# Analysis of Target Location Error

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# Caveat!

- Unclassified data from flight test was used, BUT
- Range to the target is given by point number, not actual distance
- Target Location Error (TLE) is modified – given as cubits
- Otherwise, idea, analysis approach are OK

# Aside: Analysis idea

- The approach makes ‘engineering’ sense
- TLE should decrease as distance to the target decreases *i.e.*
- $\frac{d(TLE)}{d(Range)} \;\; \alpha \;\; k \times (\mu - Range)$, *$\mu$ a long-term bias*
- Further, TLE exhibits a random component also a function of range - $\sigma \times Range \times \; dW$ where W is a random process, in this case a Weiner process with scale $\sigma$

# Basic Idea- Preliminaries

- Model Target Location Error (TLE) as a *stochastic* process:
  - We observe a collection of random variables indexed by distance to a target
  - Random variables, TLE(r), take values in a set commonly referred to as a *state space.*
  - In the present case, TLE, the state space is continuous
  - Change of the process at range r depends on TLE(r)- a Markov Process.
- Simple (deterministic) ordinary differential equation (ODE) description:

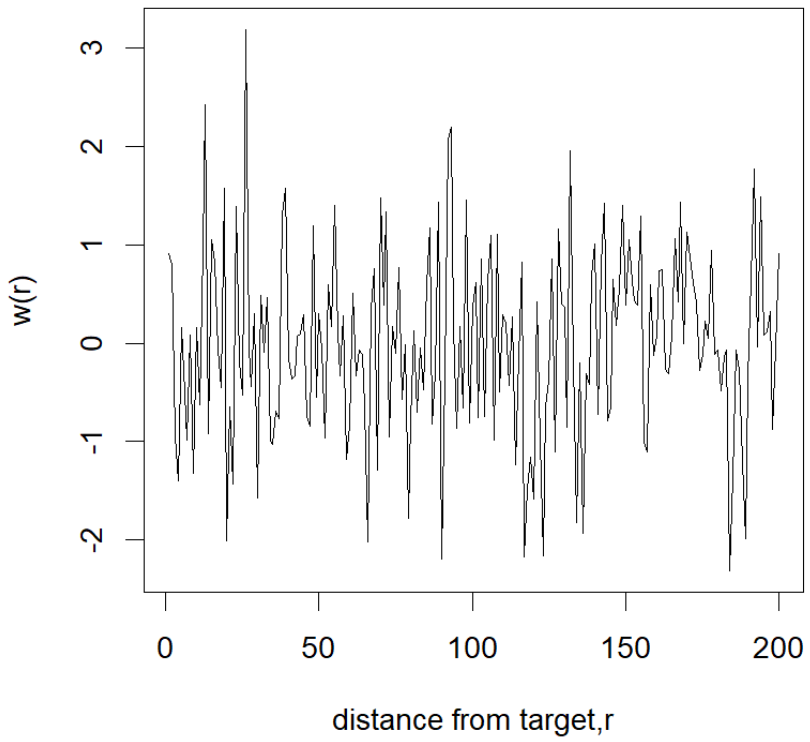
  $dTLE_r = f(r, TLE_r)dr$ and thus $TLE_r = \int_0^r f(r, TLE_r)dr$

  This plays a major role in the stochastic differential equation (SDE) model of TLE

# Bownian Motion..Weiner process

- A Weiner process is a generalization of a random walk: *viz.,* each increment is independent. This gives rise to a stochastic differential equation:
  - $\frac{d\beta(t)}{dt} = w(t)$ w(t) is a white-noise process, *i.e.* w(t) is independent of w(t') for $t \neq t'$
  - Brownian motion is the solution to this SDE
- Combine an ordinary differential equation with driving white noise:
  - dx/dt = f(x,t) + L(x,t)w(t) where w(t) is a forcing function, modeled as white noise
  - Since the forcing function is random, the solution, x(t) is a random process

# Realization of white noise




- White noise, w(t) driving TLE
- w(t) is independent of w(t'), $t \neq t'$,
- w(t) is a Gaussian process, zero mean, and dirac delta correlation: $\delta\ (t - s)Q$
- w(t) is continuous almost everywhere and is nowhere differentiable
- General Model: dx/dt = f(x,t) + L(x,t)w(t) , f(x,t) is the drift function, L(x,t) is dispersion function

# So… $\beta(t)$ a Weiner process…

- If f(x,t) is linear we get a solution, usual calculus stuff:

$$x(t) - x(t_0) = \int_{t_0}^{t} f(x(t), t)dt + \int_{t_0}^{t} L(x(t), t)\beta(t)dt$$

  - The first integral on the RHS is OK, a Riemann integral
  - The second integral on the RHS.. not so. The white noise term $\beta(t)$ is unbounded and discontinuous almost everywhere. Recall that the Riemann integral relies on the limit of the integrand < ∞ as of the mesh of the partition, $\Delta \to 0$, but everywhere the upper and lower $\beta(t)$ terms do not converge.

  WE NEED a new definition of the integral! $It\hat{o}$

# Itô integral- get around discontinuous $\beta(t)$

- Fix the choice of the $t_k$ terms, then $\int_{t_0}^{t} L(x(t),t)d\beta(t)$

$$= \lim_{n\to\infty} \sum_k L(x(t_k), t_k)\ |\beta(t_{k+1} - \beta(t_k)|, \text{ and}$$

$$0 = t_0, t_1, \dots, t_n = t$$

- One detail to work out, $\int_0^t \beta(t)d\beta(t)$ ; we expect this to be $\frac{\beta^2(t)}{2}$ ; but- $\int_0^t \beta(t)d\beta(t)$ = $\lim_{n\to\infty} \sum_k \beta(t_k)|\beta(t_{k+1} - \beta(t_k)|$
- Multiply out, collect terms, use $\lim_{n\to\infty}\ (\beta(t_{k+1}) - \beta(t_k))^2$ = t
- $\int_0^t \beta(t)d\beta(t)$ = -t/2+ $\beta^2$(t)/2

# Aside (again)

- Why is $\lim_{n \to \infty} (\beta(t_{k+1}) - \beta(t_k))^2$ = t ?
- $\beta(t)$ a Gaussian process, random walk, each step independent of the others; variance of the difference between two observations is variance of the Gaussian process, step size, t

# Back to the derivative

- From $\int_0^t \beta(t)d\beta(t)$ = -t/2+ $\beta^2$(t)/2,
- d $\frac{\beta^2(t)}{2}$ = $\beta(t)d\beta(t)$ + t/2
- Yikes! chain rule is different in $It\hat{o}$ calculus- the expected (Stieltjes) form: $\beta(t)d\beta(t)$ is not the derivative, as one would expect.
- This is a consequence of fixing $t_k$
- This has a significant impact on solution of a SDE and on the process of fitting an SDE to data

# On to TLE!

- So, analysis of system TLE performance consists of three steps:
- First, determine the appropriate form of a SDE for TLE analysis
  - $dTLE = f(TLE, r)dr + L(TLE, r)d\beta$, f() the 'drift' term from an ODE- rate of reversion to the long-term mean of the process, and L(), the uncertainty or dispersion of the Weiner process
- Secondly, fit the chosen model to the observed TLE data
  - Estimate the parameters of f() and L(),
  - Use multiple runs
- Thirdly, determine TLE characteristics from the fitted model

# What model

- Much of the literature in SDEs is devoted to two non-linear SDE processes: Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR)
- In both models, $\theta$ is the rate of reversion, $\mu$ the long-term mean, and $\sigma$ the Weiner process uncertainty
- OU: Model form: $dTLE = \theta(\mu - TLE)dr + \sigma d\beta$
- CIR: Model form: $dTLE = \theta(\mu - TLE)dr + \sigma\sqrt{TLE}\, d\beta$
- Main differences:
  - OU admits negative values of TLE; CIR requires TLE >0
  - OU appropriate for along- and cross track errors / CIR appropriate for radial errors
  - CIR model more difficult to fit to data

# Fit the CIR model to data

- CIR model was selected; TLE is typically a 2-dimensional (radial) error
- Maximum likelihood estimate of the parameters:
  - Minimize –log likelihood($\boldsymbol{\theta}$ | data) where $\boldsymbol{\theta}$ = $(\theta, \mu, \sigma)$, parameters from the CIR model
- log-likelihood – Bayesian approach posterior distribution of parameters, due to the Markov nature of SDEs:
  - p(TLE(1), TLE(2), … TLE(n)|$\boldsymbol{\theta}$) = $\prod_{k=0}^{n-1} p(TLE(k+1)|TLE(k), \boldsymbol{\theta})$ where $\boldsymbol{\theta}$ the parameters, and $p(TLE(k+1)|TLE(k), \boldsymbol{\theta})$ the transition density from the process
  - Likelihood of $\boldsymbol{\theta}$ is $l(\theta)$= -$\sum_{k=0}^{n} \log(p(TLE(k+1) \,\big|\, TLE(k), \boldsymbol{\theta})$

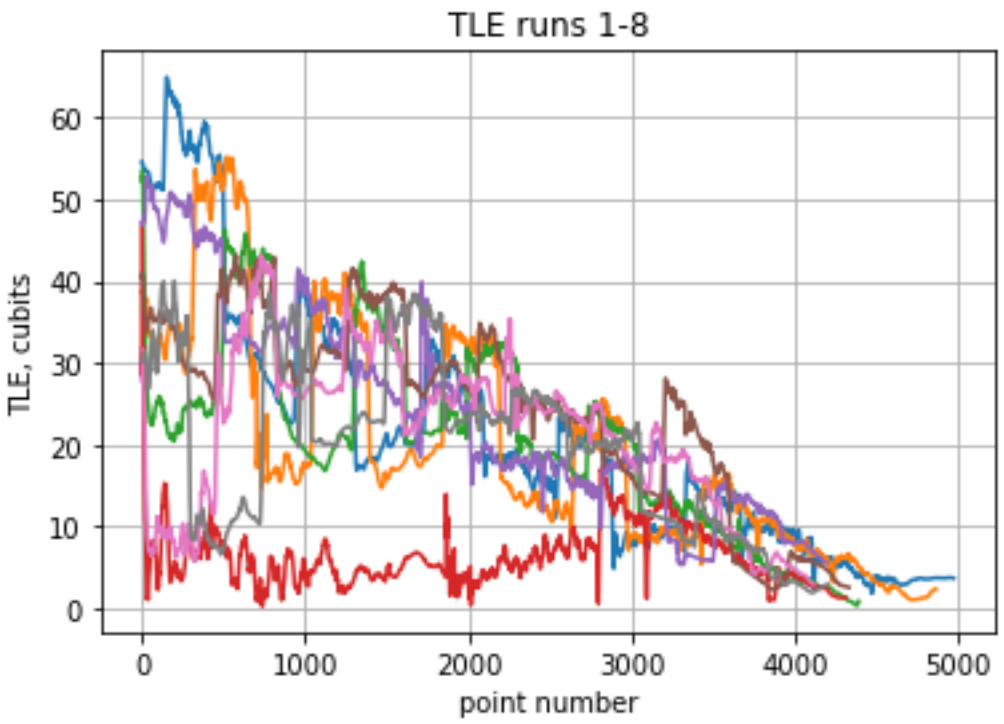
# Fit the CIR model to data, continued

- The MLE of $\theta \ = \arg\min_{\boldsymbol{\theta}} l(\boldsymbol{\theta})$; thus, the unnormalized negative log posterior distribution is $l(\boldsymbol{\theta})$- log(p($\boldsymbol{\theta}$)) where p is the prior distribution for $\boldsymbol{\theta}$
- $l(\boldsymbol{\theta})$- log(p($\boldsymbol{\theta}$)) can be used in MCMC or Maximum A-posterori to estimate $\boldsymbol{\theta}$
- Alternatively, $l(\boldsymbol{\theta})$ can be *approximated* by $\log(\hat{p}(TLE(k+1) \,\big|\, TLE(k), \boldsymbol{\theta}) \approx N(TLE(k+1)|\mu(TLE(k), dt, \widehat{\boldsymbol{\theta}}), \Sigma(TLE(k), dt, \widehat{\boldsymbol{\theta}}))$
- This approximation was implemented in the Python code for this presentation

# Analysis of TLE from the SDE model

- The data set used for this example is an unclassified collection of eight flight-test runs from a targeting pod
- Two SDE realizations are presented; first an OU fit to the data shown with only the first run, and second the CIR fit show with all eight data runs

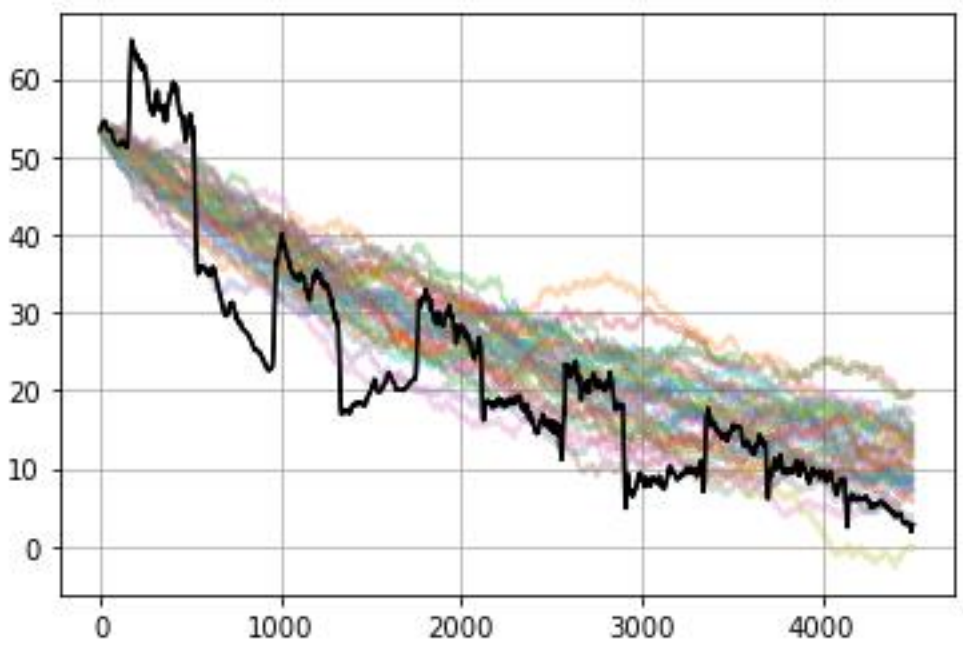
# Ensemble: 8 runs




The graph shows eight runs of TLE data from flight test. The ensemble of these runs was used in both the OU fit and the CIR model fit

Both the OU and CIR models were fit for continuous range.

# Example data run: OU model fit




Using the Ornstein-Uhlenbeck model:

$$dTLE = \theta\ (\mu - TLE)dr + \sigma d\beta$$

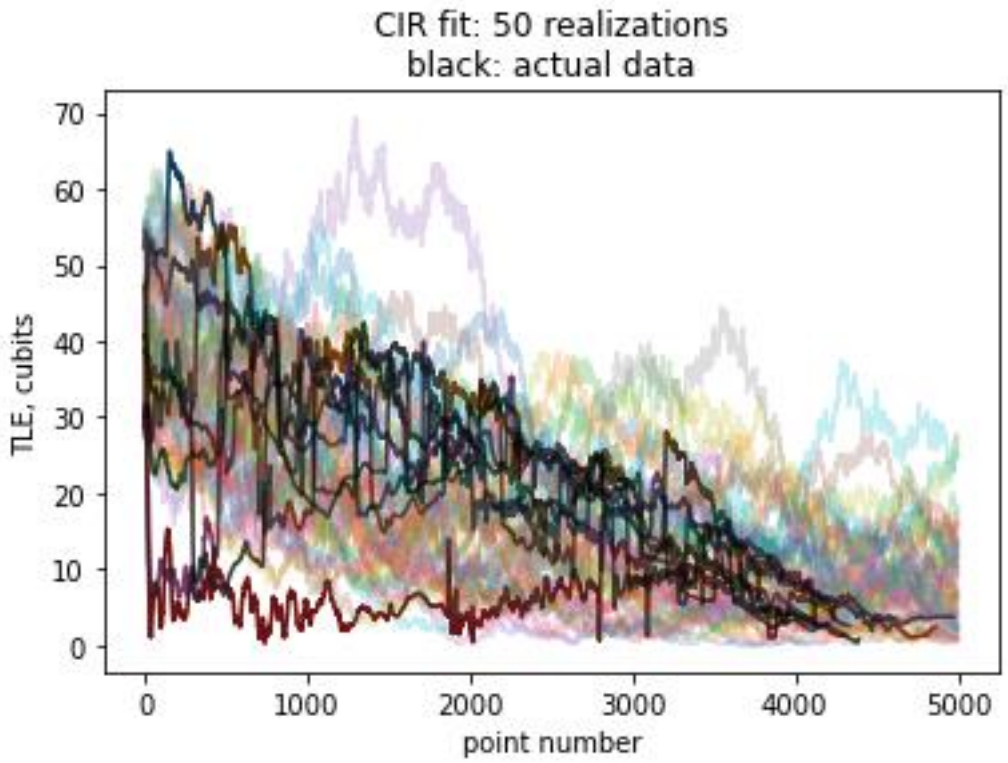
Where $\theta$ is the rate of reversion, $\mu$ the long-term mean, and $\sigma$ the dispersion of the Weiner process

Graph (black) shows one run of the TLE data and a collection of OU realizations

Notice the ‘stepping’ feature is not captured, suggesting that an additional term in the ODE may be required.

# CIR model of the TLE system

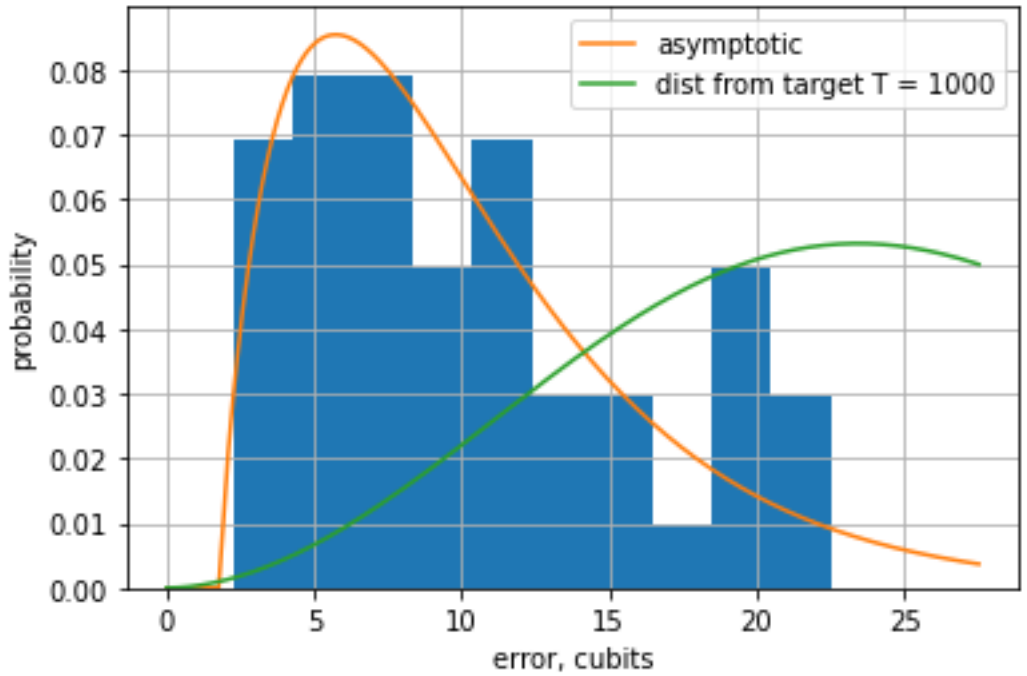
- Data with simulated runs




Black are data runs, behind them are simulated runs from the CIR model fit.

# Asymptotic, intermediate results

- Histogram: error at target; predicted error distribution 1000 data points from the target:



# Summary:

- Based on eight runs and a fitted stochastic differential equation, system performance under test conditions:
- Use terminal distribution to characterize performance- using fitted SDE model
  - Terminal (at target) TLE – 90% CI (4, 18) cubits
  - Error coming into target: at 1000 points out there is a 50% chance of TLE being greater than 25 Cubits.
  - Use model to estimate earliest ‘release point’ for weapon

# Questions?