ARL-MR-1037 • AUG 2021

DEVCOM ARMY RESEARCH LABORATORY

Instantaneous Axisymmetric Line Explosion in Incompressible Fluid-like Media

by Steven B Segletes

NOTICES

Disclaimers

ARL-MR-1037 • Aug 2021

Instantaneous Axisymmetric Line Explosion in Incompressible Fluid-like Media

by Steven B Segletes
Weapons and Materials Research Directorate, DEVCOM Army Research Laboratory

REPORT DOCUMENTATION PAGE			*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)* August 2021	**2. REPORT TYPE** Memorandum Report	**3. DATES COVERED (From - To)** July 2021
4. TITLE AND SUBTITLE Instantaneous Axisymmetric Line Explosion in Incompressible Fluid-like Media		**5a. CONTRACT NUMBER**
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Steven B Segletes		**5d. PROJECT NUMBER**
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) DEVCOM Army Research Laboratory ATTN: FCDD-RLW-TC Aberdeen Proving Ground, MD 21005-5066		**8. PERFORMING ORGANIZATION REPORT NUMBER** ARL-MR-1037
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		**10. SPONSOR/MONITOR'S ACRONYM(S)**
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES
This report is made in direct reference to ARL-TR-9247.

14. ABSTRACT
This report examines the problem of an instantaneous line explosion along the axis of an incompressible fluid shell. The derivation mirrors that of spherical point explosion, but applied to an alternate geometry. The analysis takes into account the nonzero, but finite, radial extent of the incompressible fluid domain at both the inner and outer boundaries. While numerical integration is required for the solution in time, analytical representations of the velocity and pressure fields are achieved, when expressed not as a function of time, but as a function of the instantaneous crater size.

15. SUBJECT TERMS
line explosion, point explosion, axisymmetry, energy conservation

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON** Steven B Segletes
a. REPORT Unclassified	**b. ABSTRACT** Unclassified	**c. THIS PAGE** Unclassified	UU	19	**19b. TELEPHONE NUMBER (Include area code)** 410-278-6010

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

Contents

List of Figures

Acknowledgments

The author is grateful to his friend and colleague Dr Michael Grinfeld, who not only provided the technical review of this report, but also introduced the author in 2016 to the family of mathematical problems involving instantaneous explosion. To Ms Carol Johnson, the author expresses profound thanks for her incisive, prompt editing of this report.

1. Introduction

In a recent report, ARL-TR-9247,[1] the classical concept of point explosion in an incompressible fluid was revisited, but generalized to include a nonzero-sized initial crater ($r = R_*^\circ$) and finite-sized outer dimension to the incompressible fluid shell ($r = R_{\mathrm{out}}^\circ$). In this report, we mirror (literally, by equation number) the derivations of ARL-TR-9247. However, instead of considering the explosion to originate from a point, we now consider the instantaneous explosion along the axis of symmetry of an infinitely long, hollow cylinder of incompressible fluid and the subsequent radial growth of the axisymmetric crater. We call this situation a "line explosion".

As in the case of the point explosion, the boundary conditions associated with the crater wall are as follows: the pressure on the inner crater wall is 0, for time $t > 0$, as the line explosion converts its specified internal energy (per unit length) into a purely kinetic response at $t = 0$, and the axisymmetric crater expands under its own inertia, subject to the outer boundary condition. The outer boundary will expand in accordance with continuity requirements.

Beyond the outer boundary of the fluid is an idealized reservoir at fixed pressure, p_∞. The resulting kinematic motion is one in which the cylindrically radial inertial flow of the incompressible fluid shell monotonically diminishes under the work applied by the reservoir, until such time that the expansion halts.

2. Formulation of the Model

Consider an instantaneous explosion along the axis of symmetry of a hollow cyclindrical shell, as shown in Fig. 1. We assume that the fluid shell has an initial outer radius R_{out}°. A cylindrical empty hole occupies the center of the cylinder, with radius R_*°. The cylinder is immersed in an reservoir of fixed pressure p_∞. The reservoir is assumed to have no inertia, nor is it capable of storing internal energy.

As in the case of the spherically symmetric point explosion, the axisymmetric instantaneous line explosion converts its chemical potential energy into a kinetic response at time $t = 0$ and the inner wall of the cavity is subjected to zero pressure load for $t > 0$ (since the line explosion has undergone an infinite *relative* expansion). The inner crater wall, of radius $R_* = R_*(t)$, expands under its inertia and, because the fluid cylinder is incompressible, the outer boundary of the cylinder likewise undergoes a time-dependent radial expansion, such that $R_{\mathrm{out}} = R_{\mathrm{out}}(t)$.

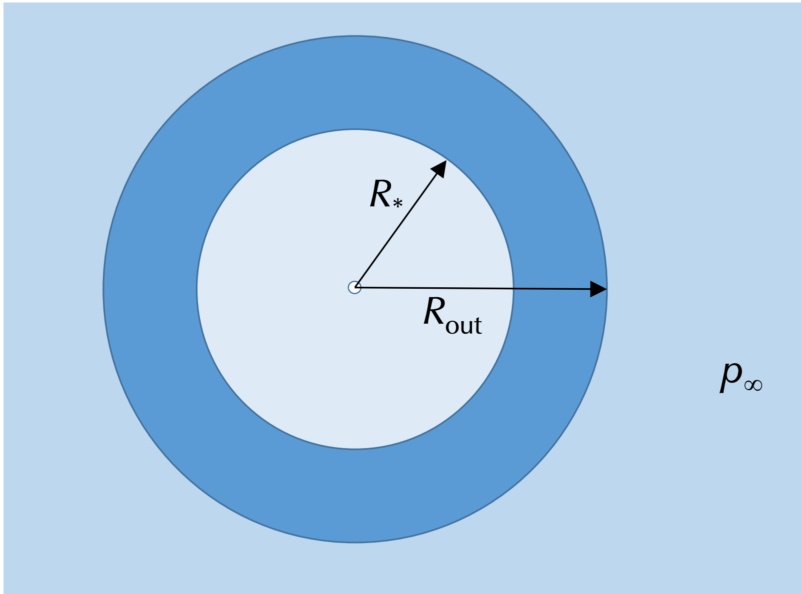

Fig. 1 Cross section of axisymmetric line explosion

Per our definition at $t = 0$,

$$R_*(0) = R_*^\circ \tag{1}$$

$$R_{\text{out}}(0) = R_{\text{out}}^\circ \quad . \tag{2}$$

Since there is no energy dissipation considered in the fluid or the reservoir, the specified total energy per unit length introduced by the explosion, dE_c/dz, is transferred to the fluid in the form of kinetic energy dK/dz, likewise expressed per unit axial length (our coordinate system is oriented such that the axis of symmetry is aligned with the z coordinate axis). Over time, the reservoir pressure performs work on the expanding fluid cylinder, eventually arresting its radial expansion.

For the fixed total energy per unit length, we define the balance in which work performed by the reservoir* diminishes the kinetic energy of the system:

$$\frac{dE_c}{dz} = \pi(R_{\text{out}}^2 - R_{\text{out}}^{\circ 2})p_\infty + \frac{dK}{dz} \quad . \tag{3}$$

The fluid's incompressibility will dictate the following axisymmetric mass-conser-

*In the axisymmetric case, the differential increment of work (per unit length of axis) performed by the reservoir is $p_\infty \cdot 2\pi r\, dr$, integrated over the motion of the outer boundary of the fluid (*i.e.*, from $r = R_{\text{out}}^\circ$ to $r = R_{\text{out}}$). This integrates to $\pi(R_{\text{out}}^2 - R_{\text{out}}^{\circ 2})p_\infty$.

vation requirement:

$$R_{\mathrm{out}}^2(t) - R_*^2(t) = R_{\mathrm{out}}^{\circ 2} - R_*^{\circ 2} \quad . \tag{4}$$

Combine Eqs. 3 and 4 to obtain

$$\frac{dE_c}{dz} = \pi(R_*^2 - R_*^{\circ 2})p_\infty + \frac{dK}{dz} \quad . \tag{5}$$

Consider the axisymmetric radial flow of incompressible fluid with the radial velocity $v(r,t)$ and pressure distribution $p(r,t)$. On the basis of momentum and mass conservation in cylindrical coordinates, these two functions obey the following system of partial differential equations:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{6}$$

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0 \quad . \tag{7}$$

This pair of equations permits the following solution:

$$v = \frac{A(t)}{r} \tag{8}$$

$$\frac{p}{\rho} = -\frac{dA}{dt}\ln r - \frac{1}{2}\frac{A^2}{r^2} + B(t) \quad , \tag{9}$$

where $A(t)$ and $B(t)$ are functions of time that are to be determined from initial and boundary conditions. Equation 8 follows directly from solving the continuity Eq. 7. To get Eq. 9, the velocity, Eq. 8, is substituted into the momentum equation, Eq. 6, and integrated once with respect to r. The pressure is subject to the inner boundary condition

$$p(R_*) = 0 \quad . \tag{10}$$

Evaluating Eq. 9 at $r = R_*$, subject to the boundary condition, Eq. 10, allows us to determine that

$$B(t) = \frac{dA}{dt}\ln R_* + \frac{1}{2}\frac{A^2}{R_*^2} \tag{11}$$

Combining Eqs. 9 and 11, we arrive at

$$\frac{p}{\rho} = -\frac{dA}{dt}\ln\left(\frac{r}{R_*}\right) + \frac{A^2}{2}\frac{r^2 - R_*^2}{r^2 R_*^2} \tag{12}$$

The total kinetic energy (per unit length) of the hollow cylinder of incompressible fluid, (R_*, R_{out}), is equal to

$$\frac{dK}{dz} = \frac{\rho}{2}\int_{R_*(t)}^{R_{\text{out}}(t)} 2\pi r v^2(r,t)\,dr \quad . \tag{13}$$

Inserting the velocity, Eq. 8, into Eq. 13 gives

$$\frac{dK}{dz} = \frac{\rho A^2}{2}\int_{R_*(t)}^{R_{\text{out}}(t)} 2\pi r^{-1}\,dr = \pi\rho\ln\left(\frac{R_{\text{out}}}{R_*}\right)A^2 \quad . \tag{14}$$

The pressure at $r = R_*$ is *explicitly* equal to 0 (via Eq. 10). Thus, Eq. 9 implies

$$-\frac{dA}{dt}\ln R_* - \frac{1}{2}\frac{A^2}{R_*^2} + B(t) = 0 \quad . \tag{15}$$

Similarly, the pressure at $r = R_{\text{out}}$ is *implicitly* equal to p_∞ (via Eq. 3).[2] Therefore, Eq. 9 also implies

$$-\frac{dA}{dt}\ln R_{\text{out}} - \frac{1}{2}\frac{A^2}{R_{\text{out}}^2} + B(t) = p_\infty \quad . \tag{16}$$

The energy relationship, Eq. 5, now reads, by way of Eq. 14,

$$\frac{dE_c}{dz} = \pi\rho\ln\left(\frac{R_{\text{out}}}{R_*}\right)A^2 + \pi p_\infty(R_*^2 - R_*^{\circ 2}) \quad . \tag{17}$$

As in ARL-TR-9247, the function $W(R_*)$ may be introduced. For the axisymmetric case, it is given as

$$W(R_*) = (R_*^2 + R_{\text{out}}^{\circ 2} - R_*^{\circ 2})^{1/2} = R_{\text{out}}(R_*) \quad . \tag{18}$$

Differentiating Eq. 18 gives

$$W_{R_*}(R_*) \equiv \frac{dW}{dR_*} = \frac{1}{2}(R_*^2 + R_{\text{out}}^{\circ 2} - R_*^{\circ 2})^{-1/2}\cdot 2R_* = \frac{R_*}{W(R_*)} \quad . \tag{19}$$

The explosive–fluid boundary moves with a velocity equal to that of the fluid particles located at $r = R_*$. Therefore, for the velocity C_* of the boundary, we obtain the following relation:

$$C_*(t) = v(R_*, t) \quad . \tag{20}$$

Because of the symmetry of our problem, we may state

$$C_* = \frac{dR_*(t)}{dt} \quad . \tag{21}$$

Combining Eqs. 20 and 21, we equate the velocity of the inner boundary to that of the fluid particles located at the same position,

$$\frac{dR_*(t)}{dt} = v(R_*, t) \quad . \tag{22}$$

Inserting the velocity, Eq. 8, into Eq. 22 gives us

$$\frac{dR_*}{dt} = \frac{A(t)}{R_*} \quad , \tag{23}$$

which can be rearranged as

$$A(t) = R_* \frac{dR_*(t)}{dt} \quad . \tag{24}$$

We may resolve Eq. 17 to isolate A, obtaining

$$A = Q(R_*) \quad \text{, where}$$
$$Q(R_*) = \sqrt{\frac{dE_c/dz - p_\infty(R_*^2 - R_*^{\circ 2})\pi}{\pi\rho} \cdot \frac{1}{\ln(W/R_*)}} \quad . \tag{25}$$

Substituting Eq. 25 into Eq. 23, we obtain the governing, nonlinear ordinary differential equation:

$$\frac{dR_*}{dt} = \frac{1}{R_*} Q(R_*) \quad , \tag{26}$$

which may be expressed in full as

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{dE_c/dz - p_\infty(R_*^2 - R_*^{\circ 2})\pi}{\pi\rho} \cdot \frac{R_*^{-2}}{\ln(R_{\text{out}}/R_*)}} \quad . \tag{27}$$

Using $W(R_*)$, defined by Eq. 18, we may rewrite Eq. 27 in the form, substituting for $R_{\text{out}}(t)$ its equivalent value, $W(R_*)$:

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{dE_c/dz - p_\infty(R_*^2 - R_*^{\circ 2})\pi}{\pi\rho} \cdot \frac{R_*^{-2}}{\ln(W(R_*)/R_*)}} \quad . \tag{28}$$

The term $W(R_*)$ may be eliminated by substituting Eq. 18, to obtain the full explicit form of dR_*/dt, in terms of $R_*(t)$:

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{dE_c/dz - p_\infty(R_*^2 - R_*^{\circ 2})\pi}{\pi\rho} \cdot \frac{R_*^{-2}}{\ln\left(\sqrt{R_*^2 + R_{\text{out}}^{\circ 2} - R_*^{\circ 2}}/R_*\right)}} \quad . \tag{29}$$

The ordinary differential equation, Eq. 29, should be solved using the initial condition of Eq. 1 for R_*°. Further, quantities such as dE_c/dz, p_∞, ρ, and R_{out}° are all fixed boundary or initial conditions that are fully specified in advance. To obtain a *time-dependent* solution of $R_*(t)$, numerical integration of Eq. 29 is required in the form of $\int dt = \int \frac{dR_*}{dR_*/dt}$ (the right side is expressible solely in terms of R_*). However, if it is sufficient for the application, both the velocity and pressure fields may be analytically expressed, in lieu of time t, as a function of the crater radius R_* through Eqs. 8 and 9, if the quantity $A(t)$ is alternately expressed as $A(R_*)$, as is shown below.

Only R_* is a function of time in Eq. 29. Therefore, as in the ARL-TR-9247, it proves convenient to use the shortcut defined as

$$\frac{dR_*(t)}{dt} = G(R_*) \quad , \tag{30}$$

where

$$G(R_*) = \sqrt{\frac{dE_c/dz - p_\infty(R_*^2 - R_*^{\circ 2})\pi}{\pi\rho} \cdot \frac{R_*^{-2}}{\ln\left(\sqrt{R_*^2 + R_{\text{out}}^{\circ 2} - R_*^{\circ 2}}/R_*\right)}} \quad . \tag{31}$$

One may determine the maximum expansion of the crater by setting Eq. 31 to zero and solving for R_*. One finds that

$$R_*^{\text{max}} = \sqrt{R_*^{\circ 2} + \frac{dE_c/dz}{\pi p_\infty}}$$

(note the typo in ARL-TR-9247, where the negative sign in the corresponding equation should have been a plus sign). This value represents the magnitude of crater expansion at which the work performed by the exterior pressure, p_∞, proves sufficient to overcome the energy dE_c/dz that had been imparted to the system at $t = 0$.

Using Eq. 24, 30, and 31, one may present the function $A(t)$ solely in terms of R_*, such that $A(t) \equiv A\big(R_*(t)\big)$:

$$A = R_* G(R_*) \quad . \tag{32}$$

Combining Eqs. 12 and 32, we arrive at the formula describing the pressure distribution in the incompressible fluid as an analytical function of r and R_*:

$$\frac{p\big(r, R_*(t)\big)}{\rho} = -\ln\Big(\frac{r}{R_*}\Big)\frac{d\big(R_*\, G(R_*)\big)}{dR_*} G(R_*) + \frac{1}{2}\frac{r^2 - R_*^2}{r^2} G^2(R_*) \tag{33}$$

Likewise, through the use of Eq. 32, the velocity field, Eq. 8, throughout the fluid shell is fully described as an analytical function of r and the instantaneous crater radius $R_*(t)$, as $v\big(r, R_*(t)\big) = G(R_*)\, R_*/r$.

An application of Eq. 33 is performed to verify the result. In this example, the fluid is taken to occupy a range of r, from $R_*^\circ = 1$ to $R_{\text{out}}^\circ = 100$. The exterior (reservoir) pressure, p_∞, is set to a value of unity, while the initial energy per unit axial length, dE_c/dz, that is introduced by the explosive into the system at $t = 0$ is set to a value of 20. The result is shown in Fig. 2, which can be compared to the corresponding result presented in ARL-TR-9247 for a spherical point explosion.

In Fig. 2, each of the five curves represents the pressure profile in the cylindrical fluid shell at a different level of crater expansion (*i.e.*, different values of R_*), ranging from the initial to the final state of expansion. The legend notes the value of dR_*/dt associated with each level of expansion. As promised, the pressure field has been analytically characterized as a function of r and the crater radius R_*.

We see for all cases that the external boundary pressure matches that of the reservoir, at a value of $p_\infty/\rho = 1$. Further, the pressure distribution at the maximum expansion, $R_* = 2.71$, is linear on the semi-log scale shown, implying (at the point of maximum expansion) that $p \propto \ln(r/R_*^{\max})$. This limiting relation occurs because, at maximum expansion $R_* = R_*^{\max}$, we have $dR_*/dt = 0$; thus, $G = 0$ from Eq. 30. The pressure equation, Eq. 33, evaluated when $G = 0$ and $R_* = R_*^{\max}$, leaves only the term that is proportional to $\ln(r/R_*^{\max})$.

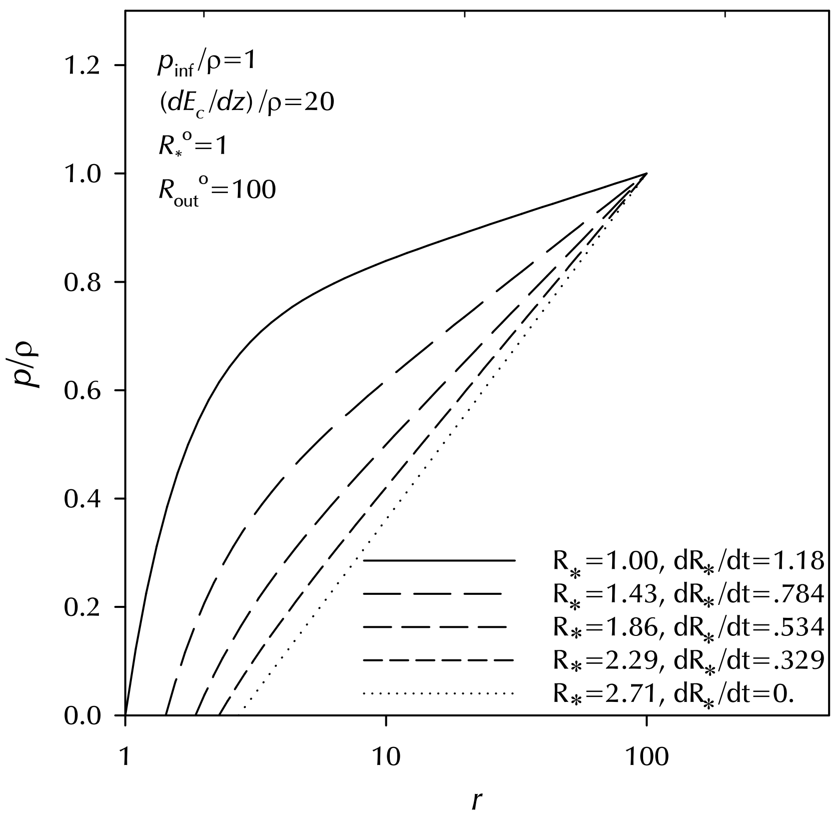

Fig. 2 Axisymmetric pressure as a function of radius

3. Solution for Unbounded Fluid Media is Disallowed

Like its point-explosion counterpart in ARL-TR-9247, Eq. 12 possesses a singularity at $r = 0$, which would come into play if the initial condition of the cylindrical shell were considered as $R_*(0) = 0$. However, unlike the point-explosion solution, this axisymmetric line-explosion solution *also* has a singularity at $r = \infty$ (addressing the case of unbounded fluid media).

The singularity arises because of the presence of the logarithmic term (and a function $A(t)$ that is not constant). In addition, the last term of Eq. 12 limits at large r to a value of $A^2/2R_*^2$, which will be neither zero nor constant in time (a necessary outer-pressure boundary condition of the reservoir). Thus, this present solution disallows consideration of the limiting case of a cylindrical fluid shell that is infinite in radial extent. We note also, in light of our observation at the moment of maximum expansion, that the relation $p \propto \ln(r/R_*^{\max})$ is, likewise, incompatible with an outer boundary $R_{\text{out}} \to \infty$, unless the proportionality constant were to approach zero in a properly prescribed manner (which is not addressed here).

4. Conclusion

In this report, the axisymmetric problem of an instantaneous line explosion along the axis of a cylindrical shell of incompressible fluid is examined. This report is specifically intended as the axisymmetric analog to the spherical point-explosion study conducted in ARL-TR-9247, even to the extent of synchronized equation numbering.

The explosion is caused by the instantaneous release of energy, distributed along the symmetry axis at a rate of dE_c/dz. As with the point-explosion solution offered in ARL-TR-9247, this line-explosion solution reduces to a system of algebraic and ordinary differential equations.

If the cavity expansion is required as a function of time, then the numerical integration of Eq. 30 is required. However, it is shown that the pressure distribution in the fluid can be analytically expressed in terms of the cavity radius R_* by way of Eq. 33. Likewise, through the use of $A\big(R_*(t)\big)$ given in Eq. 32, the velocity field may also be analytically expressed as a function of the instantaneous cavity radius $R_*(t)$.

Unlike the case of the spherical point explosion, which approaches the self-similar solution of Sedov as the fluid radius becomes unbounded, the axisymmetric line explosion (at least the solution offered here) does not allow study of the limiting case of a fluid shell of infinite radial extent because of the presence of a mathematical singularity.

5. References

1. Grinfeld MA, Segletes SB. Instantaneous point explosion in incompressible fluid-like media. Army Research Laboratory (US); 2021 Aug. Report No.: ARL-TR-9247.

2. Segletes SB. Verifying the outer boundary condition of the point explosion. Army Research Laboratory (US); 2021 Aug. Report No.: ARL-TN-1074.

1 (PDF) DEFENSE TECHNICAL
INFORMATION CTR
DTIC OCA

1 (PDF) DEVCOM ARL
FCDD RLD DCI
TECH LIB

1 (PDF) LOS ALAMOS NATIONAL LAB
L G HILL

3 (PDF) DE TECHNOLOGIES
R CICCARELLI
W FLIS
W CLARK

4 (PDF) SANDIA NATL LAB
J NIEDERHAUS
A ROBINSON
C SIEFERT
A RODRIGUEZ

2 (PDF) DREXEL UNIVERSITY
B FAROUK
P GRINFELD

72 (PDF) DEVCOM ARL
FCDD RLD
P BAKER
FCDD RLD D
T ROSENBERGER
FCDD RLD FC
T SANO
FCDD RLD FR
M TSCHOPP
FCDD RLW
S SCHOENFELD
FCDD RLW B
R BECKER
C HOPPEL
A TONGE
FCDD RLW MB
B LOVE
D HOPKINS
G GAZONAS
B POWERS
FCDD RLW MG
J ANDZELM
FCDD RLW T
R FRANCART
J HOGGE
R YEAGER
FCDD RLW TA
SR BILYK
C ADAMS
P BERNING
M COPPINGER
M J GRAHAM
M GREENFIELD
W C UHLIG
FCDD RLW TB
S SATAPATHY
T WEERASOORIYA
FCDD RLW TC
J CAZAMIAS
R BECKER
D CASEM
J CLAYTON
M FERMEN-COKER
B LEAVY
J LLOYD
S SEGLETES
L SHANNAHAN
A TONGE
C WILLIAMS
FCDD RLW TD
A BARD
R DONEY
M KEELE
D KLEPONIS
F MURPHY
D PETTY
C RANDOW
S SCHRAML
K STOFFEL
G VUNNI
V WAGONER
M ZELLNER
FCDD RLW TE
P SWOBODA
P BARTKOWSKI
D GALLARDY
D HORNBAKER
E HORWATH
J HOUSKAMP
E KLIER
C KRAUTHAUSER
M LOVE
FCDD RLW TF
T EHLERS
E KENNEDY
L MAGNESS
C MEYER
B SORENSEN
R SUMMERS

FCDD RLW TG
N GNIAZDOWSKI
C CUMMINS
E FIORAVANTE
D FOX
R GUPTA
S HUG
S KUKUCK
C PECORA
J STEWART

2 (PDF) DEVCOM DAC
FCDD DAA A
D HOWLE
FCDD DAW S
J AUTEN