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APPROXIMATION OF OPTIMAL SOLUTIONS FOR INFINITE HORIZON LINEAR PROGRAMS

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APPROXIMATION OF OPTIMAL SOLUTIONS FOR INFINITE HORIZON LINEAR PROGRAMS†

by

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University of California, Berkeley

DECEMBER 1974

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ABSTRACT

This paper defines infinite horizon linear programs and presents a procedure that will approximate the optimal solution of almost any infinite horizon linear program that has a finite optimal value. In addition, it is demonstrated that other procedures for calculating optimal solutions will not, in general, approximate the optimal solution.

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0. INTRODUCTION

This paper examines long range planning models that can be presented as linear programs over an infinite planning horizon. The main results characterize problems that have finite optimal values and establish procedures for approximating optinal solutions by solving a T period linear program. Three possible approaches to this task are examined and it is demonstrated that only one procedure leads to solvable T period problems. The T period problem is designed by decoupling the infinite problem into the sum of a T period problem and another infinite problem that commences at time T; call these problems 1 and 2. Any feasible solution of problem 1 produces an input into problem 2. The scheme calculates an approximate *salvage value* of the input from problem 1 to problem 2. This salvage value is then included in the objective of problem 1. As T increases the error in calculating the salvage value becomes less and less significant. The main conclusion of the paper are easily stated: (1) the only defensible way to solve infinite horizon linear programs is by the approximation technique outlined in section four; (2) duality is not the primary consideration in the study of infinite horizon linear programs; (3) the calculation of an equilibrium optimal policy, if one in fact exists, does not, in general, assist in the solution of a discounted criterion problem.

The remainder of this section introduces the problem, summarizes results, relates this work to others, and describes the notational conventions used in the paper.

Time is discrete $t = 0,1,2, \ldots,$. At any time t the state of the system is s . Decisions x_t are constrained by the relations $Ax_t = s$, $x_t \geq 0$. If decision x_t is taken, then a reward with time zero value $\alpha^t px_t$ is received and the state at time $t + 1$ is $b + Kx_t$. A and K are $m \times n$ matrices, p an n vector, b an m vector, s an m vector and α a positive scalar less than 1.

If the initial (time zero) state is s then the infinite sequence of decisions $\{x_t\}$ is constrained by

$$
\text{(1)} \qquad \begin{aligned} &Ax_o = s \ , \ x_o \geq 0 \\ &Ax_t = b + Kx_{t-1} \ , \ x_t \geq 0 \qquad t \geq 1 \end{aligned}
$$

Let $X(s)$ be the set of $\{x_t\}$ that satisfy (1). For any $\{x_t\}$ define

$$
\text{(2)} \qquad p(\{x_t\}) = \liminf_{T \to \infty} \sum_0^T \alpha^t px_t \ .
$$

The optimization problem under consideration is

$$
\text{(3)} \qquad \begin{aligned} &\text{Maximize} \quad p(\{x_t\}) \\ &\text{subject to} \quad \{x_t\} \in X(s) \ . \end{aligned}
$$

Let e be a vector of ones, a summation vector. A sequence $\{x_t\}$ is called α-*convergent* if the series of increasing nonnegative terms $\sum_0^T \alpha^t ex_t$ converges to a finite limit.

Denote $\tilde{X}(s)$ as the set of α-convergent $\{x_t\} \in X(s)$, and note that $p(\{x_t\}) = \sum_0^\infty \alpha^t px_t$ for $\{x_t\} \in \tilde{X}(s)$.

Section one presents assumptions made in the paper and comments on their immediate consequences and the problem of verification. These assumptions hold throughout and are not restated for each result. Section two defines the optimal value of problem (3) as a function of s. The consequences of the assumptions not holding are investigated in sections two and three. Section three contains an important result: if a program $\{x_t\}$ is feasible but not α-convergent, then $\{x_t\}$ is a bad program in the sense that $p(\{x_t\}) = -\infty$.

Section four describes a sequence of solvable T period linear programs that can be used to solve (3). The optimal solutions and optimal values of the T period problems converge to an optimal solution and the optimal value of the infinite problem. Section five presents sufficient optimality criterea and comments on duality and the establishment of necessary conditions for optimality. Section six examines the question of calculating optimal equilibrium policies; i.e., $x_t = x$ for all t. The appendix contains the statement and proof of two lemmas that are used in the main body of the study.

This paper is based on previous work by Manne [13], Hopkins [11],[12], Hopkins and Grinold [18], and Evers [3]. The work of Evers, [3] motivated this study and is the genesis for the important assumption II in section 1. Assumptions like II are implicitly made in [13] and [8], however, Evers was the first to state and exploit fully this type of assumption. The assumptions as stated in section 1 are satisfied by the class of problems considered in [3]. In fact, in sections 2 and 3 it is demonstrated that the assumptions are nearly the most general possible. Except for certain boundary cases, failure of assumption I or II implies the optimal value of (3) is either $+\infty$ or $-\infty$. The theorem in section 2 generalizes a result, theorem 5.2, [3]. Evers [3], with more restrictive assumptions, is able to establish several attractive results that characterize optimal solutions. However, since the main purpose of this paper is to delineate the class of solvable problems and to construct a viable solution procedure; the more restrictive assumptions of [3] are not necessary to accomplish this objective.

The approximation procedure used in section four can be found in [2], and in [6]-[8], and [10]-[13]. In [8] and [2], the procedure solves the infinite problem exactly when $T = 0$. There are, however, no general convergence results available prior to the theorem in section four.

Work on optimality conditions can be found in [3],[5],[6],[11]-[13],[15]. The sufficiency result is identical in spirit to [12], and the remarks on duality are consistent with [9]. In section six optimal equilibrium policies, $x_t = x$ for all t, are discussed. This question has been studied in [1],[4] and [10].

The following notational conventions are used in the paper. The symbol $\geqq 0$ means nonnegative; $\geq$ semi-positive; and $>$ strictly positive. The vector e is used for summmations; each element of e equals one. Script S is used as a subsequence of integers and $S_1 \subseteqq S_0$ means that S_1 is a refinement of S_0. Frequent use is made of the Heine-Borel theorem; every sequence in a closed bounded set of R^n has a limit point in that set. Equations are numbered within each section.

1. ASSUMPTIONS

This section presents and explores the assumptions used in all the following sections. The assumptions are

I: There exists an α-convergent solution $\{x_t\} \in \tilde{X}(s)$.

II: For every λ , $0 < \lambda \leqq \alpha$, there exists a solution (u,v) of the equations

$$u(A - \lambda K) = v + p , \quad v > 0 . \tag{1}$$

III: One of the following holds.

(i) $uA = v + p$, $v > 0$ has a solution

(ii) For some $0 < \lambda \leqq \alpha$, (u,v) solve (1) and either $uA \geqq p$ or $uA \geqq 0$.

(iii) For some $0 < \lambda \leqq \alpha$, (u,v) solves (1) and $uAx_t \geqq 0$ or $(v + \lambda K)x_t \geqq 0$ for all $\{x_t\} \in X(s)$.

We shall see in section 3, that if II and III hold and I does not then either the problem is infeasible or each feasible solution $\{x_t\}$ has $p(\{x_t\}) = -\infty$. A similar, not quite as strong, converse statement is true. Suppose I is true and for some $0 < \lambda \leq \alpha$, there does not exist a solution (u,v) of

$$u(A - \lambda K) = p + v , \quad v \geqq 0 .$$

In this case, as is demonstrated in section 2, the problem is unbounded for almost any value of s .

Both I and II cannot be verified in general. However, it is possible to gain some information about the assumptions. For example, if I is satisfied, then there exists a solution x of the system

$$(A - \alpha K)x = s + \frac{\alpha b}{1-\alpha} , \quad x \geqq 0 . \tag{2}$$

To see this, multiply the t^{th} constraint of (1:1) by α^t, sum and note that $x = \sum_0^\infty \alpha^t x_t$ satisfies (2). If (2) is not feasible, then I fails. There is no conclusive test to verify I. As in linear programming this is equivalent to another infinite horizon linear program.

Assumption II is difficult to verify. However, II and III (ii) will be satisfied if there exists a solution of either

$$u(A - \alpha K) - v = p \ , \ v > 0 \ , \ uA \geqq p \tag{3}$$

or

$$u(A - \alpha K) - v = p \ , \ v > 0 \ , \ uA \geqq 0 \ . \tag{4}$$

For (3) note that

$$A - \lambda K = \left(1 - \frac{\lambda}{\alpha}\right) A + \frac{\lambda}{\alpha} (A - \alpha K) \ .$$

Thus for all $0 < \lambda < \alpha$, we have

$$u(A - \lambda K) \geqq \frac{\lambda}{\alpha} v + p \ .$$

For (4) note that

$$\left(\frac{\alpha}{\lambda}\right) u(A - \lambda K) \geqq u(A - \alpha K) = v + p \ .$$

General verification of II requires that there does not exist a solution of the generalized eigenvalue problem

$$\left(\left[\begin{array}{c|c} p & -1 \\ \hline A & 0 \end{array}\right] - \lambda \left[\begin{array}{c|c} 0 & 0 \\ \hline K & 0 \end{array}\right]\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $x, y \geq 0$, and $0 < \lambda \leq \alpha$.

Assumption III is a patchwork of special cases to cover the apparent loophole in II ($\lambda = 0$). Note that III (i) will be satisfied if II holds for all λ , $0 \leqq \lambda \leqq \alpha$, or, more directly, if either $Ax = 0$, $x \geq 0$ has no solution or $Ax = 0$, $x \geq 0$ implies $px < 0$, or $Ax = 0$, $x \geq 0$, $px \geqq 0$ implies $Kx = 0$. Assumption III is only necessary if $Ax = 0$, $x \geq 0$, $px \geqq 0$, $Kx \neq 0$ has a solution. Items (ii) and (iii) are directed to this particular case. It is a reasonable conjecture that assumption III is not necessary. At present, theorem 1, section 3, cannot be proved without III, and it has not been possible to construct a counter example based on the failure of III. The details in III (ii) and (iii) are included to cover all the cases used in Evers [3], and, more significantly, to indicate the failure of a possible solution procedure: see example 3 in section 4. In particular, III (iii) covers the "primal directed" case [3] in which $A = (B,I)$, $K = (H,0)$ and $p = (q,0)$. Each row of B is either nonnegative, or the corresponding row of H and element of b are nonnegative. This assures $Ax_t \geq 0$ for all t and $\{x_t\} \in X(s)$. Moreover, $u(A - \alpha K) = v + p$ implies $u \geqq 0$; therefore III (iii) is satisfied.

2. THE OPTIMAL VALUE FUNCTION

Define $V(s)$ to be the optimal value of the optimization problem as a function of the initial state s .

$$V(s) = \sup\{p(\{x_t\}) \mid \{x_t\} \in X(s)\} \tag{1}$$

Let $F = \{s \mid X(s) \neq \phi\}$, be the set initially feasible states. If $s \notin F$, then $V(s)$ is defined to be $-\infty$. This section demonstrates that V is convex and satisfies a dynamic programming functional equation. The section closes by examining the consequence of assumption II not holding.

Let $\mathrm{dom}\ V = \{s \mid V(s) > -\infty\}$. Both F and $\mathrm{dom}\ V$ are convex; and $\mathrm{dom}\ V \subseteq F$. Let $\tilde{F} \subseteq F$ be the set of s such that I holds. $\tilde{F} \subseteq \mathrm{dom}\ V \subseteq F$.

Proposition 1:

$V(s)$ is concave.

Proof:

Following Rockafellar, [14], p. 25, note that V is concave if and only if all $\lambda, \gamma^1, \gamma^2, s^1, s^2$ satisfying

$$0 < \lambda < 1 \ , \ V(s^1) > \gamma^1 \ , \ V(s^2) > \gamma^2$$

imply that

$$V((1 - \lambda)s^1 + \lambda s^2) > (1 - \lambda)\gamma^1 + \lambda\gamma^2 \ .$$

Select $\{x_t^i\} \in X(s^i)$ for $i = 1,2,$ such that $\gamma^i < p(\{x_t^i\}) < V(s^i)$. Note that $\{(1 - \lambda)x_t^1 + \lambda x_t^2\} \in X((1 - \lambda)s^1 + \lambda s^2)$ and that

$$V((1 - \lambda)s^1 + \lambda s^2) \geqq p\{(1 - \lambda)x_t^1 + \lambda x_t^2\} \geqq$$

$$(1 - \lambda)p\{x_t^1\} + \lambda p\{x_t^2\} > (1 - \lambda)\gamma^1 + \lambda\gamma^2 \ .$$

In addition, V satisfies a dynamic programming functional equation.

Proposition 2:

On the convex set F , V satisfies the functional equation.

$$(2) \qquad V(s) = \sup\{px + \alpha V(b + Kx)\}$$

$$Ax = s \; , \; x \geqq 0 \; .$$

Proof:

If $s \in F$, and $Ax = s$, $x \geqq 0$, then $V(s) \geqq px + \alpha V(b + Kx)$. The contrary conclusion would violate (1). However, for any $\varepsilon > 0$, it is possible to find $\{x_t\} \in X(s)$ such that

$$V(s) \geqq p(\{x_t\}) = px_o + \alpha p(\{x_{t+1}\}) \geqq V(s) - \varepsilon \; .$$

Since $\{x_{t+1}\} \in X(b + Kx_o)$, $\alpha V(b + Kx_o) \geqq \alpha p(\{x_{t+1}\})$. Therefore

$$V(s) \geqq px_o + \alpha V(b + Kx_o) \geqq V(s) - \varepsilon \; ,$$

i.e., $V(s)$ solves (2). ||

This next proposition examines the implications of a failure in assumption II. Failure of II implies for some $0 < \lambda \leqq \alpha$, $u(A - \lambda K) = v + p$, $v > 0$ has no solution. To rule out a border line case, *total failure* here will mean $u(A - \lambda K) = v + p$, $v \geqq 0$ has no solution.

Proposition 3:

If I holds and, for some λ , $0 < \lambda \leqq \alpha$, the system

$$u(A - \lambda K) \geqq p$$

has no solution, then $V(s) = +\infty$ for s in the relative interior of dom V , and $V(s) = -\infty$ for $s \notin \text{dom } V$. V can only be finite on the relative boundary of $\text{dom } V$.

<u>Proof</u>:

If $u(A - \lambda K) \geqq p$ has no solution, then Farkas Lemma implies $(A - \lambda K)y = 0$, $y \geqq 0$, $py > 0$ has a solution.

Suppose $s \in \tilde{F}$ and $\{x_t\} \in \tilde{X}(s)$. Then $\{x_t + y/\lambda^t\} \in X(s + Ky/\lambda)$, and $p(\{x_t + y/\lambda^t\}) = +\infty$. This implies $V(s)$ is an improper concave function and according to Rockafellar, [14], Theorem 7.2, V can only be finite on the relative boundary of its domain. ||

The following example illustrates the result.

<u>Example 1</u>:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \alpha = \tfrac{1}{2}$$

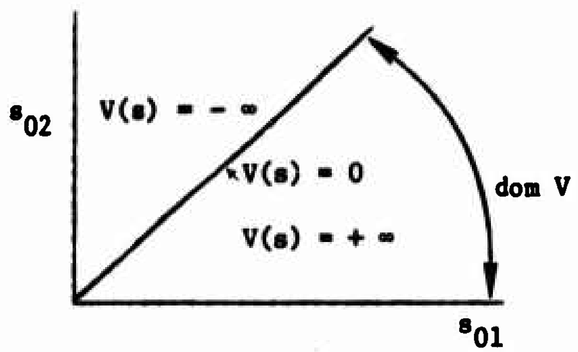
$$p = (1,-1) \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$s_o = \begin{pmatrix} s_{01} \\ s_{02} \end{pmatrix}$$

$$V(s) = -\infty \quad \text{if} \quad s \ngeqq 0$$

otherwise

$$V(s) = \liminf_{T \to \infty} T(s_{01} - s_{02}) \ .$$



If I fails, then theorem 1 of the next section indicates that $V(s) = -\infty$. Either $X(s) = \phi$, or there are no α-convergent solutions in $X(s)$. Theorem 1 states that $\{x_t\}$ *not* α-convergnet implies $p(\{x_t\}) = -\infty$.

3. THE CLASS OF OPTIMAL SOLUTIONS

The section demonstrates that, without loss of generality, problem (1:3) is equivalent to

$$\text{(1)} \qquad \operatorname{Max} \sum_{0}^{\infty} \alpha^t p x_t$$

$$\text{subject to } \{x_t\} \in \tilde{X}(s) .$$

Solutions that are not α-convergent are not potentially optimal solutions.

Theorem 1:

If $\{x_t\} \in X(s)$ is not α-convergent, then $p(\{x_t\}) = -\infty$.

Proof:

Let λ be the smallest number such that $\{x_t\}$ is γ convergent for all $0 \leq \gamma < \lambda$: λ is the *radius of convergence* of the power series $\sum_{0}^{\infty} \gamma^t e x_t$. The proof first shows that assumption II implies that any solution with radius of convergence λ , $0 < \lambda \leq \alpha$ has $p(\{x_t\}) = -\infty$. The remainder of the proof, and assumption III, are needed to deal with $\{x_t\}$ that have a radius of convergence equal to zero.

Let λ be the radius of convergence for the power series $\sum_{0}^{\infty} \gamma^t e x_t$. Assume first that $0 < \lambda \leq \alpha$. If $\lambda > \alpha$, then $\{x_t\}$ is α-convergent. If the t^{th} constraint of (1:1) is multiplied by λ^t and the first T constraints are summed the result is:

$$\text{(2)} \qquad (A - \lambda K) \sum_{0}^{T} \lambda^t x_t = s + \sum_{1}^{T} \lambda^t b - \lambda^{T+1} K x_t .$$

Let $M_T = \sum_0^T \lambda^t ex_t$, and divide (2) by M_T . Consider lemma 1 in the appendix with $a_t = \lambda^t ex_t$. From lemma 1, there exists a sequence S_o , such that $a_T/M_T = \lambda^T ex_T/M_T \to 0$. Thus on a finer subsequence $S_1 \subseteq S_o$, $\sum_0^T \lambda^t x_t/M_T \to x \geq 0$ and

$$(A - \lambda K)x = 0 \ , \ x \geq 0 \ .$$

From II, there exist (u,v) such that

$$u(A - \lambda K) = p + v \ , \ v > 0 \ .$$

It follows that

$$px + vx = 0 \ , \ vx > 0 \ , \ px < 0 \ .$$

For $T \in S_1$, $\dfrac{\sum_0^T \lambda^t px_t}{M_T} \to px$, thus $\liminf_{T \to \infty} \sum_0^T \lambda^t px_t = -\infty$. From lemma 2 in the appendix, with $\rho = \frac{\alpha}{\lambda}$, it follows that $p(\{x_t\}) = -\infty$.

Now consider the case where the radius of convergence of $\sum_0^\infty \gamma^t ex_t$ is equal to zero. For any $\lambda > 0$, (2) will have a solution. If (2) is divided by $M_T = \sum_0^T \lambda^t ex_t$, then there exist a subsequence S_1 and vectors $x(\lambda)$, $y(\lambda)$ such that

$$\frac{\sum_0^\infty \lambda^t x_t}{M_T} \to x(\lambda) \geq 0 \qquad T \in S$$

$$\text{(3)} \qquad \frac{\lambda^T x_T}{M_T} \to y(\lambda) \geq 0 \qquad T \in S$$

$$(A - \lambda K)x(\lambda) = -\lambda K y(\lambda)$$

$$ex(\lambda) = 1 ,$$

$$x(\lambda) \geqq y(\lambda) .$$

For $k \geqq \frac{1}{\alpha}$, let $\lambda = 1/k$, $x^k = x(1/k)$, $y^k = y(1/k)$. There exists a subsequence S , and points x and y such that

$$x^k \to x \geq 0 \qquad k \in S$$

(4) $$y^k \to y \geqq 0 \qquad k \in S$$

$$Ax = 0 .$$

From assumption III part (i), there exists (u,v) such that

$$uA = v + p , \; v > 0 .$$

It follows that $vx > 0$, $px < 0$, and for k sufficiently large $px^k < 0$. For $\lambda = 1/k$, $M_T = \sum_0^T \lambda^t ex_t$, there exists a subsequence S such that

$$\frac{\sum_0^T \lambda^t px_t}{M_T} \to px_k < 0 .$$

It follows that $\liminf_{T \to \infty} \sum_0^T \lambda^t px_t = - \infty$, and from lemma 2, that $p(\{x_t\}) = - \infty$.

It remains to consider III (ii) and III (iii). From (2) either

(5) $$us + \sum_1^T \lambda^t ub = \sum_0^{T-1} \lambda^t vx_t + \sum_0^T \lambda^t px_t + \lambda^T(u + \lambda uK)x_T$$

or

$$(6) \qquad us + \sum_{1}^{T+1} \lambda^t ub = \sum_{0}^{T} \lambda^t vx_t + \sum_{0}^{T} \lambda^t px_t + \lambda^{T+1} uAx_{T+1} .$$

If either III (ii) or (iii) holds, then either $(v + \lambda uK)x_T \geqq 0$ or $uAx_T \geqq 0$. Thus either

$$us + \sum_{1}^{T} \lambda^t ub \geq \sum_{0}^{T-1} \lambda^t vx_t + \sum_{0}^{T} \lambda^t px_t$$

or

$$us + \sum_{1}^{T+1} \lambda^t ub \geqq \sum_{0}^{T} \lambda^t vx_t + \sum_{0}^{T} \lambda^t px_t .$$

In both cases the left hand quantity is convergent. If $\{x_t\}$ is not α-convergent, then $\sum_{0}^{T} \lambda^t vx_t \to + \infty$, therefore $\sum_{0}^{T} \lambda^t px_t \to - \infty$, and by lemma 2, $p(\{x_t\}) = - \infty$. ||

<u>Corollary</u>:

If I fails then $V(s) = - \infty$.

<u>Proof</u>:

If $X(s) = \phi$, then $V(s) = - \infty$. Otherwise every $\{x_t\} \in X(s)$ is not α-convergent, by theorem 1 $p(\{x_t\}) = - \infty$, thus $V(s) = - \infty$. ||

From (4) in the proof of theorem 1 one can conclude several things. First, if $Ax = 0$, $x \geq 0$ has no solution, then each $\{x_t\} \in X(s)$ is λ-convergent for some $\lambda > 0$. Second, if $Ax = 0$, $x \geq 0$ implies $px < 0$, then III (i) is satisfied, and the arguments following (4) apply.

Third, if $Ax = 0$, $x \geq 0$, $px \geqq 0$ implies $Kx = 0$, then II does not hold since $(A - \alpha K)x = 0$, $x \geq 0$, $px \geqq 0$ has a solution. The cases III (ii) and (iii) are only needed if $Ax = 0$, $x \geq 0$, $px \geqq 0$ and $Kx \neq 0$, has a solution.

4. FINITE HORIZON APPROXIMATIONS

This section considers a finite (T period) horizon approximation program and demonstrates that the optimal solutions of the T period problems converge to the optimal solution of the infinite horizon problem. The T period problem is

$$\text{Max} \sum_{0}^{T} \alpha^t p x_t^T$$

$$\text{subject to } Ax_0^T = s \ , \ x_0^T \geqq 0$$

(1)

$$Ax_t^T = b + Kx_{t-1}^T \ , \ x_t^T \geqq 0$$

$$(A - \alpha K)x_T^T = \frac{b}{1-\alpha} + Kx_{T-1}^T \ ; \ x_T^T \geqq 0 \ .$$

Define $V^T(s)$ as the optimal value of (1).

Proposition 4:

Problem (1) has an optimal solution for all T, and

(i) $V^T(s) \geqq V^{T+1}(s) \geqq V(s)$.

(ii) $V^T(s) = \max[px + \alpha V^T(b + Kx)]$

$Ax = s \ , \ x \geqq 0$.

Proof:

The dual of (1) is

$$\text{Minimize} \quad u_0^T s + \sum_{1}^{T-1} u_t^T b + u_T^T b/1-\alpha$$

$$\text{(2)} \qquad \text{subject to} \quad u_t^T A \geqq \alpha^t p + u_{t+1}^T K \quad \text{for} \quad 0 \leqq t \leqq T-1$$

$$u_T^T(A - \alpha K) \geqq \alpha^T p$$

Let, by II, (u,v) satisfy $u(A - \alpha K) - v = p$, $v > 0$. Then $u_t^T = \alpha^t u$ is feasible for (2). Also, suppose that $\{x_t\} \in \tilde{X}(s)$, then

$$x_t^T = \begin{Bmatrix} x_t & t < T \\ \sum_{t=T}^{\infty} \alpha^t x_t & t = T \end{Bmatrix}$$

is feasible for (1). Since (1) and (2) have feasible solutions, they have optimal solutions.

Suppose u_t^T solves the T period approximation, then

$$u_t^{T+1} = \begin{Bmatrix} u_t^T & t \leqq T \\ \alpha u_T^T & t = T+1 \end{Bmatrix}$$

is feasible for the $T + 1$ period dual, with value

$$u_0^T s + \sum_{1}^{T} u_t^T b + \alpha u_T^T b/1-\alpha = u_0^T s + \sum_{1}^{T-1} u_t^T b + u_T^T b/1-\alpha = V^T(s) \ .$$

Therefore $V^{T+1}(s) \leqq V^T(s)$.

For any $\{x_t\} \in \tilde{X}(s)$, a feasible solution of (1), using (3), can be constructed. Thus from Theorem 1, $V^T(s) \geqq V(s)$.

Item (ii) follows directly from the definition of $V^T(s)$. ||

Theorem 2:

For each T, let $\{x_t^T\}$ be the optimal solution of (1), where $x_t^T = 0, t > T$.

(i) There exists an optimal solution $\{x_t\}$ of the infinite horizon problem.

(ii) For every τ, there exists a sequence S_τ such that $x_t^T \to x_t$ for $0 \leq t \leq \tau$, $T \in S_\tau$.

(iii) $V^T(s) = \sum_0^\infty \alpha^t p x_t^T \to V(s) = \sum_0^\infty \alpha^t p x_t$.

Proof:

Let $M_T = \sum_0^\infty \alpha^t e x_t^T$ and note that

$$(A - \alpha K)\left(\sum_0^\infty \alpha^t x_t^T\right) = s + \frac{\alpha b}{1-\alpha}. \tag{5}$$

Suppose $M_T \to +\infty$ on some subsequence S, then there exists a subsequence $S_1 \subseteq S$, and a vector z such that

$$\sum_0^\infty \alpha^t x_t^T / M_T \to z \quad T \in S_1$$

(6)

$$(A - \alpha K)z = 0, \; z \geq 0.$$

Since II is valid, there exists (u,v) such that $u(A - \alpha K) = p + v$, $v > 0$. Thus, $0 = pz + vz$, $vz > 0$; $\sup z < 0$. For $T \in S_1$, $V^T(s) = M_T p z_T \geq V(s) > -\infty$. However, $M_T \to +\infty$, $pz_T \to pz < 0$. Thus contradiction indicates that $M = \limsup_T M_T < +\infty$.

For each t, $ex_t^T \leq M/\alpha^t$, $x_t^T \geq 0$. Thus there exists a sequence S_o, and vector x_o such that $x_o^T \to x_o$ for $T \in S_o$. In general there exist a sequence $S_\tau \subseteq S_{\tau-1}$ and vector x_τ such that

$$x_t^T \to x_t \quad T \in S_\tau , \; 0 \leq t \leq \tau .$$

It follows that $\{x_t\} \in X(s)$ and $\sum_0^\infty \alpha^t ex_t \leq M$. To verify this last claim note that for each T and τ,

$$\sum_0^\tau \alpha^t ex_t^T \leq M - \sum_{\tau+1}^\infty \alpha^t ex_t^T \leq M .$$

The limit over $T \in S_\tau$, yields $\sum_0^\tau \alpha^t ex_t \leq M$. Thus $\{x_t\} \in \tilde{X}(s)$; and

$$\sum_0^\infty \alpha^t px_t^T = V^T(s) \geq V(s) \geq \sum_0^\infty \alpha^t px_t .$$

In the limit

$$\lim_{T\to\infty} \sum_0^\infty \alpha^t px_t^T = \varepsilon + \sum_0^\infty \alpha^t px_t , \quad \varepsilon \geq 0 .$$

If $\varepsilon = 0$, the proof is completed. Suppose in contrast $\varepsilon > 0$. Then for each T

$$\sum_0^\infty \alpha^t p(x_t^T - x_t) \geq \varepsilon > 0 .$$

Let u,v solve $u(A - \alpha K) = p + v$, $v > 0$. From (5)

$$(6) \qquad us + \frac{\alpha ub}{1-\alpha} - \sum_0^\infty \alpha^t vx_t^T = \sum_0^\infty \alpha^t px_t^T .$$

Since $\{x_t\} \in \widetilde{X}(s)$,

(7) $$us + \frac{\alpha ub}{1-\alpha} - \sum_0^\infty \alpha^t vx_t = \sum_0^\infty \alpha^t px_t \; .$$

Subtract (7) from (6) to obtain

$$\sum_0^\infty \alpha^t p(x_t^T - x_t) = \sum_0^\infty \alpha^t v(x_t - x_t^T) \geqq \varepsilon > 0$$

or, for any τ .

(8) $$\sum_0^\tau \alpha^t v(x_t - x_\tau^T) + \sum_{\tau+1}^\infty \alpha^t vx_t \geqq \varepsilon + \sum_{\tau+1}^\infty \alpha^t vx_\tau^T \geqq \varepsilon > 0 \; .$$

However, the left hand side of (8) can be made arbitrarily small. First choose τ large enough so that

$$0 \leqq \sum_{\tau+1}^\infty \alpha^t vx_t \leq \varepsilon/3$$

Then there exists a T^* such that for all $T \in S_\tau$, $T \geqq T^*$

$$\left| \sum_0^\tau \alpha^t v(x_t - x_t^T) \right| \leq \varepsilon/3 \; . \; ||$$

<u>Corollary 2</u>:

For $s \in F$

$$V(s) = \max[px + \alpha V(b + Kx)]$$

$$Ax = s \; , \; x \geqq 0 \; .$$

One tempting variant on the approximation scheme described here is to terminate the T period problem by requiring that $x_t = x_T$ for $t \geqq T$. The example below shows this is not generally applicable since I and II may be satisfied yet $(A - K)x = b$, $x \geqq 0$ may not have a solution.

<u>Example 2</u>: $A = 1,\ K = 1.6,\ b = 1,\ p = 1,\ s = 1,\ \alpha = \frac{1}{2}$

$x_t = \dfrac{(1.6)^{t+1} - 1}{0.6}$ is feasible and $\sum\limits_0^\infty \alpha^t x_t = 2.222\ \ldots$, yet $(A - K)x = 1$, $x \geqq 0$ has no solution.

Another possible way to approximate the infinite problem is to simply truncate the decision process at time T by solving

$$\text{Max} \sum_0^T \alpha^t p x_t^T$$

$$\text{subject to}\quad Ax_0^T = s\ ,\ x_0^T \geqq 0$$

$$Ax_t^T = b + Kx_{t-1}^T\ ,\ x_t^T \geqq 0$$

$$1 \leqq t \leqq T\ .$$

However, our next example shows that problems of this sort may be unbounded for all T.

<u>Example 3</u>:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad p = (1,1,0,0)$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$K = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \alpha = \tfrac{1}{2}$$

Note that I, and II are satisfied. In addition, III (iii) is satisfied since for every $\{x_t\}$ $Ax_t = b + Kx_{t-1} \geqq 0$. In addition, $u(A - \alpha K) = v + p$, $v > 0$ has a solution with $u > 0$.

To see the problem is unbounded, note that any positive multiple of the vector $y^1 = (0,1,0,1)$ can be added to x_1^I . Thus arbitrarily large profits are possible at time T .

5. OPTIMALITY CONDITIONS; DUALITY

This section briefly consider optimality conditions, establishes sufficient conditions for optimality and comments on necessary conditions and duality.

Theorem 3:

If $\{x_t\} \varepsilon \tilde{X}(s)$ and there exists $\{u_t\}$ such that

(i) $|| u_t || \leq \alpha^t M$

(ii) $u_t A - v_t = \alpha^t p + u_{t+1} K$, $v_t \geq 0$

(iii) $v_t x_t = 0$

then $\{x_t\}$ is optimal.

Proof:

From (ii) and (iii)

$$\sum_0^\infty u_t A x_t - \sum_1^\infty u_{t+1} K x_t = \sum_0^\infty u_t (b + K x_t) - \sum_1^\infty u_t K x_t$$

$$u_o s + \sum_1^\infty u_t b = \sum_0^\infty \alpha^t p x_t$$

If $\{\tilde{x}_t\}$ is another feasible solution not in $\tilde{X}(s)$, then $p(\{\tilde{x}_t\}) = -\infty$.
If $\{\tilde{x}_t\} \varepsilon \tilde{X}(s)$, then from (i) $\sum_0^\infty u_t A \tilde{x}_t < \infty$; thus

$$\sum_0^\infty \alpha^t p x_t = u_o s + \sum_1^\infty u_t b = \sum_0^\infty v_t \tilde{x}_t + \sum_0^\infty \alpha^t p \tilde{x}_t \geq \sum_0^\infty \alpha^t p \tilde{x}_t \ . \ ||$$

It is possible to define a dual infinite horizon program.

$$\text{minimize} \quad \limsup_{T \to \infty} \; u_o s + \sum_{1}^{T} u_t b \tag{1}$$

$$\text{subject to} \quad u_t A - v_t = \alpha^t p + u_{t+1} K \;, \; v_t \geq 0 \;.$$

<u>Proposition 4</u>:

The maximum primal infinite horizon problem has a value greater than or equal to the optimal value of the minimization problem.

<u>Proof</u>:

Consider the dual linear programs, (4:2). The optimal solution $\{u_t^T\}$ determines a feasible solution of (1) with $u_t^T = \alpha^{t-T} u_T^T$ for $t \geq T$. That solution has value $V^T(s)$. Thus the infimum in (1) is less than or equal to $V(s)$. ||

Suppose for $\rho \geq 1$, $\tilde{u}(A - \rho K) \geq 0$, $\tilde{u}b < 0$ has a solution. Then if $u(A - \alpha K) \geq p$, $\{\alpha^t u + \rho^t \tilde{u}\}$ solves (1), and $\limsup_{T \to \infty} (u + \tilde{u})s + \sum_{0}^{T} (\alpha^t u + \rho^t \tilde{u})b = -\infty$. Thus to insure (1) has a finite lower bound, one must *at least* require $(A - \rho K)x = b$, $x \geq 0$ has a solution for *all* $\rho \geq 1$. This is too stringent on assumption.

If one requires that dual solutions satisfy $\| u_t \| \leq \alpha^t M$ for some M, then the infimum of (1) is equal to the maximum of the original problem. Recall the definition of u_t^T in section 4 and the construction of the optimal solution $\{x_t\}$. Let $M_T = \sup_t \alpha^t \| u_t^T \|$. If a subsequence S exists such that $M_T \leq M$ for $T \in S$, then it is possible to establish the existence of $\{u_t\}$, such that

(i) $\alpha^t \, || \, u_t \, || \leqq M$

(ii) $u_t A - v_t = \alpha^t p + u_{t+1} K \, , \; v_t \geqq 0$

(iii) $v_t x_t = 0$

(iv) There exists S_τ such that $u_t^T \to u_t$ for $T \in S_\tau \, , \; 0 \leqq t \leqq \tau$.

This fact is easy to establish. However, it is of little use unless conditions which bound M_T can be deduced from the data A,K,p,b,α .

6. EQUILIBRIUM SOLUTIONS

In [1], [4] and [10], the problem of finding an equilibrium optimal solution $x_t = x$ for all t was considered and solved in a most elegant way. This section points out that knowledge of an equilibrium optimal solution is not particularly helpful in solving problem (1:3).

Suppose the system

$$(A - K)x = b \ , \ x \geqq 0 \qquad (1)$$

has a feasible solution. Then, if x , satisfies (1), $\{x_t = x\} \in X(b + Kx)$ and $V(b + Kx) \geqq \frac{px}{1-\alpha}$. Now consider, the optimization problem

$$\begin{aligned} &\text{Max } px \\ &\text{subject to } (A - K)x = b \ , \ x \geqq 0 \ . \end{aligned} \qquad (2)$$

Suppose first that (2) has unbounded solution; i.e., $py > 0$, $(A - K)y = 0$, $y \geqq 0$ has a solution. Then for any $s \in F$

$$V(s + \mu Ky) \geqq V(s) + \frac{\mu}{1-\alpha} py \ ,$$

since $\{x_t + \mu y\} \in X(s + \mu Ky)$ for all $\{x_t\} \in X(s)$. In this case the function V is unbounded. If (2) has an optimal solution, x^* , then $V(b + Kx^*) \geqq \frac{px^*}{1-\alpha}$.

Now suppose an equilibrium optimal solution $\tilde{x}$ exists; i.e., $(A - K)\tilde{x} = b$, $\tilde{x} \geqq 0$, $V(b + K\tilde{x}) = \frac{p\tilde{x}}{1-\alpha}$. It follows that either $\tilde{x}$ solves (2) or

$$V(b + Kx^*) \geqq \frac{px^*}{1-\alpha} > \frac{p\tilde{x}}{1-\alpha} = V(b + K\tilde{x}) \qquad (3)$$

Thus the suboptimal solution $s = b + Kx^*$, and $x_t = x^*$ for all t is preferable to the optimal solution $s = b + K\tilde{x}$, $x_t = \tilde{x}$ for all t .

More general equilibrium policies can be discovered by allowing a T period cycle; i.e., $s = b + Kx_T$. Solve with s as a variable

$$\text{Max} \sum_{0}^{T} \alpha^t p x_t$$

$$\text{subject to} \quad -Is + Ax_o = 0 \ , \ x_o \geqq 0$$

$$Ax_t = b + Kx_{t-1} \ , \ x_t \geqq 0 \quad 1 \leqq t \leqq T$$

$$Is - Kx_T = b \ .$$

If W^T is the optimal value of (4), and $s, x_o, \ldots, x_T$ is the optimal solution, then

$$V(s) \geqq \frac{W^T}{1-\alpha^{T+1}} \geqq V(b + Kx^*) \geqq \frac{px^*}{1-\alpha} \ .$$

Since $b + Kx^*$, $x_t = x^*$ is feasible for (1).

Thus if optimal equilibrium solutions can be calculated then "better" suboptimal solutions can be calculated using (2) and (4). These calculations, however, are not directly useful in the solution of the original optimization problem. The more direct procedure outlined in section 4 yields approximately optimal value and an approximately optimal first period decision that is consistent with the initial state s .

7. APPENDIX

This appendix contains two lemmas used in the proof of theorem 1.

Lemma 1:

If

(i) $a_t \geq 0$ all t

(ii) $\sum_0^\infty \lambda^t a_t < +\infty$ if $0 \leq \lambda < 1$

(iii) $\lim \sum_0^T a_t = +\infty$

then $\liminf\limits_{T \to \infty} \dfrac{a_T}{\sum_0^T a_t} = 0$.

Proof:

If the result is false, there exists an $1 > \varepsilon > 0$, and T^* such that for all $T \geq T^*$

(1) $$a_T \geq \varepsilon \sum_0^T a_t \ .$$

Let $b_o = \sum_0^{T^*} a_t$ and $b_t = a_{T^*+t}$. Therefore

(2) $$b_T \geq \varepsilon \sum_0^T b_t \quad \text{for all } T \ .$$

or

(3) $$b_T \geq \left(\frac{\varepsilon}{1-\varepsilon}\right) \sum_0^{T-1} b_t = \mu \sum_0^{T-1} b_t \ .$$

It can be established by induction that

(4) $$b_T \geq \mu(1+\mu)^{T-1} b_o \quad \text{for} \quad T \geq 1 .$$

(3) demonstrates that (4) is true for $T = 1$. To check $T + 1$ note that

$$b_{T+1} \geq \mu \sum_0^T b_t \geq \mu\left(1 + \sum_0^T \mu(1+\mu)^{t-1}\right) b_o$$

$$= \mu\left(1 + \mu \frac{(1-(1+\mu)^T}{-\mu}\right) b_o$$

$$= \mu(1+\mu)^T b_o .$$

This indicates that

$$\sum_1^T \lambda^t b_t > T\left(\frac{\mu}{1+\mu}\right) b_o$$

where $\lambda = \frac{1}{1+\mu} < 1$.

However

$$\sum_{T^*+1}^{T^*+T} \lambda^t a_t = \lambda^{T^*} \sum_1^T \lambda^t b_t$$

which implies $\sum_0^\infty \lambda^t a_t$ diverges. ||

<u>Lemma 2</u>:

If $\liminf\limits_{T \to \infty} \sum_0^T a_t = -\infty$ and $\rho \geq 1$, then $\liminf\limits_{T \to \infty} \sum_0^T \rho^t a_t = -\infty$.

Proof:

Trivial if $\rho = 1$. Let $r_T = \sum_0^T \rho^t a_t$, and $\mu = 1/\rho < 1$. Then,

$$\sum_0^T a_t = (1 - \mu) \sum_0^T \mu^t r_t + \mu^{T+1} r_T . \tag{5}$$

If $\liminf_{T \to \infty} r_T > -\infty$, then r_T must be bounded below, i.e., $r_T \geqq M$ for all T . Thus from (5)

$$\sum_0^T a_t \geqq \left[(1 - \mu) \sum_0^T \mu^t + \mu^{T+1}\right] M = M$$

which contradicts the hypotheses. ||

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