting of the following 15 classes of manpower.

Class	Description
1	Nontenure
2	Tenure - Age 30 to 34
3	Tenure - Age 35 to 39
4	Tenure - Age 40 to 44
5	Tenure - Age 45 to 49
6	Tenure - Age 50 to 54
7	Tenure - Age 55 to 58, Low Salary
8	Tenure - Age 59 to 61, Low Salary
9	Tenure - Age 62 to 64, Low Salary
10	Tenure - Age 55 to 58, Medium Salary
11	Tenure - Age 59 to 61, Medium Salary
12	Tenure - Age 62 to 64, Medium Salary
13	Tenure - Age 55 to 58, High Salary
14	Tenure - Age 59 to 61, High Salary
15	Tenure - Age 62 to 64, High Salary

The transition matrix is given in Table II.23 and shows the estimated fractional flows when an early retirement system has been instituted.

We have broken the departure class (0) into three separate classes, early retirement (16), normal retirement (17), and others (18).

The departure fraction $\tilde{w}_i$ is the sum of the last three numbers in column i. Before, the early retirement program was instituted the fractional appointment vector a was given by

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a	.8	.028	.044	.044	.036	.01	0	0	0	0	0	0	.01	.028	0

Under the early retirement policy, all new appointments that arise from early retirements will be filled by nontenure appointments.

For each class i let $r_i = q_{16,i}$ be the fraction that retire early, and let w_i be the fraction that left the system before the early retirement plan $w_i = q_{17,i} + q_{18,i}$. In other words we assume that

$$\tilde{w}_i = r_i + w_i .$$

Now define $b_j = \begin{Bmatrix} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{Bmatrix}$ and the matrix A by

$$a_{ij} = b_i r_j + a_i w_j \tag{56}$$

A portion of the appointment matrix is presented below along with the equilibrium solution of (54) with $\lambda = 413$ (total faculty in system).

Equilibrium Solution

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s	141	23	47	47	50	46	9	3	2	9	5	2	12	9	8

The non zero elements of the first and ninth columns of A are shown.

		rows							
		1	2	3	4	5	6	13	14
column	1	0.12	.0042	.0066	.0066	.0054	.0015	.0015	.0042
	9	.556	.0034	.0053	.0053	.0043	.0012	.0012	.0034

Note that column 9 is not proportional to column 1.

State j \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.78														
2	.43	.7													
3	.02	.28	.77												
4	.007		.19	.76											
5				.20	.79										
6					.18	.8									
7						.06	.69								
8							.19	.54							
9								.2	.42						
10						.06				.71					
11										.21	.57				
12											.23	.47			
13						.06							.73		
14													.23	.64	
15														.30	.62
16							.09	.23	.46	.06	.17	.37	.01	.04	.08
17									.09			.14			.28
18	.15	.02	.04	.04	.03	.02	.03	.03	.03	.02	.03	.02	.03	.02	.02

TABLE II.23: FRACTIONAL FLOW FOR AN EARLY RETIREMENT MODEL

The equivalent simple appointment policy is given by $\tilde{a} = As/\tilde{w}s$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\tilde{a}$	0.83	.024	.037	.037	.03	.008	0	0	0	0	0	0	.008	.024	

From an initial value of $s(0)$, given below, we projected $s(t)$ for five periods using first the appointment policy A (56), then using $\tilde{a}$ above. The values of $s_1(t)$ are shown below

	Period					
	0	1	2	3	4	5
Policy A	137	144	146	147	146	145
Policy $\tilde{a}$	137	143	145	146	145	144

14* Evolution of Fixed Size Systems.

This section examines the fractional flow model

(57) $$s(t) = Q\, s(t-1) + f(t),\ f(t) \geq 0 \quad t \geq 1$$

$$\text{given} \quad s(0) \geq 0, \quad s(0)e = \lambda.$$

We make four assumptions about this system:

(i) $Q \geq 0$

(ii) $w = e - e\,Q \geq 0$

(iii) $(I-Q)$ has a nonnegative inverse.

(iv) $s(t)e = \lambda$ for all t.

Items (i) and (ii) simply identify a fractional flow model. Assumption (iii) is equivalent to $Q^t \to 0$. Thus the legacy of any initial stock levels $s(0)$ becomes negligible in the distant future. The final assumption (iv), of constant size, places a limitation on the appointment vector $f(t)$. Summing the vectors in (57),

$$es(t) = eQs(t-1) + ef(t) = es(t-1) - ws(t-1) + ef(t).$$

Thus to preserve constant size

(58) $$ef(t) = ws(t-1).$$

Equation (58) simply says the number of new appointments in t must equal the number of departures in $(t-1)$.

We can normalize the problem by defining

$$z(t) = s(t)/\lambda, \; h(t) = f(t)/\lambda\,.$$

Then (57) and (58) become

$$z(t) = Qz(t-1) + h(t), \tag{59}$$

$$eh(t) = wz(t-1),$$

$$h(t) \geq 0, \qquad t = 1,2,\ldots,$$

$$z(0) \text{ given}, \; ez(0) = 1, \;\; z(0) \geq 0.$$

Define

$$S = \{z \mid ez = 1, \; z \geq 0\}. \tag{60}$$

S is the set of all distributions of manpower in the N classes. The constraints in (59) require that $z(t) \in S$ for all t, using S we can write (59) in an alternate manner; given any sequence $\{z(t), h(t)\}$ that satisfies (59) define $a(t)$ as

$$a(t) = \begin{cases} z(t-1) \text{ if } wz(t-1) = 0, \\ \dfrac{h(t)}{wz(t-1)} \quad \text{otherwise.} \end{cases} \tag{61}$$

Notice that $a(t)$ S for all $t \geq 1$, and that

$$z(t) = (Q + a(t) \cdot w) \; z(t-1) \tag{62}$$

or

$$z(t) = P[a(t)] \; z(t-1),$$

where $P[a(t)]$ is the stochastic matrix $Q + a(t) \cdot w$ with elements $P_{ij}[a(t)] = q_{ij} + a_i(t)w_j$. Alternately, $z(0) \in S$ and a sequence $a(t) \in S$, $t \geq 1$, determines a solution $z(t) \in S$ of (62); by defining $h(t) = a(t) \cdot wz(t-1)$, we can then construct a solution of (59).

Problem 30: Given the system

$$\tilde{s}(t) = \tilde{Q}s(t-1) + \hat{f}(t), \quad \hat{f}(t) \geq 0$$
$$\tilde{s}(t)e = \theta^t\lambda \qquad t = 1,2,\ldots,$$

and $\tilde{s}(0)e - \lambda\tilde{s}(0) \geq 0$. How and under what conditions can we convert this into the form of (59), and still satisfy requirements (i) - (iv)?

Let z be a distribution of manpower at any time. An interesting set of points to examine, is all points in S that can be reached from z in a single time period. We define this set to be

(63) $$R(z) = \{y | y \geq Qz, \ y \in S\} .$$

$R(z)$ is the set of all points that can be reached from z in one period. To determine the set of points that can be reached in two periods, we must generalize our notion of R. Let A be any nonempty subset of S, and define

(64) $$R(A) = \{y | y \geq Qz, \ z \in A, \ y \in S\} .$$

$R(A)$ is the set of all points that can be reached in one step from some point in A. It follows that

(65) $$R(A) = \bigcup_{z \in A} R(z) .$$

Now define

(66) $$R^0(A) = A$$
$$R^1(A) = R(A)$$
$$R^t(A) = R[R^{(t-1)}(A)], \quad t \geq 2 .$$

It follows that $R^t(A)$ is the set of all points that can be reached in t periods starting from some $z \in A$. When A consists of a single point z, then we write $R^t(z)$.

The analysis that follows is motivated by the following problem: given an initial distribution $z(0)$ can we reach a desired distribution y in a finite number of steps? Moreover, when we reach y is it possible to remain at y or return to y? The easy question, can we remain at y, will be treated first. We can only give a partial answer to the questions, can we move from $z(0)$ to y, and can we return to y. We give an operational characterization of the set E of maintainable or equilibrium distributions. If $y \in E$, then it is possible to remain at y.

We also describe a set L of limiting distributions. If y is in the interior of L then, for any $z(0)$, it is possible to move to y in a finite number of steps, and it is obviously possible to return to y from y in a finite number of steps. However, given any y it is difficult to determine if $y \in L$. Thus, the characterization of L is not as operational as the characterizations of the set E.

<u>Example 11</u>: Given Q, $z(0)$, y below

$$Q = \begin{bmatrix} .8 & & \\ .1 & .95 & \\ & .02 & .9 \end{bmatrix} \quad z(0) = \begin{bmatrix} 33 \\ 34 \\ 33 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

it is not possible to reach y from $z(0)$, since $z_3(t) > 0$ for all t.

□

Clearly, it is possible to remain at y if and only if $y \in R(y)$, or

$$y \in \{z \mid z \geq Qy,\ z \in S\}\ .$$

This will be true if and only if $y \geq Qy$. Let us define the <u>equilibrium</u> set E as the set of distributions that can be repeated

$$E = \{y \mid y \in R(y),\ y \in S\} = \{y \mid y \geq Qy,\ y \in S\}\ . \tag{67}$$

It follows that $y \in R^t(y)$ for all t, thus it is possible to remain at y indefinitely. There is another way to describe the set E. Recall that $D = (I-Q)^{-1}$. Then

$$E = \{y \mid y = Dh,\ y \in S,\ h \geq 0\}\ . \tag{68}$$

To see this, note that if $y = Dh$ and $h \geq 0$, then $(I-Q)y = h \geq 0$ and $y \in E$. Also, if $y \in E$, then define $h = (I-Q)y \geq 0$, and note that $y = Dh$.

For $y \in E$, define $h(y) = (I-Q)y$, $a(y) = (I-Q)y/wy$, and $P[a(y)] = Q + a(y) \cdot w$. Then we obtain

$$P[a(y)]y = y\ . \tag{69}$$

If $h(y) > 0$, then for any initial distribution $z(0)$ we obtain

$$P^t[a(y)]\ z(0) \to y\ . \tag{70}$$

<u>Problem 31</u>: Prove

(i) If $y \in E$, then $wy > 0$

(ii) There is a $y \in E$, such that $y > Qy$.

(iii) If $y > Qy$, then $P[a(y)]$ is a regular Markov matrix.

<u>Problem 32</u>: Construct an example where $y \in E$, but $y > Qy$ is <u>not</u> true, and $P^t[a(y)]\, z(0)$ does <u>not converge</u> (or come close to y). □

We now turn to the long range behavior of the system (59). Let A and B be subsets of S. If $A \supset B$, then $R(A) \supset R(B)$. This is reasonable, since it tells us that if you can go to y from $z \in B$, then you can certainly reach y from $z \in B \subset A$. Notice that $R(E) \supset E$. This is true because $y \in E$ implies $y \in R(y)$. Now consider the inclusions

$$\text{(i)} \quad R(E) \supset E$$

$$\text{(ii)} \quad S \supset E$$

$$\text{(iii)} \quad S \supset R(S)$$

and repeatedly apply R. We obtain

(71) $$S \supset R^t(S) \supset R^{t+1}(S) \supset R^{t+1}(E) \supset R^t(E) \supset E \quad \text{for all } t \geq 0.$$

Since $R^t(S)$ is a contracting sequence of sets we can define

(72) $$L = \bigcap_{t=0}^{\infty} R^t(S) .$$

It is evident that L is nonempty since $L \supset E$, and it is not too difficult to show that $R(L) = L$.

<u>Problem 33</u>: Prove $R(L) = L$. □

This result can be carried one step further when we make the additional assumption that $w > 0$.

<u>Theorem</u>: If $w > 0$, then L is the unique closed set that satisfies $R(B) = B$. Moreover, if A is any closed subset of S, then $R^t(A) \to L$ geometrically.

The theorem is proved in Grinold and Stanford [1973]. Notice that the set L has several remarkable properties. First, if $z(0) \in L$, then $z(t) \in L$ for all t. Once the system enters L it cannot leave.

Consider the problem of moving from $z(0) \in S$ to y. If y is in the interior of L, then there is a t^* such that for $t \geq t^*$, $y \in R^t(z(0))$. In particular if we take $z(0) = y$, then one can return in a finite number of steps to any y in the interior of L. Thus any $z(0) \in S$ can reach y in a finite number of steps. In contrast, suppose $y \notin L$. Take $z(0) \in E$, then for each t

$$y \notin L \supset R^t(E) \supset R^t(z(0)).$$

It is not possible to reach y from $z(0)$. Moreover, if $y \notin L$, then the system cannot return to y. A return would imply $y \in R^{t \cdot n}(y)$ for $n = 1,2,\ldots$. However, $R^{t \cdot n}(y) \to L$, and $y \notin L$. This contradiction shows we cannot return to any $y \notin L$.

In general it is not possible to obtain a characterization of L. The question, "is y in L" cannot be precisely answered. However, $E \subset L$ is explicitly known. If a $z \in L$ is found such that $y \in R^t(z)$ for some t, then $y \in L$.

<u>Example 12</u>: When $n = 3$, it is possible to depict the set E.

Suppose

$$Q = \begin{bmatrix} q_{11} & & \\ q_{21} & q_{22} & \\ & q_{32} & q_{33} \end{bmatrix}$$

and $w > 0$. For Q in this form it is straightforward to show that

$$D = \begin{bmatrix} \frac{1}{(q_{21}+w_1)} & & \\ \frac{q_{21}}{(q_{21}+w_1)(q_{32}+w_2)} & \frac{1}{(q_{32}+w_2)} & \\ \frac{q_{21}\,q_{32}}{(q_{21}+w_1)(q_{32}+w_2)w_3} & \frac{q_{32}}{(q_{32}+w_2)w_3} & \frac{1}{w_3} \end{bmatrix}.$$

Now let $y(k)$ be the point in E which corresponds to the stationary policy of making all appointments into class k, $k = 1,2,3,$. The three points $y(1)$, $y(2)$, and $y(3)$ form the extreme points of the set E and are given by (using (68))

$$y(1) = \begin{bmatrix} q_{32}+w_2)w_3/K_1 \\ q_{21}w_3/K_1 \\ q_{21}q_{32}/K_1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 0 \\ w_3/K_2 \\ q_{32}/K_2 \end{bmatrix} \quad y_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where $K_1 = (q_{32}+w_2)w_3 + (q_{32}+w_3)q_{21}$, and $K_2 = q_{32} + w_3$.

Using the values of Q in example 11, $K_1 = 0.017$, $K_2 = 0.12$ and the extreme points of E are

$$y(1) = \begin{bmatrix} 0.294 \\ 0.588 \\ 0.118 \end{bmatrix} \quad y(2) = \begin{bmatrix} 0 \\ 0.833 \\ 0.167 \end{bmatrix} \quad y(3) = \begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix}.$$

The sets S, E, $R(z)$ and $R^2(z)$ for $z = [\cdot 8, \cdot 1, \cdot 1]$ are illustrated in Figure II.13.

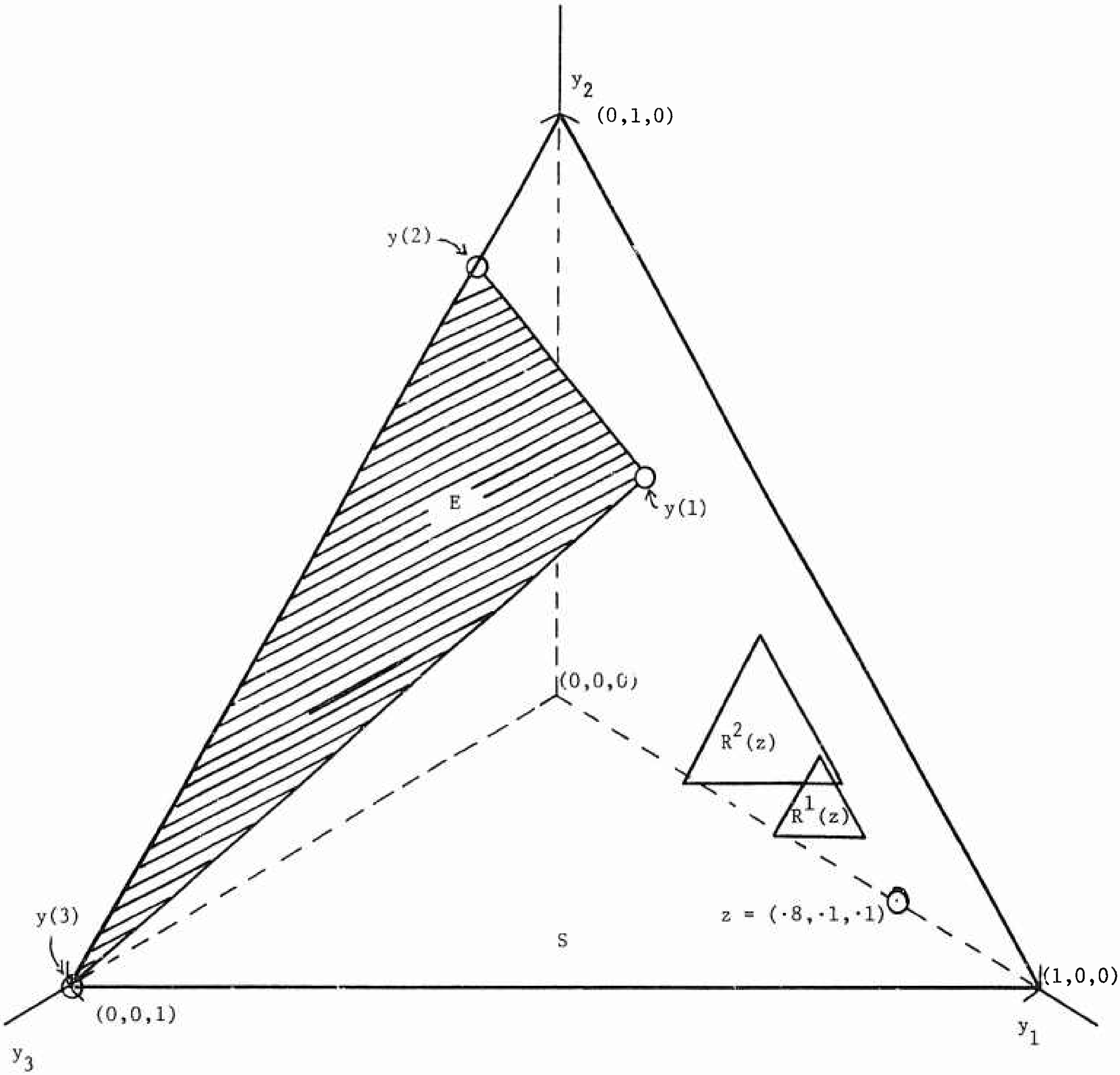

Figure II.13: Illustration of Sets S, E, R(z) and $R^2(z)$ for the 3-class System in Example 11.

15. Notes and Comments.

The fractional flow model has been discussed in numerous papers in the literature, but almost all discussions have been in the context of a Marko chain. An extensive bibliography can be found in Bartholomew [1973], pages 381-402.

The hindsight and foresight models discussed in sections 3 and 4 correspond to "closed" systems as defined in Bartholomew [1973] with (N+1) and N states respectively. The matrix theory referred to in section 6 can be found in numerous places including Debreu and Herstein [1953]. Section 8 is essentially taken from Branchflower [1970]. The reader interested in pursuing the probabilistic interpretation should consult Bartholomew [1973] and Kemeny and Snell [1960].

The data and some of the ideas in section 10 are taken from Hopkins [1974]. The early retirement scheme suggested in this section has been used on a trial basis at Stanford University. Section 11 is based on a report by Marshall, Oliver and Suslow [1970], and in some sense shows the limitations of the cross-sectional model. A close look at the data in this section is the motivation for the longitudinal models discussed in the next report. The vacancy model in section 12 was formulated by White [1970]. These models have been used to forecast flow in several strict hierarchies.

Sections 13 and 14 are more advanced and are intended for those readers more familiar with matrix theory and Markov chain theory. Section 13 owes a great deal to privateconversations with Robert Stanford. Section 14 considers the questionof long-run evolution, which has been investigated by Bartholomew [1969]. Armacost [1970], Toole [1971], Davies [1973] and Grinold and Stanford [1973].

Included in the bibliography are examples which demonstrate the many applications of cross-sectional models in manpower. Young and Almond Rowland and Sovereign, and Vroom and MacCrimmon have applied the model to the distribution of staff and management in an organization. Blumen Kogan and McCarthy have applied it to Labor Mobility. Thonstad has used it as the basis of models for national education and manpower planning as did Armitage, Smith and Alper. Clough and McReynolds, and Marshall, Oliver and Suslow have applied it to student enrollment forecasting in higher education. Charnes, Copper and Niehaus use it as a basis of their models for planning the civilian manpower in the U.S. Department of the Navy.

The report by the Naval Personnel Research Lab [1973] gives a summary of numerous manpower planning models used in the U.S. Armed Forces (with emphasis on the Navy). The basis of many of these models, though often not explicitly stated, is the cross-sectional model with its fractional flow assumptions. Finally, the proceedings of two NATO conferences on Manpower Planning are available in Smith [1971] and Wilson [1969].

Selected Bibliography

Armacost, R. L., An Analysis of the Sets of Rank Distribution in an Hierarchical Organization, M.S. Thesis, Naval Postgraduate School, Monterey, CA (1970).

Armitage, P.H., Smith, C.S., and Alper, P., Decision Models for Educational Planning, Allen Lane, The Penguin Press, London, (1969).

Bartholomew, D.J., A Mathematical Analysis of Structural Control in a Graded Manpower System, Paper P.3. Research Program in University Administration, University of California, Berkeley, CA. (1969).

Bartholomew, D.J., Stochastic Models for Social Processes, 2nd Edition, John Wiley, New York (1973).

Blumen, I., Kogan, M., and McCarthy, P.J., The Industrial Mobility of Labor as a Probability Process, Cornell University Press, Ithaca, N.Y. (1955).

Branchflower, N.H., Analysis of Rank Distribution in a University Faculty, M.S. Thesis, Naval Postgraduate School, Monterey, CA (1970).

Charnes, A., Cooper, W.W., and Niehaus, R. J., Studies in Manpower Planning, Office of Civilian Manpower Management, Dept. of the Navy, Washington, D.C. (1972).

Clough, D.J., and McReynolds, W.P., State Transition Model of an Educational System Incorporating a Constraint Theory of Supply and Demand, Ontario J. Educational Res., Vol. 9, pp 1-18 (1966).

Davies, G.S., "Structural Control in a Graded Manpower System," Man. Sci. (Theory) Vol. 20, pp 76-84 (1973).

Debreu, G. and Herstein, I. N., "Non-negative Square Matrices," Econometrics Vol. 21, pp 597-607 (1953).

Gani, J., "Formulae for Projecting Enrollment and Degrees Awarded in Universities," J. R. Stat. Soc., Ser. A, Vol. 126, pp 400-409 (1963).

Grinold, R.C., and Stanford, R.E., Limiting Distributions in a Linear Fractional Flow Model, Report ORC 73-16, Operations Research Center, University of California, Berkeley, CA (1973).

Hopkins D.S.P.,"Faculty Early Retirement Programs", Oper. Res., Vol. 22, pp 455-467 (1974).

Kemeny, J.G., and Snell, J.L., Finite Markov Chains, D. Von Nostrand, Princeton, N.J. (1960).

Marshall, K. T., and Oliver, R. M., "A Constant-Work Model for Student Attendance and Enrollment," Oper. Res., Vol. 18, pp 193-206 (1970).

Marshall, K.T., Oliver, R.M., and Suslow, S., Undergraduate Enrollment and Attendance Patterns, Report #4, Administrative Studies Project in Higher Education, Office of Institutional Research, University of California, Berkeley (1970).

Rowland, K.M., and Sovereign, M.G.,"Markov-chain Analysis of Internal Manpower Supply," Industrial Relations, Vol. 9, pp 88-99 (1969).

Smith, A.R. (Ed.) Models for Manpower Systems, English University Press, London (1971).

Suslow, S., Langlois, E., Sumariwalla, R., and Walther, C., Student Performance and Attrition and the University of California, Berkeley, Office of Institutional Research, U. of CA, Berkeley, CA (1968).

Thonstad, T. Education and Manpower: Theoretical Models and Empirical Applications, Oliver and Boyd, London (1969).

Toole, P.J., An Appointment, Promotion, and Retirement Model for a Graded Manpower System of Total Fixed Size, Report ORC-71-21, Operations Research Center, University of California, Berkeley (1971).

U.S. Navy Computer Models for Manpower and Personnel Management: State of Current Technology, NAMPS Project Report 73-2, Naval Personnel Research and Development Lab, Washington, D.C. (1973).

Vroom, V. H., and MacCrimmon, K.R., "Towards a Stochastic Model of Managerial Careers," Admin. Sci. Quart., Vol. 13, pp 26-46 (1968).

White, H.C., Chains of Opportunity, Harvard University Press, Cambridge, MA (1970).

Wilson, N.A.B., (Ed.) Manpower Research, English University Press, London (1969).

Young, A., and Almond, G., "Predicting Distributions of Staff," Comp. J., Vol. 3., pp 246-250 (1961).