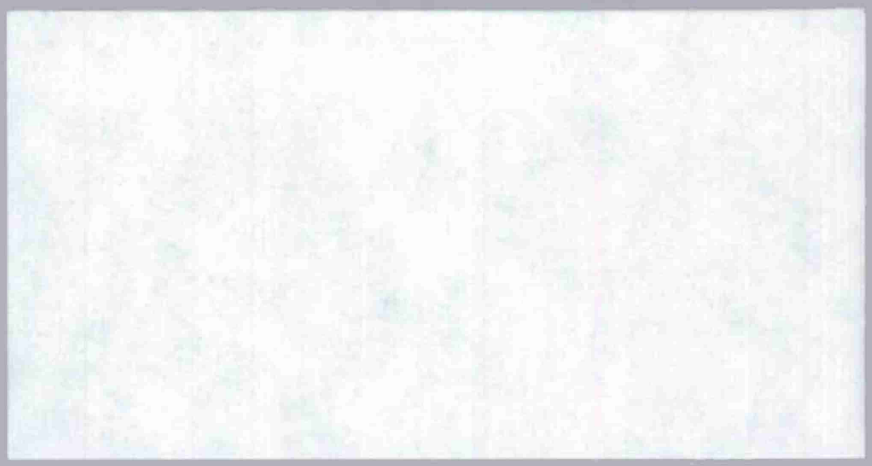
control analysis corporation.

// TR-86-16.



800 welch road • palo alto, california
telephone (415) 326-2100

CONTROL ANALYSIS CORPORATION
800 Welch Road
Palo Alto, California

Technical Report No. 86-16

December 1974

CONFIDENCE INTERVAL ESTIMATION FOR SIMULATION RESPONSE SURFACES

by

Michael A. Crane

This research was supported by the Office of Naval Research under contract N00014-72-C-0086 (NR-047-106).

CONFIDENCE INTERVAL ESTIMATION FOR SIMULATION RESPONSE SURFACES

by

Michael A. Crane

1. Introduction and Summary

In this paper, we continue the investigation of response-surface methodology in simulation, begun in [2], [3], and [6]. In [2] we showed that if g is a linear output function of an input parameter λ, over some range $a \leq \lambda \leq b$, and if statistical confidence intervals for $g(\lambda)$ are obtained at measurement points $\lambda = \lambda_1$ and $\lambda = \lambda_2$, with $a \leq \lambda_1 < \lambda_2 \leq b$, then a confidence band can be obtained for the entire function g over the interval $a \leq \lambda \leq b$ without making additional measurements. In [3], this result was extended for the case where g is a quadratic function of one variable, and in [6], g was assumed to be a general order polynomial of several variables.

One difficulty arising in the above methodology is that the width of the confidence band obtained is somewhat wider than would be necessary if one only required confidence intervals at individual points rather than a confidence band for the entire function. We are thus motivated to consider the problem of interval estimation, as opposed to band estimation and we take up this subject in this paper. We do so for the case of a linear function of one variable. Generalizations to other types of functions will be the subject of subsequent papers.

The remainder of this paper is organized as follows. The basic result is derived in Section 2. It is shown that if observations are taken at the points λ_1 and λ_2 and if certain other conditions are satisfied, then a confidence interval may be obtained for $g(\lambda)$ at any point $a \leq \lambda \leq b$. In Section 3,

it is shown that the regenerative method for analyzing simulations provides a basis for applying the result of Section 2. An illustration is given of a simulation of an M/M/1 queue and the results obtained are compared to the results obtained using the confidence band methodology of [2], [3], and [6].

2. The Result

Suppose that λ is an input parameter for a simulation and that we wish to estimate some output parameter $g(\lambda)$ which is a linear function of λ over a range $a \leq \lambda \leq b$. Independent simulation runs resulting in statistical observations are made at two particular parameter settings $\lambda = \lambda_1$ and $\lambda = \lambda_2$ where $a \leq \lambda_1 < \lambda_2 \leq b$. (These observations might be, for example, sample variates from populations with means $g(\lambda_1)$ and $g(\lambda_2)$. Alternatively, they might be observations based on random tours in an application of the regenerative approach; see Section 3 for details in this latter case.) At the parameter setting $\lambda = \lambda_1$, the simulation produces n_1 statistical observations resulting in sample statistics $\hat{r}(\lambda_1,n_1)$ and $\hat{\eta}(\lambda_1,n_1)$. Similarly, at the parameter setting $\lambda = \lambda_2$, n_2 observations result in statistics $\hat{r}(\lambda_2,n_2)$ and $\hat{\eta}(\lambda_2,n_2)$.

The following proposition forms a basis for a method to compute confidence intervals for $g(\lambda)$, where $a \leq \lambda \leq b$.

PROPOSITION 1. Suppose that $\hat{\eta}(\lambda_1,n_1) \longrightarrow \eta(\lambda_1)$ in probability as $n_1 \longrightarrow \infty$ and that $\hat{\eta}(\lambda_2,n_2) \longrightarrow \eta(\lambda_2)$ in probability as $n_2 \longrightarrow \infty$. Suppose further that

$$\lim_{n_1 \to \infty} P\{n_1^{1/2}[\hat{f}(\lambda_1,n_1) - g(\lambda_1)]/\eta(\lambda_1) \le x\} = \Phi(x) , \qquad -\infty < x < \infty$$

and

$$\lim_{n_2 \to \infty} P\{n_2^{1/2}[\hat{f}(\lambda_2,n_2) - g(\lambda_2)]/\eta(\lambda_2) \le x\} = \Phi(x) , \qquad -\infty < x < \infty ,$$

where Φ is the standard normal distribution function. Then for $a \le \lambda \le b$,

$$\lim_{\substack{n_1 \to \infty \\ n_2 \to \infty \\ n_2 = [cn_1]}} P\{[\hat{f}(\lambda,n_1,n_2) - g(\lambda)]/\hat{v}(\lambda,n_1,n_2) \le x\} = \Phi(x) , \qquad -\infty < x < \infty .$$

where c is an arbitrary positive constant,

$$\hat{f}(\lambda,n_1,n_2) = (\lambda_2-\lambda_1)^{-1} [(\lambda_2-\lambda)\hat{f}(\lambda_1,n_1) + (\lambda-\lambda_1)\hat{f}(\lambda_2,n_2)]$$

and

$$\hat{v}(\lambda,n_1,n_2) = [(\lambda_2-\lambda)^2 \hat{\eta}^2(\lambda_1,n_1)/n_1 + (\lambda-\lambda_1)^2\hat{\eta}^2(\lambda_2,n_2)/n_2]^{1/2}/(\lambda_2-\lambda_1) .$$

Proof. Let a double arrow denote convergence in distribution and let $N(\mu,\sigma)$ denote a normal random variable with mean μ and standard deviation σ. From the conditions of the proposition, we may write

$$n_1^{1/2}[\hat{f}(\lambda_1,n_1) - g(\lambda_1)] \Longrightarrow N(0, \eta(\lambda_1))$$

and

$$[cn_1]^{1/2}[\hat{f}(\lambda_2,n_2) - g(\lambda_2)] \Longrightarrow N(0, \eta(\lambda_2))$$

as $n_1 \longrightarrow \infty$ and $n_2 = [cn_1]$. Since $[cn_1]^{1/2}/(cn_1)^{1/2} \longrightarrow 1$ we have

$$(cn_1)^{1/2} [\hat{f}(\lambda_2,n_2) - g(\lambda_2)] \Longrightarrow N(0, \eta(\lambda_2))$$

or

$$n_1^{1/2}[\hat{r}(\lambda_2,n_2) - g(\lambda_2)] \Longrightarrow N(0,\ \eta(\lambda_2)/c^{1/2}) \ .$$

Since $\hat{r}(\lambda_1,n_1)$ and $\hat{r}(\lambda_2,n_2)$ are independent, we have, by Theorem 3.2 of [1], convergence of the vector process

$$\begin{pmatrix} n_1^{1/2}[\hat{r}(\lambda_1,n_1) - g(\lambda_1)] \\ \\ n_1^{1/2}[\hat{r}(\lambda_2,n_2) - g(\lambda_2)] \end{pmatrix} \Longrightarrow \begin{pmatrix} N(0,\ \eta(\lambda_1)) \\ \\ N(0,\ \eta(\lambda_2)/c^{1/2}) \end{pmatrix}$$

where the limit random variables are independent. Using the Continuous Mapping Theorem, cf. [1] Theorem 5.1, we have

$$n_1^{1/2}(\lambda_2-\lambda_1)^{-1}\ [(\lambda_2-\lambda)\ \hat{r}(\lambda_1,n_1) - (\lambda_2-\lambda)\ g(\lambda_1) + (\lambda-\lambda_1)\ \hat{r}(\lambda_2,n_2) + (\lambda-\lambda_1)\ g(\lambda_2)]$$

$$\Longrightarrow N(0,\ [(\lambda_2-\lambda)^2\ \eta^2(\lambda_1) + (\lambda-\lambda_1)^2\ \eta^2(\lambda_2)/c]^{1/2}/(\lambda_2-\lambda_1))$$

or

$$\frac{n_1^{1/2}[\hat{r}(\lambda,n_1,n_2) - g(\lambda)]}{[(\lambda_2-\lambda)^2\ \eta^2(\lambda_1) + (\lambda-\lambda_1)^2\ \eta^2(\lambda_2)/c]^{1/2}\ (\lambda_2-\lambda_1)^{-1}} \Longrightarrow N(0,1) \ .$$

Since $[cn_1]/cn_1 \longrightarrow 1$, $\hat{\eta}(\lambda_1,n_1) \longrightarrow \eta(\lambda_1)$, and $\hat{\eta}(\lambda_2,n_2) \longrightarrow \eta(\lambda_2)$ in probability, this implies

$$\frac{[\hat{r}(\lambda,n_1,n_2) - g(\lambda)]}{[(\lambda_2-\lambda)^2\ \hat{\eta}^2(\lambda_1,n_1)/n_1 + (\lambda-\lambda_1)^2\ \hat{\eta}^2(\lambda_2,n_2)/[cn_1]]^{1/2}\ (\lambda_2-\lambda_1)^{-1}} \Longrightarrow N(0,1)$$

or

$$\frac{\hat{r}(\lambda,n_1,n_2) - g(\lambda)}{\hat{v}(\lambda,n_1,n_2)} \Longrightarrow N(0,1)$$

which is the desired result.

The above proposition allows one to obtain a $(100)(1-\gamma)\%$ confidence interval for $g(\lambda)$ as follows. For n_1 and n_2 sufficiently large,

$$P\{-\Phi^{-1}(1 - \tfrac{\gamma}{2}) \leq [\hat{r}(\lambda,n_1,n_2) - g(\lambda)]/\hat{v}(\lambda,n_1,n_2) \leq \Phi^{-1}(1 - \tfrac{\gamma}{2})\} \tilde{=} 1 - \gamma .$$

This may be rewritten as

$$P\{\hat{r}(\lambda,n_1,n_2) - \hat{v}(\lambda,n_1,n_2)\,\Phi^{-1}(1- \tfrac{\gamma}{2}) \leq g(\lambda) \leq \hat{r}(\lambda,n_1,n_2) + \hat{v}(\lambda,n_1,n_2)\Phi^{-1}(1- \tfrac{\gamma}{2})\}$$

$$\tilde{=} 1 - \gamma,$$

giving the desired confidence interval.

Note that the confidence interval obtained at $\lambda = \lambda_1$ reduces to

$$\hat{r}(\lambda_1,n_1) \pm n_1^{-1/2}\,\hat{\eta}(\lambda_1,n_1)\,\Phi^{-1}(1 - \tfrac{\gamma}{2}) ,$$

which is exactly the same confidence interval which could be obtained based on the n_1 observations at $\lambda = \lambda_1$. Similarly, the confidence interval obtained at $\lambda = \lambda_2$ is

$$\hat{r}(\lambda_2,n_2) \pm n_2^{-1/2}\,\hat{\eta}(\lambda_2,n_2)\,\Phi^{-1}(1 - \tfrac{\gamma}{2}) .$$

Letting $\hat{\ell}(\lambda,n_1,n_2)$ denote the length of the confidence interval obtained at λ, we see that

$$\hat{\ell}(\lambda,n_1,n_2)$$

$$= 2\hat{v}(\lambda,n_1,n_2)\ \Phi^{-1}(1-\tfrac{r}{2})$$

$$= 2[(\lambda_2-\lambda)^2\ \hat{\eta}^2(\lambda_1,n_1)/(\lambda_2-\lambda_1)^2 n_1 + (\lambda-\lambda_1)^2\ \hat{\eta}^2(\lambda_2,n_2)/(\lambda_2-\lambda_1)^2 n_2]^{1/2}\ \Phi^{-1}(1-\tfrac{r}{2})$$

$$= [(\lambda_2-\lambda)^2\ \hat{\ell}^2(\lambda_1,n_1)/(\lambda_2-\lambda_1)^2 + (\lambda-\lambda_1)^2\ \hat{\ell}^2(\lambda_2,n_2)/(\lambda_2-\lambda_1)^2]^{1/2}$$

where $\hat{\ell}(\lambda_1,n_1)$ and $\hat{\ell}(\lambda_2,n_2)$ are the lengths obtained at λ_1 and λ_2 respectively. Hence $\hat{\ell}(\lambda,n_1,n_2)$ is the square root of a quadratic function which passes through $(\lambda_1,\ \hat{\ell}^2(\lambda_1,n_1))$ and $(\lambda_2,\ \hat{\ell}^2(\lambda_2,n_2))$.

3. Applications Using the Regenerative Method

In this section, we give examples illustrating the use of Proposition 1 for a specific class of simulations, namely, those simulations where the regenerative method is applicable. (References [2], [3], [4], [5], and [7] may be consulted for more details of the regenerative method.) A basic problem in statistical inference in simulations is to estimate the constant $g(\lambda) = E\{f(\underset{\sim}{X}(\lambda))\}$ where f is a general real-valued function, λ is an input parameter, and $\underset{\sim}{X}(\lambda)$ is the stationary random vector associated with $\{\underset{\sim}{X}(s,\lambda): s \geq 0\}$, the process being simulated. In the regenerative method, we observe the process $\{\underset{\sim}{X}(s,\lambda): s \geq 0\}$ in random cycles of lengths $\alpha_1(\lambda)$, $\alpha_2(\lambda)$, ... , $\alpha_n(\lambda)$ and record in each cycle the values $Y_1(\lambda)$, $Y_2(\lambda),\ldots,Y_n(\lambda)$ where $Y_i(\lambda)$ is the area under the curve $f(\underset{\sim}{X}(s,\lambda))$ in the i^{th} cycle.

The crucial conditions required for the regenerative method to be used are that the n pairs $\{(Y_i(\lambda), \alpha_i(\lambda)), i = 1, 2, \ldots, n\}$ are independent and identically distributed and that $g(\lambda) = E\{Y_1(\lambda)\}/E\{\alpha_1(\lambda)\}$. Now define the column vector $\underset{\sim}{U}_i(\lambda) = (Y_i(\lambda), \alpha_i(\lambda))$ and let

$$\underset{\sim}{\Sigma}(\lambda) = \begin{pmatrix} \sigma_{11}(\lambda) & \sigma_{12}(\lambda) \\ \sigma_{12}(\lambda) & \sigma_{22}(\lambda) \end{pmatrix}$$

denote the covariance matrix for $\underset{\sim}{U}_i(\lambda)$. Denote the sample mean by

$$\bar{\underset{\sim}{U}}(\lambda,n) = \begin{pmatrix} \bar{Y}(\lambda,n) \\ \bar{\alpha}(\lambda,n) \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{U}_i(\lambda)$$

and the sample covariance by

$$\underset{\sim}{s}(\lambda,n) = \begin{pmatrix} s_{11}(\lambda,n) & s_{12}(\lambda,n) \\ s_{21}(\lambda,n) & s_{22}(\lambda,n) \end{pmatrix} = \frac{1}{n-1} \sum_{i=1}^{n} [\underset{\sim}{U}_i(\lambda)-\bar{\underset{\sim}{U}}(\lambda,n)][\underset{\sim}{U}_i(\lambda)-\bar{\underset{\sim}{U}}(\lambda,n)]'$$

where the prime denotes transpose. Next, define point estimates for $g(\lambda)$ as follows:

(1) Classical estimator

$$\hat{f}_c(\lambda,n) = \frac{\bar{Y}(\lambda,n)}{\bar{\alpha}(\lambda,n)} ;$$

(2) Beale estimator

$$\hat{f}_b(\lambda,n) = \frac{\bar{Y}(\lambda,n)}{\bar{\alpha}(\lambda,n)} \cdot \frac{[1 + s_{12}(\lambda,n)/n\ \bar{Y}(\lambda,n)\ \bar{\alpha}(\lambda,n)]}{[1 + s_{22}(\lambda,n)/n\ \bar{\alpha}^2(\lambda,n)]} ;$$

(3) Tin estimator

$$\hat{r}_t(\lambda,n) = \frac{\bar{Y}(\lambda,n)}{\bar{\alpha}(\lambda,n)} \left[1 + \left(\frac{s_{12}(\lambda,n)}{\bar{Y}(\lambda,n)\,\bar{\alpha}(\lambda,n)} - \frac{s_{22}(\lambda,n)}{\bar{\alpha}^2(\lambda,n)} \right) n^{-1} \right] ;$$

(4) Jackknife estimator

$$\hat{r}_J(\lambda,n) = \frac{1}{n} \sum_{i=1}^{n} \theta_i(\lambda,n)$$

where $\theta_i(\lambda,n) = n[\bar{Y}(\lambda,n)/\bar{\alpha}(\lambda,n)] - (n-1)\left[\sum_{j\neq i} Y_j(\lambda) / \sum_{j\neq i} \alpha_j(\lambda) \right]$.

Finally, define

$$\hat{\eta}_c(\lambda,n) = [s_{11}(\lambda,n) - 2\hat{r}_c(\lambda,n)\, s_{12}(n) + \hat{r}_c^2(\lambda,n)\, s_{22}(\lambda,n)]/\bar{\alpha}(\lambda,n)$$

and

$$\hat{\eta}_J(\lambda,n) = \{ \sum_{i=1}^{n} [\theta_i(\lambda,n) - \hat{r}_J(\lambda,n)]^2/(n-1) \}^{1/2} .$$

Now let

$$Z_i(\lambda) = Y_i(\lambda) - g(\lambda)\, \alpha_i(\lambda)$$

and note that $E\{Z_i(\lambda)\} = 0$ and define $\sigma^2(\lambda) = \mathrm{var}\{Z_i(\lambda)\}$. Since the vectors $\{\underset{\sim}{U}_i(\lambda),\ i \geq i\}$ are i.i.d. it follows that $\{Z_i(\lambda),\ i \geq 1\}$ are i.i.d. By the central limit theorem for partial sums of i.i.d. random variables, it follows that

$$\lim_{n \to \infty} P\{ \sum_{i=1}^{n} Z_i(\lambda)/n^{1/2}\, \sigma(\lambda) \leq x\} = \Phi(x) , \qquad -\infty < x < \infty ,$$

which may be rewritten

$$\lim_{n \to \infty} P\{n^{1/2}[\hat{r}_c(\lambda,n) - g(\lambda)]\, \bar{\alpha}(\lambda,n)/\sigma(\lambda) \leq x\} = \Phi(x) , \qquad -\infty < x < \infty .$$

Since $\bar{\alpha}(\lambda,n) \longrightarrow E\{\alpha_1(\lambda)\}$ a.e., it follows that

$$\lim_{n \to \infty} P\{n^{1/2}[\hat{r}_c(\lambda,n) - g(\lambda)]/\eta(\lambda) \leq x\} = \Phi(x) , \qquad -\infty < x < \infty ,$$

where $\eta(\lambda) = \sigma(\lambda)/E\{\alpha_1(\lambda)\}$. Now it may be shown that

$$n^{1/2}[\hat{r}_c(\lambda,n) - \hat{r}_b(\lambda,n)] \longrightarrow 0 \quad \text{a.e.},$$

$$n^{1/2}[\hat{r}_c(\lambda,n) - \hat{r}_t(\lambda,n)] \longrightarrow 0 \quad \text{a.e.},$$

$$n^{1/2}[\hat{r}_c(\lambda,n) - \hat{r}_J(\lambda,n)] \longrightarrow 0 \quad \text{a.e.},$$

and that $\hat{\eta}_c(\lambda,n) \longrightarrow \eta(\lambda)$ and $\hat{\eta}_J(\lambda,n) \longrightarrow \eta(\lambda)$ is probability as $n \longrightarrow \infty$. Hence, the conditions of Proposition 1 are satisfied with any of the following substitutions:

$\hat{r}_c(\lambda_k,n_k)$, $\hat{r}_b(\lambda_k,n_k)$, $\hat{r}_t(\lambda_k,n_k)$ or $\hat{r}_J(\lambda_k,n_k)$

substituted for $\hat{r}(\lambda_k,n_k)$, $k = 1, 2$

$\hat{\eta}_c(\lambda_k,n_k)$ or $\hat{\eta}_J(\lambda_k,n_k)$ substituted for $\hat{\eta}(\lambda_k,n_k)$, $k = 1, 2$.

To illustrate the application of Proposition 1, consider a simulation of the customer waiting time process $\{W_n, n \geq 1\}$ in an M/M/1 queue. Suppose we wish to study the sensitivity of the mean stationary waiting time

$g(\lambda) = E\{W(\lambda)\}$ to the arrival rate λ over the range $3 \leq \lambda \leq 5$, with the service rate $\mu = 10$. In what follows, we shall illustrate the proposition for this simulation using the "classical" estimators $\hat{r}_c$ and $\hat{\eta}_c$, though we could have chosen any of the estimators given above.

In the queueing simulation, we say that the i^{th} busy cycle is initiated with the arrival of the i^{th} customer to find an empty queue. Suppose that simulation runs consisting of $n_1 = n_2 = 10{,}000$ busy cycles are made at parameter settings $\lambda = \lambda_1 = 3$ and $\lambda = \lambda_2 = 5$. Let $\alpha_i(\lambda)$ denote the number of customers served in the i^{th} busy cycle, and let $Y_i(\lambda)$ be the sum of the waiting times for those customers. Then it may be shown, cf. [2], that $\{(Y_i(\lambda), \alpha_i(\lambda), i \geq 1\}$ are independent and identically distributed, and $E\{W(\lambda)\} = E\{Y_1(\lambda)\}/E\{\alpha_1(\lambda)\}$. Hence, it is appropriate to apply the regenerative method as discussed above. In particular, we can compute, for $k = 1, 2$, $\bar{Y}(\lambda_k,n_k)$, $\bar{\alpha}(\lambda_k,n_k)$, $s_{11}(\lambda_k,n_k)$, $s_{12}(\lambda_k,n_k)$, and $s_{22}(\lambda_k,n_k)$ and from these we can compute $\hat{r}_c(\lambda_k,n_k)$ and $\hat{\eta}_c(\lambda_k,n_k)$ as defined above.

Suppose that, as the result of these computations,

$$\hat{r}_c(\lambda_1,n_1) = .042$$

$$\hat{r}_c(\lambda_2,n_2) = .098$$

$$\hat{\eta}_c(\lambda_1,n_1) = .172$$

$$\hat{\eta}_c(\lambda_2,n_2) = .430$$

Then

$$\hat{r}(\lambda,n_1,n_2) = (\lambda_2-\lambda_1)^{-1} [(\lambda_2-\lambda)\, \hat{r}_c(\lambda_1,n_1) + (\lambda-\lambda_1)\, \hat{r}_c(\lambda_2,n_2)]$$

$$= .028\lambda - .042$$

and

$$\hat{v}(\lambda,n_1,n_2) = [(\lambda_2-\lambda)^2\ \hat{\eta}_c^2(\lambda_1,n_1)/n_1 + (\lambda-\lambda_1)^2\ \hat{\eta}_c^2(\lambda_2,n_2)/n_2]^{1/2}/(\lambda_2-\lambda_1)$$

$$= [(5-\lambda)^2\ (.0296) + (\lambda-3)^2\ (.185)]^{1/2}/200\ .$$

Now assume that $E\{W(\lambda)\}$ is approximately linear for $3 \le \lambda \le 5$ and suppose that we desire a 98% confidence interval for $E\{W(\lambda)\}$ for each λ such that $3 \le \lambda \le 5$. From Proposition 1, such a confidence interval is given by

$$\hat{r}(\lambda,n_1,n_2) \pm \hat{v}(\lambda,n_1,n_2)\ \Phi^{-1}(1 - \tfrac{\gamma}{2})\ ,$$

where $\hat{r}(\lambda,n_1,n_2)$ and $\hat{v}(\lambda,n_1,n_2)$ are computed as above and $\gamma = .02$. These confidence intervals are illustrated in Figure 1. Thus, for example, a 98% confidence interval for $E\{W(4)\}$ is $.070 \pm .0054$.

It is of interest to compare the confidence intervals shown in Figure 1 to a confidence band obtained for the function $E\{W(\lambda)\}$ using the method of [2]. Figure 2 shows such a comparison. A 96% confidence band for $E\{W(\lambda)\}$ over $3 \le \lambda \le 5$ consists of two straight lines intersecting the 98% confidence intervals at $\lambda = 3$ and $\lambda = 5$ and everywhere containing the 98% confidence intervals at $3 < \lambda < 5$. Thus, if one desires confidence bands rather than confidence intervals, it is apparent that such confidence bands are obtained at a cost of lower levels of confidence in addition to wider intervals.

Suppose that after running simulations at $\lambda = 3$ and $\lambda = 5$, we are interested in estimating $E\{W(\lambda)\}$ for $2 \le \lambda \le 5$. If we assume that $E\{W(\lambda)\}$ is approximately linear over $2 \le \lambda \le 5$, then the above expression for the confidence intervals remains valid. See Figure 3 for an illustration of the confidence intervals so obtained.

Acknowledgement. It is a pleasure to acknowledge useful discussions with Donald L. Iglehart concerning the research reported here.

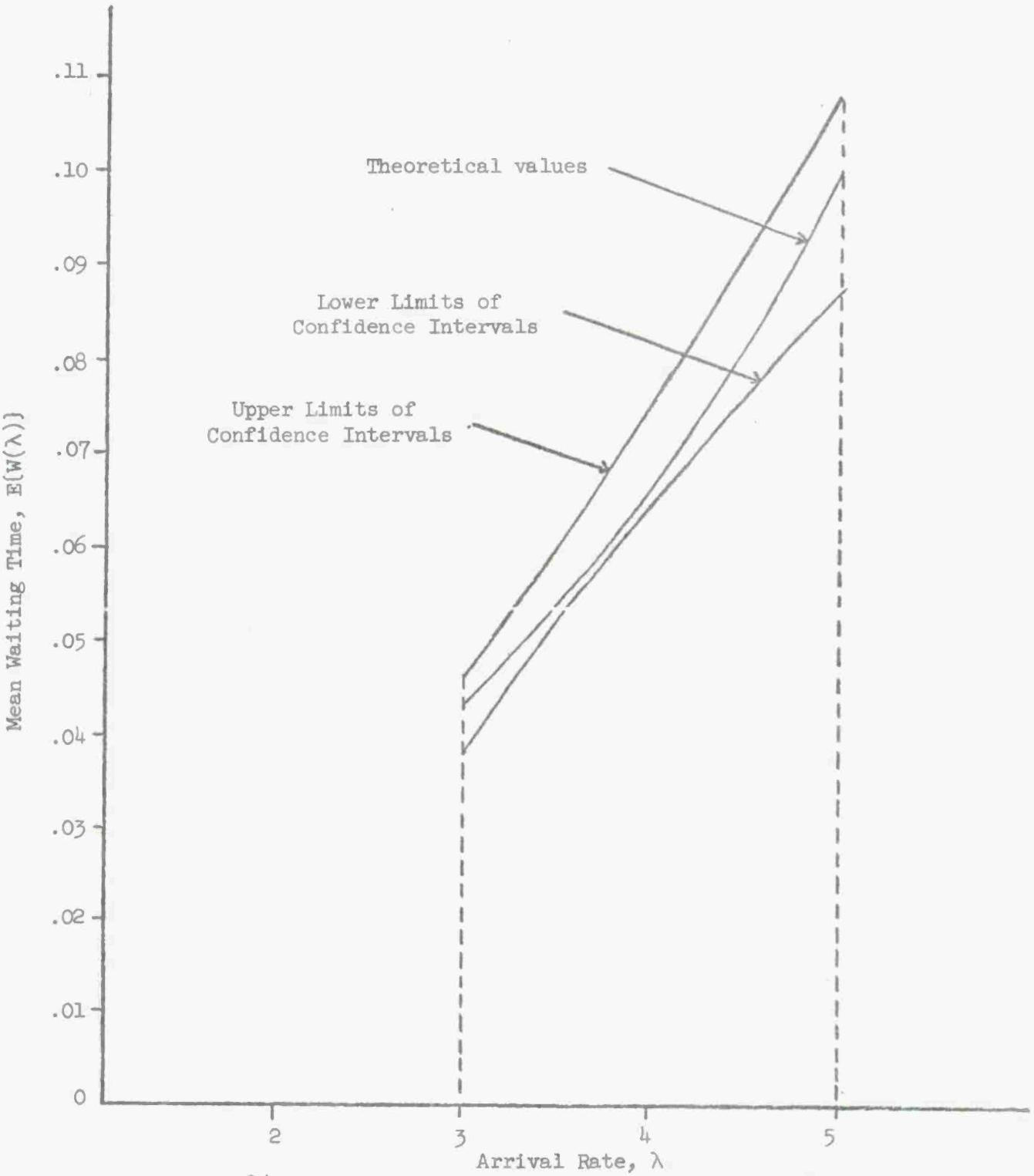


Figure 1. 98% Confidence Intervals for the Mean Waiting Time, as a Function of the Arrival Rate λ Over $3 \leq \lambda \leq 5$. (M/M/1 Queue with Service Rate $\mu = 10$.)

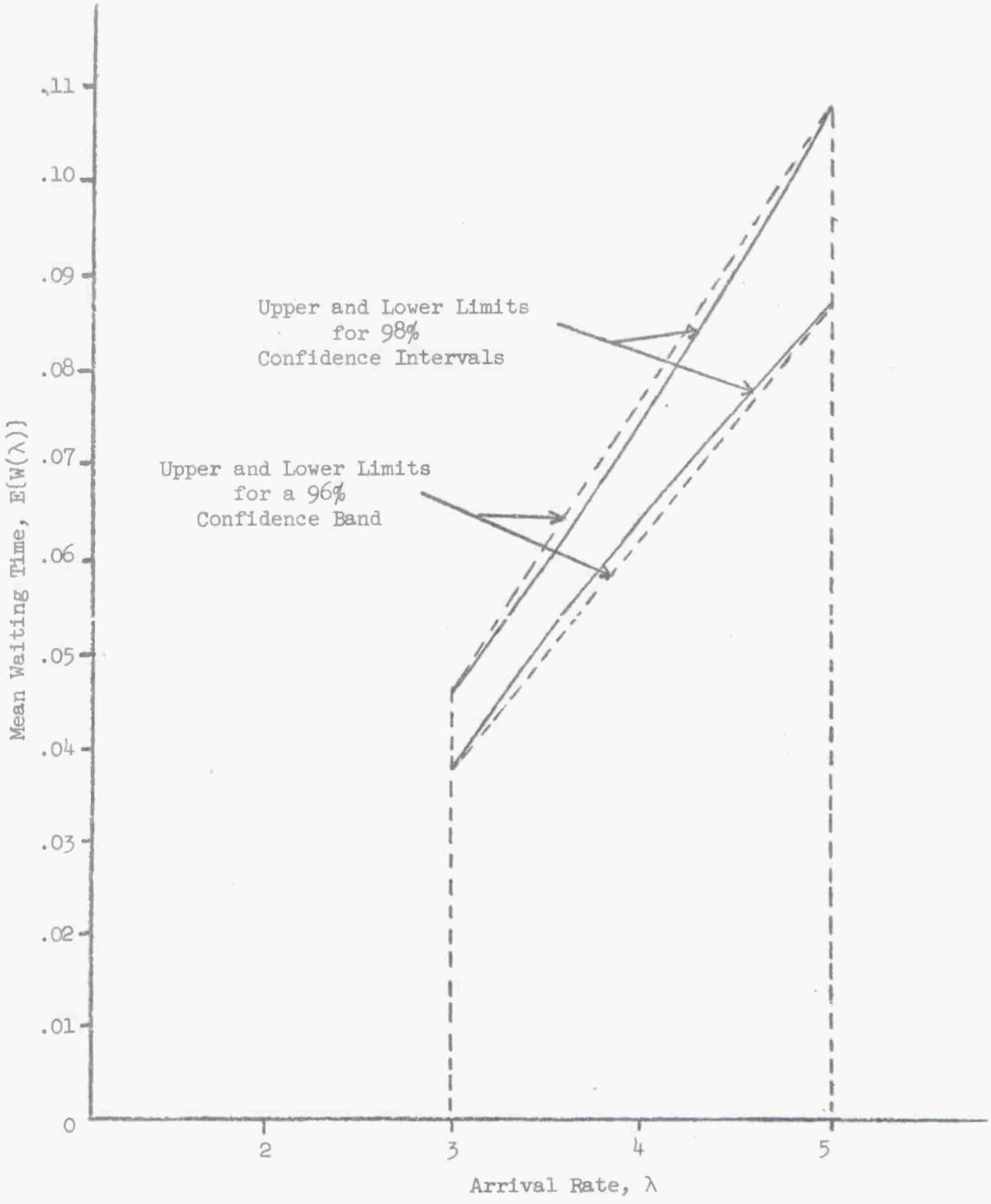


Figure 2. Comparison of Confidence Intervals vs. Confidence Band for the Mean Waiting Time, as a Function of the Arrival Rate λ Over $3 \leq \lambda \leq 5$.

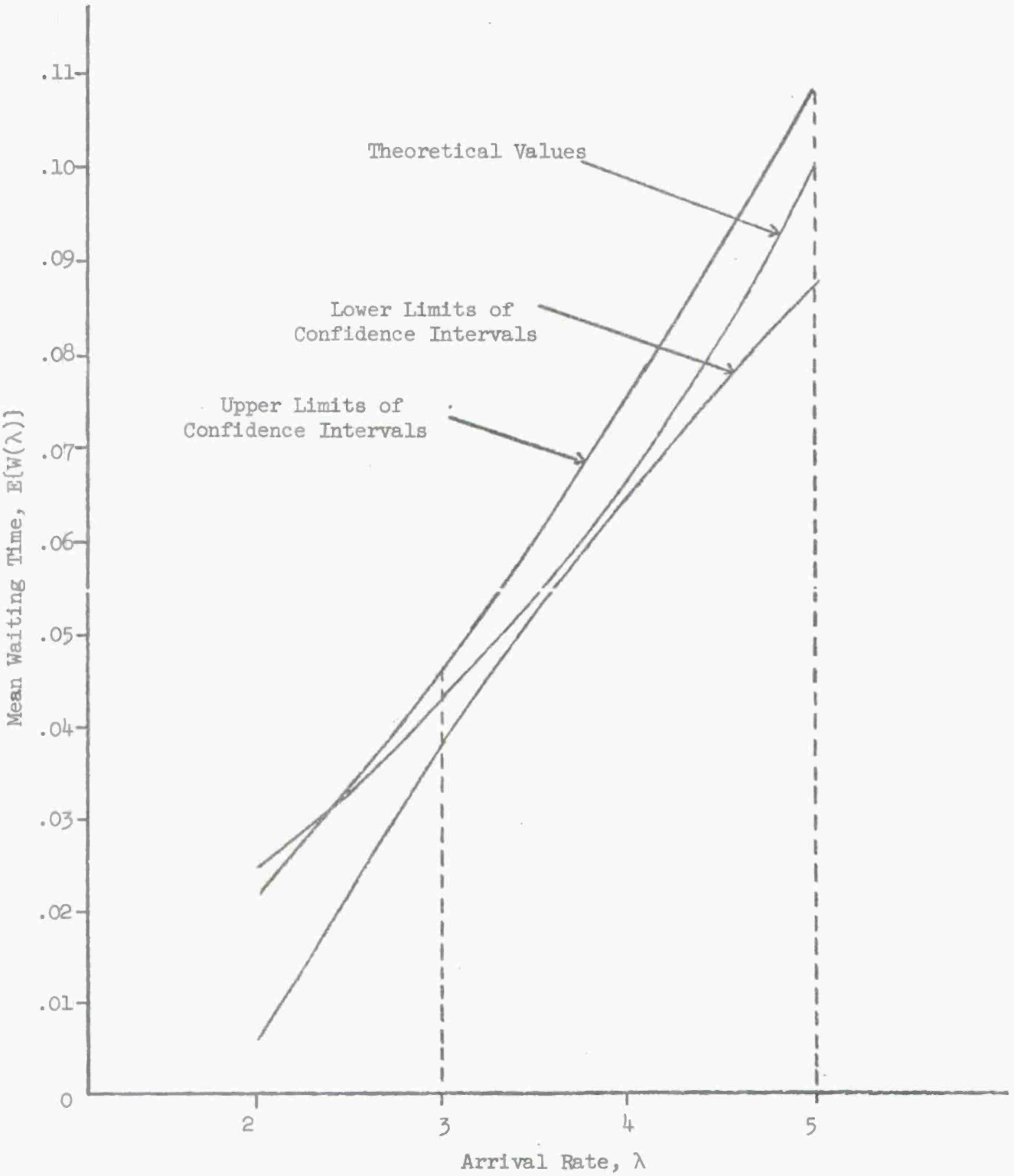


Fig. 3. 98% Confidence Intervals for the Mean Waiting Time as a Function of the Arrival Rate λ Over $2 \leq \lambda \leq 5$.

References

[1] BILLINGSLEY, P. (1968). Convergence of Probability Measures. John Wiley, New York.

[2] CRANE, M.A. and IGLEHART, D.L. (1974). Simulating stable stochastic systems, I: General multi-server queues. J. Assoc. Comput. Mach. 21, 103-113.

[3] CRANE, M.A. and IGLEHART, D.L. (1974). Simulating stable stochastic systems, II: Markov chains. J. Assoc. Comput. Mach. 21, 114-123.

[4] CRANE, M.A. and IGLEHART, D.L. (1974). Simulating stable stochastic systems, III: Regenerative processes and discrete-event simulations. To appear in Operations Res.

[5] CRANE, M.A. and IGLEHART, D.L. (1974). Simulating stable stochastic systems, IV: Approximation techniques. To appear in Management Sci.

[6] CRANE, M.A. and MOREY, R.C. (1973). Response surface estimation in simulation experiments. Technical Report No. 86-12, Control Analysis Corporation.

[7] IGLEHART, D.L. (1974). Simulating stable stochastic systems, V: Comparison of ratio estimators. To appear in Naval Research Logistics Quarterly.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author)
Control Analysis Corporation
800 Welch Road
Palo Alto, California

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3 REPORT TITLE
CONFIDENCE INTERVAL ESTIMATION FOR SIMULATION RESPONSE SURFACES

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

5 AUTHOR(S) (First name, middle initial, last name)
Crane, Michael A.

6 REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1974	15	7

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-72-C-0086	Technical Report No. 86-16
b. PROJECT NO. NR-047-106	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10 DISTRIBUTION STATEMENT
Distribution of this document is unlimited.

11 SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Operations Research Program Office of Naval Research Arlington, Virginia 22217

13 ABSTRACT

In this paper, we continue the investigation of response surface methodology in simulation, begun in references [2], [3], and [6]. Suppose that g is a linear output function of an input parameter λ, over some range $a \leq \lambda \leq b$, and that observations are taken at two parameter settings λ_1 and λ_2. It is shown that if certain conditions are satisfied, then a confidence interval may be obtained for $g(\lambda)$ at any point $a \leq \lambda \leq b$ without making any additional observations. An illustration is provided which makes use of the regenerative approach for analyzing simulations.

DD FORM 1 NOV 68 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Simulation						
Statistical analysis of simulations						
Confidence intervals in simulation						
Response surface estimation						
Linear approximation						
Regenerative approach						