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APPLICATION OF NEGATIVE BINOMIAL
PROBABILITY TO INVENTORY CONTROL

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Various inventory expressions are developed assuming demand has a negative binomial distribution. The Camp-Paulson approximation of the negative binomial variate is used in actual evaluation of the expressions.		

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1. Background

This note presents a computation technique for efficiently implementing a common inventory model for the case when lead time demand is assumed to be negative binomial. There is empirical evidence to justify use of the negative binomial for many items.⁽¹⁾

The model, which is described in the next section, is currently being used by the Department of Defense in wholesale management of secondary items [4]. In implementing this model it is necessary to determine expected time-weighted backorders; to determine the probability of a stock-out; and, to determine the optimum reorder warning point (R) for a given reorder quantity (Q).

The computation technique presented uses the Camp-Paulson approximation to the negative binomial [1] to accomplish these objectives. In order to apply the Camp-Paulson approximations, the general forms of the inventory model equations must be transformed using identities peculiar to the negative binomial. These identities are proved in Appendix 1.

2. Model

We will examine an inventory system which has a central stockage point from whom customers requisition stock. The inventory system is examined using the following assumptions:

- a. As demands are received they are filled from available stock.
- b. When stock is insufficient to fill a requisition, the unfilled portion is backordered.
- c. When the assets (stock on-hand plus on-order) reach the reorder warning point, R, a procurement action is initiated for the amount Q. The check of the reorder warning point is done on a continuous review basis.

(1) See for example "Evaluation of Several VSL/EOQ Models," R.L. Deemer and W.K. Kruse, AMCIRO, ALMC, Ft. Lee, Va., May 1974, AD 781948, and reference [3] of this report.

d. Procurements will raise the asset level to the requisition objective $(R+Q)$.

e. Negative reorder points are not allowed, i.e., $R \geq 0$, and the item will be procured, i.e., $Q \geq 1$.

f. The procurement lead time is deterministic.

g. Demands during the procurement lead time follow a negative binomial probability function.

We wish to minimize the annual cost of holding and ordering stock at the stockage point subject to a constraint on the time-weighted requisitions short. The holding cost is expressed as

$$H \left(R + \frac{Q}{2} \right)$$

where H is the holding cost, expressed in dollars per unit per year.

There will be D/Q orders per year and hence the ordering cost is

$$P \frac{D}{Q}$$

where P is the variable cost associated with initiating a procurement action.

Hadley and Whitin [6] give the general expression for expected units backordered as

$$B(R) = \frac{1}{Q} \sum_{i=R}^{\infty} (i-R) [G(i+1;t) - G(i+Q+1;t)] \quad (1)$$

where $G(y;t)$ is the probability of having y or more demands in time t . Expected requisitions short are approximated as the expected units backordered divided by the average requisition size, S . [7]

The objective of the model is to minimize total annual variable cost. Mathematically, this will be:

$$\text{minimize: } TVC(R) = H \left(R + \frac{Q}{2} \right) + P \frac{D}{Q} + \lambda \frac{B(R)}{S} \quad (2)$$

where λ is an implied requisition short cost. This value, λ , can be determined in one of two ways. For example, when the maximum average number of requisitions short is specified for a group of items, this will imply a shortage parameter and this affects the inventory funds; or, if the total funding level for safety level investment is fixed for a group of items, this implies a shortage parameter and establishes the performance that can be achieved.

The optimal reorder quantity, Q , is computed in some non-optimal manner. For example, the optimum Q could be approximated by the Wilson Q or some other approximate expression.

Only the optimal R will be solved for exactly. Since TVC is convex in R , difference equations must be solved for optimum R . Thus, the optimality condition is

$$TVC(R) - TVC(R-1) \leq 0 \leq TVC(R+1) - TVC(R) \quad (3)$$

3. Fundamental Identities

The general form of the negative binomial probability function is

$$f(y;r) = \frac{\Gamma(r+y)}{\Gamma(r) \Gamma(y+1)} p^r q^y \quad y \geq 0 \quad (4)$$

$$f(y;r) = 0 \quad y < 0$$

where $0 \leq p \leq 1$, $q = 1 - p$, $\Gamma(x+1) = x!$ if x is an integer and $\Gamma(x+1) = x \Gamma(x)$.

The mean and variance of the negative binomial distribution are given by:

$$\mu = \frac{rq}{p} \quad \sigma^2 = \frac{rq}{p^2}$$

Evaluation of (3) depends on the following identities. Let i be an integer and $R \geq 0$ for all the results. Then,

$$i f(i;r) = \frac{iq}{p} f(i-1;r+1) \quad (5)$$

$$\sum_{i=0}^R i f(i;r) = \frac{iq}{p} F(R-1; r+1) \quad (6)$$

$$\sum_{i=0}^R i^2 f(i;r) = \frac{iq}{p} \left\{ \frac{(r+1)q}{p} F(R-2; r+2) + F(R-1; r+1) \right\} \quad (7)$$

$$\sum_{i=0}^R F(i;r) = (R+1) F(R;r) - \frac{iq}{p} F(R-1; r+1) \quad (8)$$

$$\sum_{i=0}^R i F(i;r) = \frac{R(R+1)}{2} F(R;r) - \frac{1}{2} \frac{iq}{p} \frac{(r+1)q}{p} F(R-2; r+2) \quad (9)$$

where $F(y;r)$ is the cumulative distribution function. The derivation of these identities is given in Appendix 1.

Furthermore, $F(y;r)$ and $G(y;r)$ are related by

$$G(y+1;r) = 1 - F(y;r) \quad (10)$$

and

$$\sum_{i=0}^M G(i;r) = 1 + \sum_{i=0}^{M-1} [1 - F(i;r)] \quad (11)$$

4. Probability of Stock Availability

The evaluation of expression (3) requires the calculation of probability of stock availability as a function of R . In particular

$$B(R-1) - B(R) = 1 - a \quad (12)$$

where a denotes probability of availability of stock (see Appendix 2).

The general expression² for the probability of stock availability, α , is given in Hadley and Whitin, [6]. It is

$$\alpha = 1 - \frac{1}{Q} \left\{ \sum_{i=0}^{R+Q} G(i;r) - \sum_{i=0}^R G(i;r) \right\} \quad (13)$$

Transforming this to the cumulative distribution, $F(y;r) = 1 - G(y+1;r)$, and using the identities of section 3, the expression for the availability of stock is

$$\begin{aligned} \alpha &= 1 - \frac{1}{Q} \left\{ 1 + \sum_{i=0}^{R+Q+1} 1 - \sum_{i=0}^{R+Q+1} F(i;r) - 1 - \sum_{i=0}^{R-1} 1 + \sum_{i=0}^{R-1} F(i;r) \right\} \\ &= 1 - \frac{1}{Q} \left\{ (R+Q) - (R+Q) F(R+Q-1;r) + \right. \\ &\quad \left. \frac{Kq}{p} F(R+Q-2;r+1) - R + R F(R-1;r) - \frac{Kq}{p} F(R-2; r+1) \right\} \\ &= 1 - \frac{1}{Q} \left\{ R F(R-1;r) - \frac{Kq}{p} F(R-2; r+1) + Q - \right. \\ &\quad \left. (R+Q) F(R+Q-1;r) + \frac{Kq}{p} F(R+Q-2; r+1) \right\} \\ &= \frac{1}{Q} \left\{ (R+Q) F(R+Q-1;r) - \frac{Kq}{p} F(R+Q-2; r+1) - \right. \\ &\quad \left. R F(R-1;r) + \frac{Kq}{p} F(R-2; r+1) \right\} \quad (14) \end{aligned}$$

5. Expected Number of Backorders

The expression for the expected number of backorders is:

²This expression is generally only true for requisitions of unit size. An expression for non-unit requisition size was developed by E.A. Silver, Chung-Hoi Ho, R.L. Deemer, "Cost-Minimizing Inventory Control of Items Having a Special Type of Erratic Demand Pattern," *INFOR*, Vol 9, No. 3, November 1971.

$$B(R) = \frac{1}{Q} \sum_{i=R}^{\infty} (1-R) [G(i+1;r) - G(i+Q+1;r)] \quad (15)$$

Now the subtraction within the brackets is accomplished and

$$\begin{aligned} E &= \frac{1}{Q} \left\{ 1[G(R+2;r) - G(R+Q+2;r)] + \right. \\ &\quad 2[G(R+3;r) - G(R+Q+3;r)] + \\ &\quad \dots + Q [G(R+Q+1;r) - G(R+2Q+1;r)] + \\ &\quad (Q+1) [G(R+Q+2;r) - G(R+2Q+2;r)] \\ &\quad \left. + \dots \right\} \\ &= \frac{1}{Q} \left\{ G(R+2;r) + 2G(R+3;r) + \dots + \right. \\ &\quad QG(R+Q+1;r) + QG(R+Q+2;r) + \\ &\quad \left. QG(R+Q+3;r) + \dots \right\} \\ &= \frac{1}{Q} \left\{ \sum_{i=R+1}^{R+Q} (i-r) G(i+1;r) + Q \sum_{i=R+Q+2}^{\infty} G(i;r) \right\} \\ E &= \frac{1}{Q} \left\{ \sum_{i=R+1}^{R+Q} (i-R) [1-F(i;r)] + Q \left[\sum_{i=1}^{\infty} G(i;r) - \right. \right. \\ &\quad \left. \left. \sum_{i=1}^{R+Q+1} G(i;r) \right] \right\} \\ &= \frac{1}{Q} \left\{ \sum_{i=0}^{R+Q} (i-R) [1-F(i;r)] - \sum_{i=0}^R (i-R) [1-F(i;r)] \right. \\ &\quad \left. + Q \left[\frac{RQ}{p} + 1 - 1 - \sum_{i=0}^{R+Q} [1 - F(i;r)] \right] \right\} \end{aligned}$$

The mean part (RQ/p) of this expression is based on the expected value expression found in [8], i.e.,

$$\bar{X} = \sum_{i=1}^{\infty} \Pr(X \geq i) \text{ and}$$

$$\bar{X} + 1 = \sum_{i=0}^{\infty} \Pr(X \geq i).$$

Continuing the algebraic operations the expressions for the expected backorders is

$$B = \frac{1}{Q} \left\{ \begin{aligned} & \sum_{i=0}^{R+Q} i[1-F(i;r)] - R \sum_{i=0}^{R+Q} [1 - F(i;r)] - \\ & \sum_{i=0}^R i[1-F(i;r)] + R \sum_{i=0}^R [1 - F(i;r)] + \\ & Q \left[\frac{rq}{p} - \sum_{i=0}^{R+Q} [1 - F(i;r)] \right] \end{aligned} \right\}$$

$$B = \frac{1}{Q} \left\{ \begin{aligned} & \frac{(R+Q)(R+Q+1)}{2} - \frac{(R+Q)(R+Q+1)}{2} F(R+Q;r) + \\ & \frac{1}{2} \frac{rq}{p} \frac{(r+1)q}{p} F(R+Q-2;r+2) - R(R+Q+1) + \\ & R(R+Q+1) F(R+Q;r) - R \frac{rq}{p} F(R+Q-1;r+1) - \\ & \frac{R(R+1)}{2} + \frac{R(R+1)}{2} F(R;r) - \frac{rq}{2p} \frac{(r+1)q}{p} F(R-2;r+2) + \\ & R(R+1) - R(R+1) F(R;r) + R \frac{rq}{p} F(R-1;r+1) + \\ & Q \left[\frac{rq}{p} - (R+Q+1) + (R+Q+1) F(R+Q;r) - \frac{rq}{p} F(R+Q-1;r+1) \right] \end{aligned} \right\}$$

$$B = \frac{1}{2Q} \left\{ \begin{aligned} & [(R+Q)^2 + R+Q] F(R+Q;r) - R(R+1) F(R;r) + \\ & \frac{rq}{p} \frac{(r+1)q}{p} [F(R+Q-2;r+2) - F(R-2;r+2)] + \\ & 2 \frac{rq}{p} [R F(R-1;r+1) - (R+Q) F(R+Q-1;r+1)] + \\ & Q \left[2 \frac{rq}{p} - (2R+Q+1) \right] \end{aligned} \right\} \quad (16)$$

6. Reorder Warning Point Computation

To minimize the total annual variable cost it will be necessary to have a value for the shortage cost. We will assume that one of the two managerial methods described in Section 2 will be used to establish a value for λ .

In Section 2 we also assumed that a value for Q would be given. Hence, the reorder warning point is the only variable left to be evaluated. Let us proceed to solve (3) for R .

$$TVC(R) - TVC(R-1) \leq 0 \leq TVC(R+1) - TVC(R)$$

$$H(R + \frac{Q}{2}) + P\frac{D}{Q} + \lambda\frac{B(R)}{S} - H(R-1 + \frac{Q}{2}) -$$

$$P\frac{D}{Q} - \lambda\frac{B(R-1)}{S} \leq 0 \leq H(R+1 + \frac{Q}{2}) + P\frac{D}{Q} +$$

$$\lambda\frac{B(R+1)}{S} - H(R + \frac{Q}{2}) - P\frac{D}{Q} - \lambda\frac{B(R)}{S}$$

which yields

$$B(R-1) - B(R) \leq \frac{H}{\lambda\frac{Q}{S}} \leq B(R) - B(R+1) \quad (17)$$

The optimal R which satisfies (17) will be the largest R such that³

$$B(R-1) - B(R) \leq \frac{H}{\lambda\frac{Q}{S}} \quad (18)$$

From (12) we get

$$1 - \alpha \leq \frac{H}{\lambda\frac{Q}{S}} \quad (19)$$

³See [6] for derivation of this statement.

(14) gives the availability, α , as a function of R . Since R can not be solved explicitly we have to search to find the largest R such that (19) holds.

Although we are seeking the optimum reorder warning point, RWP, it will be easier to find the optimum safety level where RWP is made up of the expected demand during the procurement lead time, μ_L , plus a safety level, SL , to provide protection for the random nature of demand. The RWP can be written as

$$R = \mu_L + SL$$

where μ_L can be considered a constant for any given particular calculation of R and the SL is the portion of the RWP which will vary with the degree of protection desired.

It is possible that the optimum safety level will turn out to have a negative value. However, most managers are reluctant to implement negative safety levels. Therefore, for practical considerations a lower limit of zero will be established.

Let SL^* be the optimum safety level and SL be any non-negative value. Then,

$$TC(SL) \geq TC(SL^*) \quad (20)$$

where TC represents, as before, the total annual expected cost which includes holding, ordering and penalty costs. Thus,

$$TC(SL) \geq H(\mu_L + SL^* + \frac{Q}{2}) + \frac{P^D}{Q} + \lambda \frac{B(\mu_L + SL^*)}{S}$$

If a positive term is dropped from the right hand side of the inequality, the relationship still holds, i.e.,

$$TC(SL) \geq H(\mu_L + SL^* + \frac{Q}{2}) + \frac{P^D}{Q}$$

and

$$H(\mu_L + SL + \frac{Q}{2}) + P\frac{D}{Q} + \lambda \frac{B(\mu_L + SL)}{S} \geq H(\mu_L + SL^* + \frac{Q}{2}) + P\frac{D}{Q}$$

which yields

$$H(\mu_L + SL) + \lambda \frac{B(\mu_L + SL)}{S} \geq H(\mu_L + SL^*)$$

Since SL is evaluated at any non-negative value, let $SL = 0$. Then,

$$H\mu_L + \lambda \frac{B(\mu_L)}{S} \geq H(\mu_L + SL^*)$$

and

$$\frac{\lambda}{H} \frac{B(\mu_L)}{S} \geq SL^* \tag{21}$$

This expression provides an upper bound for the safety level.

7. Camp-Paulson Approximation

The evaluation of the upper bound of the safety level (21) and the inequality which must be satisfied for the optimal R, (19), require the evaluation of the negative binomial cumulative probability. For larger values of R+Q this could require a significant amount of time for computation purposes. Two probability distributions are also required; one for parameter r, and one for parameter r+1.

Bartko [1] has shown that a good approximation for the cumulative negative binomial is a transformation of the negative binomial variate to a standard normal variate⁴, i.e.,

$$NB(x;r,p) = \Phi\left(\frac{y}{3\sqrt{2}}\right) \tag{22}$$

where

⁴The article in Technometrics had a typographical error which has been corrected here.

$$y = \frac{-(9r-1) \left[\frac{rq}{p(x+1)} \right]^{1/3}}{r} + \frac{9x+8}{x+1}$$

$$z = \frac{\left[\frac{rq}{p(x+1)} \right]^{2/3}}{r} + \frac{1}{x+1}$$

To calculate the necessary probabilities for the optimum R now will simply require 6 evaluations to get the maximum safety level (equations 21 and 16) and 4 evaluations for each R value which is appraised in the search routine (equations 19 and 14).

APPENDIX I

IDENTITIES

Section 3 gave identities which were instrumental in the development of the solution of the optimal reorder warning point. These identities will now be derived.⁵

The negative binomial probability function is,

$$\begin{aligned} f(y;r) &= \frac{\Gamma(r+y)}{\Gamma(r) \Gamma(y+1)} p^r q^y & y \geq 0 \\ f(y;r) &= 0 & y < 0 \end{aligned} \tag{23}$$

The mean and variance of the negative binomial are,

$$\mu = \frac{rq}{p} \qquad \sigma^2 = \frac{rq}{p^2}$$

The identities are developed in sequence since most of the results depend on previously derived expressions. First, let i be an integer. Then

$$\begin{aligned} i f(i;r) &= i \frac{\Gamma(r+i)}{\Gamma(r) \Gamma(i+1)} p^r q^i \\ &= \frac{\Gamma(r+i)}{\Gamma(r)(i-1)!} p^r q^i \\ &= \frac{rq}{p} \frac{\Gamma(r+i)}{r\Gamma(r)\Gamma(i)} p^{r+1} q^{i-1} \\ &= \frac{rq}{p} \frac{i\Gamma(i)}{\Gamma(r+1)\Gamma(i)} p^{r+1} q^{i-1} \\ i f(i;r) &= \frac{rq}{p} f(i-1;r+1) \end{aligned} \tag{24}$$

⁵Chung-Mei Chern did much of the early work in the development of these identities.

The second expression is the sum of the first. Thus,

$$\begin{aligned}
 \sum_{i=0}^R i f(i;r) &= \sum_{i=1}^R i f(i;r) \\
 &= \frac{rq}{p} \sum_{i=1}^R f(i-1;r+1) \\
 &= \frac{rq}{p} \sum_{i=0}^{R-1} f(i;r+1) \\
 \sum_{i=0}^R i f(i;r) &= \frac{rq}{p} F(R-1;r+1) \tag{25}
 \end{aligned}$$

The third expression is

$$\begin{aligned}
 \sum_{i=0}^R i^2 f(i;r) &= \sum_{i=0}^R i [i f(i;r)] \\
 &= \frac{rq}{p} \sum_{i=1}^R i f(i-1;r+1) \\
 &= \frac{rq}{p} \left\{ \sum_{i=1}^R (i-1) f(i-1;r+1) + \sum_{i=1}^R f(i-1;r+1) \right\} \\
 &= \frac{rq}{p} \left\{ \sum_{i=0}^{R-1} i f(i;r+1) + \sum_{i=0}^{R-1} f(i;r+1) \right\} \\
 \sum_{i=0}^R i^2 f(i;r) &= \frac{rq}{p} \left\{ \frac{(r+1)q}{p} F(R-2;r+2) + F(R-1;r+1) \right\} \tag{26}
 \end{aligned}$$

The fourth identity uses the definition

$$\sum_{i=0}^R F(i;r) = \sum_{i=0}^R \sum_{j=0}^i f(j;r)$$

By changing the order of summation we get

$$\begin{aligned}
\sum_{i=0}^R F(i;r) &= \sum_{j=0}^R \sum_{i=j}^R f(j;r) \\
&= \sum_{j=0}^R f(j;r) \sum_{i=j}^R 1 \\
&= \sum_{j=0}^R f(j;r) (R-j+1) \\
&= (R+1) \sum_{j=0}^R f(j;r) - \sum_{j=0}^R j f(j;r)
\end{aligned}$$

$$\sum_{i=0}^R F(i;r) = (R+1) F(R;r) - \frac{Rq}{p} F(R-1;r+1) \quad (27)$$

The last identity of concern is

$$\sum_{i=0}^R i F(i;r) = \sum_{i=0}^R i \sum_{j=0}^i f(j;r)$$

By changing the order of summation we get

$$\begin{aligned}
\sum_{i=0}^R i F(i;r) &= \sum_{j=0}^R \sum_{i=j}^R i f(j;r) \\
&= \sum_{j=0}^R f(j;r) \sum_{i=j}^R i \\
&= \sum_{j=0}^R f(j;r) \left\{ \sum_{i=1}^R i - \sum_{i=1}^{j-1} i \right\} \\
&= \sum_{j=0}^R f(j;r) \left\{ \frac{R(R+1)}{2} - \frac{(j-1)j}{2} \right\} \\
&= \frac{R(R+1)}{2} \sum_{j=0}^R f(j;r) - \frac{1}{2} \sum_{j=0}^R (j^2-j) f(j;r) \\
&= \frac{R(R+1)}{2} F(R;r) - \frac{1}{2} \frac{Rq}{p} \left\{ \frac{(R+1)q}{p} F(R-2;r+2) + \right. \\
&\quad \left. F(R-1;r+1) \right\} + \frac{1}{2} \frac{Rq}{p} F(R-1;r+1) \\
\sum_{i=0}^R i F(i;r) &= \frac{R(R+1)}{2} F(R;r) - \frac{1}{2} \frac{Rq}{p} \frac{(R+1)q}{p} F(R-2;r+2) \quad (28)
\end{aligned}$$

APPENDIX II

RELATIONSHIP OF AVAILABILITY OF STOCK AND EXPECTED BACKORDERS

Consider two inventory systems which differ only by their reorder warning points, one using R , the other $R-1$. Let these systems start with 0 stock at time $t=0$ and subject them to demands starting at $t=0$ to $t=\infty$, and have them replenish according to the rules stated earlier (Section 2) for $t \geq 0$. If $I_R(t)$, and $I_{R-1}(t)$ are the net on hand values of the systems with reorder points R , $R-1$ respectively, then $I_R(t) = I_{R-1}(t) + 1$ for $t \geq \text{PLT}$ and $I_R(t) = I_{R-1}(t)$ for $t < \text{PLT}$ where PLT is the procurement lead time. Define a stockout period as the time from when $I_R(t)$ reaches zero from a previous positive value until it next returns to 0. Let $t_i(R)$ be the length of the i^{th} stockout period. Apart from the backorders which occur for $t < \text{PLT}$ (which are the same for both systems) no backorders occur in either system except during stockout periods. Further if $b_i(R)$, $b_i(R-1)$ are the time weighted backorders during $t_i(R)$ for systems with R , $R-1$ respectively, then

$$b_i(R) = b_i(R-1) - t_i(R) * 1 \text{ unit}$$

and

$$B(R) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^{N(T)} b_i(R)$$

where $N(T)$ = number of stockout periods during the interval $(\text{PLT}, \text{PLT} + T)$. Then

$$B(R-1) - B(R) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=1}^{N(T)} t_i(R)$$

But this is just the proportion of time no stock is available or $1-a$.

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