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# A NOTE ON ADMISSIBLE EXCHANGES IN SPANNING TREES

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# A NOTE ON ADMISSIBLE EXCHANGES IN SPANNING TREES

by

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April, 1974



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Spanning trees appear in numerous guises and applications throughout graph theory, ranging from the minimum spanning tree problem [1, 8, 9, 10, 13] to the network "basis structures" of linear programming whose exploitation has resulted in efficient algorithms for minimum cost flow problems [2, 3, 7, 12]. Spanning trees have also recently been shown to have relevance for solving traveling slaesman problems [5,6].

Uses of spanning trees are frequently based on "swaps" involving the deletion of an "in-tree" edge and the addition of an "out-of-tree" edge, such that the result is also a spanning tree. This type of swap, which we will call an <u>admissible exchange</u>, corresponds precisely to a linear programming basis exchange step (i.e., "pivot operation") in a network. Thus, it is well known (and immediately apparent), for example, that given any two distinct spanning trees, T and T', one can always select any edge of T that is not in T', and thereupon find some edge of T' not in T, so that swapping these edges (deleting the first and adding the seocnd) will give an admissible exchange relative to T.

The purpose of this note is to establish a considerably stronger result, concerning the ability to "pair" all edges of T - T' with all edges of T' - T (treating these trees as edge sets) so that each pair gives an admissible exchange relative to T. Furthermore, we will provide a constructive procedure in the proof of this result for identifying precisely such a pairing.

Letting T'(e) denote the unique edge-simple path in T' joining the endpoints of the edge e (or alternatively, the collection of edges on this path) we make the following preliminary observation.

<u>Lemma</u>: Let $e_1$, $..e_k$ be any k edges of T - T'. Then the union of $T'(e_1),...T'(e_k)$ contains at least k distinct edges of T' - T.

<u>Proof</u>: The union of $T'(e_i)$, $i = 1,...,k$ is a forest in T'. The endpoints of each edge $e_i$ lie in exactly one tree of this forest. Now add the edges $e_1,...e_k$ to this forest and delete all edges of T' - T. The result is a subgraph of T. If the forest contains fewer than k edges of T' - T to be deleted, then a cycle is introduced in at least one tree of this forest by adding $e_1,...,e_k$. Since T has no cycles, this is impossible, completing the proof.

Note that the sets $T'(e_i)$ in the foregoing lemma can be also replaced with the sets $T^*(e_i) = T'(e_i) - T$. Utilizing this lemma, we now establish the following result.

<u>Theorem</u>: There is a one-one pairing (matching) of the edges of T - T' with the edges of T' - T so that each pair gives rise to an admissible exchange in T.

<u>Proof</u>: We establish the theorem by a constructive process that generates the desired pairing.* Assume that the edges in a subset E of T - T' have been paired with the edges in a subset E' of T' - T. Select any edge $e_o \in (T - T')-E$. Let $U_o = \emptyset$, and for $i \geq 1$ define $V_i$ = the union of $T^*(e_o)$ with all sets $T^*(e)$ such that $e \in U_{i-1}$, and define $U_i = \{e \in E\colon\ e \text{ pairs with some } e' \in V_i\}$. As long as $V_i \not\subseteq E'$ we have $|U_i| = |V_i| \geq |U_{i-1}| + 1$, where the inequality follows from the definition of $V_i$ in terms of $U_{i-1}$ and the lemma. Thus there must be a finite least k such that $V_k \not\subset E'$. Identify any $e'_k \in V_k-E'$, and for each $i < k$ identify $e_i$ such that $e_i \in U_i \cup \{e_o\}$ and $e'_{i+1} \in T^*(e_i)$, and identify $e'_i$ such that $e'_i$ currently pairs with $e_i$. It follows that the pairing $(e_o, e'_1), (e_1, e'_2),...,(e_{k-1},e'_k)$, with all other pairs in E and E' unchanged, creates a 1-1 correspondence of the edges in $E \cup \{e_o\}$ and $E' \cup \{e'_k\}$.

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*We are indebted to Hal Gabow for pointing out that a nonconstructive proof of the theorem can be based on applying Hall's theorem for distinct set representatives to the foregoing lemma (see, e.g.,[4a]p.53).

The ability to identify and construct the one-one matching of the theorem in the manner indicated is a useful tool for characterizing various types of optimality criteria involving spanning trees. If, for example, one seeks to minimize F(T), where T is a spanning tree, then the theorem establishes the validity of an algorithm that utilized sequential admissible exchanges, all of which are "improving," provided the difference $F(T) - F(T')$ can be expressed in terms of an appropriate function of the edges of $T - T'$ and of $T' - T$. We shall not attempt to fully specify the conditions of an "appropriate function," which can be of a highly complex character, but note that the objective function evaluation for feasible linear programming bases falls in the proper category. As demonstrated in [4], the theorem is also useful for establishing optimality conditions in certain problems where not all improving moves can be restricted to single admissible exchanges.

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