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This paper considers and unifies two search problems which have been extensively discussed. A class of sequential problems is proposed that includes both. A theorem is proved, under arbitrary partial ordering constraints, characterizing a strategy to minimize the expected cost of a successful search. The main tool is a set of functional equations in strategy space.

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1. Two Important Search Problems

The first search problem considers an object hidden in the kth of n boxes with probability p_k. A search strategy for finding it is a permutation of a subset of the first n integers saying what to do next if the object has not yet been found. Thus (9,2,3,...) is interpreted to mean that box 9 is to be searched first; if the object is not found then box 2 is searched, etc. In this section we consider the simplified model in which a search of a box containing the object is sure to be successful, although this assumption is later relaxed. A search of box k costs c_k if it is unsuccessful and x_k if it is successful.

Thereare at least two kinds of such searches. In a detection search, the goal is to find an object in some search of some box. In a whereabouts search, the goal is to state correctly at the end of a search which box contains an object. This can be accomplished either by finding an object in the search, as in the detection case, or, alternatively by guessing correctly at the end of an unsuccessful search which box contains an object. See Kadane (1971) for a treatment of optimal whereabouts search.

In this paper, the first search problem is to determine a search strategy that includes each of the boxes and minimizes the expected cost of a detection search. An earlier paper (Kadane (1968)), deals with maximizing the probability of a successful detection search spending no more than some budget B (when $x_k \leq c_k$ for all k).

Let $\sigma_i (i=1,2)$ be any two disjoint strategies. Then $\sigma = \sigma_1 \sigma_2$ is a strategy which looks first at the boxes specified by σ_1, in the order specified by σ_1, and then at the boxes specified by σ_2, in the order specified by σ_2, until the object is found or σ is exhausted. Also let σ_k^* be the strategy consisting of a search at box k only.

For any strategy σ, let $X(\sigma)$ be the expected cost of σ, $P(\sigma)$ be the probability that σ is successful and $C(\sigma)$ be the cost of σ if σ is unsuccessful. Then we have the initial conditions

$$(1.1) \quad \begin{cases} X(\sigma_k^*) = p_k x_k + (1 - p_k) c_k \\ C(\sigma_k^*) = c_k \\ P(\sigma_k^*) = p_k \end{cases}$$

and the recurrence relations

$$(1.2) \quad \begin{cases} X(\sigma_1 \sigma_2) = X(\sigma_1) + X(\sigma_2) - P(\sigma_1) C(\sigma_2) \\ C(\sigma_1 \sigma_2) = C(\sigma_1) + C(\sigma_2) \\ P(\sigma_1 \sigma_2) = P(\sigma_1) + P(\sigma_2). \end{cases}$$

The first equation in (1.2) arises because $X(\sigma_1) + X(\sigma_2)$ is the cost of going ahead with σ_2 even if the object was found using σ_1. The probability of its being found in σ_1 is $P(\sigma_1)$ and if it was it is sure not to be found in σ_2, so $C(\sigma_2)$ is the appropriate cost.

For consistency, if Λ is the empty strategy, define

$$(1.3) \qquad C(\Lambda) = P(\Lambda) = X(\Lambda) = 0$$

Using these definitions, C, X, and P are associative and C and P are commutative. Problems of this type are considered by Bellman (1957), Black (1965), Blackwell (n.d., see Matula (1964)), Denby (1967), Greenberg (1964), Kadane (1968), Matula (1964), and Staroverov (1963), among others.

In the second search problem considered here, the event E_k that an object is hidden in the kth of n boxes again has probability p_k, but E_k now is <u>independent</u> of $E_{k'}$, $(k \neq k')$, where in the first problem it was <u>disjoint</u>. Again a strategy, σ is a permutation of a subset of the first n integers specifying the order in which the boxes are attempted until an object is found or σ is exhausted. Again there is a cost c_k for an unsuccessful search of box k, and a cost x_k for a successful one, and once again the problem is to find a search strategy σ that includes all boxes and minimizes the expected cost of the search.

For any strategy σ, let $V(\sigma)$ be the expected cost of using σ and $S(\sigma)$ be the probability that the strategy is <u>not</u> successful in finding the object. Then we have the initial conditions

$$(1.4) \qquad \begin{cases} V(\sigma_k^*) = p_k x_k + (1 - p_k) c_k \\ \\ S(\sigma_k^*) = 1 - p_k \end{cases}$$

and the recurrence relations

$$(1.5)\qquad \begin{cases} V(\sigma_1\sigma_2) = V(\sigma_1) + S(\sigma_1)V(\sigma_2) \\ S(\sigma_1\sigma_2) = S(\sigma_1)S(\sigma_2) \end{cases}$$

For consistency, if Λ is the empty strategy, define

$$(1.6)\qquad V(\Lambda) = 0, \quad S(\Lambda) = 1.$$

Problems of this type are considered by Bellman (1957), Dean (1966), Garey (1973), Joyce (1971), Kadane (1969), Mitten (1960), Simon and Kadane (1975), and Sweat (1970).

2. A Convenient Class of Problems Embracing Both Search Problems

This section owes a large, but not transparent, debt to the paper of Rau (1971). The class of problems proposed below is a proper subset of the class proposed by Rau; these relationships are not pursued in this paper. Suppose that three functions f, F and G are defined on strategies σ_k^* consisting of a single search of box k. Suppose also that f, F and G are extended to arbitrary strategies by the recurrence relations

$$F(\sigma_1\sigma_2) = F(\sigma_1) + F(\sigma_2) + G(\sigma_1)f(\sigma_2) \tag{2.1}$$

$$f(\sigma_1\sigma_2) = f(\sigma_1) + [1 + mG(\sigma_1)]f(\sigma_2) \tag{2.2}$$

$$G(\sigma_1\sigma_2) = G(\sigma_1) + [1 + mG(\sigma_1)]G(\sigma_2) \tag{2.3}$$

where m is a fixed number. For an empty strategy Λ, we take

$$F(\Lambda) = f(\Lambda) = G(\Lambda) = 0 \tag{2.4}$$

First we establish a basic theorem about the system 2.1-3:

<u>Theorem 1</u>

With the above definitions, F, F and G are defined consistently on strings of arbitrary length. In particular

$$F((ab)c) = F(a(bc)) \tag{2.5}$$

$$f((ab)c) = f(a(bc)) \tag{2.6}$$

$$G((ab)c) = G(a(bc)) \tag{2.7}$$

Proof:

$$F((ab)c) = F(ab) + F(c) + G(ab)f(c)$$
$$= F(a) + F(b) + G(a)f(b) + F(c) + f(c)[G(a) + G(b) + mG(a)G(b)]$$

$$F(a(bc)) = F(a) + F(bc) + G(a)f(bc)$$
$$= F(a) + F(b) + F(c) + G(b)f(c) + G(a)[f(b) + [1 + mG(b)]f(c)]$$

thus $F((ab)c) = F(a(bc))$, proving (2.5).

The proofs of (2.6) and (2.7) are similar, and are therefore omitted.

QED

Next we establish that the two search problems of section 1 are special cases of the system 2.1-3.

Theorem 2

(a) When $m = 0$, associating f with $-C$, G with P, and F with X, the system (2.1-3) yields the recurrence relations (1.2).

(b) When $m = -1$, associating F and $-f$ with V, and G with $1-S$ yields a consistent set of recurrence relations identical with (1.5).

Proof:

(a) Let $m = 0$, and make the substitutions indicated. (1.2) is immediate.

(b) Let $m = -1$, and consider (2.3):

$$1 - S(\sigma_1\sigma_2) = 1 - S(\sigma_1) + S(\sigma_1)[1 - S(\sigma_2)]$$
$$= 1 - S(\sigma_1)S(\sigma_2).$$

Now the second equation of (1.5) is immediate.

Next consider (2.1):

$$V(\sigma_1\sigma_2) = V(\sigma_1) + V(\sigma_2) - G(\sigma_1)V(\sigma_2)$$

$$= V(\sigma_1) + S(\sigma_1)V(\sigma_2)$$

which reproduces the first equation of (1.5)

Finally, consider (2.2):

$$- V(\sigma_1\sigma_2) = -V(\sigma_1) - S(\sigma_1)V(\sigma_2)$$

which again reproduces the first equation of (1.5). This shows that the substitutions yield a consistent set of equations identical with (1.5).

QED.

The fact that the first equation of (1.5) has two (identical) generalizations in (2.1) and (2.2) causes no problem in the sequel.

Thus the system (2.1) to (2.3) is a class of sequential problems including both search problems proposed in section 1.

3: Constraints

Reconsider the first search problem of section 1 where now there is a probability $\alpha_{j,k}$ of overlooking the object in the jth search of box k given that it is in box k and has not been found before the jth search of box k. Then the unconditional probability $p_{j,k}$ that the jth search of box k is successful (if it is in the search strategy) satisfies

$$(3.1) \qquad p_{j,k} = p_k(1-\alpha_{j,k}) \prod_{0<j'<j} \alpha_{j',k}.$$

Additionally the jth search of box k can be supposed to cost some amount $c_{j,k}$ if it is unsuccessful and $x_{j,k}$ if it is successful. The notation can be simplified by denoting the jth search of box k by a single index, say i. Thus p_i is the probability of success, c_i the cost if unsuccessful and x_i the cost if successful, of some search. If the object is found in the jth search of box k, it is found in no other search of any box. Hence the events $E_{j,k}$ that the object is found in the jth search of box k are disjoint. In effect this observation allows $\alpha_{j,k} = 0$ without loss of generality, at the cost of introducing a constraint on the optimal strategy. A strategy is called <u>feasible</u> if the jth search of box k is preceeded by the $(j-1)$st search of box k for every k and every $j > 1$. Clearly feasible strategies are the only ones which make sense.

Constraints of this type are called "parallel" because they can be graphed as n parallel lines, one for each box, indicating that the jth search of box k must be preceeded by the j-1st of box k and must precede the $(j+1)$st search of box k.

A similar generalization of the second problem would have the jth search of box k cost $c_{j,k}$ if unsuccessful, $x_{j,k}$ if successful, and have probability $p_{j,k}$ of success. In order for this to be a valid generalization of the second problem, the event $E_{j,k}$ must be independent of $E_{j',k'}$ provided $(j,k) \neq (j',k')$. Again only feasible strategies are interesting. See Kadane (1969) for a discussion.

More generally, suppose S is a set of searches and C is a set of constraints, a subset of $S \times S$. Thus if $c = (s_1, s_2) \in C$, then search s_1 must be conducted before search s_2. The pair (S,C) form a graph. The <u>transitive closure</u> C^* of C is the subset of $S \times S$ such that $(s_1, s\ ) \in C^*$ iff there exist $s_1, s_2, \ldots, s$ such that $(s_1,s_2) \in C$, $(s_2,s_3) \in C, \ldots$. Thus (S,C^*) is again a graph, and has all the constraints implied by C^* and transitivity. If $(s_1,s_2) \in C^*$ then s_1 is a <u>predecessor</u> of s_2 and s_2 is a <u>successor</u> of s_1.

We now restrict the discussion to graphs such that, if $(s_1,s) \in C^*$, there is a finite sequence $(s_1,s_2,\ldots,s_r,s)$ such that $(s_1,s_2) \in C, (s_2,s_3) \in C, \ldots, (s_r,s) \in C$. Notice that in the case of parallel constraints above this restriction is satisfied. A case where it would not be satisfied is where all searches of box 1 had to be completed before any searches of box 2 could be undertaken.

With this restriction, if s_1 is a predecessor of s_2, and no other predecessor of s_2 is a successor of s_1 then s_1 is an <u>immediate predecessor</u> of s_2 and s_2 is an <u>immediate successor</u> of s_1. The immediate graph C^- is formed by $(s_1,s_2) \in C^-$ if s_1 is an immediate successor of s_2.

The case of parallel constraints is then seen to satisfy the restriction that every search has no more than one immediate predecessor and no more than one immediate successor.

A <u>cycle</u> is a sequence of arcs

$$u = (u_1,\ldots,u_q)$$

such that

(1) each arc u_k; $1 < k < q$, has one endpoint in common with u_{k-1} and the other endpoint in common with u_{k+1}.

(2) the same arc does not appear twice

(3) the endpoint u_1 does not share with u_2 is the same as the endpoint u_q does not share with u_{q-1}.

A <u>chain</u> satisfies the first condition above only. Note that in a cycle the endpoint u_k shares with u_{k+1} need not be the successor in u_k. Thus $(s_1,s_2)(s_3,s_2)(s_3,s_1)$ is a cycle.

A <u>connected</u> graph is a graph which contains, for every two nodes x and y, a chain from x to y. Since the relation, $x = y$ or there is a chain from x to y, is an equivalence relation, the equivalence classes divide S into <u>connected components</u>. Finally a <u>tree</u> is a connected graph without cycles, and a <u>forest</u> is a graph without cycles, i.e., a graph whose connected components are trees.

A forest is thus a more general structure than parallel constraints. The theory of sections 4 and 5 applies to an arbitrary graph of constraints on S. However the Garey reduction algorithm of Section 5 applies especially well to finite forests. Further details about graph theory may be found in many books, for example those of Berge (1962,1973).

4: Search Over a Partially Ordered Set

In this section we prove a theorem about the optimal strategy for every member of the class of problems introduced in Section 2 under arbitrary partial ordering constraints. Thus this section generalizes the main result of Simon and Kadane (1975) from the second search problem to the entire class.

Before stating and proving the theorem, a few lemmas are necessary. Let $S(a) = 1 + mG(a)$, where G is defined in 2.3

Lemma 1 $G(ab) = G(ba)$ and $S(ab) = S(a)S(b)$

Proof

$$\begin{aligned} G(ab) &= G(a) + [1 + mG(a)]G(b) \\ &= G(a) + G(b) + mG(a)G(b) \\ &= G(b) + [1 + mG(b)]G(a) \\ &= G(ba) \end{aligned}$$

$$\begin{aligned} S(ab) = 1 + mG(ab) = 1 + mG(a) + m(Gb) + m^2G(a)G(b) &= [1 + mG(a)]\ [1 + mG(b)] \\ &= S(a)S(b). \end{aligned}$$

Q.E.D.

Lemma 2: Then

$$F(abcd) - F(acbd) = [1 + mG(a)]\ \{f(c)G(b) - f(b)G(c)\}$$

Proof:

$$\begin{aligned} F(abcd) - F(acbd) &= F(abc) + F(d) + G(abc)f(d) - F(acb) - F(d) - G(acb)f(d) \\ &= F(abc) - F(acb) \\ &= F(a) + F(bc) + G(a)f(bc) - F(a) - F(cb) - G(a)f(cb) \end{aligned}$$

$$= F(bc) - F(cb) + G(a)[f(bc) - f(cb)]$$

$$= F(b) + F(c) + G(b)f(c) - F(b) - F(c) - G(c)f(b)$$

$$+ G(a)[f(b) + f(c) + mG(b)f(c) - f(b) - f(c) - mG(c)f(b)]$$

$$= G(b)f(c) - G(c)f(b) + mG(a)[G(b)f(c) - G(c)f(b)]$$

$$= [1 + mG(a)][G(b)f(c) - G(c)f(b)]$$

Q.E.D.

We seek to minimize F over strategies. Let $\phi(a) = f(a)/G(a)$, where a is a strategy.

A strategy on a set of nodes T is any ordering of the nodes of T that satisfies the order constraints on those nodes. Let A and B be two mutually exclusive sets of nodes, and C their set sum. Then A and B are interchangeable iff there exist a strategy $c = (ab)$ and a strategy $c' = (b'a')$ where c and c' are strategies on C, a and a' strategies on A and b and b' strategies on B. If A and B are interchangeable, if a is any strategy of A and if b is any strategy of B, then (ab) and (ba) are strategies of C.

<u>Theorem 3</u>

Suppose $G(a) > 0$ and $1 + mG(a) > 0$ for all a. If b and c are interchangeable in $(abcd)$ and if $\phi(c) > \phi(b)$, then $(abcd)$ can be improved by interchanging b and c, and hence is not optimal.

Proof:

Using Lemma 2,

$$F(abcd) - F(acbd) = [1 + mG(a)]\{f(c)G(b) - f(b)G(c)\}$$

$$= [1 + mG(a)]G(b)G(c)[\phi(c) - \phi(b)] > 0$$

Q.E.D.

Let A be a partially ordered set of nodes, and let it contain B and $C = A - B$. Iff there exist strategies b on B and c on C such that $a = (bc)$ is a strategy on A, then B is an initial subset of A and C is a terminal subset of A.

A strategy of a set of nodes D for which ϕ assumes its greatest value over strategies on D is called the best strategy on D and is designated $t(D)$. An initial set, D of the set T for which $\phi(t(D))$ is maximal over all initial sets of T is called a best set of T.

<u>Theorem 4</u>:

Suppose D is a best set of T. Suppose σ is an arbitrary strategy n T having the form eh where e and h are strategies on the non-overlapping sets E and H. E and H may be chosen without loss of generality so that

(i) $D \subseteq E$

(ii) The last element of e is a member of D. Thus e consists of $t(D)$ possibly interspersed with nodes of $T - D$, and the last element of e is a node in D.

If e contains any nodes not belonging to D, then e can be improved (weakly) by moving these "intruding" nodes beyond the last node of D, that is, by bringing the nodes of D to the front of e with the remaining nodes of e following them.

If D is contained in no best set of T and $F(eh) < \infty$ then the improvement above is strict.

<u>Lemma 3</u> Let $\underline{A}$ and $\underline{A \cup B}$ be initial sets of $\underline{T}$ such that:

(1) $\underline{A}$ is the best set of $\underline{T}$, with best strategy $\check{\underline{t}}(\underline{\bar{A}}) = \underline{a}$ and

(2) $\underline{b}$ is any strategy for $\underline{B}$.

Then $\phi' \geq \phi(\underline{b})$.

If $A \cup B$ is not a best set of T, then $\phi' = \phi(a) > \phi(ab)$

<u>Proof</u>:

Since $\underline{A}$ is the best set of $\underline{T}$, $\phi' = \phi(a) \geq \phi(ab)$.

If $A \cup B$ is not a best set of T, then $\bar{\phi} = \phi(a) > \phi(ab)$

Now

$$\begin{aligned}\phi(ab)G(ab) &= r(ab)\\ &= r(a) + [1 + mG(a)]r(b)\\ &= \phi(a)G(a) + [1 + mG(a)]\phi(b)G(b).\end{aligned}$$

But $G(ab)\ \phi(a) \geq G(ab)\phi(ab)$.

Thus

$$G(ab)\ \phi(a) \geq \phi(a)G(a) + [1 + mG(a)]\phi(b)G(b).$$

Expanding the left-hand side,

$$(G(a) + [1 + mG(a)]G(b))\ \phi(a) \geq \phi(a)G(a) + [1 + mG(a)]\ \phi(b)G(b).$$

i.e., $\phi(a) \geq \phi(b)$.

If $A \cup B$ is not a best set of T, then $G(ab)\phi(a) > G(ab)\phi(ab)$ implies $\phi(a) > \phi(b)$ by the same argument.

Q.E.D.

Lemma 4: Let $\underline{A}$ be a set consisting of the mutually exclusive subsets of nodes $\underline{B}$, $\underline{C}$, and $\underline{D}$, where $\underline{B}$ is an initial bloc of $\underline{A}$, while $\underline{C}$ and $\underline{D}$ are interchangeable, hence also both terminal blocs. Let the best strategy, $\check{t}(A)$ be:

$$\check{t}(A) = (b, c_1, d_1, \ldots, c_k d_k).$$

where $\underline{b}$ is a strategy for $\underline{B}$, $\underline{c} = (\underline{c}_1, \ldots, \underline{c}_k)$ is a strategy for $\underline{C}$ and $\underline{d} = (\underline{d}_1, \ldots, \underline{d}_k)$ is a strategy for D. Then $\phi(c_1) \geq \phi(d_1) \geq \ldots \geq \phi(\underline{c}_k) \geq \phi(\underline{d}_k)$.

Proof:

Suppose $\phi(d_i) < (c_{i+1})$. Then by Lemma 2, $\underline{\check{t}}(\underline{A})$ could be improved by exchanging $\underline{d}_i$ and $\underline{c}_{i+i}$, contrary to the hypothesis that $\phi(\underline{A})$ is maximal. But the exchange is admissible, since $\underline{C}$ and $\underline{D}$ are exchangeable. Similarly the supposition that $\phi(c_i) < \phi(d_i)$ leads to a contradiction.

Q.E.D.

Lemma 5: Given $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ as in Lemma 4, with $\underline{c} = (\underline{c}_1, \ldots, \underline{c}_k)$ and $\underline{d} = (\underline{d}_1, \ldots, d_k)$, suppose that A is a best set of T, so that $\check{\phi}(\underline{A}) = \phi'$. Then $\phi(d) \geq \check{\phi}(A)$, and therefore $\check{\phi}(\underline{D}) \geq \check{\phi}(A) = \phi'$.

Proof:

Define
$$e = (\underline{b}\underline{c}_1\underline{d}_1 \ldots \underline{c}_k).$$

Then

$$\phi' = \phi(ed_k) = \frac{f(ed_k)}{G(ed_k)} = \frac{f(e) + S(e)f(d_k)}{G(e) + S(e)G(d_k)}.$$

If $\phi(d_k) = f(\underline{d}_k)/G(\underline{d}_k) < \phi'$, then $\phi(e) > \phi'$. But $\underline{e}$ is a strategy for an initial bloc of $\underline{A}$, and also of $\underline{T}$. Since ϕ' is maximal over all such blocs, the inequality is a contradiction. Therefore $\phi(d_k) \geq \phi'$.

By Lemma 4,

$$\phi(\underline{d}_k) \leq \phi(\underline{c}_k) \leq \cdots \geq \phi(s_1) \geq \phi(c_1).$$

Then

$$\phi(d) = \frac{f(d)}{G(d)} = \frac{f(d_1) + S(d_1)f(d_2) + \cdots + S(d_1)\cdots S(d_{k-1})f(d_k)}{G(d_1) + S(d_1)f(d_2) + \cdots + S(d_1)\cdots S(d_{k-1})G(d_k)}$$

$$= \frac{\phi(d_1)G(d_1) + S(d_1)G(d_2)\phi(d_2) + \cdots + S(d_1)\cdots S(d_{k-1})G(d_k)\phi(d_k)}{G(d_1) + S(d_1)G(d_2) + \cdots + S(d_1)\cdots S(d_{k-1})G(d_k)}$$

$$\geq \phi'.$$

Q.E.D.

<u>Lemma 6</u>: Let $(a\ c\ d)$ and $(ac'd)$ be strategies over the same set of nodes. Then

$$F(a\ c\ d) \geq F(a\ c'\ d).$$

if c' is optimally ordered according to ϕ.

Proof.

By repeated application of Theorem 2.

Q.E.D.

We now proceed to the proof of Theorem 3, by introducing some new notation. Suppose $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_r)$ are strategies. Then let

$$B_i = (b_1 \ldots b_i)$$

$$C_i = (b_1 a_1 b_2 a_2 \ldots b_i a_i)$$

$$B_i^* = (b_{i+1} \ldots b_r)$$

$$C_i^* = (b_{i+1} a_{i+1} \ldots b_r a_r)$$

$$\check{A}_i = \text{the best permutation of } (a_1 \ldots a_i)$$

$$A_i = (\check{A}_{i-1} a_i)$$

$$\check{A}_i^* = \text{the best permutation of } (a_{i+1} \ldots a_r)$$

$$A_i^* = (a_{i+1} \check{A}_{i-1}^*)$$

For consistency define

$$C_0 = C_r^* = A_0 = A_r^* = B_0 = B_r^* = \Lambda, \quad \text{the null strategy.}$$

Designate $t(D)$ by A_r, and e by C_r, so that B_r is a strategy on the intruding nodes. The strategy on T asserted by Theorem 3 to be an improvement over eh is then $A_r B_r h$. If $F(eh) = \infty$ there is nothing to prove. Thus to prove the theorem we suppose $F(eh) = F(C_r h) < \infty$ and must show

$$F(C_r h) - F(A_r B_r h) \geq 0$$

Note that <u>a</u>'s may be moved forward, interchanging them with b's, since D is an initial bloc of T.

$$F(C_r h) - F(A_r B_r h) =$$
$$F(C_r) + F(h) + G(C_r) f(h) - F(A_r B_r) - F(h) - G(A_r B_r) f(b).$$

Now since the set of C_r is the same as the set of A_rB_r, and in view of Lemma 1,

$$G(C_r) = G(A_rB_r)$$

Then

$$F(C_rh) - F(A_rB_rh) = F(C_r) - F(A_rB_r).$$

Now $A_rB_r = A_0^*B_0^*$. Also, by definition of D, $\check{A}_0^* = A_0^*$.

Then

$$F(C_r) - F(A_rB_r) = F(C_r) - F(A_0^*B_0^*)$$

$$= \sum_{i=1}^{r} [F(C_i\check{A}_i^*B_i^*) - F(C_{i-1}\check{A}_{i-1}^*B_{i-1}^*)].$$

Considering the individual terms of the summation, we have:

$$F(C_i\check{A}_i^*B_i^*) - F(C_{i-1}\check{A}_{i-1}^*B_{i-1}^*)$$

$$= F(C_{i-1}b_iA_{i-1}^*B_i^*) - F(C_{i-1}\check{A}_{i-1}^*b_iB_i^*)$$

$$= F(C_{i-1}b_iA_{i-1}^*B_i^*) - F(C_{i-1}b_i\check{A}_{i-1}^*B_i^*)$$

$$+ F(C_{i-1}b_i\check{A}_{i-1}^*B_i^*) - F(C_{i-1}\check{A}_{i-1}^*b_iB_i^*)$$

But, by Lemma 6, $F(C_{i-1}b_iA_{i-1}^*B_i^*) - F(C_{i-1}b_i\check{A}_{i-1}^*B_i^*) \geq 0$.

Therefore,

$$F(C_i\check{A}_i^*B_i^*) - F(C_i\check{A}_{i-1}^*B_{i-1}^*)$$

$$\geq F(C_{i-1}b_i\check{A}_{i-1}^*B_i^*) - F(C_{i-1}\check{A}_{i-1}^*b_iB_i^*).$$

Applying Lemma 2, with $\underline{C}_{i-1} = \underline{a}$,

$$\underline{b}_i = \underline{b},\ \underline{\check{A}}^*_{i-1} = \underline{c},\ \underline{B}^*_i = \underline{d},$$

we get,

$$F(C_i\check{A}^*_iB^*_i) - F(C_i\check{A}^*_{i-1}B^*_{i-1})$$

$$\geq S(C_{i-1})[F(\check{A}^*_{i-1})G(b_i) - F(b_i)G(\check{A}^*_{i-1})], \quad \text{whence}$$

$$F(C_r) - F(A^*_0B^*_0) \geq \sum_{i=1}^{r} S(C_{i-1})G(b_i)G(\check{A}^*_{i-1})(\phi(\check{A}^*_{i-1}) - \phi(b_i))$$

$$= T_1 - T_2$$

where,

$$T_1 = \sum_{i=1}^{r} S(C_{i-1})G(b_i)G(\check{A}^*_{i-1})\phi(\check{A}^*_{i-1})$$

$$T_2 = \sum_{i=1}^{r} S(C_{i-1})G(b_i)G(\check{A}^*_{i-1})\phi(b_i)$$

Consider $\underline{T}_1$. $\underline{\check{A}}_{i-1}$ is the best strategy of a terminal bloc of $\underline{D}'$. Hence, by Lemma 5, $\phi(\underline{\check{A}}^*_{i-1}) \geq \phi'$, so that

$$T_1 \geq \sum_{i=1}^{r} S(C_{i-1})G(b_i)G(\check{A}^*_{i-1})\phi'$$

Next, consider $\underline{T}_2$. Factoring $\underline{S}(\underline{C}_{i-1}) = \underline{S}(\underline{B}_{i-1})\underline{S}(\underline{A}_{i-1})$,

above, we obtain,

$$T_2 = \sum_{i=1}^{r} S(B_{i-1})G(b_i)\phi(b_i)S(A_{i-1})G(\check{A}^*_{i-1})$$

Since $\underline{G}(\underline{A}_r^*) = 0$, we have the identity:

$$S(A_{j-1})G(\check{A}^*_{i-1}) = \sum_{j=1}^{i} Z_j, \quad \text{where}$$

$$Z_j = S(A_{j-1})\, G(\check{A}^*_{j-1}) - S(A_j)G(\check{A}^*_j)$$

Using this equation in the previous one, and then changing the order of summation, we find,

$$T_2 = \sum_{i=1}^{r} \sum_{j=1}^{r} S(B_{i-1})G(b_i)\phi(b_i)Z_j$$

$$= \sum_{j=1}^{r} Z_j \sum_{i=1}^{j} S(B_{i-1})f(b_i)$$

$$= \sum_{j=1}^{r} Z_j f(B_j) = \sum_{j=1}^{r} Z_j G(B_j)\phi(B_j)$$

But $\underline{B}_j$ satisfies the conditions of $\underline{B}$ of Lemma 5 with $\check{\underline{A}}_r$ as $\underline{A}$.

Therefore, by that lemma, $\phi(\underline{B}_j) \leq \phi'$. Hence,

$$T_2 \leq [\sum_{j=1}^{r} Z_j G(B_j)]\phi'$$

$$\leq [\sum_{j=1}^{r} Z_j \sum_{i=1}^{j} S(B_{i-1})G(b_i)]\phi'$$

$$\leq [\sum_{i=1}^{r} S(B_{i-1}) \sum_{j=1}^{r} Z_j\, G(b_i)]\phi'$$

$$\leq [\sum_{i=1}^{r} S(B_{i-1})G(b_i)S(A_{i-1})G(\overset{\vee}{A}{}^*_{i-1})\phi'$$

$$\leq [\sum_{i=1}^{r} S(C_{i-1})G(b_i)G(\overset{\vee}{A}{}^*_{i-1})\phi'$$

Combining, we have finally:

$$V(C_r) - V(A_0^*B_0^*) \geq T_1 - T_2$$

$$\geq \sum_{i=1}^{r} [S(C_{i-1})G(b_i)G(\overset{\vee}{A}{}^*_{i-1})](\phi' - \phi') = 0.$$

This is the result we want, and the first statement in the theorem is proved.

If D is contained in no best set of T, then Lemma 3 implies $\phi(B_j) < \phi'$. This in turn implies

$$T_2 < \sum_{i=1}^{r} S(C_{i-1})G(b_i)G(\overset{\vee}{A}{}^*_{i-1})\phi', \quad \text{and hence}$$

$V(C_r) - V(A_0^*B_0^*) \geq T_1 - T_2 > 0$, which concludes the proof of theorem 4.

Q.E.D.

Theorem 4 implies the following structure for an optimal strategy: Let B_1 be a best bloc of T, and σ_1 a best strategy of B_1. Let B_2 be a best bloc of $T - B_1$, and σ_2 the best strategy of $B_2, \ldots$. Then $\sigma = (\sigma_1\sigma_2\ldots)$. Note that $\phi(B_1) \geq \phi(B_2) \geq \cdots$, by construction.

<u>Corollary 1</u>. If restrictions of the type "b_i precedes b_j" where $b_i \in B_i$, $b_j \in B_j$ and $i < j$ are added to the problem, σ is still optimal.

Proof: Let σ_R be a best strategy available in the more restricted problem. Since σ_R is the best strategy in a more restricted problem, $F(\sigma) \leq F(\sigma_R)$. However since σ satisfies the added restrictions, it is also true that $F(\sigma) \geq F(\sigma_R)$. Then $F(\sigma) = F(\sigma_R)$ and σ is optimal.

Q.E.D.

Corollary 2: If restrictions of the type above are removed, σ is still optimal.

Proof: Immediate from Corollary 1.

Q.E.D.

5. Garey Reduction Theorems

In a recent paper Garey (1973) gives some theorems and an algorithm that reduces every problem that has a partial ordering restriction, and finds an optimal strategy for problems where the partial ordering graph C has an immediate graph C^- that forms a forest. Garey's results were proved for the second example of section 1. The purpose of this section is to show that Garey's reduction theorems and reduction algorithm apply to the whole class of problems developed in section 2.

A search is called <u>terminal</u> iff it has no successors, and <u>initial</u> if it has no predecessors. A search s is a <u>maximal successor</u> of search s_1 iff it is an immediate successor of s_1 and satisfies, if s' is any immediate successor of s_1, $\phi(s) \geq \phi(s')$. [For readers comparing this treatment with Garey's, note that Garey's R satisfies $R(s) = -\phi(s)$.] A search is a <u>minimal predecessor</u> of search s_1 iff it is an immediate predecessor and satisfies, if s' is any immediate predecessor of s_1, $\phi(s') \geq \phi(s)$.

Theorem 5: For any problem of the class considered here that has an optimal strategy, let t_i be a nonterminal search having only terminal successors. If t_j is a maximal successor of t_i satisfying $\phi(t_j) \geq \phi(t_i)$ and t_j has no other immediate predecessors, then there is an optimal solution in which the subsequence $t_i t_j$ occurs.

Proof:

Let σ be an optimal strategy. Of necessity t_i occurs somewhere in σ, and each of the successors of t_i, say $t_1^0 t_2^i, \ldots,$ including t_j occur in σ after t_i. Let, without loss of generality

$$\sigma = a_0 t_i a_1 t_1^i a_2 t_2^i \; . \; . \; . \; a_r t_j a_{r+1}$$

where every a_k except possibly a_{r+1}, contains no successor of t_i. Then every non-empty a_k, except a_{r+1} and a_0 is interchangeable with t_{k-1}^i and t_k^i. Therefore, using the optimality of σ if a_k is non-empty $\phi(t_{k-1}^i) \geq \phi(a_k) \geq \phi(t_k^i)$. If a_k is empty, t_{k-1}^i and t_k^i are exchangeable and again by the optimality of σ $\phi(t_{k-1}^i) \geq \phi(t_k^i)$.

Therefore

$$\phi(t_1^i) \geq \phi(a_2) \geq \phi(t_2^i) \geq \ldots \geq \phi(t_j)$$

(where empty a_i's can be dropped from the above string of inequalities).

Since t_j is maximal among successors to t_i,

$$\phi(t_j) \geq \phi(t_1^i),$$

so equality obtains throughout the above expression.

$$\sigma' = a_0 t_i a_1 t_j t_1^i a_2 t_2^i \ldots a_r a_{r+1}$$

is a strategy, and Theorem 2 implies

$$V(\sigma) = V(\sigma').$$

Now if a_1 is empty the theorem is proved. If not, it is exchangeable with both t_i and t_j. Then

$$\phi(t_i) \geq \phi(a_1) \geq \phi(t_j)$$

by Theorem 2. Now $\phi(t_j) \geq \phi(t_i)$ by assumption, so equality obtains in the above. Hence

$$V(\sigma) = V(a_0 a_1 t_i t_j t_1^i a_2 t_2^i . . . a_r a_{r+1})$$

and the theorem is proved by optimality of σ.

Q.E.D.

Theorem 6: Let t_j be a terminal search having an immediate predecessor t_i such that $\phi(t_i) > \phi(t_j)$. Consider the modified problem which is identical to the given problem except that the constraint graph C of the modified problem is formed from the original constraint graph by replacing the constraint from t_i to t_j by a constraint from each immediate predecessor of t_i to t_j. Then every optimal solution to the modified problem is also an optimal solution to the original problem.

Proof:

Let σ be an optimal solution to the modified problem. Suppose that t_j preceeds t_i in σ. Then we can write $\sigma = (a_0 t_j a_1 t_i a_2)$, where a_i's may be empty for i=0,1,2. Suppose a_1 is not empty. All predecessors of t_j must be in a_0 since σ is a solution. Hence all predecessors of t_i are in a_0, also. Finally, since t_j is terminal, all predecessors of a_1 are in a_0. Hence t_j and a_1 are interchangeable, and a_1 and t_i are interchangeable. Then

$$\phi(t_j) \geq \phi(a_1) \geq \phi(t_i).$$

If a_1 is empty, t_j and t_i are interchangable, leading to

$$\phi(t_j) \geq \phi(t_i)$$

by the optimality of σ. But these inequalities are impossible by the assumption of the theorem that $\phi(t_j) < \phi(t_i)$. Hence t_i precedes t_j in σ. Hence σ is also a solution to the original problem.

Let σ_R be an optimal solution to the original problem. Since the original problem is the more restricted,

$$V(\sigma_R) \geq V(\sigma).$$

Then $V(\sigma) = V(\sigma_R)$ and σ is optimal for the original problem.

Q.E.D.

The following theorems are duals to theorems 5 and 6.

Theorem 7: For any problem of the class considered here which has an optimal strategy, let t_j be a non-initial task having only initial predecessors. If t_i is a minimal predecessor of t_j satisfying $\phi(t_i) \leq \phi(t_j)$ and t_i has no other immediate successors, then there is an optimal strategy in which the strategy $t_i t_j$ occurs.

Theorem 8: Let t_j be a terminal search having an immediate predecessor t_i such that $\phi(t_i) > \phi(t_j)$. Consider the modified problem which is identical to the given problem except that the constraint graph C of the modified problem is formed from the original constraint graph by replacing the constraint that t_i precede t_j by constraints that t_i precede each immediate successor of t_j. Then every optimal solution for the modified problem is also a solution to the original problem.

The proofs of Theorems 7 and 8 are the same as those of Theorems 5 and 6, respectively, with the sense of each constraint reversed, each inequality reversed, and each strategy reversed.

Garey then proposes the following reduction algorithm.

Algorithm 1.

Step (a) Select a connected component, containing at least one constraint from the current reduced precedence graph. If none exists, go to step (i).

Step (b) Depending upon whether the component under consideration has no multiple immediate predecessors or no multiple immediate successors, go to either step (c) or step (f), respectively.

Step (c) Choose any nonterminal task t_i', having only terminal immediate successors, from the current reduced version of the component under consideration. If no such task exists, go to step (a), having completely reduced the chosen component

Step (d) Find a maximal successor t_j' of t_i'. If $\phi(t_j') < \phi(t_i')$, go to step (e). Otherwise reduce the component by deleting t_j' and the constraint from t_i' to t_j', and replace t_i' by a new strategy $[t_i', t_j']$. If the new task is terminal, go to step (c). Otherwise repeat step (d).

Step (e) For each immediate successor t_k' of t_i', replace the constraint t_i' to t_k' by a constraint from the immediate predecessor of t_i' to t_k'. Go to step (c).

Step (f) Choose any noninitial task t'_j, having only initial immediate predecessors, from the current reduced version of the component under consideration. If no such task exists, go to step(a), having completely reduced the chosen component.

Step (g) Find a minimal predecessor t'_i of t'_j. If $\phi(t'_j) < \phi(t'_i)$, go to step (h). Otherwise reduce the component by deleting t'_i and the constraint from t'_i to t'_j by a new strategy $[t'_i, t'_j]$. If the new strategy is initial go to step (f). Otherwise repeat step (g) with $[t'_i, t'_j]$ acting as the strategy t'_j.

Step (h) For each immediate predecessor t'_k of t'_j, replace the arc from t'_k to t'_j by an arc from t'_k to the immediate successor of t'_j. Go to step (f).

Step (i) Let $t'_1, t'_2, \ldots, t'_m$ denote the remaining strategies in the completely reduced precedence graph. Order them as $t'_{k_1}, t'_{k_2}, \ldots, t'_{k_m}$ so that $\phi(t'_{k_i}) \geq \phi(t'_{k_{i+1}})$, for all i, $1 \leq i \leq m-1$. Removing the brackets from this sequence results in an optimal solution to the original problem.

Garey's algorithm completely reduces a forest, and will be of benefit in an arbitrary partially ordered graph. Garey concludes his investigation by saying that in the partially-ordered case "the proper choice may depend somehow on the overall likelihood of success for the complete set of tasks or certain large subsets thereof, a non-local property which may be difficult to use in an efficient algorithm." (Garey, 1973, p. 55). We believe that Theorem 4 is the non-local theorem sought by Garey. It's efficient use in algorithms depends on the exploitation of special structure to reduce the number of sets over which the best set of T must be searched for.

imposed in Kadane (1968), that $\frac{p_{ij}}{c_{ij}}$ be non-increasing in j for each i, is thus seen to be the condition that each best bloc consist of only a single element.

In this sense Theorem 4 generalizes the result proposed, but not proved, in Kadane (1968).

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