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Convergence Behavior of Series Solutions of the Lambert Problem

J. D. Thorne, Project Leader

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CONVERGENCE BEHAVIOR OF SERIES SOLUTIONS OF THE LAMBERT PROBLEM[1]

James D. Thorne[2]

ABSTRACT

Lambert's problem, to find the unique conic trajectory that connects two points in a spherical gravity field in a given time, is represented by a set of transcendental equations due to Lagrange. The associated Lagrange equations for the orbital transfer time may be expressed as series expansions for all cases. Power series solutions have been published that reverse the functionality of the Lagrange equations to provide direct expressions for the unknown semi-major axis as an explicit function of time. The convergence behavior of the series solutions is examined over the range of possible transfer angles and flight times. The effect of arbitrary precision calculations is shown on the generation of the series coefficients.

INTRODUCTION

Lambert's problem, to find the arc of a unique conic trajectory that connects two points in a spherical gravity field in a given time, is represented by a set of transcendental equations originally derived by Lagrange. Many iterative algorithms have been developed to solve the Lagrange equations since they give the time of flight on a trajectory as a function of the semi-major axis of a conic section, while it is the semi-major axis that is normally desired as a function of the time of flight. Power series have been published that reverse the functionality of the Lagrange equations to provide direct expressions for the unknown semi-major axis as an explicit function of the given time of flight. This paper examines the convergence behavior of the series solutions to the Lambert problem over the range of possible transfer angles and flight times that approach the full period of the orbit. Also, the effect of arbitrary precision calculations is shown on the generation of the series coefficients.

BACKGROUND - SERIES SOLUTIONS OF THE LAMBERT PROBLEM

For convenience, the series solutions of the Lambert problem[2] are repeated here for the discussion of convergence properties. Lambert's theorem states that the orbital transfer time (t) between two known positions in the 2-body orbital problem is dependent only on the semi-major axis (a) given two fixed position vectors and a known gravitational constant. Lagrange proved this theorem and derived elegant equations that show this functional dependence.[1] Apart from the limiting cases of straight-line and parabolic transfers, the possible orbital paths fall into three categories: hyperbolic arcs (1H, 2H); elliptical arcs with transfer times that are either less than (1A, 2A) or greater (1B, 2B) than a minimum-energy transfer time. The basic problem geometry is shown in Figure 1 below.

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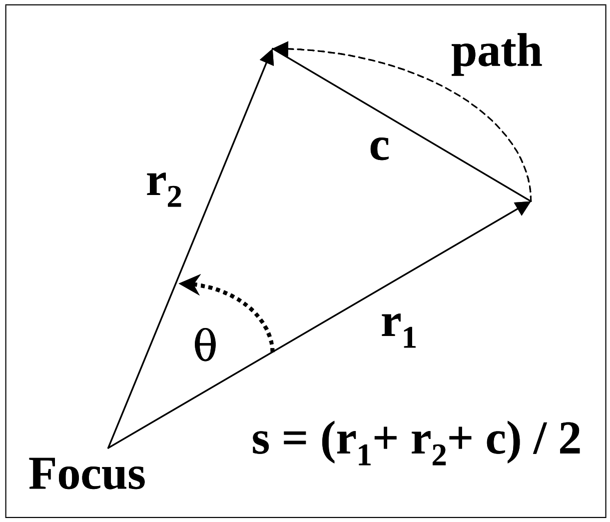

Figure 1: Problem Geometry

If the transfer angle is less or more than π radians, there is a corresponding sign change ($\mp$) in the Lagrange equation, as shown below:

$$\text{1H, 2H:} \quad t = \frac{\sqrt{-a^3}}{k} \left\{ (\sinh \alpha' - \alpha') \mp \left(\sinh \beta' - \beta' \right) \right\} \tag{1}$$

$$\text{1A, 2A:} \quad t = \frac{\sqrt{a^3}}{k} \left\{ (\alpha - \sin \alpha) \mp (\beta - \sin \beta) \right\} \tag{2}$$

$$\text{1B, 2B:} \quad t = \frac{\sqrt{a^3}}{k} \left\{ 2\pi - (\alpha - \sin \alpha) \mp (\beta - \sin \beta) \right\} \tag{3}$$

Where

$$\alpha' = 2 \sinh^{-1} \sqrt{\frac{-s}{2a}}, \; \beta' = 2 \sinh^{-1} \sqrt{\frac{c-s}{2a}} \tag{4}$$

and

$$\alpha = 2 \sin^{-1} \sqrt{\frac{s}{2a}}, \; \beta = 2 \sin^{-1} \sqrt{\frac{s-c}{2a}} \tag{5}$$

In Eqs. (4) and (5), c is the chord which is a line segment connecting the two observed position vectors, and s is one-half the sum of the lengths of the position vectors and the chord, $s = (r_1 + r_2 + c) \, / \, 2$ as shown in Figure 1. The gravitational constant is k. These quantities are known from the given observations and are not dependent on the transfer time, t.

Lagrange's Eqs. (1) - (3) give a set of closed-form relationships between the observed transfer time and the semi-major axis a of the conic arc. However, the functional dependence is the opposite of what would be desired, since t is known from observations, but a is not. If values of a are plotted as a function of t as shown in Figure 2, it is evident that semi-major axis is a single-valued function of time-of-flight. The plot may be divided into three regions, representing the hyperbolic (H), elliptic short-way (A) and elliptic long-way (B) cases. The hyperbolic region is characterized by negative values of semi-major axis. The parabolic transfer time t_p forms the boundary between the hyperbolic and short-way elliptic cases. The minimum energy transfer time t_{me} forms the boundary between the second and third regions, which correspond to the short-way and long-way elliptic cases. The third region is the primary focus of this paper, where there are two separate

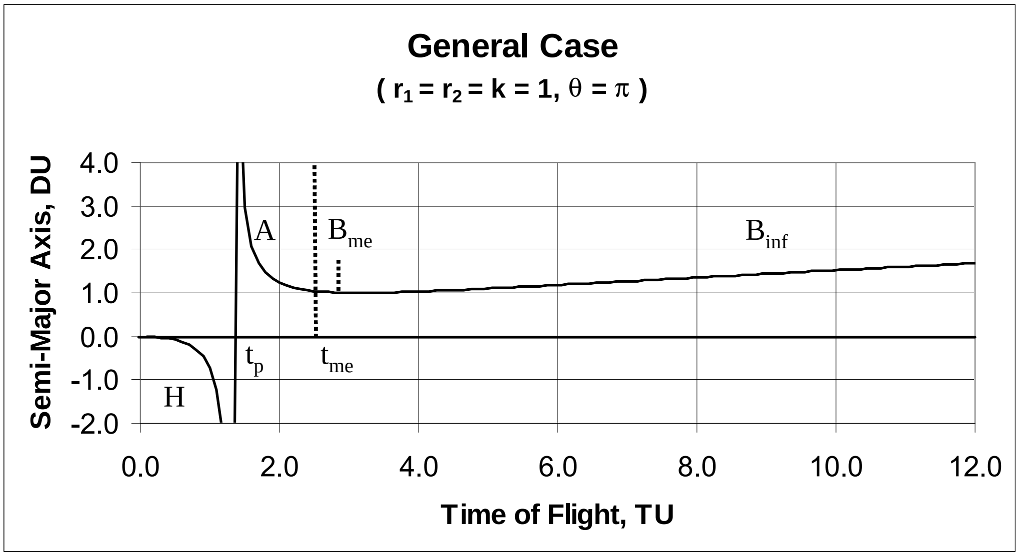

Figure 2: General semi-major axis plot, canonical units

boundaries (B_{me}, B_{inf}) to examine. The first boundary is the minimum-energy transfer time, and the second boundary is region where the transfer time approaches infinity.

Physically, as the semi-major axis of a long-way elliptic transfer becomes very large, the transfer time will approach the value of the period of the complete closed orbit since the complementary time on the short-way arc becomes very small. So, the plot of the semi-major axis in the third region of Figure 2 will approach a function that is proportional to $t^{2/3}$, as may be seen from solving the orbital period for the semi-major axis a:

$$\text{period} = \frac{2\pi}{k} a^{3/2} \simeq t, \Longrightarrow a \simeq \left(\frac{k}{2\pi}\right)^{2/3} t^{2/3} \tag{6}$$

Many excellent iterative techniques exist to solve the Lagrange equations.[3] However, since a is the unknown quantity to be found, it would be quite useful to find a direct solution to avoid the need for any type of root-finding technique.

Since the Lagrange equations are case dependent, they take different forms for hyperbolic short-way, and long-way elliptical transfers as described above. The series solutions also take three forms, but they do not correspond exactly to the cases of the Lagrange equations. In particular, the hyperbolic and short-way elliptical cases are solved by the same power series because they both result from an expansion about the parabolic case. As the flight time increases for a given, fixed geometry, there is another series solution where the flight time is near to the minimum-energy case. Finally, as the flight times grow much larger than the minimum-energy case, there is a long-way series solution which is another expansion about the limiting case where the time of flight approaches the period of the orbit.

Hyperbolic and Short-Way Elliptic Series Solution (H, A)

For reference, the three series solutions of the Lagrange equations are presented for the purpose of examining the convergence properties. It has been shown previously[2] that the series solution for both the hyperbolic and short-way elliptic classes of transfer arcs is given by:

$$a = \left(\frac{s}{2}\right) \sum_{n=0}^{\infty} \underset{[\mathrm{H,A}]}{B_n} \left(\frac{t}{t_p} - 1\right)^{(n-1)} \tag{7}$$

The B_n coefficients in Eq. (7) may be determined through the construction of a matrix equation, where the matrix $\mathbf{Q}$ transforms the original series coefficients A_n from Eq. (**??**) to produce the required simultaneous reversion and inversion. Once the $\mathbf{Q}$ matrix has been constructed, the new series coefficients are generated by a simple matrix multiplication:

$$\vec{B} = \mathbf{Q}\vec{A} \tag{8}$$

where $B_1 = A_1$, $\vec{A} = [A_2, A_3, ..A_n]$ and $\vec{B} = [B_2, B_3, ..B_n]$ in Eq. (8).

The matrix $\mathbf{Q}$ is defined by:

$$Q = \begin{bmatrix} A_1^{-1} & 0 & 0 & 0 & 0 & \dots \\ -\frac{A_2}{A_1^3} & A_1^{-2} & 0 & 0 & 0 & \dots \\ \frac{2A_2^2 - A_1A_3}{A_1^5} & -\frac{2A_2}{A_1^4} & A_1^{-3} & 0 & 0 & \dots \\ \frac{-5A_2^3 + 5A_1A_2A_3 - A_1^2A_4}{A_1^7} & \frac{5A_2^2 - 2A_1A_3}{A_1^6} & -\frac{3A_2}{A_1^5} & A_1^{-4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & A_1^{-i} \end{bmatrix} \tag{9}$$

Although the $\mathbf{Q}$ matrix was originally derived to accomplish the *simultaneous* reversion and inversion of a general series, it is interesting to notice that the first column of $\mathbf{Q}$ is composed of the expressions needed for a series reversion only. Thus, $\mathbf{Q}$ may also be used in the solution of Eq. (3) as will be shown. The recursion relationships needed to produce the elements Q_{ij} of the $\mathbf{Q}$ matrix have been derived previously,[2] but they are presented again here for convenience:

$$Q_{(1,1)} = A_1^{-1} \tag{10}$$

$$Q_{(i,j)} = \sum_{k=1}^{i-1} Q_{(i-k,j-1)} Q_{(k,1)}, \quad (i = 2, 3, 4, ...), \quad 1 < j \leq i \tag{11}$$

$$Q_{(i,1)} = \sum_{k=1}^{i-1} \left(\frac{-1}{A_1}\right) Q_{(i,k+1)} A_{(k+1)} \tag{12}$$

Eq. (10) gives the upper left element of $\mathbf{Q}$, and the following two series expressions complete the $\mathbf{Q}$ matrix to the desired order. Finally, the series coefficients to solve the hyperbolic (H) and short-way elliptic (A) cases are given by Eq. (13).

$$\underset{[H,A]}{B_n} = \sum_{m=1}^{n-1} Q_{(n-1,m)} \underset{[\mathrm{H,A}]}{A_{(m+1)}} \tag{13}$$

Although the recursive relationships needed to produce the series coefficients in Eq. (13) are quite simple algebraically, numerical problems can arise with the evaluation of them if one is using a fixed-precision computer language. Based on numerical experience, as the index value gets near to the number digits of machine precision, the calculations can lose significance because the algorithm produces small differences between very large numbers when calculating $Q_{(i,1)}$ from Eq. (12). For example, the coeffiecients given by Eq. (7) appear to diverge after an index value of about 20 terms if the machine precision is about 17 decimal digits.

Long-Way Elliptic Series Solution near Minimum Energy (B_{me})

To solve Eq. (3) in the neighborhood of the minimum-energy case, one may perform a straightforward series reversion to get the new series coefficients using only the first column of elements taken from the **Q** matrix in Eq. (10) once it has been formed. The series solution of Eq. (3) for long-way transfers (B) near the minimum-energy time is given by:

$$a = \frac{s}{2} + x^2,\ x = \sum_{n=1}^{\infty} \underset{[\text{B-me}]}{B_n} (t - t_{me})^n \tag{14}$$

In this case, the series coefficients tend to increase with the index, but if the argument is less than unity the magnitude of the full term will decrease with the index. In canonical units, this series should be used when the absolute value of the difference between the given flight time and the minimum-energy flight time is less than unity.

Long-Way Elliptic Series Solution (B_{inf})

As shown previously,[2] this is the form of the solution for the elliptic case as the transfer time approaches infinity:

$$a = \sum_{n=0}^{\infty} \underset{[\text{B- inf}]}{B_n} t^{\left(\frac{2-n}{3}\right)} \tag{15}$$

As with the previous case, the series coefficients tend to increase with the index, but if the argument is less than unity, the magnitude of the full term will decrease with the index. For this series the argument is the flight time, which tends to be large for long-way cases, so the series convergence behavior improves with increasing flight times for index values of $n \geq 3$ because the exponents are all negative.

In summary, the complete set of series solutions to the Lagrange equations is given by Eqs. (7), (14) and (15).

NUMERICAL RESULTS

Naturally, as one tries to use each of these three solutions for examples that depart from the expansion points, the error can grow based on the limits of the radius of convergence of each series. However, the series coefficients are generated using a recursive algorithm, so there is no explicit formula for each coefficient as a function of the series index. For this reason, traditional means to analyze convergence properties of the power series do not apply, so the behavior will be examined numerically. Of the three cases, the hyperbolic and short-way elliptic solution, Eq. (7) has shown the most problematic convergence behavior for high index values. The other two solutions, Eqs. (14) and (15) have shown more consistent convergence behavior especially in the limiting cases near the expansion points of minimum-energy flight time and flight times that approach infinity. For this reason, the hyperbolic and short-way elliptic solution, Eq. (7) will be examined for convergence properties in this paper.

Reversion and inversion for the hyperbolic and short-way elliptic series

In the derivation of the series solution for the hyperbolic and short-way elliptic cases, series reversion was combined with algebraic inversion to improve the convergence properties over the reversion process only.[3] Performing a reversion operation only of the series expansion of Lagrange equations for hyperbolic and short-way elliptic cases leads to poor convergence behavior. However, if the reversion process is combined with an algebraic inversion of the series, the radius of convergence is significantly improved. The difference between reversion of the HA case and reversion followed

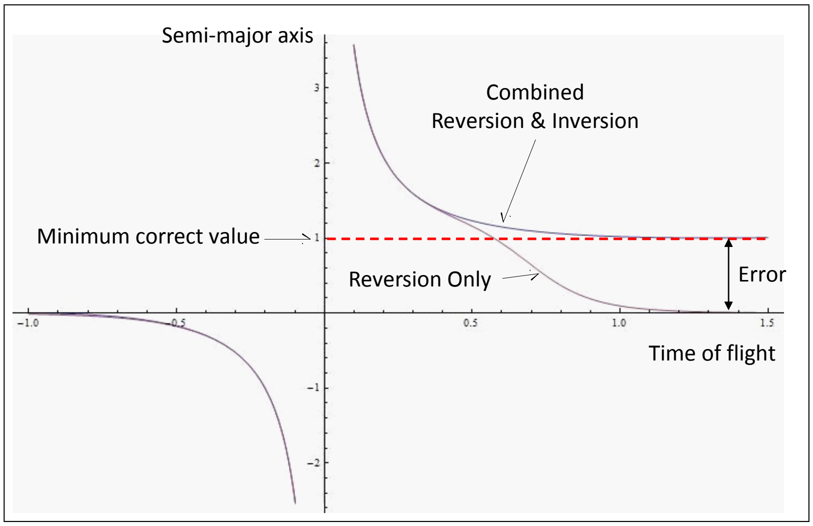

Figure 3: Series reversion combined with algebraic inversion

by inversion may be seen in Figure 3. The dashed line shows the correct value of semi-major axis for the minimum-energy case, which is only found using the combination of reversion and inversion. The numerical values shown in Figure 3 were calculated using arbitrary precision arithmetic for both approaches.

Effect of machine precision on the hyperbolic and short-way elliptic series

The series solution for the hyperbolic and short-way elliptic cases, Eq. (7), originally appeared to be asymptotically convergent and thus would be a divergent series. However, based on recent numerical experiments using arbitrary precision, the series appears to be convergent. This effect of arbitrary machine precision may be seen in Figure (4).

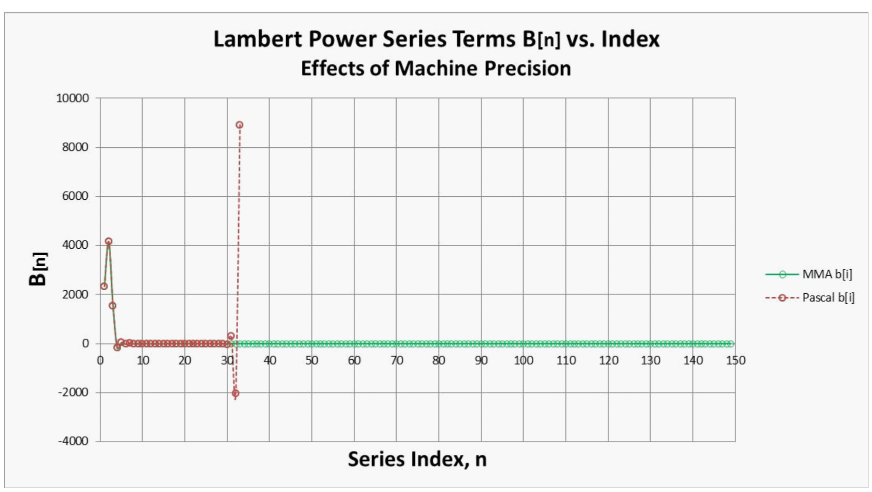

Figure 4: Effect of machine precision on convergence of short-way (HA) series

In Figure 4, the magnitude of the series coefficients is plotted against the index value. For the case in consideration where the position vectors are unity, and the transfer angle is π, the coefficients appear to achieve a minimum at approximately 20 terms using a fixed precision of about 17 decimal digits. However, by using arbitrary precision with a symbolic manipulation program, the magnitude of the coefficients continues to decrease uniformly out to 300 terms based on numerical experiments. Although this does not constitute a mathematical proof, it strongly suggests that the series is convergent. As mentioned previously, as the index value gets near to the number digits of machine precision, the calculations can lose significance because the algorithm produces small differences between very large numbers in Eq. (12). Arbitrary precision arithmetic addresses this problem, so there is no loss in precision.

Error map of series solutions of the Lambert problem

Figure 5 shows an error map for all transfer angles within one revolution on the right axis, and a range of flight times from hyperbolic to well past the minimum energy case on the left axis. The vertical dimension shows the error calculated by taking the difference between the given time of flight and the time given by using the series' value of semi-major axis in the corresponding Lagrange equation, divided by the given time of flight.

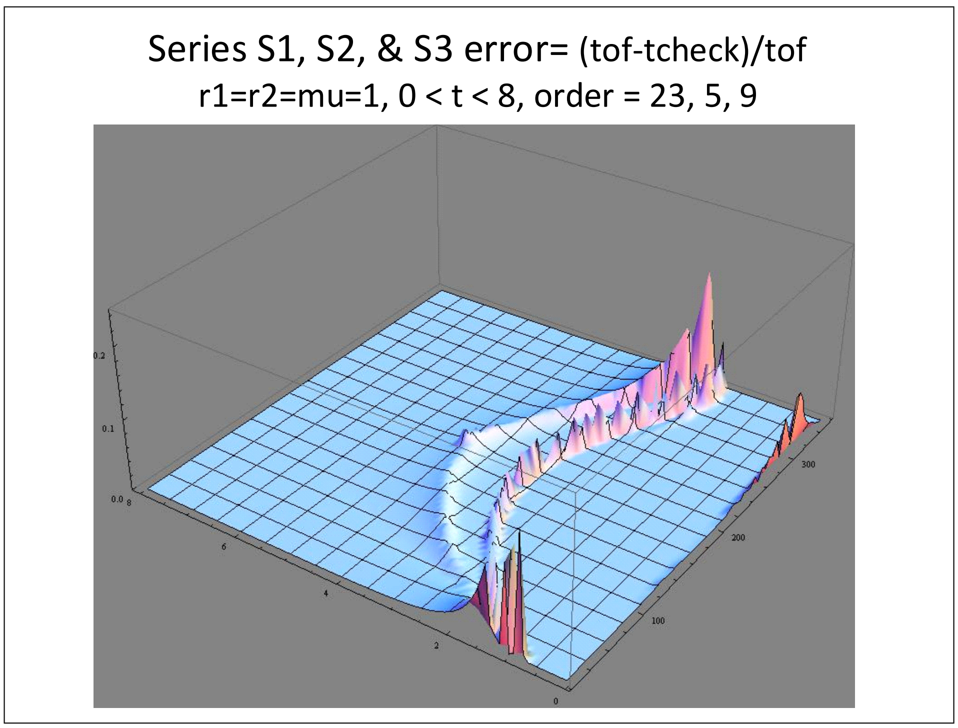

Figure 5: Series error vs. time and transfer angle

One may see that the errors are largest at the boundaries near the minimum-energy case for each transfer angle, which correspond to transitions between the various series solutions. Arbitrary precision arithmetic was used to produce all results shown in Figure 5, and was most effective for the hyperbolic and short-way elliptic cases at the lower right side of the map. Given the behavior at the series solution boundaries, it may be worthwhile to investigate more accurate modeling of the minimum-energy case in particular, perhaps by including other terms beyond the quadratic in Eq. (14).

CONCLUSIONS

The series solutions for the Lambert problem show good convergence properties near their expansion points, as would be expected. The first series solution to the Lagrange equations, Eq. (7), uses an expansion about the region where the semi-major axis approaches infinity on a parabolic transfer, so the accuracy of the series solution is best near the parabolic time, as would be expected. The second series solution to the Lagrange equations, Eq. (14), uses an expansion about the point where the semi-major axis reaches its minimum positive value, which corresponds physically to a minimum-energy transfer arc. This series solution provides a reasonably accurate solution for the range from the minimum-energy transfer time up to a transfer time that is approximately 1.5 times the minimum-energy transfer time, based on numerical investigation. The third series solution to the Lagrange equations, Eq. (15), uses an expansion about the region where the semi-major axis approaches infinity as the transfer time also approaches infinity. Physically, this means that the transfer time approaches the period of the closed orbit.

For the series solution for the hyperbolic and short-way elliptic cases, Eq. (7), a combination of series reversion and inversion results in better convergence properties than reversion alonge. However, this series will appear to be asymptotically convergent if the coefficients are calculated using finite precision arithmetic. Based on numerical investigation, this divergent behavior completely disappears out to 300 terms when using arbitrary precision calculations, which would suggest that the series solution is actually convergent.

The error map of all three series solutions in Figure 5 shows very small error values for a broad range of parameter values of transfer angle and time of flight. A few boundary regions show larger errors at the transitions between series solutions, so this would be a natural area to consider for future research. The minimum-energy series solution in particular might be improved with alternate functional forms. The significance of this analysis is that arbitrary precision can be an important tool for initial orbit determination processes.

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14. ABSTRACT

Lambert's problem, to find the unique conic trajectory that connects two points in a spherical gravity field in a given time, is represented by a set of transcendental equations due to Lagrange. The associated Lagrange equations for the orbital transfer time may be expressed as series expansions for all cases. Power series solutions have been published that reverse the functionality of the Lagrange equations to provide direct expressions for the unknown semi-major axis as an explicit function of time. The convergence behavior of the series solutions is examined over the range of possible transfer angles and flight times. The effect of arbitrary precision calculations is shown on the generation of the series coefficients.

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