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RECENTLY DEVELOPED FORMULATIONS OF THE INVERSE PROBLEM IN ACOUSTICS AND ELECTROMAGNETICS

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Recently developed formulations of the inverse problems in acoustics and electromagnetics are described. There are two types of formulations, one in the geometrical optics limit and the other, an exact formulation for the inverse source problem. Both basic formulations are extended to include the realistic problem of a "limited aperture" of observations. It is also shown that the inverse source formulation can be applied to the problem of reconstruction of media inhomogeneities from remotely sensed field data. The basic physical optics result is that the characteristic function of the scattering obstacle and the phase and range normalized scattering amplitude are a Fourier transform pair. All other formulations lead to Fredholm integral equations of the first kind.

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Recently Developed Formulations of the Inverse Problem in Acoustics and Electromagnetics

by

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and

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Abstract

Recently developed formulations of the inverse problems in acoustics and electromagnetics are described. There are two types of formulations, one in the geometrical optics limit and the other, an exact formulation for the inverse source problem. Both basic formulations are extended to include the realistic problem of a "limited aperture" of observations. It is also shown that the inverse source formulation can be applied to the problem of reconstruction of media inhomogeneities from remotely sensed field data. The basic physical optics result is that the characteristic function of the scattering obstacle and the phase and range normalized scattering amplitude are a Fourier transform pair. All other formulations lead to Fredholm integral equations of the first kind.

I. INTRODUCTION

The objective of inverse scattering methods is to obtain information about a source distribution, a scattering obstacle or inhomogeneity of a medium from remotely sensed field data. Recently, we have formulated such inverse problems in acoustics and electromagnetics in terms of Fredholm integral equations of the First Kind. Furthermore, some of these formulations take account of the realistic difficulty that the f lds are usually observed only over some "limited aperture", both spatially and temporally, rather than in all directions and for all time.

In this paper we shall describe these formulations and present the available evidence of their validity. This evidence is admittedly sparse at this time, but is also extremely promising. Our purpose is to collect these recent results and to communicate them because we believe there is a wealth of research directions to be explored well beyond our own capacities and interests.

The list of applications, non-military as well as military, is quite extensive. Among these are the following.

(1) Construction of images of tumors.

(2) Analysis of subsurface strata for resource identification and recovery.

(3) Location and identification of discharges in storms for the analysis of the storms themselves and prediction of tornadoes from charasteristic source patterns.

(4) Location of buried bodies--an important problem in law enforcement.

(5) Construction of images of airplanes, missiles, surface vessels and submarines.

(6) Landmine detection.

(7) Analysis of soundspeed variation (and consequent sonar degradation) in the water and sea bed sedimentary layer.

We shall see below that the method of deriving the integral equations is not peculiar to the wave equation or Maxwell's equations. Thus, it is likely that for wave phenomena governed by other equations, completely analogous formulations of the inverse problem are possible.

There are two distinct categories of problem formulation to discuss, a physical optics (approximate) formulation and an exact integral equation formulation. The former evolved from a basic identity derived a number of years ago by the second author (Bojarski, 1967, and Lewis, 1969) for the time-harmonic acoustic case. The latter evolved from more recent work of his (Bojarski, 1973) for time harmonic wave propagation.

The physical optics identity relates the range and phase normalized back scattered (monostatic) far-field cross-section to the Fourier transform of the characteristic function† of the scattering body. This result has now been extended to the bistatic case. In this form, much more information is extracted with only two stationary transmitters and an array of receivers. These results are described in Section II and a numerically synthesized check on the method reported by the second author (Bojarski, April 1973, February 1974) are presented in Section IV.

In another extension discussed in Section III, we dispense with the far-field approximations and obtain an integral equation for the characteristic function. This formulation would

†The characteristic function is unity inside the scatterer and zero outside, thus describing the scattering body.

be appropriate when the distance between the observed remote field and the obstacle was comparable to the length scale of the obstacle.

The consequences of limited aperature for far-field physical optics are discussed in Section IV. Again, numerical results by Bojarski (April 1973) are described.

In the exact formulation, one obtains an inhomogeneous integral equation for a source distribution with the inhomogeneous term being an integral of the data observed remotely. Bojarski's basic integral equation for wave propagation governed by the Hemoltz equation is described in Section V. Again, numerically synthesized checks on the basic formulation are presented. Significant among these is an example in which two point sources, one-half wave length apart are resolved.

Again, the limited aperature problem is of great interest and the formulation which takes account of this feature is presented in Section VI.

Problems in which one seeks an inhomogeneity of a medium can be reformulated as inverse source problems amenable to analysis by this exact formulation. Inverse scattering problems for "translucent" obstacles (neither acoustically hard nor soft) can also be viewed as inhomogeneity problems and are thus reduced to inverse source problems. The "source" here is not to be confused with "equivalent sources" which are physically unrealizable mathematical idealizations. These ideas are expanded upon and clarified in Section VII.

Extensions of the inverse source problem formulation to the time domain are discussed in Section VIII. These time domain formulations have the inherent advantages noted above. Furthermore, there are two distinct formulations depending on whether or not the source distribution decays to zero sufficiently rapidly in time. When this is not the case, it is seen that the time harmonic case is not necessarily equivalent to a transform of the time dependent formulation.

All of the integral equations obtained here are Fredholm integral equations of the First Kind. The problems are mathematically ill-posed, suffering both from non-uniqueness and non-continuous dependence on the data.

As to the former difficulty, we believe that the non-uniqueness resides in a class of non-physically realizable solutions such as source distributions "at infinity" or scattered waves containing eigenmodes of annular regions, which modes are not "outgoing". This is a conjecture to be verified or contradicted.

The ill-posedness has as practical implication, the fact that the solutions are "noise-limited". That is, relatively small amounts of noise can degrade the solution so much that the "true" solution is unrecognizable. The quantification of this observation is well worth studying.

The extension of the formulations presented here to Maxwell's equations is straightforward. They can be found in the above cited Bojarski papers. Since the objective of this report is to collect the basic formulations, we do not repeat those extensions here.

II. PHYSICAL OPTICS FAR-FIELD TIME HARMONIC INVERSE SCATTERING

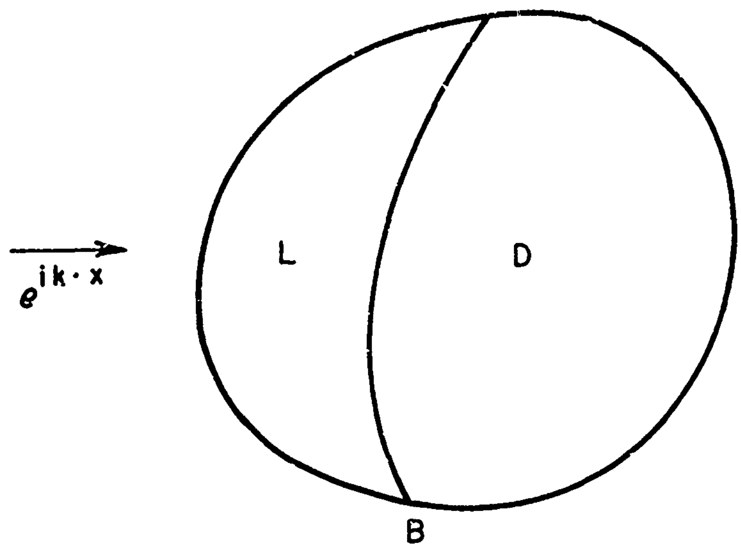


FIGURE 2.1

We shall briefly derive here the monostatic and bistatic physical optics far-field identities. This derivation is based on the earlier monostatic identity (Bojarski, 1967, Lewis, 1969). We suppose that the incident field

$$u_I = e^{i\underline{k}\cdot\underline{x}} \quad ; \quad \underline{k} = (k_1, k_2, k_3) \quad , \quad \underline{x} = (x_1, x_2, x_3) ; \quad (2.1)$$

illuminates the acoustically soft† convex scatterer B in Figure 2.1. We define the total field u_T by

$$u_T = u_I + u_S \qquad (2.2)$$

†The acoustically hard case can be treated by this method as well. For other cases, the method of Section VII must be employed.

with u_S the scattered field. It is required that u_T satisfy the equation

$$(\nabla^2 + k^2)\, u_T = 0 \qquad (2.3)$$

and the boundary condition

$$u_T = 0 \quad \text{on } \partial B \qquad (2.4)$$

The scattered field must satisfy the same equation (2.3), must be such that (2.4) is satisfied and be "outgoing" at infinity. One can then derive the following integral equation for u_S (Baker and Copson, 1950).

$$u_S(\underline{x},\underline{k}) = \int_{\partial B} \left[u_S(\underline{x}',\underline{k}) \frac{\partial g(\underline{x}-\underline{x}')}{\partial n'} - g(\underline{x}-\underline{x}') \frac{\partial u_S(\underline{x}')}{\partial n'} \right] ds' \,. \qquad (2.5)$$

Here ∂B denotes the boundary of B, $\partial/\partial n'$ denotes the normal derivative and

$$g(\underline{x}) = \frac{e^{ikx}}{4\pi x} \quad ; \; x = |\underline{x}| \qquad (2.6)$$

is the free space outgoing Green's function.

The physical optics approximation consists of making the following assumptions about the fields. The incident plane wave illuminates a portion of ∂B called the lit side and denoted by L in Figure 2.1. We may characterize L analytically by the condition $\underline{k}\cdot\hat{n} > o$ on L, with $\hat{n}$ the outward unit normal to B. The remainder of ∂B is denoted by D in that Figure. It is then assumed that u_S satifies the following approximate boundary conditions

$$\left.\begin{aligned} u_S &= -u_I = -\exp\{i\,\underline{k}\cdot\underline{x}\} \\ \frac{\partial u_S}{\partial n} &= \frac{\partial u_I}{\partial n} = -i\underline{k}\cdot\hat{n}\exp\{i\underline{k}\cdot\underline{x}\} \end{aligned}\right\} \text{ on L}, \quad u_S = \frac{\partial u_S}{\partial n} = 0, \text{ on D} \tag{2.7}$$

We may then substitute (2.7) into (2.5) to obtain

$$u_S(\underline{x},\underline{k}) = \int \exp\{\, i\underline{k}\cdot\underline{x}' \,\}\left\{ \frac{\partial g}{\partial n'}(\underline{x}-\underline{x}') + i\underline{k}\cdot\hat{n}\;\; g\;(\underline{x}-\underline{x}')\right\} dS \tag{2.8}$$

where we have used the explicit form (2.1) for u_S.

We now proceed to introduce the far-field approximations into this result. Firstly, we set

$$u_S(\underline{x},\underline{k}) = \rho(\underline{k}) \;\; \exp\;\{i\; k\; x\}\,/4\pi x. \tag{2.9}$$

Here, the function $\rho(\underline{k})$ is known as the range-and-phase normalized far-field amplitude.

Secondly, we expand $|\underline{x}-\underline{x}'|$ for x>>x' and use this result in g:

$$g(\underline{x}-\underline{x}') = \exp\;\{ik\;[x-i\hat{x}\cdot x']\}\;/_{4\pi x}\;;$$

$$\frac{\partial}{\partial n'}\;g\;(\underline{x}-\underline{x}') = ik\hat{x}\cdot\hat{n}\;\;\exp\;\{\,ik\;[x-i\hat{x}\cdot x']\}\;/4\pi x \tag{2.10}$$

Here $\hat{x}$ is a unit vector in the direction of $\underline{x}$.

The results (2.9) and (2.10) are substituted into (2.8) and the common factor exp $\{ikx\}/4\pi x$ is removed to yield

$$\rho(\underline{k},\hat{x}) = -\int_L i\hat{n}\cdot\left[\underline{k} - k\hat{x}\right] \exp\{i\underline{x}'\cdot\left[\underline{k} - k\hat{x}\right]\}\, dS' \qquad (2.11)$$

Let us now illuminate the scatterer with the incident wave,

$$u_I\ (\underline{x}, -\underline{k}) = \exp\{-i\,\underline{k}\cdot\underline{x}\}, \qquad (2.12)$$

and observe at the point $-\underline{x}$. The range-and-phase normalized scattering amplitude for this case is then $\rho\,(-\underline{k}, -\hat{x})$. In this case, it is D that is illuminated and $\rho(-\underline{k},-\hat{x})$ has a representation similar to (2.1) except that the $\underline{k}$ and $\hat{x}$ have changed sign:

$$\rho(-\underline{k}, -\hat{x}) = \int_D i\hat{n}\cdot\left[\underline{k} - k\hat{x}\right] \exp\{-i\underline{x}'\cdot\left[\underline{k} - k\hat{x}\right]\}\, dS' \qquad (2.13)$$

We take complex conjugate (denoted by *) in this equation and add to (2.11) to obtain

$$\rho(\underline{k},\hat{x}) + \rho^*(-\underline{k}, -\hat{x}) = -\int_{L\,\&\,D} i\hat{n}\cdot\left[\underline{k} - k\hat{x}\right] \exp\{i\underline{x}\cdot\left[\underline{k} - k\hat{x}\right]\}\, dS' \qquad (2.14)$$

The integrand here can be continued to the interior of B as a regular function. Thus we may apply the divergence theorem to rewrite the right side as a volume integral:

$$\rho(\underline{k},\hat{\ell}) + \rho^{*}(-\underline{k},\hat{\ell}) = \left[\underline{k}-k\hat{\ell}\right]^2 \int_B \exp\{i\underline{x}'\cdot\left[\underline{k}-k\hat{\ell}\right]\}dV' \tag{2.15}$$

Let us denote by $\gamma(\underline{x})$ the charaacertistic function of the scatterer B and simply rewrite (2.15) as

$$\frac{\rho(\underline{k},\hat{x}) + \rho^{*}(-\underline{k},-\hat{x})}{(\underline{k}-k\hat{x})^2} = \int_B \gamma(x') \exp\{i\underline{x}'\cdot\left[\underline{k}-k\hat{\ell}\right]\} dV' \tag{2.16}$$

where now the domain of integration is all of space.

We observe from (2.11) and (2.13) that both $\rho(\underline{k},\hat{\ell})$ and $\rho^{*}(-\underline{k},-\hat{\ell})$ are functions of $\underline{k}-k\hat{\ell}$. Thus, let us define

$$\underline{\kappa} = \underline{k}-k\hat{\ell} \tag{2.17}$$

and

$$\hat{\gamma}(\underline{\kappa}) = \frac{\rho(\underline{k},\hat{\ell}) + \rho^{*}(-\underline{k},-\hat{\ell})}{\kappa^2} \tag{2.18}$$

Then, (2.16) can be rewritten as

$$\hat{\gamma}(\underline{\kappa}) = \int \gamma(\underline{x}') \exp\{i\underline{x}'\cdot\underline{\kappa}\} dV' \tag{2.19}$$

We see here that the observed fields have produced the Fourier transform of the characteristic function of the scatterer.

If $\gamma(\underline{\kappa})$ is known for all $\underline{\kappa}$, then we can invert (2.19) to yield

$$\gamma(\underline{x}) = \frac{1}{(2\pi)^3} \int \hat{\gamma}(\underline{\kappa}) \exp\{i\underline{x}\cdot\underline{\kappa}\} \, d\kappa^3 \tag{2.20}$$

In the earlier back scattering derivation,

$$k\hat{R} = -\underline{k}; \ \underline{\kappa} = 2\underline{k} \tag{2.21}$$

Thus, for this case one illuminates and receives at all wave numbers and in all directions in order to cover all $\underline{\kappa}$. For the general result in which $\underline{\kappa}$ is given by (2.17) rather than (2.21), the obstacle is illuminated from only two directions, $\pm\hat{R}$ and at one wave number k. Then, observations in all directions fill out a sphere in κ space with center at $\underline{k}$ and radius k, as shown in Figure 2.2.

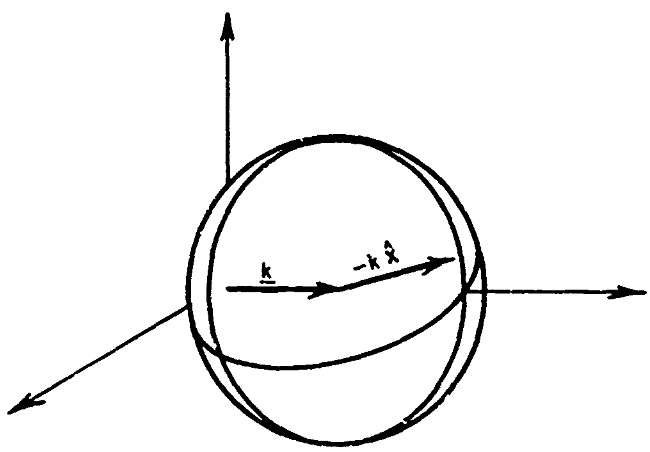


FIGURE 2.2

It is well known in radar analysis that the physical optics approximation is poor in the bistatic case if the directions of transmission and observation are too widely separated; i.e., if the angle between $\underline{k}$ and $-\hat{R}$ becomes too large. Nonetheless, for a cone of small angle, such as is shown in Figure 2.3, the bistatic result (2.17, 19) still remains valid. In conjunction with the limited aperture discussion in Section IV, this observation should prove quite useful.

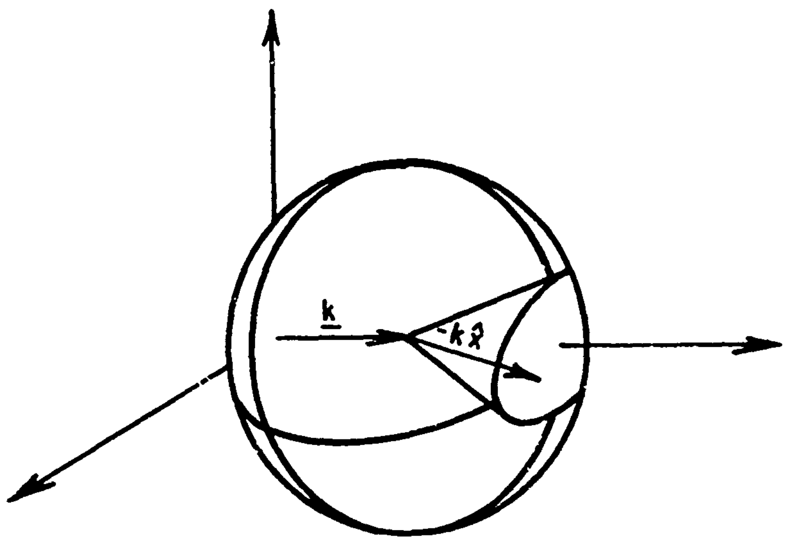


FIGURE 2.3

III. PHYSICAL OPTICS NEAR-FIELD INVERSE SCATTERING

In this section we shall develop an integral equation for the characteristic function of a scattering obstacle with the aid of the physical optics approximation but under the constraint that the fields are observed too near the scatterer for the far-field approximations to be valid.

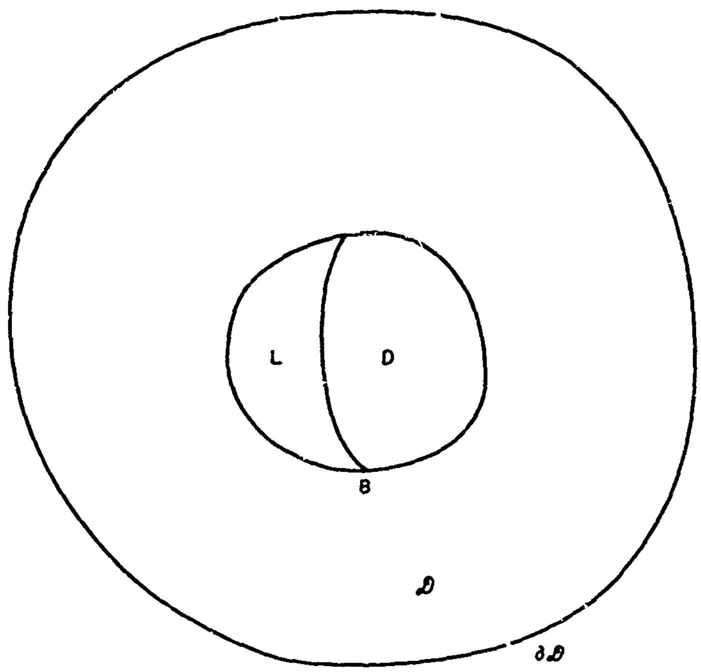


FIGURE 3.1

We suppose that the scattered field and its normal derivative are observed on the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$ of Figure 3.1. As in the previous section, we assume that u_S is the response to the incident field u_I of (2.1) subject to the conditions (2.2, 3, 4). The result (2.5) is still valid as well. Now, however, we also write down the solution representation

$$-u_S(\underline{x},\underline{k}) = \int_{\partial B} \{u_S(\underline{x}',\underline{k}) \frac{\partial g^*}{\partial n'}(\underline{x}-\underline{x}') - g^*(\underline{x}-\underline{x}') \frac{\partial u_S}{\partial n'}(\underline{x}',\underline{k}\} dS + \Theta(\underline{x},\underline{k}), \qquad \underline{x} \in \mathcal{D} \tag{3.1}$$

Here (*) denotes complex conjugate and

$$\Theta(\underline{x},\underline{k}) = -\int_{\partial\mathcal{D}} \{u_S(\underline{x}',\underline{k}')\ \frac{\partial g^*}{\partial n'}\ (\underline{x}-\underline{x}') - g^*(\underline{x}-\underline{x}')\ \frac{\partial u_S}{\partial n'}\ (\underline{x}',\underline{k})\}\ dS' \tag{3.2}$$

The integral Θ arises in this representation because g^* does not satisfy the radiation condition that g does. We note, however, that Θ contains in it the data observed on $\partial\mathcal{D}$ and is thus a known quantity.

By subtracting (3.1) from (2.5) we obtain

$$\int_{\partial B} \{u_S(\underline{x}',\underline{k})\ \frac{\partial g_I}{\partial n'}\ (\underline{x}-\underline{x}') - g_I(\underline{x}-\underline{x}')\ \frac{\partial u_S}{\partial n'}\ (\underline{x}',\underline{k})\}\ dS' = \Theta(\underline{x},\underline{k}),\ \underline{x} \in \mathcal{D} \tag{3.3}$$

Here

$$g_I(\underline{x}) = g(\underline{x}) - g^*(\underline{x}) = \frac{i \sin k x}{2\pi x} \tag{3.4}$$

We now make the physical optics approximations (2.7) and obtain

$$\int_L \hat{n} \cdot \nabla \left\{g_I(\underline{x}-\underline{x}')\ \exp\ \{i\underline{k}\cdot\underline{x}'\}\right\}\ dS' = \Theta(\underline{x},\underline{k}),\quad \underline{x} \in \mathcal{D} \tag{3.5}$$

As in Section II we now carry out the same analysis for the incident wave

$$u_I\ (\underline{x},-k) = u_I^*\ (\underline{x},\underline{k}) = \exp\ \{-i\ \underline{k}\cdot\underline{x}\} \tag{3.6}$$

and obtain

$$-\int_{D} \hat{n}\cdot\nabla \left\{ g_I(\underline{x}-\underline{x}') \exp\{-i\,\underline{k}\cdot\underline{x}\} \right\} dS' = \Theta(\underline{x},-\underline{k}), \quad \underline{x} \in \mathcal{D} \tag{3.7}$$

the domain of integration being D here because this is the surface illuminated by the incident field (3.6).

We take complex conjugate in (3.7), using (3.4), and subtract this result from (3.5) to obtain

$$\int_{\partial B} \hat{n}\cdot\nabla \left\{ g_I(x-x') \exp\ \{i\underline{k}\cdot\underline{x}'\} \right\} dS' = \Theta_I(\underline{x},\underline{k}), \ \underline{x} \in \mathcal{D} \tag{3.8}$$

Here

$$\Theta_I(\underline{x},\underline{k}) = \Theta^*(\underline{x},-\underline{k}) - \Theta(\underline{x},\underline{k}) \tag{3.9}$$

As in Section II, we can now apply the divergence theorem to rewrite the left side of (3.8) as a volume integral. Indeed, here and in the previous section, it is the physical optics approximation which makes this crucial step straightforward. The result is

$$\int_{B} \nabla^2 \left\{ g_I(\underline{x}-\underline{x}') \exp\ \{i\underline{k}\ \underline{x}'\} \right\} dV' = \Theta_I(\underline{x},\underline{k}), \quad x \in \mathcal{D} \tag{3.10}$$

We may rewrite this equation in the form

$$\int_{\mathcal{D}} \nabla^2 \left\{ g_I(\underline{x}-\underline{x}') \exp\{i\underline{k}\cdot\underline{x}'\} \right\} \gamma(\underline{x}')\, dV' = \Theta_I(\underline{x},\underline{k}), \quad \underline{x} \in \mathcal{D} \tag{3.11}$$

This is a Fredholm integral equation of the First Kind, the first of many to be encountered here. It is well known that these equations are ill-posed, in general, with all three, existence, uniqueness and continuous dependence on the data open to question. We expect that these difficulties can only be resolved when $\gamma(\underline{x})$ is assumed to be non-zero only over a finite domain interior to $\mathcal{D}$ and $\mathcal{D}$ itself is a finite domain. Indeed, because of this latter constraint, it is expected that the result of Section II cannot be derived by simple limiting arguments from (3.12).

IV. LIMITED APERTURE OF OBSERVATIONS

We return now to (2.19) and assume that we have only a "limited aperture" of observations; that is, we assume that $\hat{\gamma}(\underline{\kappa})$ is known only over some finite domain as shown in Figure 4.1.

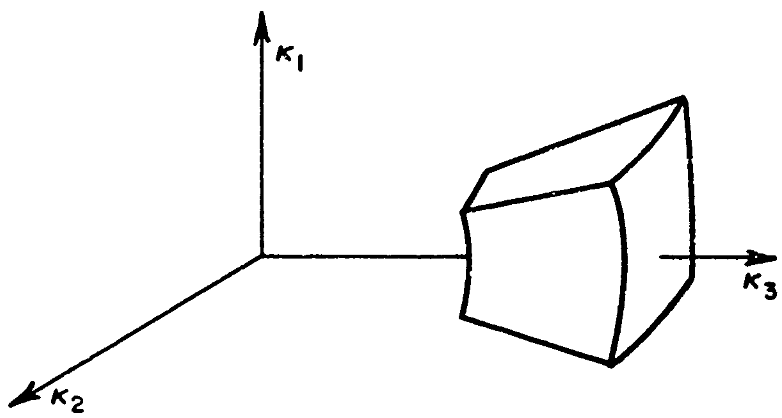


FIGURE 4.1
Domain in which $\hat{\gamma}(\underline{\kappa})$ is known.

Let us define the characteristic function of that domain to be

$$\hat{a}(\underline{\kappa}) = \begin{cases} 1 \;, & \hat{\gamma}(\underline{\kappa}) \text{ is known} \\ 0 \;, & \hat{\gamma}(\underline{\kappa}) \text{ is not known} \end{cases} \tag{4.1}$$

and denote by $a(x)$ its Fourier transform. Then, in fact, what we really know is the function $\hat{a}(\underline{\kappa})\hat{\gamma}(\underline{\kappa})$. By the Fourier convolution theorem, this product is the Fourier transform of the convolution of $\gamma(\underline{x})$ with $a(\underline{x})$:

$$\frac{1}{(2\pi)^3}\int \hat{a}(\underline{\kappa})\;\hat{\gamma}(\underline{\kappa})\;\exp[-i\underline{\kappa}\cdot\underline{x}]\;d\kappa^3 = \int a(\underline{x}-\underline{x}')\;\gamma(\underline{x}')\;dx'^3 \tag{4.2}$$

Here, the left side is known and this is a Fredholm integral equation of the First Kind with the kernel $a(x)$ being a generalization of the sin x/x kernel which would arise in one dimension. Again, we note that, in general, these integral equations are known to be ill-posed with difficulties as to all three aspects of existence, uniqueness and continuous dependence on the data. Nonetheless, for this particular equation, the second author Bojarski (1973, 1974) has been successful at obtaining solutions.

We remark further that once the idea of multiplication of $\hat{\gamma}(\underline{\kappa})$ is introduced, it may be exploited in another way. In particular, we note that is is not $\gamma(\underline{x})$ that we seek really, but its derivatives. Indeed, it is these functions that describe the surface of the scatterer rather than the scatterer itself. In Bojarski (1974), this idea is exploited to develop a check on the method. The function $\rho(\underline{\kappa})$ is calculated from the exact Mie series solution for scattering by a sphere. The inverse transform of $i\kappa_3\hat{\gamma}(\underline{\kappa})$ is calculated, this function yielding $\frac{\partial}{\partial x_3}\gamma(x)$. Figure 4.2 is a graph of this function in the (x_1, x_2) plane.

The formulation of Section III can also be modified to take account of limited aperture. To this end, let us suppose that $\partial\mathfrak{D}$ in Figure 3.1 is decomposed into two parts $\partial\mathfrak{D}_1$, $\partial\mathfrak{D}_2$ as in Figure 4.3.

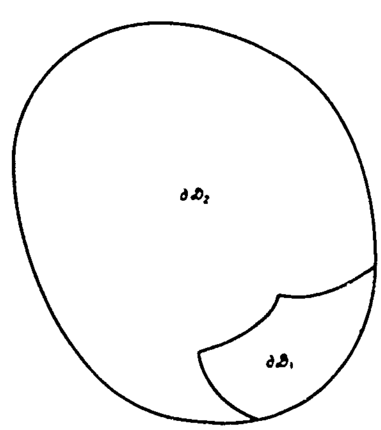
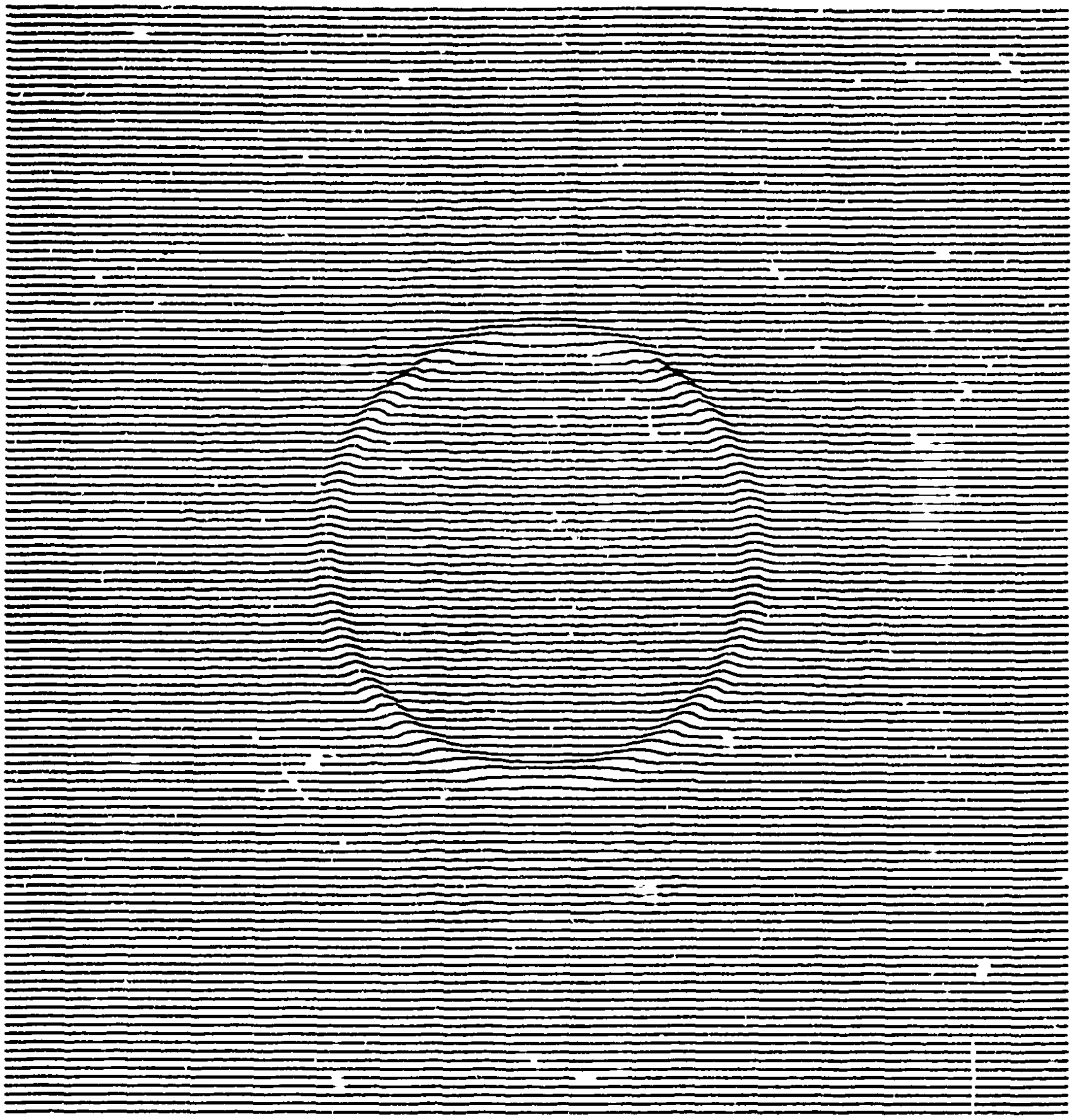


FIGURE 4.3



TWO-DIMENSIONAL DISPLAY OF RECONSTRUCTED SPHERE FROM SYNTHETIC SCATTERING DATA

FIGURE 4.2

On $\partial\mathcal{D}_1$ the data u and $\frac{\partial u}{\partial n}$ are observed, on $\partial\mathcal{D}_2$ they are not. We return to the definition of Θ in (3.9) and decompose the domain of integration into $\partial\mathcal{D}_1 + \partial\mathcal{D}_2$; that is, rewrite (3.2) as

$$\Theta(\underline{x},\underline{k}) = \Theta^{(1)}(\underline{x},\underline{k}) + \Theta^{(2)}(\underline{x},\underline{k}) \tag{4.3}$$

Here

$$\Theta^{(j)}(\underline{x},\underline{k}) = \int_{\partial D_j}\left\{u_S(\underline{x}',\underline{k}')\frac{\partial g^*}{\partial n'}(\underline{x}-\underline{x}') - g^*(\underline{x}-\underline{x}')\frac{\partial u_S}{\partial n'}(\underline{x}',\underline{k})\right\}dS',\ j = 1, 2. \tag{4.4}$$

In $\Theta^{(2)}$, where we do not know the data, we replace u_S and $\frac{\partial u_S}{\partial n}$ by

$$u_S(\underline{x}',\underline{k}) = \int_{\partial B}\left\{u_S(\underline{x}'',\underline{k})\frac{\partial g}{\partial n''}(\underline{x}''-\underline{x}') - g(\underline{x}''-\underline{x}')\frac{\partial u_S}{\partial n''}(\underline{x}'',\underline{k})\right\}dS'' \tag{4.5}$$

and

$$\frac{\partial u_S}{\partial n'}(\underline{x}',\underline{k}) = \int_{\partial B}\left\{u_S(\underline{x}'',\underline{k})\frac{\partial^2 g}{\partial n'\partial n''}(\underline{x}''-\underline{x}') - \frac{\partial g}{\partial n'}(\underline{x}'',-\underline{x}')\frac{\partial u_S}{\partial n''}(\underline{x}'',\underline{k})\right\}dS'' \tag{4.6}$$

As in Section III, we can make the physical optics approximations here and repeat for $\Theta^*(\underline{\mathbf{x}},-\underline{k})$. The result is

$$\begin{aligned}\Theta^{(2)*}(\underline{x},\underline{k}) + \Theta^{(2)}(\underline{x},-\underline{k}) &= \Theta_I^{(2)}(\underline{x},\underline{k})\\ &= -\int_{\partial\mathcal{D}_2}dS'\int_{L\&D}dS''\left[\frac{\partial g^*}{\partial n'}(\underline{x}-\underline{x}')\frac{\partial}{\partial n''}\left\{u_I(\underline{x}'',\underline{k})\,g(\underline{x}''-\underline{x}')\right\}\right.\\ &\qquad\left.-g^*(\underline{x}-\underline{x}')\frac{\partial}{\partial n''}\left\{u_I(\underline{x}'',\underline{k})\frac{\partial g}{\partial n'}(\underline{x}''-\underline{x}')\right\}\right]\end{aligned} \tag{4.7}$$

Again, we can replace the surface integral by a volume integral. The result is

$$\Theta_I^{(2)}(\underline{x},\underline{k}) = \int_D \gamma(\underline{x}'') \, K(\underline{x}'',\underline{x}) \, dV'' \tag{4.8}$$

Here

$$K(\underline{x}'',\underline{x}) = \int_{\partial D_2} \left\{ g(\underline{x}''-\underline{x}') \frac{\partial g^*}{\partial n'} (\underline{x}-\underline{x}') - g^*(\underline{x}-\underline{x}') \frac{\partial g}{\partial n'} (\underline{x}''-\underline{x}') \right\} dS' \tag{4.9}$$

Now the integral equation (3.11) becomes

$$\int_D \left[\nabla^2 \left\{ g_I(\underline{x}-\underline{x}') \exp(i\underline{k}\cdot\underline{x}') \right\} + K(\underline{x}',\underline{x}) \right] \gamma(\underline{x}') = \Theta_I^{(1)}(\underline{x},\underline{k}), \tag{4.10}$$

$$\Theta_I^{(1)}(\underline{x},\underline{k}) = \int_{\partial D_1} \left\{ u_S(\underline{x}') \frac{\partial g_I^*}{\partial n'} (\underline{x}-\underline{x}') - g_I^* (\underline{x}-\underline{x}') \frac{\partial u_S}{\partial n'} (\underline{x}') \right\} dS' \tag{4.11}$$

Equation (4.10) is again a Fredholm integral equation of the First Kind for $\delta(\underline{x})$. It is expected that the equation becomes more and more ill-posed as $\partial\mathcal{D}_2$ is the entire surface $\partial\mathcal{D}$, $\Theta_I^{(1)} = 0$ and the equation is homogeneous. In the opposite limit $\partial\mathcal{D}_1 = \partial\mathcal{D}$ and this equation reduces to (3.11).

V. THE INVERSE SOURCE PROBLEM FOR THE REDUCED WAVE EQUATION

We consider now the inhomogeneous equation

$$Lu = (\nabla^2, + k^2)\; u = -\rho(\underline{x}) \tag{5.1}$$

for some outgoing wave u. We assume that the source $\rho(\underline{x})$ is confined to the interior of some domain $\mathcal{D}$ (Figure 5.1) and that both u and $\frac{\partial u}{\partial n}$ are

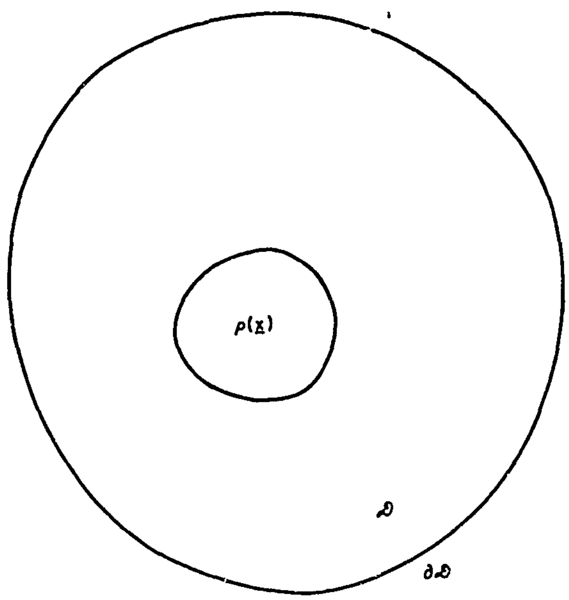


FIGURE 5.1

observed over the entire boundary $\partial\mathcal{D}$. We seek $\rho(\underline{x})$ from this data. Here we shall derive an integral equation for $\rho(\underline{x})$.

We begin with the observation that

$$u(\underline{x}) = \int_{\mathcal{D}} \rho(\underline{x}')\; g(\underline{x}-\underline{x}')\quad dV', \tag{5.2}$$

with g the free space Green's function (2.6). We are justified in restricting the domain of integration to $\mathfrak{D}$ here because $\rho(\underline{x})$ is non-zero only inside of $\mathfrak{D}$. Let us now consider a volume integral similar to the one used to derive (5.2), but with g replaced by g^*, namely we set

$$\Theta(\underline{x}) = \iint_{\mathfrak{D}} \left[g^*(\underline{x}-\underline{x}') \, Lu(\underline{x}') - u(\underline{x}') \, Lg^*(\underline{x}-\underline{x}') \right] dV' \tag{5.3}$$

By using the property that $Lg^*(\underline{x}) = \delta(\underline{x})$, we find that

$$\Theta(\underline{x}) = u(\underline{x}) - \int_{\mathfrak{D}} g^*(\underline{x}-\underline{x}') \, \rho(\underline{x}') \, dV', \quad \underline{x} \in D \tag{5.4}$$

By using Green's theorem, we also find that

$$\Theta(\underline{x}) = -\iint_{\partial\mathfrak{D}} \left\{ u(\underline{x}') \frac{\partial g^*}{\partial n'} (\underline{x}-\underline{x}') - g^*(\underline{x}-\underline{x}') \frac{\partial u}{\partial n'} (\underline{x}') \right\} dS'. \tag{5.5}$$

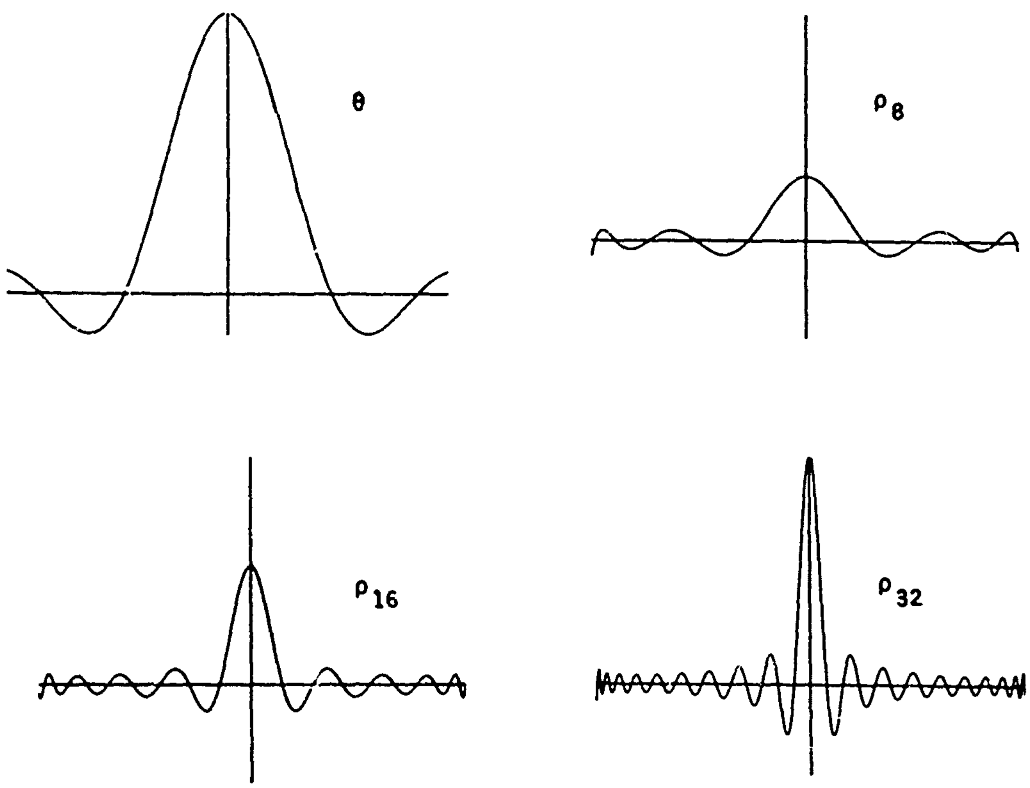
Here, the integrand is known for u and $\frac{\partial u}{\partial n}$ observed on $\partial\mathfrak{D}$. We remark that a similar integral appears in the derivation of (5.2). However, by invoking the outgoing property for both u and g, one can show that the surface integral is zero in that case; see, for example, Morse & Feshbach (1953) or Baker & Copson (1950). Thus, the process that yields so simple a result, (5.2), for the direct problem, annihilates the data that is known for the inverse problem, while the same process applied to (5.3) does not do this.

We use (5.2) and (5.5) in (5.4) now to obtain

$$\int_{\mathfrak{D}} g_I(\underline{x}-\underline{x}') \quad \rho(\underline{x}') \, dV' = \Theta(x), \tag{5.6}$$

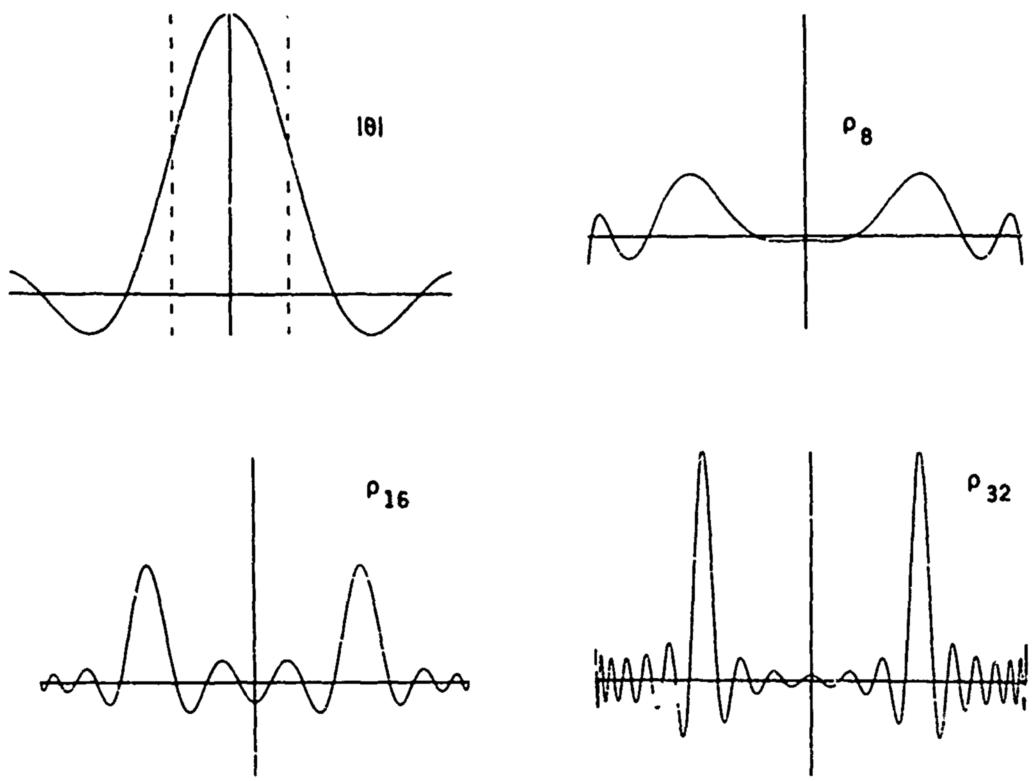
and this is a Fredholm integral equation of the First Kind for $\rho(\underline{x})$.

As a check on (5.6), Bojarski (1974) has carried out two examples. In each, the function $\Theta(\underline{x})$ as defined by (5.5) is generated from a known source. The check then consists of solving (5.6) for $\rho(\underline{x})$ and comparing with the known source. The technique of solution is a relaxation iteration method. The sources treated are a point source and then two point sources, one-half wave length apart. Results after 8, 16 and 32 iterations are shown in Figures 5.2 and 5.3, respectively. The second example strongly suggests that the method is not Raleigh diffraction limited.



RECONSTRUCTED POINT SOURCE

FIGURE 5.2



RECONSTRUCTED POINT-SOURCE-PAIR

FIGURE 5.3

VI. LIMITED APERTURE FOR THE INVERSE SOURCE PROBLEM

We now suppose that u and $\frac{\partial u}{\partial n}$ are observed only on a portion of $\partial\mathcal{D}$, namely on $\partial\mathcal{D}_1$ in Figure 4.3. As in Section VI, we define $\Theta^{(1)}$ and $\Theta^{(2)}$ by (4.3) and (4.4) with $\Theta(\underline{x},\underline{k})$ and u_S $u_S(\underline{x}',\underline{k})$ replaced by $\Theta(\underline{x})$ and $u(\underline{x}')$, respectively. In the integral over $\partial\mathcal{D}_2$ we use (5.2) to obtain an expression completely analogous to (4.8), namely that

$$\Theta^{(2)}(\underline{x}) = \int_{\mathcal{D}} \rho(\underline{x}'') \; K(\underline{x}'',\underline{x}) \; dV'', \tag{6.1}$$

and $K(\underline{x}'',\underline{x})$ is given by (4.9). We then obtain instead of (5.3), the modified integral equation

$$\iint_{\mathcal{D}} \left[g_I(\underline{x}-\underline{x}') + K(\underline{x}',\underline{x})\right] \rho(\underline{x}') \; dV' = \Theta^{(1)}(\underline{x}); \tag{6.2}$$

$$\Theta^{(1)}(x) = \int_{\partial\mathcal{D}_1} \left[u(\underline{x}')\frac{\partial g^*}{\partial n'}(\underline{x}-\underline{x}') - g^*(\underline{x}-\underline{x}')\frac{\partial u(\underline{x}')}{\partial n'}\right] dS'.$$

VII. ANALYSIS OF INHOMOGENEITY OF A MEDIUM

Let us suppose that in an otherwise homogeneous medium there is a finite region of inhomogeneity.

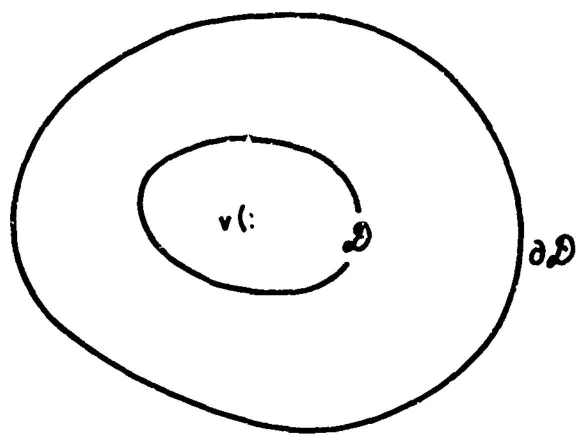


FIGURE 7.1

Our objective is to analyze the inhomogeneity from remotely sensed data. In this section we shall show how this problem can be reduced to the inverse source problem (5.1) and thus that the inverse problem here reduces to the study of equation (5.6).

We suppose that the wave (2.1) (exp $\{i\underline{k}\cdot\underline{x}\}$) incident on the region of inhomogeneity and that the total field $u_I + u_S$ satisfies the equation

$$(\nabla^2 + k^2)\ (u_I + u_S) = -v(\underline{x})\ (u_I + u_S) = -\rho(\underline{x}) \tag{7.1}$$

Here $v(\underline{x})$ is assumed to have finite support.

In fact, u_I satisfies the homogeneous Helmholtz equation in the uniform medium. Thus, (7.1) reduces to

$$(\nabla^2 + k^2)\, u_S = -\rho(\underline{x}) \tag{7.2}$$

We observe u_S and $\frac{\partial u_S}{\partial n}$ on $\partial \mathcal{D}$ and obtain the integral equation (5.6) for ρ. When ρ is known, then u_S can be calculated by using (5.2). Since u_I is known, we can now use the right-hand equation in (7.1) to find $v(\underline{x})$.

In problems in which one medium is imbedded in another (tumors, land mines, sea mines, buried objects in general), $v(\underline{x})$ is discontinuous and (7.1) is valid everywhere except on the interface between the media. On that interface, certain jump conditions must be satisfied. These conditions are simply the integral form of (7.1) across the interface. As a consequence of this, one can verify that even in this case, $\rho(\underline{x})$ must satisfy (5.6).

VIII. THE TIME-DEPENDENT INVERSE SOURCE PROBLEM

We suppose now that $U(\underline{x},t)$ is a solution of the wave equation

$$\mathcal{L}U = \nabla^2 U - \frac{1}{c^2} U_{tt} = -P(\underline{x},\underline{t}), \tag{8.1}$$

with U, U_t and P assumed to be zero initially (sufficiently far in the past). We assume that U and $\frac{\partial U}{\partial n}$ are observed on $\partial\mathcal{D}$ of Figure 5.1 and that P is confined to the interior of $\mathcal{D}$. We shall derive an integral equation for P.

We begin by observing, analogous to (5.2) that

$$U(\underline{x},\underline{t}) = \int_0^T \int_{\mathcal{D}} P(\underline{x}',t')\, G(\underline{x}-\underline{x}',t-t')\, dV'\, dt', \quad t \leq T. \tag{8.2}$$

Here G is the "causal" Green's function. We also introduce the "effectal" Green's function G*:

$$G(\underline{x},t) = \frac{\delta(t-x/c)}{4\pi x}, \quad G^*(\underline{x},t) = \frac{\delta(t+x/c)}{4\pi x} \tag{8.3}$$

As in Section V, we consider the function

$$\Psi(\underline{x},t,T) = \int_0^T \int_{\mathcal{D}} \Big\{ U(x',t')\mathcal{L}G^*(\underline{x}-\underline{x}',t-t') - G^*(\underline{x}-\underline{x}',t-t')\,\mathcal{L}U(\underline{x}',t') \Big\}\, dV'dt', \tag{8.4}$$

and observe that

$$\Psi(x,t,T) = -U(x,t) + \int_0^T \int_{\mathcal{D}} G^*(\underline{x}-\underline{x}',t-t')\; P(\underline{x}',t')\; dV'dt'; \qquad \underline{x} \in \mathcal{D},\; 0 \leq t \leq T. \tag{8.5}$$

As in Section V, we apply the divergence theorem (now in $\underline{x}$, t) and obtain

$$\Psi(\underline{x},t,T) = \Theta(\underline{x},t,T) + \Phi(\underline{x},t,T) \tag{8.6}$$

Here

$$\Theta(x,t,T) = \int_0^T \int_{\partial\mathcal{D}} \Big\{ U(\underline{x}',t')\,\frac{\partial G^*}{\partial n'}(\underline{x}-\underline{x}',t-t') - G^*(\underline{x}-\underline{x}',t-t')\frac{\partial U}{\partial n'}(\underline{x}',t') \Big\}\, dS'dt' \tag{8.7}$$

and

$$\Phi(\underline{x},t,T) = -\frac{1}{c^2} \int_{\mathcal{D}} \Big\{ U(\underline{x}',T)\frac{\partial}{\partial T}\, G^*(\underline{x}-\underline{x}',t-T) - G^*(\underline{x}-\underline{x}',t-T)\,\frac{\partial U}{\partial T}(\underline{x}',T) \Big\}\, dV'. \tag{8.8}$$

We use (8.2, 6, 7, 8) in (8.5) and obtain

$$\int_0^T \int_{\mathcal{D}} G_I(\underline{x}-\underline{x}', t-t') P(\underline{x}', t')\, dV' dt'$$

$$= \Theta(\underline{x}, t, T) + \Phi(\underline{x}, t, T),\ \underline{x} \in \mathcal{D},\ 0 \le t \le T. \tag{8.9}$$

Here

$$G_I(\underline{x}, t) = G(\underline{x}, t) - G^*(\underline{x}, t) \tag{8.10}$$

If the source is "shut off" after a finite time or decays "sufficiently rapidly", then $\lim_{T \to \infty} \Phi = 0$.
In this case, we may let $T \to \infty$ in (8.9) and obtain the inverse transform ($k = \omega/c$) of (5.6) as the equation for the source in space time. When P does not vanish for $T \to \infty$ we are faced with the additional difficulty of having only a limited aperture (0, T) of observations in time. We deal with this difficulty as in previous sections by using (8.2) in (8.8). The result is that $P(\underline{x}, t)$ must satisfy the integral equation

$$\int_0^T \int_{\mathcal{D}} \left[G_I(\underline{x}-\underline{x}', t-t') + K(\underline{x}, \underline{x}', t, t'; T)\right] P(\underline{x}', t')\, dV' dt'$$
$$= \Theta(\underline{x}, t, T),\ \underline{x} \in \mathcal{D},\ 0 \le t \le T \tag{8.11}$$

and

$$K(\underline{x}, \underline{x}', t, t'; T) = -\frac{1}{c^2} \int_{\mathcal{D}} \left\{ G(\underline{x}'-\underline{x}'', T-t') \frac{\partial}{\partial T} G^*(\underline{x}-\underline{x}'', t-T) \right.$$

$$\left. - G^*(\underline{x}-\underline{x}'', t-T) \frac{\partial}{\partial T} G(\underline{x}'-\underline{x}'', T-t') \right\} dV''. \tag{8.12}$$

One can certainly modify this equation further to take account of a limited aperture of observations in space. However, this modification is carried out exactly as in Section IV and VI and we shall not do so here.

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