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STATISTICAL THEORY OF LIGHT PROPAGATION  
IN A TURBULENT MEDIUM (REVIEW)

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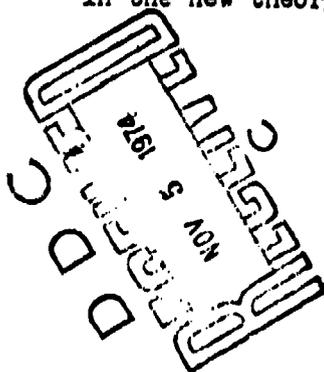
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ABSTRACT: A study of the statistical theory of light propagation in a turbulent medium. A parabolic equation which was examined was only valid when back scattering could be disregarded. The theory proposed uses the small value of the longitudinal scale of heterogeneities, as compared with other longitudinal scales of the problem. Many difficulties encountered in the range of strong fluctuations are eliminated in the new theory.



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## Introduction

In the propagation of electromagnetic waves, in a medium with random large-scale (in comparison with the long wave) irregularities due to the multiple forward scattering effect, fluctuations of the wave field rapidly increase with distance. Beginning at a certain distance, they are unsuitable for calculation by the perturbation theory in any of its forms (range of strong fluctuations)<sup>1</sup>. This effect was observed experimentally by Gracheva and Gurvich [2] in experiments on the propagation of light in a turbulent atmosphere, and in further detailed research in work by Gurvich, Kallistratova, Time [3], Gracheva, Gurvich, Kallistratova [4], and Mordukovich [5].

Recently, a number of books have appeared in which equations have been obtained by different methods, and which describe the strong fluctuation region of a field [6 - 17]. The method for obtaining these equations, used in works [10 - 17], is based on the approximation of the wave propagation process in a nonuniform medium by the random diffusion process. In this approximation,

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<sup>1</sup>The range of fluctuations of intensity, which is described by the first approximation of the even perturbations method, is called the weak fluctuations range. The basic results of theoretical and experimental research in this range have been shown in some detail in a book by Tatarskiy [1] (see also below Section 6 of the first chapter).

closed equations, suitable for the strong fluctuations range, can be obtained for all moments of a field and equation of the Einstein-Fokker type can be obtained for the characteristic function of a field.

In this work, we try to show in sequence the basic results obtained in this direction. The work consists of two chapters.

The first chapter is devoted to a general examination of the propagation process of a light wave in a turbulent medium. The first section examines the stochastic equation, describing propagation of a wave in a random medium and several accurate conclusions of this equation are given. The second section examines a model in which one can disregard the longitudinal radius of the dielectric constant correlation in comparison with all the longitudinal scales of the problem. This supposition, equivalent to the substitution of an actual correlation function of the refractive index on the delta-function in a longitudinal direction, allows one to obtain closed integral equations for all moments of the wave field. Throughout the work, apart from the supposition above, there is also a supposition on the Gauss distribution of probabilities for fluctuations of the refractive index. Here, one can reduce the integral equations to differential ones and show that the characteristic functional of a field is satisfied by an equation of the Einstein-Fokker type, and shows that the propagation of a wave is a diffusion process. The fourth section shows a method of successive approximations for solving the stochastic equation of a wave propagation, in which the diffusion approximation shown above is the first approximation. Investigation of the second approximation allows one to obtain boundaries for the applicability of the diffusion approximation and show that the latter can also be used in the strong fluctuation range of a field. In the fifth section there is an examination of an actual example, devoted to calculating the mutual coherence function in a turbulent

medium and examining the comparison of results of calculation with experimental data. The final section of the first chapter investigates the amplitude-phase characteristics of a light wave.

In a whole series of works describing wave propagation in a medium with random irregularities, the equation for ray diffusion is used [18 - 21] (an approximation of geometrical optics). The diffusion equation itself (the Einstein-Fokker-EEF) is normally written on the basis of intuitive consideration relying on the analogy with well-known problems, leading to this equation. The dynamic equation of the problem (in this case -- the equation of rays) is only used for calculating coefficients in the EEF. As was said in work [22, 23], it is still not clear in what conditions one can justifiably use the EEF (a number of such conditions was suggested in [23] from physical considerations). At the same time, there must be certain conditions superimposed on functions entering the dynamic equations, and certain limitations on parameters, on which the solution depends, in which the EEF will be the logical result of dynamic equations. These conditions are examined in work [24], where it was shown that the EEF can only be valid in narrow angle approximation. Statistical characteristics of amplitude and phase of a light wave, as shown in work [25], are determined by the statistical characteristic of rays.

In the second chapter of this work, light propagation in a turbulent medium is investigated in the approximation of geometrical optics. The first section examines the problem on the diffusion of rays in random irregular media, in the second and the third sections the amplitude-phase fluctuations of a light wave are examined.

## I. General Examination

### 1. Initial stochastic equations and some of their consequences

The propagation of monochromatic light in a medium with large-scale irregularities, when depolarization is small [26], can be quite accurately described by the scalar wave equation

$$\Delta\psi + k^2[1 + \tilde{\varepsilon}(r)]\psi = 0. \quad (1)$$

Here  $\psi$  is linked with a component of the electric field  $E$  by the ratio  $E = \psi \exp(-i\omega t)$ ,  $k^2 = \frac{\omega^2}{c^2} \langle \varepsilon \rangle$ ,  $\tilde{\varepsilon} = \frac{\varepsilon - \langle \varepsilon \rangle}{\langle \varepsilon \rangle}$  -- which is the fluctuating part of the dielectric constant. If one disregards large angle scatter, then instead of (1), one can use the parabolic equation for function  $u$  linked with  $\psi$  by the ratio  $\psi = u \exp(ikx)$ :

$$2ik \frac{\partial u}{\partial x} - \Delta_{\perp} u + k^2 \tilde{\varepsilon}(x, \rho) u(x, \rho) = 0, \quad (2)$$

$$\rho = \{y, z\}, \quad \Delta_{\perp} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

During the transition from (1) to (2), the term  $d^2u/dx^2$  is rejected. The initial condition for (2) is

$$u(0, \rho) = u_0(\rho). \quad (3)$$

Further, we shall proceed from (2).

Equation (2) with the boundary condition (3) can be written in the form of an integral-differential equation

$$u(x, \rho) = u_0(\rho) \exp\left[i \frac{k}{2} \int_0^x d\xi \tilde{\varepsilon}(\xi, \rho)\right] + \quad (4)$$

$$+ \frac{i}{2k} \int_0^x d\xi \exp\left[i \frac{k}{2} \int_{\xi}^x d\eta \tilde{\varepsilon}(\eta, \rho)\right] \Delta_{\perp} u(\xi, \rho),$$

which, in several cases, is more suitable.

If one writes the solution of equation (4) in the form of an interative series, one can easily be persuaded that function  $u(x, \rho)$  will only depend functionally in the previous values along  $x$  of  $u(\xi, \rho)$  from the interval

$0 < \xi < x$ . It follows that  $u(x, \rho)$  does not change when varying the function  $\varepsilon(\xi, \rho)$  outside of this interval, that is, in sections  $\xi < 0$ ,  $\xi > x$ . Consequently, the variation derivative  $\delta u / \delta \varepsilon$  satisfies the condition

$$\frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')} = 0 \quad (x' < 0, x' > x), \quad (5)$$

which we shall call the causality condition. Furthermore, a more suitable value for us is  $\frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')}$  where  $x = x'$ . This value can be found from equation (2), if it is integrated in respect to  $x$  within limits  $(0, x)$ , and afterwards functioned by the operator  $\frac{\delta}{\delta \varepsilon(x', \rho')}$ :

$$2ik \frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')} + \int_0^x d\xi [\Delta_\rho + k^2 \tilde{\varepsilon}(\xi, \rho)] \frac{\delta u(\xi, \rho)}{\delta \varepsilon(x', \rho')} + k^2 \delta(\rho - \rho') u(x', \rho') = 0.$$

Here we calculated that  $\frac{\delta \tilde{\varepsilon}(\xi, \rho)}{\delta \varepsilon(x', \rho')} = \delta(\xi - x') \delta(\rho - \rho')$ , and the lower limit for integrating 0 was replaced by  $x'$ , since according to (5)  $\frac{\delta u(\xi, \rho)}{\delta \varepsilon(x', \rho')} = 0$  when  $x' > \xi$ . Assuming that  $x' = x$ , we obtain the ratio (10)

$$\frac{\delta u(x, \rho)}{\delta \varepsilon(x, \rho')} = i \frac{k}{2} \delta(\rho - \rho') u(x, \rho). \quad (6)$$

In most cases, the value  $\frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')}$  when  $0 < x' < x$  can be expressed by the function of the Green equation (2), linking  $u(x, \rho)$  and  $u(x', \rho')$  when  $0 < x' < x$ :

$$u(x, \rho) = \int d\rho' G(x, \rho; x', \rho') u(x', \rho').$$

by using the ratio [1]

$$\frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')} = i \frac{k}{2} G(x, \rho; x', \rho') u(x', \rho').$$

The solution of equation (2) with the initial condition (3) can, by using the methods suggested by Fradkin [27, 28], be written in operational form or in the form of a Feynman continual integral:

$$\begin{aligned}
u(x, \rho) &= \exp \left[ \frac{i}{2k} \int_0^x d\xi \frac{\partial^2}{\partial \tau^2(\xi)} \right] \left\{ u_0 \left( \rho + \int_0^x d\tau v(\xi) \right) \times \right. \\
&\quad \left. \times \exp \left[ i \frac{k}{2} \int_0^x d\xi \tilde{\epsilon}(\xi, \rho + \int_0^x d\tau v(\tau)) \right] \right\} \Big|_{\tau=0} = \\
&= \int Dv \exp \left\{ i \frac{k}{2} \int_0^x d\xi \left[ v^2(\xi) + \tilde{\epsilon}(\xi, \rho + \int_0^x d\tau v(\tau)) \right] \right\} u_0 \left( \rho + \int_0^x d\tau v(\tau) \right),
\end{aligned} \tag{7}$$

where

$$Dv = \prod_{\xi=0}^x dv(\xi) \left\{ \int \dots \int \prod_{\xi=0}^x dv(\xi) \exp \left[ i \frac{k}{2} \int_0^x d\xi v^2(\xi) \right] \right\}^{-1}.$$

In the first of the equalities, (7) after carrying out the function of the operator in the exponent for the brace which follows it, one must place  $\tau = 0$ . A clear presentation of the solution in form (7), in several cases is suitable for investigation (see [12] and [16]).

## 2. Approximation for delta-correlated fluctuations of a dielectric constant along the propagation direction

As was said above, field  $u(x, \rho)$  only functionally depends on preceding values of  $\tilde{\epsilon}(\xi, \rho)$ . However, there can be a statistical link between  $u(x, \rho)$  and subsequent values  $\tilde{\epsilon}(x', \rho')$ , since values  $\tilde{\epsilon}(x', \rho')$  when  $x' > x$  are correlated with values of  $\tilde{\epsilon}(\xi, \rho)$  when  $\xi < x$ . It is obvious that the correlation of field  $u(x, \rho)$  with the subsequent values of  $\tilde{\epsilon}(x', \rho')$  is marked when  $x' - x \leq l_{\parallel}$ , where  $l_{\parallel}$  -- is the longitudinal radius of correlation of  $\tilde{\epsilon}$ . At the same time, the characteristic radius of correlation of field  $u(x, \rho)$  in a longitudinal direction has a value in the order of  $x$  (see for example [1]). Therefore, in the problem examined, there is a small parameter of  $l_{\parallel}/x$ , which can be used for an approximate solution. One can place  $l_{\parallel} = 0$  in the first approximation. In this case, values of fields  $u(\xi_1, \rho_1)$  when  $\xi_1 < x$  will be not only functionally but also statistically

independent of values of  $\tilde{\epsilon}(\eta_j, \rho_j)$  when  $\eta_j > x$ , that is

$$\left\langle \prod_{i,j} u(\xi_i, \rho_i) \tilde{\epsilon}(\eta_j, \rho_j) \right\rangle = \left\langle \prod_i u(\xi_i, \rho_i) \right\rangle \left\langle \prod_j \tilde{\epsilon}(\eta_j, \rho_j) \right\rangle \quad (8)$$

( $\xi_i < x, \eta_j > x$ ).

By using the feature of (8), one can easily find equations for the statistical moments of field  $u(x, \rho)$ . Let us indicate this on an example  $\langle u \rangle$ . Let us use equation (4) for this. By averaging it, we take into account that in the second term in the right part the value  $\tilde{\epsilon}(\eta, \rho)$  in the exponent is always taken when values of  $\eta > x$ , that is, statistically independent of the second factor  $\Delta_1 u(\xi, \rho)$ . Therefore, when averaging (4), these factors can be averaged independently:

$$\begin{aligned} \langle u(x, \rho) \rangle = & u_0(\rho) \left\langle \exp \left[ i \frac{k}{2} \int_0^x d\xi \tilde{\epsilon}(\xi, \rho) \right] \right\rangle + \\ & + \frac{i}{2k} \int_0^x d\xi \left\langle \exp \left[ i \frac{k}{2} \int_\xi^x d\eta \tilde{\epsilon}(\eta, \rho) \right] \right\rangle \Delta_1 \langle u(\xi, \rho) \rangle. \end{aligned} \quad (9)$$

The equation obtained is closed, since it does not contain other known functions, apart from  $\langle u \rangle$ . In a similar manner, one can obtain an equation for the moment of the arbitrary order:

$$\begin{aligned} M_{n,m}(x, \rho_1, \dots, \rho_n; \rho'_1, \dots, \rho'_m) = & \langle u(x, \rho_1), \dots, u(x, \rho_n) \times \\ & \times u^*(x, \rho'_1), \dots, u^*(x, \rho'_m) \rangle. \end{aligned}$$

For this, one must first of all write the differential equation for  $u(x, \rho_1), \dots, u^*(x, \rho'_m)$  and then convert it into an integral-differential equation of type (4). After this, by using (8), one can carry out averaging. An equation for  $M_{n,m}$  obtained in this way in [11], has the form

$$\begin{aligned} M_{n,m}(x, \{\rho_i\}, \{\rho'_i\}) = & M_{n,m}^0(\{\rho_i\}, \{\rho'_i\}) \\ & \left\langle \exp \left\{ i \frac{k}{2} \int_0^x d\xi \left[ \sum_{i=1}^n \tilde{\epsilon}(\xi, \rho_i) - \sum_{j=1}^m \tilde{\epsilon}(\xi, \rho'_j) \right] \right\} \right\rangle \dots \\ & - \frac{i}{2k} \int_0^x d\xi \left\langle \exp \left\{ i \frac{k}{2} \int_\xi^x d\eta \left[ \sum_{i=1}^n \tilde{\epsilon}(\eta, \rho_i) - \sum_{j=1}^m \tilde{\epsilon}(\eta, \rho'_j) \right] \right\} \right\rangle \dots \end{aligned} \quad (10)$$

$$\left( \sum_{l=1}^n \Delta \rho_l - \sum_{k=1}^m \Delta \rho_k' \right) M_{n,m}(\xi, \{\rho_l\}, \{\rho_k'\}) \quad (10)$$

where  $\{\rho_l\}$  is the aggregate of all  $\rho_l$ , and the value of

$$M_{n,m}^0(\{\rho_l\}, \{\rho_k'\}) = u_0(\rho_1), \dots, u_0(\rho_n) u_0'(\rho_1'), \dots, u_0'(\rho_m').$$

is designated by  $M_{n,m}^0$ .

### 3. Equations for moment of a wave field in a medium with a Gauss distribution of fluctuations of the dielectric constant.

When deriving equations (10), only the characteristics of the delta-correlation of fluctuations of the dielectric constant along the direction of wave propagation were used; the distribution law itself was not defined concretely. Let us now examine a particular case of the Gauss law of distribution for  $\tilde{\epsilon}$ . In this case, the statistical characteristics of  $\tilde{\epsilon}$  are completely described by the correlation function

$$B_\epsilon(x, \rho; x', \rho') = \langle \tilde{\epsilon}(x, \rho) \tilde{\epsilon}(x', \rho') \rangle.$$

The delta-correlation condition is equivalent to substituting  $B_\epsilon$  by an effective correlation'

$$B_\epsilon^{\text{eff}}(x, \rho; x', \rho') = \delta(x - x') A(x, \rho, \rho'), \quad (11)$$

here  $\hat{A}$  is determined from the equality of integrals from  $B_\epsilon$  and  $B_\epsilon^{\text{eff}}$  in respect to  $x'$ :

$$A(x, \rho, \rho') = \int_{-\infty}^{\infty} dx' B_\epsilon(x, \rho; x', \rho').$$

In a case of statistical uniformity of the field  $\tilde{\epsilon}$ , which we shall examine for simplicity,  $A = A(\rho - \rho')$ , that is  $A$  does not depend on  $x$ . In the case examined, one can easily find the value

$$\left\langle \exp \left\{ i \frac{k}{2} \int_0^L dq \left| \sum_{l=1}^n \tilde{\epsilon}(q, \rho_l) - \sum_{j=1}^m \tilde{\epsilon}(q, \rho_j') \right| \right\} \right\rangle$$

and after calculating it reduce equation (10) to a differential equation [11]

$$\begin{aligned} \frac{\partial}{\partial x} M_{n,m} = & \frac{l}{2k} [\Delta_1 + \dots + \Delta_n - \Delta'_1 - \dots - \Delta'_m] M_{n,m} - \\ & - \frac{k^2}{8} Q(\rho_1, \dots, \rho'_m) M_{n,m}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} Q_{n,m} = & \sum_{l=1}^n \sum_{j=1}^n A(\rho_l - \rho_j) - 2 \sum_{l=1}^n \sum_{k=1}^m A(\rho_l - \rho'_k) + \\ & + \sum_{k=1}^m \sum_{l=1}^m A(\rho'_k - \rho'_l). \end{aligned} \quad (13)$$

In the case examined of Gauss delta-correlation fluctuations, the aggregate of equations (12) and also their interconnecting equation for the characteristic functional of a field  $u$  can be obtained by another method [10], which we shall illustrate on an equation for  $\langle u \rangle$ . By averaging (2), we obtain

$$\left( 2lk \frac{\partial}{\partial x} + \Delta_1 \right) \langle u(x, \rho) \rangle + k^2 \langle \tilde{\varepsilon}(x, \rho) u(x, \rho) \rangle = 0. \quad (14)$$

To find the last term, we use the formula obtained by Furutsu [29] and Novikov [30]

$$\langle \tilde{\varepsilon}(r) R[\tilde{\varepsilon}] \rangle = \int dr' \langle \tilde{\varepsilon}(r) \tilde{\varepsilon}(r') \rangle \langle \delta R[\tilde{\varepsilon}] / \delta \tilde{\varepsilon}(r') \rangle, \quad (15)$$

allowing us to calculate the correlation of the Gauss random function  $\tilde{\varepsilon}(r)$  ( $\langle \tilde{\varepsilon} \rangle = 0$ ) with the functional  $R[\tilde{\varepsilon}]$  from it. Formula (15) can be proved, for example, by expanding  $R[\tilde{\varepsilon}]$  into a functional Taylor series.

Since the solution of  $u(x, \rho)$  of equation (2) is a functional of  $\tilde{\varepsilon}$ , one can write

$$\begin{aligned} \langle \tilde{\varepsilon}(x, \rho) \tilde{\varepsilon}(x', \rho') \rangle & = B_0(x, \rho, x', \rho'), \\ \langle \tilde{\varepsilon}(x, \rho) u(x, \rho) \rangle & = \int dx' \int d\rho' B_0(x, \rho, x', \rho') \times \\ & \times \left\langle \frac{\delta u(x, \rho)}{\delta \tilde{\varepsilon}(x', \rho')} \right\rangle. \end{aligned} \quad (16)$$

By substituting instead of  $B_0(x, \rho, x', \rho')$  the effective correlation function (11), we obtain

$$\langle \tilde{\varepsilon}(x, \rho) u(x, \rho) \rangle = \frac{1}{2} \int d\rho' A(\rho - \rho') \left\langle \frac{\partial u(x, \rho)}{\partial \tilde{\varepsilon}(x, \rho')} \right\rangle,$$

where, by integration with respect to  $x$ , the parity of the  $\delta$ -function is taken into account, as a result of which the  $1/2$  factor appears. By substituting the averaged ratio (6), we obtain

$$\langle \tilde{\varepsilon}(x, \rho) u(x, \rho) \rangle = l \frac{k}{4} A(0) \langle u(x, \rho) \rangle,$$

and equation (14) takes on the form

$$\left( 2ik \frac{\partial}{\partial x} + \Delta_{\perp} \right) \langle u(x, \rho) \rangle + l \frac{k^2}{4} A(0) \langle u(x, \rho) \rangle = 0. \quad (17)$$

Equation (17) coincides with equation for  $M_1, 0$ , obtained from (12).

A complete statistical description of field  $u(x, \rho)$  can be obtained from the characteristic functional

$$W_x[u, v^*] = \langle \exp \left[ i \int d\rho [u(x, \rho)v(\rho) + u^*(x, \rho)v^*(\rho)] \right] \rangle.$$

By differentiating  $W_x$  with respect to  $x$ , by using equation (2) and ratio (6), an equation for  $W_x$  can also be obtained. It has the form [10]

$$\begin{aligned} \frac{\partial W_x}{\partial x} = \frac{l}{2k} \int d\rho \left[ v(\rho) \Delta_{\perp} \frac{\partial W_x}{\partial v(\rho)} - v^*(\rho) \Delta_{\perp} \frac{\partial W_x}{\partial v^*(\rho)} \right] - \\ - \frac{k^2}{8} \int d\rho d\rho' A(\rho - \rho') \hat{M}(\rho) \hat{M}(\rho') W_x, \end{aligned} \quad (18)$$

where

$$\hat{M}(\rho) = v(\rho) \frac{\partial}{\partial v(\rho)} - v^*(\rho) \frac{\partial}{\partial v^*(\rho)}.$$

Equation (18), as can easily be shown [10, 15] is equivalent to the aggregate of equations (12). Equation (18) is the infinite dimensional analog of the Einstein-Fokker equation, and in this respect the described approximation of wave propagation in a medium with Gauss delta-correlation fluctuations of  $\tilde{\varepsilon}$  can be called diffused.

Let us write in clear form the equation for the function  $G_n(x, R, \rho)$

$$\left\langle u \left( x, R + \frac{1}{2} \rho \right) u^* \left( x, R - \frac{1}{2} \rho \right) \right\rangle \quad \text{and for function } G_n(x, \rho_1, \rho_2, \rho'_1, \rho'_2) \\ \langle u(x, \rho_1) u(x, \rho_2) u^*(x, \rho'_1) u^*(x, \rho'_2) \rangle, \text{ resulting from (12), when } n = m = 1 \text{ and}$$

$n = m = 2$ :

$$\frac{\partial G_2}{\partial x} = \frac{i}{k} \frac{\partial^2}{\partial R \partial \rho} G_2 - \frac{k^2}{4} D(\rho) G_2, \quad D(\rho) = A(0) - A(\rho), \quad (19)$$

$$G_2(0, R, \rho) = u_0 \left( R + \frac{1}{2} \rho \right) u_0^* \left( R - \frac{1}{2} \rho \right);$$

$$\frac{\partial G_4}{\partial x} = \frac{i}{2k} [\Delta_1 + \Delta_2 - \Delta'_1 - \Delta'_2] - \frac{k^2}{4} [D(\rho_1 - \rho'_1) + D(\rho_2 - \rho'_2) + D(\rho_1 - \rho'_2) + D(\rho_2 - \rho'_1) - D(\rho_2 - \rho_1) - D(\rho'_2 - \rho'_1)] G_4, \quad (20)$$

$$G_4(0, \rho_1, \rho_2, \rho'_1, \rho'_2) = u_0(\rho_1) u_0(\rho_2) u_0^*(\rho'_1) u_0^*(\rho'_2).$$

Equation (19) was obtained by Dolin [6], and Beran [31]. Equation (20) was first obtained by Shishov [7]. Equation (12) for moments of arbitrary order, apart from works mentioned above, was also obtained by another method in a work by Chernov [8].

As Dolin noted [6], equation (19) is equivalent to the so-called narrow angle approximation of a radiation transfer equation. In fact, by substituting (19)

$$G_2(x, R, \rho) = \int dx' J(x, R, x') \exp(ix\rho),$$

one can obtain an equation for function for  $J$

$$\frac{\partial J}{\partial x} = \frac{x}{k} \nabla_R J - \gamma J + \int dx' f(x - x') J(x, R, x'), \quad (21)$$

where  $\gamma = \frac{k^2 A(0)}{4} = \int dx f(x)$  -- is the coefficient of extinction,

$f(x) = \frac{\pi k^2}{2} \Phi_s(0, x)$  -- is the scattering indicatrix,  $\Phi_s(x_1, x_2, x_3)$  -- is the three-dimensional spectral density of the correlation function  $B_g(r)$ .

Equation (19) for  $G_2$  can be solved in a general case for the arbitrary function  $D(\rho)$  and arbitrary initial conditions. As for an equation for  $G_4$ , it cannot be solved analytically and one must resort to numerical methods.

Apart from equations for the mean values for the product of fields  $u(x, \rho_1), \dots, u^*(x, \rho'_m)$ , where all the arbitrary  $x$  are identical. Equations can also be obtained for functions  $\langle u(x_1, \rho_1), \dots, u^*(x'_m, \rho'_m) \rangle$  for non-concurrent

values of  $x$  [14]. The boundary conditions for these equations will contain functions of  $M_{n,m}$  for concurrent values of arbitrary arguments.

Let us also note that presenting the solution in an operational form or in the form of a continual integral (7) when using suppositions on the Gauss distribution of  $\xi$  and the delta-correlation along an arbitrary coordinate makes it possible to calculate  $\langle u(x, \rho) \rangle$  and  $G_2(x, R, \rho)$ . In these cases, it is also possible to solve equation (12). In cases when the solution of equations (12) is not possible, formulation (7) also makes it impossible to find a clear expression for  $M_{n,m}$ , or however, it is suitable for investigating the asymptotic behavior of moments [16].

4. A method of successive approximations and a condition for the applicability of a diffused approximation

Let us now examine a more general method for obtaining equations for moments, a particular case of which is the approximation of a diffused random process. Let us illustrate this method by an example of an equation for an average field.

We shall reckon that the fluctuations of  $\xi$  have a Gauss distribution, but the correlation function  $B_\xi(x, \rho, x', \rho')$  is not changed to  $B_\xi^{\text{eff}}$ . Let us average equation (2), and to find  $\langle \xi u \rangle$  we use the Furutsu-Novikov formula (15). As a result, we obtain the expression (16), the substitution of which into the equation for  $\langle u \rangle$  gives

$$\left( 2ik \frac{d}{dx} + \Delta \right) \langle u \rangle + k^2 \int_0^L dx' \int d\rho' B_\xi(x, \rho, x', \rho') \dots \quad (22)$$

$$\left\langle \frac{\partial u(x, \rho)}{\partial a(x', \rho')} \right\rangle = 0.$$

Equation (22) is not closed, since it contains a new unknown function  $\left\langle \frac{\partial u}{\partial a} \right\rangle$ . To obtain an equation for this function, we shall act on equation (2) by the

operator  $\frac{\delta}{\delta \varepsilon(x', \rho')}$  when  $x' < x$  and we shall average. For finding value of  $\left\langle \frac{\delta u(x, \rho) \delta u(x, \rho)}{\delta \varepsilon(x', \rho')} \right\rangle$  we again use formula (15). As a result, we obtain

$$\left( 2ik \frac{\partial}{\partial x} + \frac{\partial^2}{\partial \rho^2} \right) \left\langle \frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')} \right\rangle + k^2 \int_0^x dx'' \int d\rho'' B_1(x, \rho, x'', \rho'') \times \quad (23)$$

$$\times \left\langle \frac{\delta^2 u(x, \rho)}{\delta \varepsilon(x', \rho') \delta \varepsilon(x'', \rho'')} \right\rangle = 0.$$

Equation (23) is again closed, since it contains a second variational derivative from  $u$ . By continuing this process, one can obtain an infinitely linked chain of equations. A closed equation in approximation of a diffused random process is obtained when converting in equation (22) function

$E_g$  to  $B_g^{\text{eff}} = \mathcal{C}(x - x') A(\rho - \rho')$ , since, in this case, there is a value  $\langle \delta u(x, \rho) / \delta \varepsilon(x', \rho') \rangle$  for concurrent values  $x = x'$ , which according to (6) is expressed through  $\langle u \rangle$ . Converting  $E_g$  to  $B_g^{\text{eff}}$  can be done not in the first equation of the chain (22), but in one of the subsequent equations. For example, if this is done in (22), value  $\left\langle \frac{\delta^2 u(x, \rho)}{\delta \varepsilon(x, \rho'') \delta \varepsilon(x', \rho')} \right\rangle$  will appear under the symbol of the integral. It can be obtained from equation (6) (in which  $\rho'$  is converted to  $\rho''$ ), if one acts with the operator on this equality

$$\frac{\delta}{\delta \varepsilon(x', \rho')} \quad \text{and average} \quad \left\langle \frac{\delta^2 u(x, \rho)}{\delta \varepsilon(x, \rho'') \delta \varepsilon(x', \rho')} \right\rangle = i \frac{k}{2} \delta(\rho - \rho'') \left\langle \frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')} \right\rangle \quad (24)$$

By substituting this expression into (23), we obtain a closed equation for  $\left\langle \frac{\delta u(x, \rho)}{\delta \varepsilon(x', \rho')} \right\rangle$ . After this, the system of equations (22), (23) can be solved, and we shall find a second approximation for  $\langle u \rangle$ .

A second approximation is much more accurate than an approximation of a diffused random process. It is quite difficult to show it in general form, but this question can be investigated in an example permitting an accurate solution. As an example, let us examine equation

$$2ik \frac{du}{dx} + k^2 \tilde{\epsilon}(x)u(x) = 0 \quad (u(0) = u_0),$$

which differs from (2) by the lack of term  $\Delta_{\perp} u$ . If one introduces  $B_1(x, x')$   
 $= \sigma_1^2 \exp(-|x - x'|/l)$ , then

$$\bar{u}(x) = u_0 \exp[-\mu(\tau - 1 + e^{-\tau})],$$

where  $\tau = x/l$ ,  $\mu = \frac{k^2}{4} l^2 \sigma_1^2$ ,  $l$  is the radius of correlation  $\bar{\epsilon}$ .

The solution of the first and second approximations respectively have  
the form

$$\bar{u}_1(x) = u_0 \exp(-\mu\tau),$$

$$\bar{u}_2(x) = \frac{u_0}{1-\mu} [\exp(-\mu\tau) - \mu \exp(-\tau)].$$

By comparing  $\bar{u}_1$ ,  $\bar{u}_2$ , and  $\bar{u}$  when  $\tau \gg 1$ , one can easily determine that the  
approximated solutions can only have a good approximation when  $\mu \ll 1$ . Apart  
from this, function  $\bar{u}_2$ , as opposed to  $\bar{u}_1$ , at zero has the same form as an  
accurate solution. Figure 1 shows functions of  $\bar{u}_1$ ,  $\bar{u}_2$  and  $\bar{u}$  when  $\mu = 0.2$   
(●●● -  $\bar{u}/u_0$ , ○○○ -  $\bar{u}_1/u_0$ , ××× -  $\bar{u}_2/u_0$ ). In this case, the difference in  
the second approximation from an accurate solution does not exceed 2.5%.

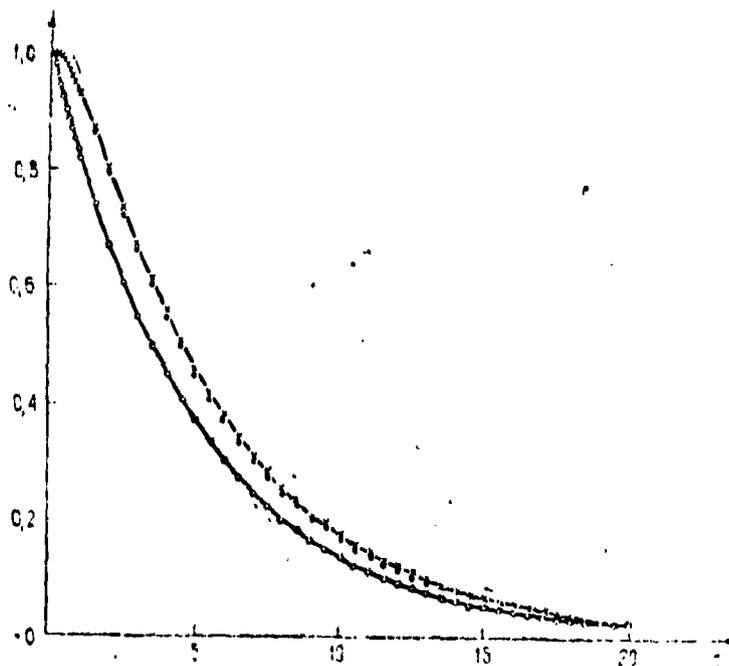


Figure 1

Equations of a second approximation can be solved for  $\langle u \rangle$  and  $Q_2$ . A comparison of the first and second approximations makes it possible to find conditions for the applicability of approximation of a diffused random process. Here, it appears that the conditions for the applicability for  $\langle u \rangle$  have the form

$$\gamma l \ll 1, \quad x \gg l, \quad kl \gg 1, \quad (25)$$

where  $\gamma = \frac{k^2}{4} A(0)$  -- is the coefficient of extinction,  $l$  -- is the radius of correlation of a field  $\tilde{E}$ . At the same time, conditions for the applicability of a first approximation for  $Q_2(x, R, \rho)$  have the form

$$\rho \ll x, \quad kx |\gamma_1 A(\rho)| \ll 1 \quad (D_{\varphi\varphi}(x, \rho) \ll k\rho), \quad (26)$$

where  $D_{\varphi\varphi}(x, \rho)$  -- is the structural function of a complex phase, calculated by the continuous perturbation method [1].

It must be stressed that conditions (25) and (26) are practically independent, since they apply limitations to various parameters. In particular, it can be shown that conditions (26) are fulfilled when condition  $\gamma l \ll 1$  is violated. Let us state that condition (26) sets limitations only on the local characteristics of fluctuations of  $\tilde{E}$  and therefore can also be written for a turbulent medium.

Conditions for the applicability of a parabolic equation (2) are investigated in work [12], where comparison of equations solutions (1) and (2), written in operational form (7), were done. Here, it appeared that conditions for the applicability of a parabolic equation were done throughout the whole range of applicability of a diffused approximation.

##### 5. Solution of an equation for the function of the mutual coherence and some experimental data

Equation (17) for the average field can be easily solved. Its solution has the form

$$\langle u(x, \rho) \rangle = u_0(x, \rho) \exp\left(-\frac{\gamma}{2} x\right),$$

where  $u_0(x, \rho)$  -- is the solution where there are no fluctuations,  $\gamma = \frac{k^2}{4} A(0)$  -- is the coefficient of extinction.

In a general case, one can also solve an equation for function  $G_2$ . This solution was obtained by Bremmer [32] and Dolin [33] when investigating the equation for the transfer of radiation in narrow angular approximations (21), equivalent to equation (19), before an equation was obtained for  $G_2$ . Later, a similar solution was investigated in works [13, 34]. Whereas in (19) one carries out Fourier conversion on variable  $R$ , which is not included in coefficients of the equation, we obtained a linear differential equation in partial derivatives of the first order, which can be easily solved by the characteristic method. The solution has the following form

$$G_2(x, R, \rho) = \frac{k^2}{4\pi^2 \lambda^2} \int dR' d\rho' G_2(0, R - R', \rho - \rho') \times \exp\left[\frac{ikR'\rho'}{x} - \frac{k^2}{4} \int_0^x d\xi D\left(\rho - \rho' \frac{\xi}{x}\right)\right], \quad (27)$$

where  $G_2(0, R, \rho)$  -- is the initial value of the coherency function when  $x = 0$ . Formula (27) can also be used for partially coherent sources [35], when  $G_2(0, R, \rho)$  do not have the form

$$u_0\left(R + \frac{1}{2}\rho\right)u_0^*\left(R - \frac{1}{2}\rho\right).$$

For a partial case of a plane incident wave when  $G_2(0, R, \rho) = \text{const} = |u_0|^2$ , the integral along  $R'$  in (27) is calculated and produces  $\delta(k\rho'/x)$ , after which the formula takes on the simple form of:

$$G_2(x, R, \rho) = |u_0|^2 \exp\left[-\frac{k^2 x}{4} D(\rho)\right]. \quad (27a)$$

If one examines the fluctuations of a dielectric constant, caused by turbulent fluctuations of temperature, in the considerable interval of wave numbers the three-dimensional spectral density  $\Phi_\epsilon(\mathbf{k})$  has the form

$$\Phi_1(x) = AC_1^2 x^{-11/3} \quad (x_{\min} \leq x \leq x_{\max}),$$

where  $A = 0.033$  -- is the numerical constant,  $C_1^2$  -- is the structural characteristic of fluctuations of a dielectric constant, depending on meteorological conditions [1]. In this case, function  $D(\rho)$  is calculated and equals

$$D(\rho) = 2\pi \int dx \Phi_1(x) [1 - \exp(-ix\rho)] = NC_1^2 \rho^{5/3},$$

where  $N = 36\pi^2 A^2(7/6)/5 \cdot 2^{2/3} Q(11/6) = 1.46$ . By substituting this expression into (27), one can calculate both as a function  $G_2(x, \rho, R)$ , and as average intensity of a wave  $\langle I(x, R) \rangle = G_2(x, R, 0)$ .

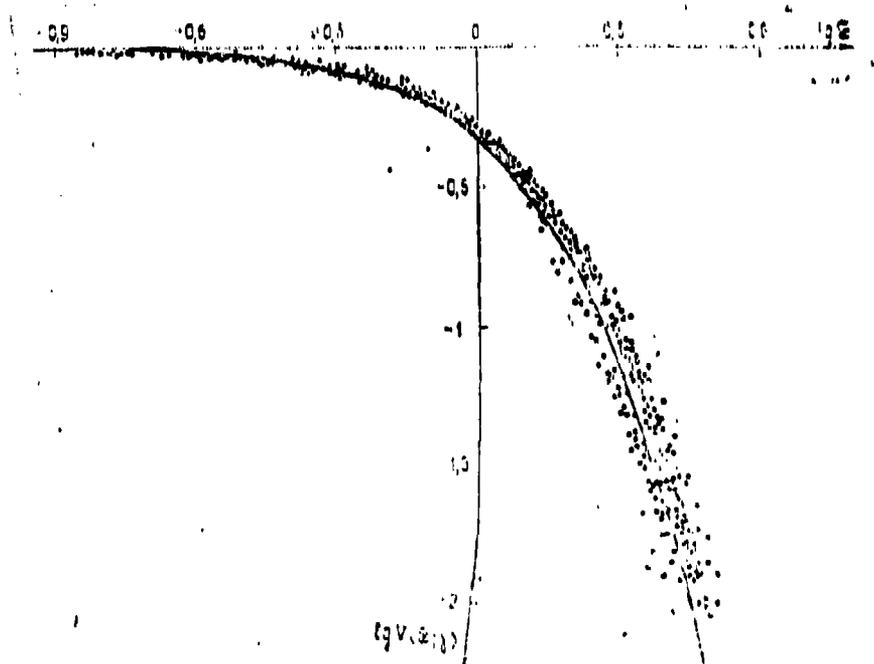


Figure 2. The space spectrum of the coherency function of the second order. Points -- experimental data according to [36], the continuous curve was plotted on the basis of formula (27) with respect to the effect of the internal scale of turbulence, described by parameter  $\gamma$  (cm [36]).

Figure 2 shows the space spectrum of the coherency function obtained in work [36]. On the same graph is the theoretical dependence, plotted on the basis of (27). In work [37] the behavior of space-limited beams in the

atmosphere was experimentally studied. In Figure 3, the results of measurement of average intensity are compared with a curve, plotted on the basis of formula (27).

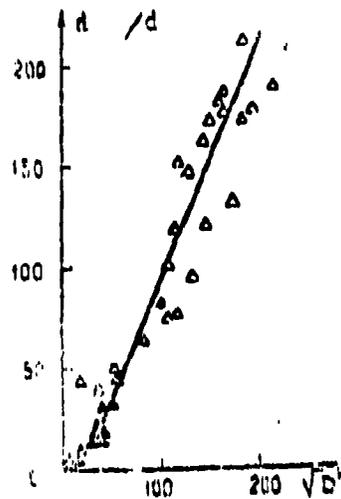


Figure 3

As for the equation for  $Q_{\mu}$ , it was not possible to solve it analytically. An approximated solution of equation (20) (with respect to single scattering in the sense of the theory of radiation transfer) is shown in [13, 15] for turbulent fluctuations of a dielectric constant. In work by Dagkesamanskaya and Shishov [38] there are results of a numerical solution of this equation for the correlation function of a dielectric constant in the form of a Gauss curve. In this work they obtained saturation of intensity fluctuation, which agreed qualitatively with results described in [2 - 5].

Let us state that function  $Q_{\mu}$  also describes fluctuations of the shift of space-limited beams in random irregular media [13, 39].

#### 6. Amplitude-phase fluctuations of a light wave

Let us now examine the statistical description of amplitude-phase fluctuations of a light wave.

By introducing a complex phase of wave  $\varphi = \chi + iS$ , where  $\chi = \ln A/A_0$  -- is the amplitude level, and  $S$  -- are the fluctuations of the phase of a wave relative to the phase of an incident wave  $(x, k)$ , one can write the equation for the continuous perturbation method:

$$2ik \frac{\partial \varphi}{\partial x} + \Delta_{\perp} \varphi + (\nabla_{\perp} \varphi)^2 + k^2 \tilde{\epsilon} = 0, \quad u = \exp(\varphi). \quad (28)$$

Accurate solutions of equations (2) and (28) are equivalent. In the first approximation of the continuous perturbation method in equation (28), the term  $(\nabla_{\perp} \varphi)^2$  is omitted, that is, the equation examined is

$$2ik \frac{\partial \varphi_0}{\partial x} + \Delta_{\perp} \varphi_0 + k^2 \tilde{\epsilon}(x, \rho) = 0. \quad (29)$$

In this case, the solution is a Gauss random field and the statistical properties of amplitude functions are described by the parameter  $\sigma_0^2 = \langle \varphi_0^2(x, \rho) \rangle$  [1]. Here, the validity condition of the first approximation of the continuous perturbation method for amplitude function is the condition  $\sigma_0^2 \ll 1$ . In the range  $\sigma_0^2 \gg 1$  (which is called the range of strong fluctuations) one must study the full equation (28).

Let us state that the diffused approximation for equation (2) does not impose limitations on the amplitude fluctuations and, consequently, equations for moments of field  $u(x, \rho)$  (12) are also valid in the range of strong fluctuations of amplitude.

Statistical properties of the solution (28) can also be described in diffused approximation. However, on the strength of the nonlinearity of the equation (28), equations for moments of function  $\varphi(x, \rho)$  are incomplete.

By separating the imaginary part in (28), we obtain equations for the level of amplitude  $\chi(x, \rho)$  and wave intensity  $I = \exp(2\chi)$ :

$$2k \frac{\partial \chi}{\partial x} + \Delta_{\perp} S + 2\nabla_{\perp} \chi \nabla_{\perp} S = 0; \quad (30)$$

$$k \frac{dI}{dx} + \nabla_{\perp}(I \nabla_{\perp} S) = 0. \quad (31)$$

Experimental study of the distribution of probabilities for amplitude level [2] showed that this distribution, both in the weak range and in the range of strong fluctuations is close to a normal distribution, although "tails" of this distribution, at the present time, have been poorly studied and they also determine the distribution of probabilities for wave intensity. As for fluctuations of the angle of arrival of a wave at the observation point, linked with fluctuations of  $\nabla_{\perp} S$ , they are well described by the first approximation of the continuous perturbation method [40].

For a plane incident wave, boundary conditions for (30), (31) are conditions  $\chi(0, \rho) = 0, I(0, \rho) = 1$  and the solution of these equations will be similar to random fields in plane  $x = \text{const.}$

By averaging equation (31), we obtain the ratio

$$\langle I(x, \rho) \rangle = 1, \quad (32)$$

expressing the law of conservation of energy for the examined problem.

By multiplying (31) by  $\chi$  and averaging, we obtain, by allowing for (32) and space homogeneity, the ratio

$$\langle \chi(x, \rho) I(x, \rho) \rangle = \frac{1}{2k} \int_0^x d\xi \langle \nabla_{\perp} I(\xi, \rho) \nabla_{\perp} S(\xi, \rho) \rangle. \quad (33)$$

Values in the left part (32), (33), are determined by the single-point distribution of probabilities for field  $\chi(x, \rho)$ , and the value in the right part of (33) is linked with the correlation of value  $\nabla_{\perp} I$  and  $\nabla_{\perp} S$ .

If the single-point distribution of probabilities for  $\chi$  is of the Gauss type, then according to (32) it will be the expression

$$\langle \chi(x, \rho) \rangle = \exp(-\sigma^2/2), \quad (34)$$

where  $\sigma^2$  -- is the dispersion of the amplitude level, in this case  $\langle I \rangle = \exp(-\sigma^2/2)$ .

If, when calculating the amplitude fluctuations, the supposition on the possibility for using the expression from the first approximation of the continuous perturbation method for  $\nabla_1 S$  is valid, the right part of (33) can be calculated in a diffused approximation [41], and we obtain

$$\sigma_x^2 \sim \ln \sigma_0 \quad (\sigma_0^2 \gg 1). \quad (35)$$

In this way, the result of the combined use of suppositions on the normality of a single-point distribution of probabilities for amplitude level and the possibility of substituting  $\nabla_1 S$  into the expression from the first approximation of the continuous perturbation method is expression (35), which, however, does not coincide with experimental data relative to the behavior of  $\sigma_x^2$  depending on  $\sigma_0^2$ . Fuller information on the amplitude-phase fluctuations of a plane light wave can be obtained, which is limited by the validity of approximation of geometrical optics.

## II. Approximation of the Geometrical Optics

### 1. Diffusion of Beams in a Medium with Random Irregularities

From the strictly formal point of view, the following is reduced to the Einstein-Fokker equation. Let,  $\xi = \{\xi_1(s), \dots, \xi_n(s)\}$  satisfy the system of dynamic equations

$$\frac{d\xi_i(s)}{ds} = v_i(\xi, s) + f_i(\xi, s), \quad (36)$$

where  $v_i(\xi, s)$  -- are the determinate functions, and  $f_i(\xi, s)$  -- are the random functions ( $n + 1$ ) of the variant having properties

- $f_i(\xi, s)$  -- a Gauss random field in ( $n + 1$ ) -dimensional space  $(\xi, s)$ ,
- $\langle f_i(\xi, s) \rangle = 0$ ,
- $\langle f_i(\xi, s) f_k(\xi', s') \rangle = 2\delta(s - s') F_{ik}(\xi, \xi', s)$ .

In this case, the probability density for solving system (36), that is, function

$$P_x(x) = \langle \delta(\xi(s) - x) \rangle$$

(here  $\xi(s)$  -- is solution of (36), corresponding to the specific realization of  $f(\xi, s)$ , and averaging is done on the population of all realizations of  $f$ ) satisfies the Einstein-Fokker equation:

$$\frac{\partial P_s(x)}{\partial s} + \frac{\partial}{\partial x_k} \{ [v_k(x, s) + A_k(x, s)] P_s(x) \} - \frac{\partial^2}{\partial x_k \partial x_l} [F_{kl}(x, x, s) P_s(x)] = 0. \quad (38)$$

Here, the notation was introduced

$$A_k(x, s) = \left. \frac{\partial F_{kl}(x, y, s)}{\partial x_l} \right|_{y=x}$$

and summation is done on the recurrent indices.

These facts are well known (see, for example, [42, 43]). For practical use, a suitable deduction for equation (38), based on the use of the Furutsu-Novikov formula (15), is shown in work [24]. In the same work there is a method for finding corrections to equation (38), allowing for a finiteness of a radius of correlation of field  $f$  along the  $s$  axis, which, by the same token, allows one to determine limits for the applicability of the equation (38) for actual physical systems.

Let us turn to equations of beams. Normally, beam length  $l$  is used as an independent variable  $s$ . By accepting  $l$  as the independent variable, we shall write equations for beams in a form

$$\frac{dr_k(l)}{dl} = v_k(l), \quad \frac{d\tau_i(l)}{dl} = (v_{ik} - v_i \tau_k) \frac{\partial \mu(r)}{\partial r_k}, \quad (39)$$

where  $\mu = \ln n$ ,  $n$  -- is the refractive index. If one introduces 6-vectors  $\xi = (r, \tau)$ ,  $\nu = (\tau, 0)$ ,  $f = (0, a)$ , where  $a_i = (v_{ik} - v_i \tau_k) \frac{\partial \mu(r)}{\partial r_k}$ , the system of equations (39) can be written in form (36). Conditions (37a, b) can be accepted without reservations. However, as for the transfer to approximations of the correlation function for  $f$  by using the  $\delta$ -function, we meet insurmountable difficulties. The fact is that  $a_i = a_i(r, \tau) = a_i(\xi)$  and does not depend on  $s = l$ . Formally, one can reckon that the function  $B_{ij}(\xi, \xi', s, s')$

(in this case, not depending on  $s, s'$ ) has according to variable  $s - s'$  an infinite interval of correlation for with as large an increase as possible in the value of  $s - s'$ , but with  $\xi, \xi'$  fixed, this function does not diminish. In this way, it is not possible to write the EEF corresponding to the dynamic system (39). However, one can show equations of beams, by taking the independent variable to coordinate  $x$ . If one looks for the equation of a beam in form  $R_{\perp} = R_{\perp}(x)$ , where  $R_{\perp} = \{y, z\}$  -- is the transverse shift, instead of (39), we shall have a dynamic system

$$\frac{d}{dx} R_{\perp}(x) = \frac{\tau_{\perp}}{\sqrt{1 - \tau_{\perp}^2}}, \quad \frac{d}{dx} \tau_{\perp} = \frac{a_{\perp}(R_{\perp}, \tau_{\perp}, x)}{\sqrt{1 - \tau_{\perp}^2}}, \quad (40)$$

where  $\tau_{\perp} = (\tau_y, \tau_z)$ ,  $a_{\perp} = (a_y, a_z)$ . However, we notice that in this form the equation of beams can be used only until the first point of turn, where the denominator  $\sqrt{1 - \tau_{\perp}^2}$  is reduced to zero. It follows that in a statistical problem (40) can only be used in a range where the probability of negative  $\tau_x$  is small, that is, in the range of narrow angular deviations of beam. Since in this range  $\sqrt{1 - \tau_{\perp}^2} \approx 1$ , instead of (40) we obtain an approximate system of equations for beams in narrow-angle approximations:

$$\frac{d}{dx} R_{\perp}(x) = \tau_{\perp}(x), \quad \frac{d}{dx} \tau_{\perp}(x) = \nabla_{\perp} a_{\perp}(R_{\perp}, x). \quad (41)$$

For system (41)  $s = x$  is already included in the number of arguments  $f$ , and therefore, here one can transfer to the EEF. The corresponding EEF, according to (38) has the form

$$\frac{\partial P_{\perp}}{\partial x} + \tau_{\perp} \frac{\partial P_{\perp}}{\partial R_{\perp}} - D \frac{\partial^2 P_{\perp}}{\partial \tau_{\perp}^2}, \quad (42)$$

where  $D$  -- is the coefficient of diffusion, which occurs when calculating  $F_{kl}$  and in a model of statistically uniform and isotropic fluctuations of  $\mu$ :

$$D = \pi^2 \int_0^{\infty} dx x^2 \Phi(x). \quad (43)$$

Here,  $\Phi(k)$  -- is the three-dimensional spectral density of the correlation function  $\mu$ . Expression (43) for  $D$  coincides with those shown in works [8 - 21]. Equation (42) is easily solved, and its solution, corresponding to the initial condition  $P_0(R_{\perp}, \tau_{\perp}) = \delta(R_{\perp}) \delta(\tau_{\perp})$ , has the form of a Gauss distribution with moments

$$\begin{aligned} \langle R_{\perp i}(x) R_{\perp k}(x) \rangle &= \frac{2}{3} D \delta_{ik} x^3, \\ \langle R_{\perp i}(x) \tau_{\perp k}(x) \rangle &= D \delta_{ik} x^2, \\ \langle \tau_{\perp i}(x) \tau_{\perp k}(x) \rangle &= 2D \delta_{ik} x. \end{aligned} \quad (44)$$

These expressions are well known. On the basis of system (41) it is also easy to obtain the longitudinal correlation function of beams displacement [24]:

$$\frac{\langle R_{\perp i}(x) R_{\perp i}(x') \rangle}{V \langle R_{\perp i}^2(x) \rangle \langle R_{\perp i}^2(x') \rangle} = \left(1 + \frac{3}{2} \frac{x}{x_{\text{min}}}\right) (1 + \frac{3}{2} \frac{x'}{x_{\text{min}}})^{-3/2}, \quad \frac{x - x'}{x_{\text{min}}} \quad (45)$$

Let us now examine the problem of simultaneous diffusion of two beams. In this case we have the following dynamic system of eighth order:

$$\frac{dR_{\perp i}}{dx} = \tau_{\perp i}, \quad \frac{d\tau_{\perp i}}{dx} = \frac{\partial P_{\perp i}(R_{\perp i}, \lambda)}{\partial R_{\perp i}} \quad (46)$$

$i = 1, 2$  -- number of beams. The corresponding EEF for the system (46), according to (38), has the form

$$\begin{aligned} \frac{\partial}{\partial x} P_{\perp}(R_{\perp 1}, R_{\perp 2}, \tau_{\perp 1}, \tau_{\perp 2}) &+ \tau_{\perp 1} \frac{\partial P_{\perp}}{\partial R_{\perp 1}} + \tau_{\perp 2} \frac{\partial P_{\perp}}{\partial R_{\perp 2}} \\ &= D \left( \frac{\partial^2 P_{\perp}}{\partial \tau_{\perp 1}^2} + \frac{\partial^2 P_{\perp}}{\partial \tau_{\perp 2}^2} \right) + 2 \frac{\partial^2 P_{\perp}}{\partial \tau_{\perp 1} \partial \tau_{\perp 2}} W_{\perp}(R_{\perp 1} - R_{\perp 2}) P_{\perp} \end{aligned} \quad (47)$$

where

$$\begin{aligned} W_{\perp} &= \frac{1}{2} \frac{\partial^2 A(\rho)}{\partial x_1 \partial x_2}, \quad \rho = R_{\perp 1} - R_{\perp 2}, \quad x_1 = y_1, \quad x_2 = z \text{ and} \\ A(\rho) &= 2\pi \int d^2k \Phi(k) \exp(i k \rho), \quad \rho = R_{\perp 1} - R_{\perp 2}. \end{aligned}$$

In equation (47) one can introduce new variables  $\rho = R_{\perp 1} - R_{\perp 2}$ ,  $R = \frac{1}{2}(R_{\perp 1} + R_{\perp 2})$ ,  $l = \tau_{\perp 1} - \tau_{\perp 2}$ ,  $\tau = \frac{1}{2}(\tau_{\perp 1} + \tau_{\perp 2})$ . After transferring to new variables, it is possible to carry out integration on  $R$  and  $\tau$  and obtain an equation for the function:

$$P_{\perp}(\rho, l) = \langle \delta(R_{\perp 1} - R_{\perp 2} - \rho) \delta(\tau_{\perp 1} - \tau_{\perp 2} - l) \rangle,$$

describing the relative diffusion of two beams,

$$\frac{\partial P_A(\rho, l)}{\partial x} + l \frac{\partial P_A}{\partial \rho} = D_{rel}(\rho) \frac{\partial^2 P_A}{\partial l^2} \quad (48)$$

Here

$$D_{rel}(\rho) = 2[D\epsilon_{03} - W_{03}(\rho)] = 2\pi \int dx x \epsilon_0 \Phi(x) [1 - \cos \kappa \rho]. \quad (49)$$

If  $l_0$  -- is the radius of correlation of gradients of the refractive index, when  $\rho \gg l_0$   $D_{rel}(\rho) = 2D\epsilon_{03}$ . This ratio shows that then there are large initial distances between beams, in comparison with  $l_0$ , their relative diffusion occurs with a two-fold diffusion coefficient, which corresponds to the independent diffusion of each beam. The joint distribution of probabilities in this case can be reckoned as Gauss.

In a general case it was not possible to solve equation (48). It is only obvious that when there is a variable diffusion coefficient (49) the solution is not a Gauss distribution. This can indicate [24] that the mean square of the distance between beams increases proportionally to the cube of the distance traveled, but the coefficient in this formula is different in small and reciprocal distances. These two ranges are separated from each other by a small range of exponential increase, the beginning of which coincides with the beginning of the range of strong fluctuations of wave intensity.

Let us examine the limits for the applicability of a diffused approximation for beams shown above. As was already said, the EEF for diffusion of beams could only be substantiated in narrow-angle approximation. Hence, according to (44) the condition arises

$$\langle \sigma^2 \rangle \ll 1 \quad \text{or} \quad l \ll \frac{\sigma^2 x}{l_0} \quad (50)$$

where  $\sigma_\mu^2$  -- is the dispersion of fluctuations of the refractive index. This condition imposes a small limitation on the transverse displacement of the

beam:  $\langle R^2(x) \rangle \ll x^2$ . An examination of corrections linked with the finiteness of the longitudinal radius of correlation, does not lead to new limitations.

Let us state that in another extreme case (when the beam goes far off into a heterogeneous medium) there is an isotropic distribution of probabilities for the beam in space similar to the Boltzmann distribution in statistical physics [44].

## 2. Amplitude-phase fluctuations of a light wave in random heterogeneous media

Let us now examine the amplitude-phase fluctuations of a plane light wave.

In an approximation of the geometrical optics, the system of equations for the amplitude level and wave phase takes on the form

$$\begin{aligned} \frac{\partial \chi}{\partial x} + \frac{1}{k} \nabla_{\perp} S \nabla_{\perp} \chi &= -\frac{1}{2k} \Delta_{\perp} S(x, \rho) \quad (\chi(0, \rho) = 0), \\ \frac{\partial S}{\partial x} + \frac{1}{2k} (\nabla_{\perp} S)^2 &= \frac{k}{2} \tilde{\epsilon}(x; \rho) \quad (S(0, \rho) = 0). \end{aligned} \quad (51)$$

The system of equations (51) was investigated in works [45 - 47, 25].

Solution of an equation for  $\chi$  can be written in the form [45]<sup>1</sup>.

$$\chi(x, \rho) = -\frac{1}{2k} \int_0^x d\xi \Delta_{\perp} S(\xi, R(x, \xi; \rho)), \quad (52)$$

where  $R(x, \xi; \rho) = R(x, x'; \rho)$  are the transverse displacements of beams, determining the beam tube, which satisfy the equation

$$R(x, x'; \rho) = \rho - \frac{1}{k} \int_{x'}^x d\xi \nabla_{\perp} S(\xi, R(x, \xi; \rho)). \quad (53)$$

Here, on the strength of the supposed singularity of beams,

$$R(\xi_1, \xi; R(x, \xi_1; \rho)) = R(x, \xi; \rho). \quad (54)$$

The paths of beams arriving at the observation point  $(x, \rho)$  are determined by the value  $\nabla_{\perp} S$  the equation for which, in its turn, has the form

<sup>1</sup>Ratios (52) - (54) are also valid when there are diffraction effects.

$$\frac{\partial}{\partial x} \nabla_{\perp} S + \frac{1}{k} (\nabla_{\perp} S \nabla_{\perp}) \nabla_{\perp} S = \frac{k}{2} \nabla_{\perp} \tilde{\epsilon}(x, \rho). \quad (55)$$

The solution of equation (55) can also be written in the form of an integral with respect to the same beam:

$$\nabla_{\perp} S(x, \rho) = \frac{k}{2} \int_0^x d\xi \nabla_{\perp} \tilde{\epsilon}(\xi, R(x, \xi; \rho)). \quad (56)$$

By substituting (56) into (53), one can discard the wave phase from examination, and we shall arrive at an equation for the path [25]

$$\frac{d^2 R(x')}{dx'^2} = \frac{1}{2} \nabla_{\perp} \tilde{\epsilon}(x', R(x')) \quad (57)$$

with limiting conditions

$$R(x) = \rho, \quad \left. \frac{dR(x')}{dx'} \right|_{x'=0} = 0. \quad (58)$$

Here, the expression (52) for the amplitude level assumes the form

$$\chi(x, \rho) = \frac{1}{4} \int_0^x d\xi \int_0^{\xi} d\eta \frac{\partial^2}{\partial \rho \partial \rho_0} \tilde{\epsilon}(\eta, R(x, \eta; \rho)) \times \left. \frac{\partial R_k(\xi, \eta, \rho)}{\partial \rho_1} \right|_{\rho} R(\eta, \xi; \rho). \quad (59)$$

In this way, both the amplitude level and the phase gradient, determining the angle of arrival of the wave at the observation point for a specific realization of field  $\tilde{\epsilon}(x, \rho)$ , is determined by the single "dynamic" equation (57) with limiting conditions (58). Beams, determined by equation (57), vary from beams in Section 1 since the limiting conditions for them are set in different space points (when  $x' = 0$  and  $x' = x$ ). Hence, the propagation of beams described by equation (57) with conditions (58), does not have the nature of a diffused random process (even during the condition of  $\delta$ -correlation of field  $\tilde{\epsilon}(x, \rho)$  along the  $x$  axis), and the probability density for these beams is the arbitrary probability density for a whole aggregate of beams investigated in Section 1 of this chapter.

Beams, determined by equation (57) with conditions (58), describe fluctuations of the beam tube, which determine fluctuations of wave intensity at the observation point. In work [48] there is a calculation of fluctuations of

the sectional area of a beam tube when one can use a successive approximation method for equation (57).

Let us note that, as was shown in work [47], both the solution itself of the equation for  $\chi$ , and all the statistical characteristics of it are determined by solving the equation for the wave phase  $S(x, \rho) \equiv S[\bar{z}]$ .

### 3. The applicability of diffused approximation for describing amplitude-phase wave fluctuations

An equation for the amplitude level can be written in the form

$$\frac{\partial I}{\partial x} + \frac{1}{k} \nabla_{\perp} (I \nabla_{\perp} S) = 0 \quad (I(0, \rho) = 1), \quad (60)$$

where  $I = \exp(2\chi) = u(x, \rho) u^*(x, \rho) / |u_0|^2$  -- is the wave intensity. On the strength of the space homogeneity of field  $\bar{E}(r)$  random fields  $u(x, \rho)$ ,  $I(x, \rho)$ ,

$S(x, \rho)$  will be homogeneous fields in plane  $x = \text{const}$ . By using equations (51), (60) one can obtain an expression for the single-point intensity correlation and wave phase:

$$\frac{\partial}{\partial x} \langle IS \rangle = \frac{k}{2} \langle \bar{z} I \rangle + \frac{1}{2k} \langle I (\nabla_{\perp} S)^2 \rangle; \quad (61)$$

on the other hand, the ratio takes place

$$\nabla_{\perp} G(x, \rho_1, \rho_2) \Big|_{\rho_1 = \rho_2} = \langle I [(\nabla_{\perp} \chi)^2 - (\nabla_{\perp} S)^2] \rangle, \quad (62)$$

where  $G(x, \rho_1, \rho_2) = \langle u(x, \rho_1) u^*(x, \rho_2) \rangle$  -- is the mutual coherency function examined in the first chapter. In an approximation of geometrical optics, the expression (62) is simplified and takes on the form

$$\nabla_{\perp} G(x, \rho_1, \rho_2) \Big|_{\rho_1 = \rho_2} = \langle I (\nabla_{\perp} S)^2 \rangle. \quad (63)$$

The left portion (63) can be calculated in an approximation of the diffused random process. In this approximation, function  $G$  for the plane wave is described by formula (27 a), and  $\langle \bar{z}(x, \rho) / (x, \rho) \rangle = 0$ . Consequently, in an approximation of geometrical optics and diffused approximation<sup>1</sup>,

<sup>1</sup> Let us state that formulae (64) are also valid in the case of non-Gauss fluctuations of  $\bar{E}$ .

$$\langle I(\nabla, S)^2 \rangle = \frac{k^2}{4} \Delta_{\perp} D(0) x = k^2 \gamma, \quad (64)$$

$$\langle IS \rangle = \frac{k}{16} \Delta_{\perp} D(0) x^2,$$

where  $\gamma = \pi^2 \int_0^{\infty} dx x^2 \Phi_s(x) x = \frac{1}{k^2} \langle (\nabla_{\perp} S_0)^2 \rangle$  -- is the dispersion of the angle of arrival of a wave at the observation point in the first approximation of a continuous perturbation method (the parameter, characterizing the intensity of phase fluctuations [46],  $S_0(x, \rho) = \frac{k}{2} \int_0^x d\xi \tilde{s}(\xi, \rho)$ ).

The correlation  $\langle IS \rangle = \frac{1}{2l} \langle uu^* \ln u/u^* \rangle$  is determined by the single-point distribution of probabilities of field  $u(x, \rho)$  and generally speaking, is conditioned by all moments of  $u(x, \rho)$ . The expression (64) for this correlation was calculated in a diffused approximation.

By following Section 4 of the first chapter, one can obtain conditions for the applicability for formulae (64), which, to some degree, will characterize the condition of applicability of the diffused approximation for a single-point distribution of probabilities in an approximation of geometrical optics. These conditions for a turbulent medium will have the form [25],

$$K_m x \gg 1, \quad \gamma \ll 1, \quad (65)$$

where the value of  $K_m$  characterizes the internal scale of turbulence.

In this way, when carrying out conditions (65) calculations for the amplitude-phase fluctuations in an approximation of the diffused random process in a geometrical-optics approximation are valid. Conditions (65) coincide with conditions (50), describing the applicability of a diffused approximation for beams in a turbulent medium, which, as was shown in the previous section, also determine all the statistical characteristics of a light wave.

Let us try to analyze the results obtained from the point of view of their applicability for the propagation of light in a turbulent atmosphere.

A parabolic equation (quasi-optical approximation) was suggested as a basis for examination. This equation is only valid when one can disregard back scattering. For light, scattered in turbulent fluctuations of the refraction index, angles  $\theta \sim \lambda/l_0$ , in which the basic part of the scattering field is included, have an order of  $10^{-5} - 10^{-4}$ . Since fluctuations of the dielectric constant in the atmosphere are also very small ( $\delta\epsilon \sim 10^{-6}$ ), one can almost always disregard back scattering of light in turbulent heterogeneities. This condition is only expected to be violated in the millimeter wave band, and even in this case deviations are very small.

On the other hand, the small value of the scattering angle  $\theta$  causes a strong interaction of scattered waves among themselves and with the incident wave. This effect, in spite of the small value of  $\delta\epsilon$  also causes strong field fluctuations. As a result, as was said above, calculation methods, based on one or another form of the perturbation method, are invalid. The suggested theory uses the small value of another parameter -- the characteristic longitudinal scale of heterogeneities compared with all other longitudinal scales of the problem. Here, there is an analogy with the nonequilibrium kinetic theory of gases, where the small value of the interaction time of molecules (in our case, the longitudinal coordinate plays a time role) is used for separation. As a result, in spite of the fact that conditions for the applicability of the theory occur and contain limitations on the size of fluctuations of  $\delta\epsilon$ , these limitations are so small that they are carried out in an actual atmosphere even for relatively long routes. Unfortunately, due to the expansion of the applicability range, one must contend with a relatively complicated theory and the complication of its results, that is, one must have recourse to numerical methods of calculation (for example, for function  $G_H$ ). Apart from this, since the proposed theory operates with coherency

functions of different orders, complications occur when phase fluctuations must be calculated. However, the basic difficulty with which one must contend in the continuous perturbation method -- of not being able to describe a light field in the strong fluctuation range -- is eliminated in the new theory. Apparently, in all cases when the statistical characteristics of a light field in a turbulent atmosphere which interested us could be expressed by coherency functions, the suggested theory almost always gave us an acceptable result.

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