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APPROXIMATE PROBABILITY DISTRIBUTIONS FOR THE EXTREME SPREAD

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The extreme spread, or greatest distance between all pairs of points on a target, is often used as a rapid measure of dispersion or precision of shot groups on a target. It is therefore desirable to know its statistical properties. Since the exact theoretical distribution has not yet been worked out, this report examines the accuracy of several approximations which are checked against large sample monte carlo values.

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I. INTRODUCTION AND SUMMARY

In ballistics work, and especially in small arms firings at targets to determine accuracy, it is quite natural to measure the closeness of a group of shots by the "extreme spread", or the greatest distance between any two shots of a group. Moreover, the extreme spread may be determined quickly with a ruler and does not require any detailed or involved computation, as does the round-to-round standard deviation in each direction, or the mean radius of the shots of a group, for example. It is for these reasons that ballisticians, riflemen and others have long had a great interest in the extreme spread, for it is truly the most rapid measure of dispersion of shots on a target. Sometimes the extreme spread, or the maximum distance between pairs of points on the target, is called the "group diameter", but there is a very subtle difference between the two when one delves into the general problem on a statistical basis. We do not intend to cover all the pertinent details relating to the statistical analysis of patterns of shots on a target here, but interested readers might well study the booklet of Grubbs [1964]. Rather, we intend to develop in this paper the properties of the extreme spread more extensively than has been done in the past, and thereby contribute to an improved understanding of the statistical characteristics of the probability distribution of the extreme spread, which is required in any first-class or overall analysis of target accuracy studies.

In our introduction of the subject, we point out that the extreme spread is a random variable which follows some kind of statistical or probability distribution. Indeed, the amount of random variation from one group of shots to another depends markedly on the sample size, or the number of shots in a group, and the underlying unknown, population round-to-round standard deviation, which we will call σ. The population standard deviation, σ, is a one-directional or "linear" quantity, say for the x or horizontal direction (as well as the y or vertical direction), and for a very large number of shots it may be found as the square root of the sum of squares of deviations in the x-direction from the mean divided by the number of rounds. In rifle firing, and in many other types of weapon studies, the population standard deviations in the two directions are equal or very nearly so. Hence, it may be assumed in our following analysis that $\sigma_x = \sigma_y = \sigma$. The exact theoretical probability distribution of the extreme spread, or bivariate range, as it is often called, has not been determined as of this date,

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although many of the key properties of the distribution are fairly accurately known from previous studies. In the following, we report on the results of a Monte Carlo type of computer simulation, along with the necessary statistical analyses, to find approximate statistical distributions which for all the practical purposes result in the degree of accuracy needed to round out sufficiently our understanding of properties of the distribution of the extreme spread, at least for the very important practical cases involving small sample sizes of predominant interest.

Our acknowledgements must go to Mr. Philip G. Rust, retired industrialist of the Winnstead Plantation, Thomasville, Georgia, for his great interest in critical analyses of accuracy firings of rifles, which provided much of the motivation for this investigation, as well as for the booklet by Grubbs [1964], which are of importance to ballistic analyses generally.

II. SOME ANALYTICAL PRELIMINARIES

Consider a random sample $X_1, \ldots, X_n$ $(X_i = (x_i, y_i))$ from a bivariate normal distribution with probability density function (p.d.f.) given by

$$f(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}; \qquad (\sigma_x=\sigma_y=\sigma). \qquad (1)$$

The extreme spread (ES), or bivariate range, is defined as $ES = \max_{i,j} |X_i - X_j|$. This ES is of course a random variable, as previously pointed out, and we seek its probability distribution, realizing that it will be dependent upon the sample size n. For the case n=2, for example, we have

$ES = |X_1 - X_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = 2\chi\sigma$, where the random variate chi has two degrees of freedom. In this case, the mean value of the extreme spread, $E(ES) = 1.77245\sigma$, and the variance $V(ES) = (.9265\sigma)^2$. This result does not have a direct extension to higher sample sizes, $(n>2)$ however, and the distribution of the extreme spread ES has not been determined analytically.

Some earlier work of Wilks and Grubbs in the last reference [1964] have led to Monte Carlo estimates of the first four moments of the three-dimensional or trivariate range, the trivariate midrange, the extreme spread or bivariate range, and the bivariate midrange. Cacoullos and DeCicco [1967] have investigated two approximations of the ES distribution based on the Monte Carlo moment data of Wilks and Grubbs. In this paper we present improved Monte Carlo moment estimates (by virtue of greatly increased sample size), as well as quantile estimates, or percentage points, which were not available previously and propose some approximate distributions, which will suffice for most analytical studies in practice. The large-sample moments tabulated below include the mean μ, standard deviation σ, the skewness measure α_3, and "kurtosis" or peakedness measure α_4. (See any standard textbook on statistics for further definitions and formulas.)

III. MONTE CARLO RESULTS

The moments (Table 1) and various quantiles of interest (Table 2) are based upon 10^4 Monte Carlo samples of the random variable ES for each value of sample size, n. The value of n specifies the number of points sampled from the circular normal distribution to determine a single value of ES.

The quantiles of Table 2 are, of course, subject to standard definitions and interpretations. For example, for a sample size of n = 7 and known population standard deviation σ, the lower 1% point is 1.842σ and therefore in random sampling from a bivariate normal population with standard deviation σ we would expect that only 1% of the extreme spreads for a sample of size seven would fall below 1.842σ. Similarly, for the 99% point, $P_{.9900}$, (or upper 1% significance level) we would expect only 1% of the extreme spreads for sample size n = 10 to exceed 5.75σ. The mean value of the extreme spread for a sample of size 10 from Table 1 is 3.813σ.

IV. APPROXIMATE PROBABILITY DENSITIES

Candidates for approximating the probability distribution of the extreme spread were chosen from three well known families of distributions, specifically, the chi distribution, the lognormal distribution, and the two-parameter Weibull distribution. Some particulars of our findings are detailed in the following paragraphs.

Chi Distribution Approximation

In considering the Chi distribution, we made use of the fact that $2mw_n^2/v$ is approximately distributed as χ^2 with $\nu = \frac{2m^2}{v}$ degrees of freedom (see, for example, Grubbs et al [1966]), where w_n is the sample range for random samples of size n from a univariate normal population. In our notation, $m = E(w_n^2)$ and $v = V(w_n^2)$. We used this same type of approximation for the extreme spread ES and interpolated linearly to evaluate chi-square for fractional degrees of freedom. Using the sample moments of ES to estimate m and v, this family provided a rather good fit to the sample quantiles over the entire range of n considered, although the dimension of the sample space has increased from one to two. The results are summarized in Table 3 where the italicized value is the fitted value juxtaposed to the Monte Carlo quantile estimate.

Lognormal Distribution Approximation

The lognormal distribution provides an excellent fit for large values of n. Following recommendations of Aitchison and Brown [1966], we used the method of quantiles (specifically, the 10th and 90th percentiles) for purposes of estimating μ and σ of the associated normal distribution. Possibly, a different choice of quantiles might lead to a better fit for small n; however, a summary of results for the larger values of n is included as Table 4 with the same format as Table 3.

Weibull Distribution Approximation

In fitting a two parameter Weibull distribution, $F(x) = 1-e^{-(x/\alpha)^\beta}$, to our data, we were precluded from obtaining maximum likelihood estimates since we have at our disposal only the Monte Carlo moments of the distribution and not the individual sample values. In lieu of these, we used a moment estimate suggested by Cohen [1965], but with somewhat less than satisfying results. In fact, both the chi approximation and the lognormal seem preferable to the two parameter Weibull, although a three parameter Weibull with the introduction of a location parameter would probably offer some improvement. We felt that this exceeded our charter of consideration of a few commonly encountered distributions however, and postponed any further inquiry for a later date, especially since the chi and lognormal distributions gave very satisfactory results.

Data Summary and Example of Fitting Procedure

The chi variate is less descriptive in terms of absolute difference between the Monte Carlo and fitted value as we go further out in the tails of the distribution and the parameter n increases. As a matter of fact, the upper tail (perhaps of most interest) is described somewhat better than the lower tail, although the percentage error between the Monte Carloed and fitted value rarely exceeds 4% and then only in the extreme percentiles of the lower tail. It is also worthy of note that for hypothesis testing the region of rejection will be slightly larger than that indicated for the fitted χ variate.

The lognormal variate as previously stated provides a good fit for the larger values of $n (15 \leq n \leq 34)$, with a percentage error in excess of 2% occurring only in the most extreme percentiles.

For practical situations either fit is adequate, the chi fit being more versatile over the range of n considered and the lognormal variate offering a closer approximation over a restricted range of the sample size, n.

Suppose we take a random sample $x_1, \ldots, x_n$ of size n from an univariate normal distribution and determine the sample range $w_n = \max_{i,j} |x_i - x_j|$. Since $\frac{2m_1}{v_1} \cdot w_n^2$ is approximately distributed as $\chi^2(\frac{2m_1^2}{v_1})$ where $m_1 = E(w_n^2)$ and

$v_1 = V(w_n^2)$, we wanted to see if the extreme spread, ES, which acts like a bivariate sample range did not also closely follow a chi distribution (or, equivalently, ES^2 follow a χ^2 distribution).

To determine, for example, the 95th percentile $P_{.95}$ of ES^2 corresponding to some value of n we must satisfy the relation

$$Pr\{ES^2 \leq P_{.95}\} = .95 = Pr\{ES \leq \sqrt{P_{.95}}\}$$

or equivalently,

$$Pr\{\frac{2m}{v} ES^2 \leq \frac{2m}{v} P_{.95}\} = .95$$

To test our approximation, we assume

$$\frac{2m}{v} ES^2 \approx \chi^2(\frac{2m^2}{v}),$$

where m and v are the mean and variance of ES^2, so that we may substitute to obtain

$$Pr\{\chi^2(\frac{2m^2}{v}) \leq \frac{2m}{v} P_{.95}\} = .95$$

and interpolate in the chi square table to determine $\frac{2m}{v} P_{.95}$. Finally, multiplication of this quantity by $\frac{v}{2m}$ yields $P_{.95}$ for ES^2, and $\sqrt{P_{.95}}$ is the 95% point for ES.

It is easy to show that the mean, m, and variance v, for the extreme spread squared, i. e. ES^2, may be expressed in terms of the moments of the extreme spread ES as follows:

$$m = \sigma^2 + \mu^2,$$

and

$$v = \alpha_4\sigma^4 + 4\alpha_3\sigma^3\mu + 4\sigma^2\mu^2 - \sigma^4,$$

where the μ, σ, α_3 and α_4 are moments of ES.

Hence, referring to Table 1 for n = 10, we have

$$m = (0.745)^2 + (3.813)^2 = 15.094,$$

$$v = 3.288(0.745)^4 + 4(0.388)(0.745)^3(3.813) + 4(0.745)^2(3.813)^2 - (0.745)^4 = 35.430,$$

and we want to determine $P_{.95}$, where

$$Pr\{\chi^2(12.86) \leq 0.852\, P_{.95}\} = .95$$

Interpolating in the chi square table for 12.86 d. f. yields $0.852P_{.95} = 22.20$, or $P_{.95} = 26.06$ for ES^2, and $\sqrt{P_{.95}} = 5.11$ which is the 95% point for ES, corresponding to the entry in Table 2.

Example. Compare the relative precision of the extreme spread ES and radial standard deviation RSD for 15 rounds.

For n=15 rounds and from Table 1, $\hat{\sigma} = ES/4.190$ gives an unbiased estimate of σ, and quantity .694/4.190 = .166 is the relative precision for the extreme spread. In a like manner, the precision of the RSD is found from Grubbs (1964) Table 4 for 15 rounds to be .1817/1.354 = .134. Therefore .166 vs. 134 indicates that the RSD is slightly more precise than the ES. (For the relative precision, we compare standard errors for unbiased estimates.)

To use Table 2, suppose from previous firings we established that σ = 3 inches. Then for n=15 rounds, the chance that the extreme spread, ES, exceeds $5.396\sigma = (5.396)(3) = 16.19$ inches is .05.

MOMENT CONSTANTS OF THE EXTREME SPREAD

TABLE 1

n	μ_{ES}	σ_{ES}	α_3	α_4
2	1.766	0.932	0.632	3.294
3	2.406	0.887	0.451	3.143
4	2.787	0.856	0.393	3.163
5	3.066	0.828	0.390	3.171
6	3.277	0.806	0.374	3.194
7	3.443	0.783	0.373	3.177
8	3.582	0.771	0.392	3.231
9	3.710	0.754	0.382	3.215
10	3.813	0.745	0.388	3.288
15	4.190	0.694	0.395	3.255
20	4.452	0.668	0.400	3.240
25	4.639	0.650	0.439	3.307
28	4.734	0.642	0.426	3.357
30	4.788	0.635	0.463	3.441
31	4.822	0.631	0.434	3.321
34	4.891	0.623	0.422	3.318

Note: The numbers in the second column are $E(ES)/\sigma$, and those of the third column are $SD(ES)/\sigma$.

PERCENTAGE POINTS OF THE EXTREME SPREAD

TABLE 2

n	$P_{.0005}$	$P_{.0010}$	$P_{.0050}$	$P_{.0100}$	$P_{.0250}$	$P_{.0500}$	$P_{.1000}$
3	0.339	0.383	0.578	0.687	0.882	1.066	1.313
4	0.653	0.710	0.946	1.076	1.283	1.477	1.725
5	0.885	0.983	1.260	1.400	1.611	1.801	2.046
6	1.137	1.227	1.491	1.636	1.853	2.043	2.278
7	1.348	1.452	1.710	1.842	2.043	2.243	2.477
8	1.525	1.608	1.863	1.998	2.208	2.403	2.636
9	1.607	1.709	2.030	2.167	2.373	2.563	2.786
10	1.798	1.884	2.140	2.277	2.482	2.669	2.896
15	2.295	2.372	2.656	2.772	2.963	3.129	3.340
20	2.630	2.721	2.972	3.095	3.276	3.438	3.626
25	2.894	2.965	3.220	3.329	3.504	3.652	3.845
28	2.952	3.044	3.312	3.424	3.605	3.759	3.953
30	3.084	3.170	3.402	3.511	3.678	3.834	4.017
31	3.149	3.216	3.429	3.541	3.712	3.868	4.055
34	3.216	3.297	3.517	3.630	3.797	3.946	4.127

TABLE 2 (CONTINUED)

n	$P_{.9000}$	$P_{.9500}$	$P_{.9750}$	$P_{.9900}$	$P_{.9950}$	$P_{.9990}$	$P_{.9995}$
3	3.588	3.984	4.318	4.746	5.002	5.595	5.834
4	3.916	4.285	4.602	5.010	5.290	5.938	6.190
5	4.156	4.519	4.832	5.207	5.461	6.057	6.288
6	4.336	4.670	4.973	5.361	5.655	6.221	6.431
7	4.480	4.805	5.110	5.471	5.728	6.245	6.427
8	4.595	4.937	5.227	5.582	5.848	6.379	6.621
9	4.702	5.029	5.308	5.672	5.930	6.398	6.658
10	4.786	5.118	5.409	5.750	6.004	6.552	6.742
15	5.101	5.396	5.668	6.000	6.235	6.727	6.897
20	5.336	5.630	5.880	6.205	6.436	6.890	6.998
25	5.494	5.790	6.049	6.364	6.578	7.012	7.198
28	5.575	5.860	6.113	6.453	6.664	7.138	7.323
30	5.619	5.898	6.170	6.476	6.711	7.205	7.386
31	5.651	5.927	6.180	6.503	6.719	7.146	7.317
34	5.706	5.979	6.224	6.523	6.731	7.218	7.389

CHI APPROXIMATION

TABLE 3

n	$P_{.0050}$	$P_{.0100}$	$P_{.0250}$	$P_{.0500}$	$P_{.1000}$
3	0.578 *0.562*	0.687 *0.676*	0.882 *0.868*	1.066 *1.057*	1.313 *1.297*
4	0.946 *0.905*	1.076 *1.038*	1.283 *1.254*	1.477 *[illegible].456*	1.725 *1.707*
5	1.260 *1.182*	1.400 *1.325*	1.611 *1.550*	1.8[illegible]1 *1.758*	2.046 *2.011*
6	1.491 *1.410*	1.636 *1.560*	1.853 *1.787*	2.043 *1.995*	2.278 *2.249*
7	1.710 *1.608*	1.842 *1.756*	2.043 *1.984*	2.243 *2.191*	2.477 *2.442*
8	1.[illegible]53 *1.755*	1.998 *1.904*	2.208 *2.136*	2.403 *2.341*	2.636 *2.592*
9	2.030 *1.910*	2.167 *2.060*	2.373 *2.288*	2.563 *2.495*	2.786 *2.740*
10	2.140 *2.027*	2.277 *2.176*	2.482 *2.404*	2.669 *2.609*	2.896 *2.853*
15	2.656 *2.496*	2.772 *2.642*	2.963 *2.864*	3.129 *3.061*	3.340 *3.299*
20	2.972 *2.808*	3.095 *2.953*	3.276 *3.173*	3.438 *3.359*	3.626 *3.584*
25	3.220 *3.027*	3.329 *3.171*	3.504 *3.383*	3.652 *3.571*	3.845 *3.800*
28	3.312 *3.144*	3.424 *3.284*	3.605 *3.493*	3.759 *3.686*	3.953 *3.903*
30	3.402 *3.209*	3.511 *3.351*	3.678 *3.560*	3.834 *3.746*	4.017 *3.961*
31	3.429 *3.256*	3.541 *3.398*	3.712 *3.601*	3.868 *3.785*	4.055 *4.003*
34	3.517 *3.342*	3.630 *3.485*	3.797 *3.688*	3.946 *3.871*	4.127 *4.085*

CHI APPROXIMATION

TABLE 3 (CONTINUED)

	$P_{.9000}$	$P_{.9500}$	$P_{.9750}$	$P_{.9900}$	$P_{.9950}$	$P_{.9990}$
3	3.588	3.984	4.318	4.746	5.002	5.595
	3.593	*3.975*	*4.306*	*4.717*	*4.997*	*5.578*
4	3.916	4.285	4.602	5.010	5.290	5.938
	3.923	*4.289*	*4.597*	*4.981*	*5.233*	*5.786*
5	4.156	4.519	4.832	5.207	5.461	6.057
	4.164	*4.516*	*4.813*	*5.177*	*5.424*	*5.938*
6	4.336	4.670	4.973	5.361	5.655	6.221
	4.352	*4.677*	*4.966*	*5.318*	*5.559*	*6.052*
7	4.480	4.805	5.110	5.471	5.728	6.245
	4.483	*4.799*	*5.083*	*5.417*	*5.647*	*6.127*
8	4.595	4.937	5.227	5.582	5.848	6.379
	4.608	*4.921*	*5.196*	*5.521*	*5.752*	*6.222*
9	4.702	5.029	5.308	5.672	5.930	6.398
	4.704	*5.012*	*5.280*	*5.599*	*5.820*	*6.277*
10	4.786	5.118	5.409	5.750	6.004	6.552
	4.799	*5.105*	*5.363*	*5.680*	*5.893*	*6.343*
15	5.101	5.396	5.668	6.000	6.235	6.727
	5.109	*5.387*	*5.627*	*5.915*	*6.115*	*6.524*
20	5.336	5.630	5.880	6.205	6.436	6.890
	5.333	*5.598*	*5.833*	*6.105*	*6.293*	*6.688*
25	5.494	5.790	6.049	6.364	6.578	7.012
	5.499	*5.758*	*5.980*	*6.246*	*6.429*	*6.809*
28	5.575	5.860	6.113	6.453	6.664	7.138
	5.582	*5.835*	*6.056*	*6.317*	*6.497*	*6.875*
30	5.619	5.898	6.170	6.476	6.711	7.205
	5.626	*5.877*	*6.100*	*6.359*	*6.533*	*6.906*
31	5.651	5.927	6.180	6.503	6.719	7.146
	5.654	*5.902*	*6.122*	*6.378*	*6.553*	*6.920*
34	5.706	5.979	6.224	6.523	6.731	7.218
	5.715	*5.959*	*6.173*	*6.424*	*6.598*	*6.958*

LOG NORMAL APPROXIMATION

TABLE 4

n	$P_{.0005}$	$P_{.0010}$	$P_{.0050}$	$P_{.0100}$	$P_{.0250}$	$P_{.0500}$	$P_{.1000}$
15	2.295	2.372	2.656	2.772	2.963	3.129	3.340
	2.397	*2.478*	*2.697*	*2.811*	*2.986*	*3.146*	*3.340*
20	2.630	2.721	2.972	3.095	3.276	3.438	3.626
	2.679	*2.761*	*2.984*	*3.098*	*3.274*	*3.433*	*3.626*
25	2.894	2.965	3.220	3.329	3.504	3.652	3.845
	2.907	*2.989*	*3.211*	*3.325*	*3.499*	*3.655*	*3.845*
28	2.952	3.044	3.312	3.424	3.605	3.759	3.953
	3.020	*3.102*	*3.323*	*3.436*	*3.609*	*3.765*	*3.953*
30	3.084	3.170	3.402	3.511	3.678	3.834	4.017
	3.088	*3.170*	*3.391*	*3.504*	*3.676*	*3.830*	*4.017*
31	3.149	3.216	3.429	3.541	3.712	3.868	4.055
	3.127	*3.209*	*3.430*	*3.542*	*3.714*	*3.869*	*4.056*
34	3.216	3.297	3.517	3.630	3.797	3.946	4.127
	3.202	*3.284*	*3.504*	*3.617*	*3.788*	*3.942*	*4.127*

LOG NORMAL APPROXIMATION

TABLE 4 (CONTINUED)

15	5.101	5.396	5.668	6.000	6.235	6.727	6.897
	5.101	*5.416*	*5.706*	*6.062*	*6.317*	*6.877*	*7.108*
20	5.336	5.630	5.880	6.205	6.436	6.890	6.998
	5.336	*5.636*	*5.910*	*6.246*	*6.485*	*7.008*	*7.222*
25	5.494	5.790	6.049	6.364	6.578	7.012	7.198
	5.494	*5.779*	*6.038*	*6.354*	*6.578*	*7.067*	*7.266*
28	5.575	5.860	6.113	6.453	6.664	7.138	7.323
	5.575	*5.853*	*6.106*	*6.413*	*6.632*	*7.105*	*7.298*
30	5.619	5.898	6.170	6.476	6.711	7.205	7.386
	5.619	*5.893*	*6.141*	*6.442*	*6.656*	*7.120*	*7.309*
31	5.651	5.927	6.180	6.503	6.719	7.146	7.317
	5.651	*5.923*	*6.170*	*6.469*	*6.682*	*7.142*	*7.329*
34	5.706	5.979	6.224	6.523	6.731	7.218	7.389
	5.706	*5.974*	*6.217*	*6.511*	*6.720*	*7.171*	*7.355*

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