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Invariant Vector Representation of a Time-Dependent Qutrit in Xi-Configuration

by Vinod Mishra

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14. ABSTRACT

The mixed qutrit state density matrix is of order 3 and depends on eight parameters. Visualization of this 8-D state space is practically impossible using 8-D vectors commonly used in the standard approach. Recently a 3-D vector representation of the qutrit state space (also called invariant vector representation, or IVR) has been proposed. In this report, we present temporal dynamics and the visualization of the mixed qutrit in cascade configuration.

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1. Introduction

Quantum sensing[1] uses quantum properties of particles and states to improve the sensitivity and accuracy of measurements beyond classical limits. Currently qubits are used as the central resource for most of this task and other ones in quantum computing and communication. Advantages of higher-dimensional objects like qutrits[2–6] are still under investigation. One of the obstacles in this direction is the difficulty in visualizing qutrit states.

The qubit density matrix is of order 2 and it depends on three parameters for the most general mixed states, but only on two parameters for pure states. In addition, it can be easily visualized using Bloch sphere representation in which the pure states are represented by points on the Bloch sphere and mixed states by the points inside. On the other hand, the mixed qutrit density matrix depends on eight parameters. Its visualization in the 8-D state space is practically impossible using commonly used representation based on the Gell–Mann matrices. Recently, another 3-D vector representation of the qutrit state space based on density matrix invariants was proposed.[7] It is based on spin-1 representation matrices[2–6] and density matrix invariants. These vectors also reside on the surface of a sphere that is not a Bloch sphere.

In this report, we solve a simple model of the time-dependence of the invariant vector representation (IVR) of a qutrit in cascade or Xi-configuration. The time-dependence of the resulting IVR vectors for both pure and mixed qutrits are given, showing the utility of this approach.

2. IVR Density Matrix of a Qutrit

The density matrix ρ based on the spin-1 representation of a qutrit is given as

$$\rho_{IVR} = \begin{bmatrix} \omega_0 & Q_2 & \overline{Q_1} \\ \overline{Q_2} & \omega_1 & -Q_0 \\ Q_1 & -\overline{Q_0} & \omega_2 \end{bmatrix} \tag{1}$$

Here,

$$Q_i = \frac{1}{2}(q_i + ia_i), \overline{Q_i} = \frac{1}{2}(q_i - ia_i); i = 0,1,2 \tag{2}$$

The parameters of the density matrix in the IVR, or ρ_{IVR}, are related to the expectation values of expressions involving spin-1 components and their combinations.

$$\omega_i = \langle S_i^2 \rangle = Tr(\rho S_i^2) \tag{3a}$$

$$a_i = \langle S_i \rangle = Tr(\rho S_i) \tag{3b}$$

$$q_k = \langle S_i S_j + S_j S_i \rangle = Tr\{\rho(S_i S_j + S_j S_i)\}, k \neq i, j \tag{3c}$$

The qutrit in IVR is represented by three vectors and they are given in Table 1.

Table 1 IVR vectors for a general qutrit

IVR vectors	**Underlying trace relation**	**Cartesian components**	**Comments**
$\vec{u}$	$Tr(\rho_\Xi^2)$ $= \sum_{i=1}^{3} u_i^2 \leq 1$	$u_1 = \sqrt{\widehat{u_1^2} + 2K_1^2}$, $u_2 = \sqrt{\widehat{u_2^2} + 2K_2^2}$, $u_3 = \sqrt{\widehat{u_3^2} + 2K_3^2}$ $4\lvert Q_i\rvert^2 = q_i^2 + a_i^2$	Here, $K_1^2 = \omega_2\omega_3 - \lvert Q_1\rvert^2$, $K_2^2 = \omega_3\omega_1 - \lvert Q_2\rvert^2$, $K_3^2 = \omega_1\omega_2 - \lvert Q_3\rvert^2$ $\widehat{u_1^2} = \omega_1^2 + 2\omega_2\omega_3$, $\widehat{u_1^2} = \omega_1^2 + 2\omega_2\omega_3$ $\widehat{u_1^2} = \omega_1^2 + 2\omega_2\omega_3$, For pure states, $K_1^2 = 0, K_2^2 = 0, K_3^2 = 0$ $\sum_{i=1}^{3} u_i^2 = \sum_{i=1}^{3} \widehat{u_i^2} = 1$
$\vec{v}$	$3Tr(\rho_\Xi^2) - 2Tr(\rho_\Xi^3)$ $= \sum_{i=1}^{3} v_i^2 \leq 1$	$v_1 = \sqrt{\widehat{u_1^2} - 2(\tilde{\Delta} + 3\omega_1 K_1^2)}$ $v_2 = \sqrt{\widehat{u_2^2} - 2(\tilde{\Delta} + 3\omega_2 K_2^2)}$ $v_3 = \sqrt{\widehat{u_3^2} - 2(\tilde{\Delta} + 3\omega_3 K_3^2)}$	Here, $\tilde{\Delta} = \Delta - 2\omega_1\omega_2\omega_3$ $\Delta = -2\,\mathrm{Re}(Q_0 Q_1 Q_2)$ $= \frac{1}{4}(a_2 a_3 q_1 + a_3 a_1 q_2 + a_1 a_2 q_3 - q_1 q_2 q_3)$ For pure states, $v_1^2 \to \widehat{u_1^2} = \omega_1^2 + 2\omega_2\omega_3$, etc. $\Delta \to 2\omega_1\omega_2\omega_3$, $\tilde{\Delta} \to 0$ $\sum_{i=1}^{3} v_i^2 = \sum_{i=1}^{3} \widehat{u_i^2} = 1$
$\vec{w}$	$Tr(\rho_\Xi) = \sum_{i=1} w_i^2$ $= \omega_1 + \omega_2 + \omega_3$ $= 1$	$w_1 = \sqrt{\omega_1}, w_2 = \sqrt{\omega_2}$, $w_3 = \sqrt{\omega_3}$	Follows from unit trace relation of density matrix. It always has unit length.

The vectors $\vec{u}$ and $\vec{v}$ represent the second and third density matrix invariants of a qutrit. The bounds on the vector-norms are, in general, $\sum_{i=1}^{3} u_i^2 \leq 1$ and $\sum_{i=1}^{3} v_i^2 \leq 1$, with equality signs holding for a pure state. For a pure state, the eight density matrix parameters are related via the following four relations:

$$\text{(i) } 4\lvert Q_0\rvert^2 = q_0^2 + a_0^2 = 4\omega_1\omega_2, \tag{4a}$$

$$\text{(ii) } 4\lvert Q_1\rvert^2 = q_1^2 + a_1^2 = 4\omega_2\omega_0, \tag{4b}$$

$$\text{(iii) } 4|Q_2|^2 = q_2^2 + a_2^2 = 4\omega_0\omega_1, \text{ and} \tag{4c}$$

$$\text{(iv) } a_1a_2q_0 + a_2a_0q_1 + a_0a_1q_2 - q_0q_1q_2 = 8\omega_0\omega_1\omega_2. \tag{4d}$$

These relations lead to only four independent parameters, which is the correct number for a pure qutrit state. For a mixed or general qutrit state, the eight parameters become two general and one unit vector.

3. Hamiltonian for the Qutrit Ξ-Model with Equidistant States

The undressed energy levels for the qutrit Ξ-model are $-\varepsilon, 0$, and ε. The Hamiltonian for the system interacting with the external field ϕ with interaction strength g is given as

$$H_\Xi = \begin{bmatrix} -\varepsilon & \phi\bar{g} & 0 \\ \phi g & 0 & \phi\bar{g} \\ 0 & \phi g & \varepsilon \end{bmatrix} \tag{5}$$

Here, $g = Ge^{i\delta}$. Define $\phi = \pi p$ (ϕ and p are dimensionless) and $G = \varepsilon b$ (G and ε have dimensions of energy, b is dimensionless). Then we get

$$H_\Xi = \varepsilon \begin{bmatrix} -1 & \pi bpe^{-i\delta} & 0 \\ \pi bpe^{i\delta} & 0 & \pi bpe^{-i\delta} \\ 0 & \pi bpe^{i\delta} & 1 \end{bmatrix} \tag{6}$$

Define $f = \sqrt{1 + 2(\pi bp)^2}$, and $\omega = \varepsilon f$ then the roots of the characteristic equation are $-\omega$, 0, and ω as the dressed energy levels. We use trigonometric functions $= sin\theta = \frac{\pi bp\sqrt{2}}{f}$ and $c = cos\theta = 1/f$, and then one can rewrite H_Ξ as

$$H_\Xi = \omega \begin{bmatrix} -cos\theta & \left(\frac{sin\theta}{\sqrt{2}}\right)e^{-i\delta} & 0 \\ \left(\frac{sin\theta}{\sqrt{2}}\right)e^{i\delta} & 0 & \left(\frac{sin\theta}{\sqrt{2}}\right)e^{-i\delta} \\ 0 & \left(\frac{sin\theta}{\sqrt{2}}\right)e^{i\delta} & cos\theta \end{bmatrix} =$$

$$\omega \begin{bmatrix} -c & \left(\frac{s}{\sqrt{2}}\right)e^{-i\delta} & 0 \\ \left(\frac{s}{\sqrt{2}}\right)e^{i\delta} & 0 & \left(\frac{s}{\sqrt{2}}\right)e^{-i\delta} \\ 0 & \left(\frac{s}{\sqrt{2}}\right)e^{i\delta} & c \end{bmatrix} = \omega M \tag{7}$$

We use the von Neumann equation for density-matrix given as

$$\frac{d\rho}{dt} = -i[H_\Xi, \rho] \tag{8}$$

Its formal solution for time-dependent density matrix is

$$\rho(t) = e^{-iH_\Xi t}\rho(0)e^{iH_\Xi t} \tag{9}$$

The exponential leads to

$$e^{-iH_\Xi t} = 1 - itH_\Xi + \frac{1}{2}(-itH_\Xi)^2 + \cdots . = 1 - (1 - cos\omega t)M^2 - i(sin\omega t)M \tag{10}$$

4. Relation between the Density Matrix and IVR: Pure Qutrit

Initially let the qutrit be in the ground state so that

$$\begin{pmatrix} \psi_0(t=0) \\ \psi_1(t=0) \\ \psi_2(t=0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

This is a pure state, and it gives the initial density matrix as

$$\rho(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

Let $x = sin\omega t$ and $y = cos\omega t$. We further define some more functions as

$$\beta = \frac{s^2}{2}(1-y), \tag{13a}$$

$$\alpha = 1 - \beta, \tag{13b}$$

$$B = \frac{1}{\sqrt{2}}s\{c(1-y) + ix\}e^{i\delta}, \tag{13c}$$

and

$$K = (\beta + y - ixc)e^{2i\delta} \tag{13d}$$

Then the solution for the density matrix of the qutrit Ξ- model is given as

$$\rho(t) = \begin{bmatrix} \alpha^2 & -\alpha\bar{B} & -\beta\bar{K} \\ -\alpha B & 2\alpha\beta & -\beta\bar{B} \\ -\beta K & -\beta B & \beta^2 \end{bmatrix} \tag{14}$$

This density matrix satisfies $\rho^2 = \rho$, so this is a pure state. This is to be expected because the initial state is pure and the time-development is unitary, so this property is preserved. Further, we have

$$\rho_{IVR} = \begin{bmatrix} \omega_0 & Q_2 & \overline{Q_1} \\ \overline{Q_2} & \omega_1 & -Q_0 \\ Q_1 & -\overline{Q_0} & \omega_2 \end{bmatrix} \tag{15}$$

We equate the two density matrices representing the same qutrit entity (i.e., $\rho_{IVR} = \rho(t)$. Then their density matrix elements are related by

$$\omega_0 = \alpha^2,\ \omega_1 = 2\alpha\beta,\ \omega_2 = \beta^2 \tag{16a}$$

$$Q_0 = \frac{1}{2}(q_0 + ia_0) = \beta\bar{B};\ Q_1 = \frac{1}{2}(q_1 + ia_1) = -\beta K,\ Q_2 = \frac{1}{2}(q_2 + ia_2) = -\alpha\bar{B} \tag{16b}$$

For pure states, we get $|Q_0|^2 = \omega_1\omega_2$, $|Q_1|^2 = \omega_2\omega_0$, $|Q_2|^2 = \omega_0\omega_1$, and similar other results as given earlier. The pure-state invariant vectors and their properties are given in the Table 2.

Table 2 IVR vectors for a pure qutrit

IVR vectors	Cartesian components	Angles	Comments
$\vec{w}$	$w_1 = \sqrt{\omega_1} = \alpha$,, $w_2 = \sqrt{\omega_2} = \sqrt{2\alpha\beta}$, $w_3 = \sqrt{\omega_3} = \beta$	Polar angle of $\vec{w}$: $\psi_1(t) = cos^{-1}(\beta)$; azimuthal angle of $\vec{w}$: $\chi_1(t) = tan^{-1}(\sqrt{2\beta/\alpha})$	Follows from unit trace relation of density matrix. It always has unit length.
$\vec{u}$	$u_1 = \sqrt{\omega_0^2 + 2\omega_1\omega_2} = \sqrt{\alpha(\alpha^3 + 4\beta^3)}$, $u_2 = \sqrt{\omega_1^2 + 2\omega_0\omega_2} = \alpha\beta\sqrt{6}$, $u_3 = \sqrt{\omega_2^2 + 2\omega_0\omega_1} = \sqrt{\beta(4\alpha^3 + \beta^3)}$	Polar angle of $\vec{u}$: $\psi_2(t) =$ $cos^{-1}\left(\sqrt{\beta(4\alpha^3 + \beta^3)}\right)$; Azimuthal angle of $\vec{u}$: $\chi_2(t) = tan^{-1}\left(\sqrt{\frac{2\alpha}{3\beta} + \frac{\beta^2}{6\alpha^2}}\right)$	Length = 1
$\vec{v}$	Same as for $\vec{u}$	Same as for $\vec{u}$	Length = 1

Plots of the IVR Vector Angles of Cascade Model: Pure Qutrit

Here we study the time-dependence of two independent IVR vectors of the pure state in Figs. 1–4. The plots show the time-dependence of the co-latitude angles (ψ_1, ψ_2) and azimuth angles (χ_1, χ_2) of the IVR vectors $\vec{w}$ and $\vec{u}$, respectively.

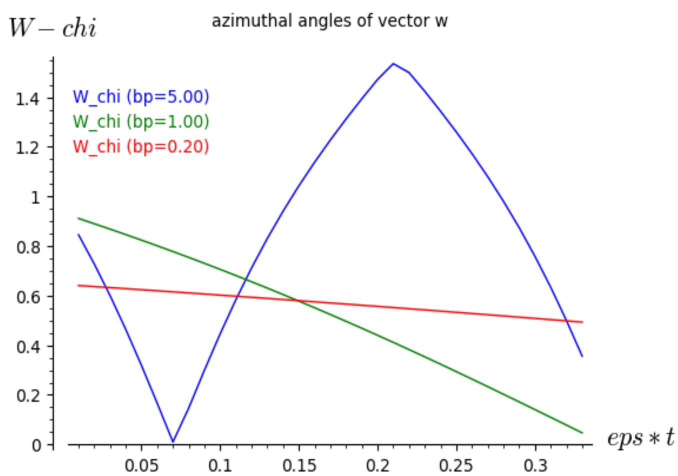

Fig. 1 **Azimuth angle of the first-order invariant vector $\vec{w}$ as a function of (εt). Three field-strength parameter *b*p* values denote weak (0.20), moderate (1.0), and strong (5.0) values.**

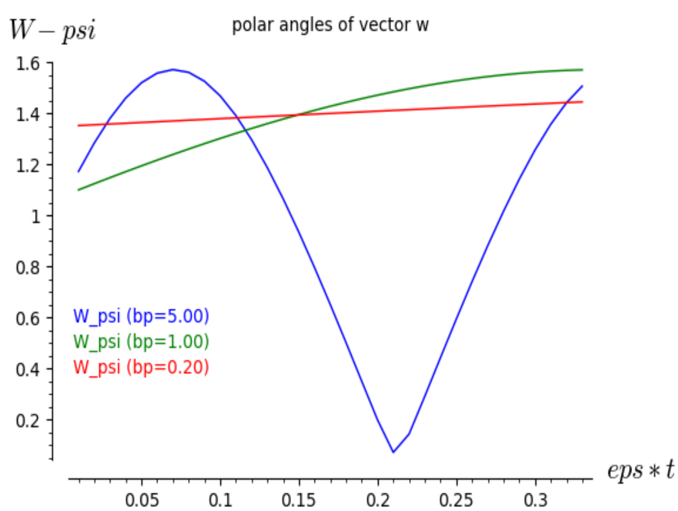

Fig. 2 **Polar angle of the first-order invariant vector $\vec{w}$ as a function of (εt). Three field-strength parameter *b*p* values denote weak (0.20), moderate (1.0), and strong (5.0) values.**

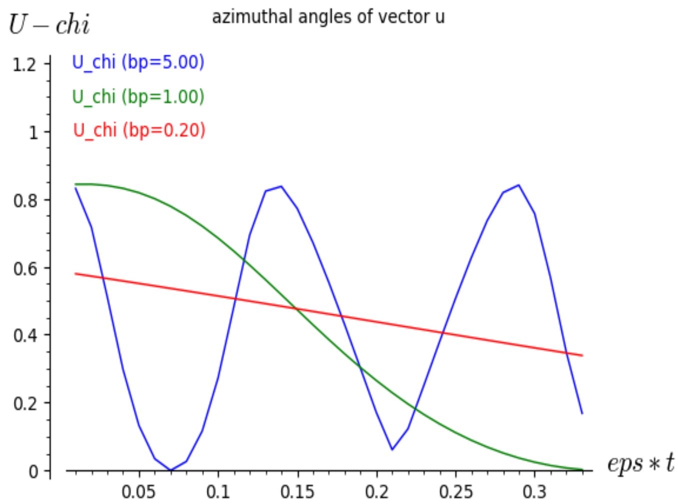

Fig. 3 **Azimuth angle of the second-order invariant vector $\vec{u}$ as a function of (εt). Three field-strength parameter *b*p* values denote weak (0.20), moderate (1.0), and strong (5.0) values.**

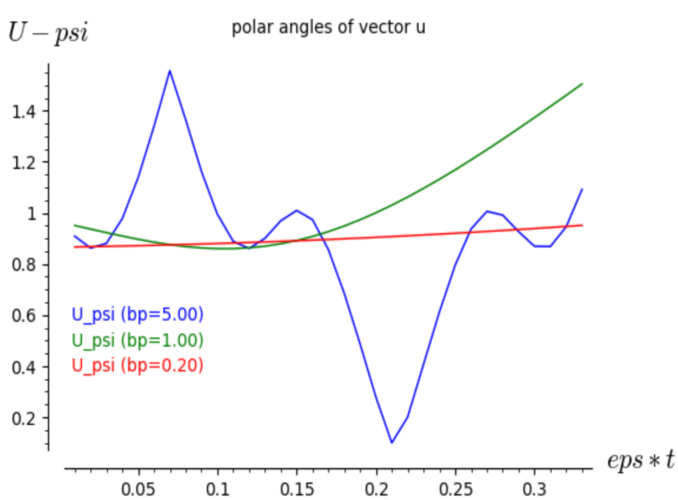

Fig. 4 **Polar angle of the second-order invariant vector $\vec{u}$ as a function of (εt). Three field strength parameter *b*p* values denote weak (0.20), moderate (1.0), and strong (5.0) values.**

The plots show that the angles have more-complicated time-dependence with increasing field strength, which is expected.

5. Relation between the Density Matrix and IVR: Mixed Qutrit

Let initially the qutrit be in a mixed state given as

$$\begin{pmatrix} \psi_0(t=0) \\ \psi_1(t=0) \\ \psi_2(t=0) \end{pmatrix} = d_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + d_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{17a}$$

It gives the initial density matrix as

$$\rho(0) = \begin{bmatrix} d_0^2 & 0 & 0 \\ 0 & d_1^2 & 0 \\ 0 & 0 & d_2^2 \end{bmatrix} \tag{17b}$$

The unit trace relation gives

$$d_0^2 + d_1^2 + d_2^2 = 1 \tag{18}$$

Then the density matrix elements given in the previous section are replaced by

$$\omega_0' = d_0^2\alpha^2 + d_1^2(2\alpha\beta) + d_2^2\beta^2 \tag{19a}$$

$$\omega_1' = (d_0^2 + d_2^2)2\alpha\beta + d_1^2\,(1\text{-}4\,\alpha\beta) \tag{19b}$$

$$\omega_2' = d_0^2\beta^2 + d_1^2(2\alpha\beta) + d_2^2\alpha^2 \tag{19c}$$

Define

$$t_0 = [d_0^2\beta + d_1^2(1-2\beta) + d_2^2\alpha] \tag{20a}$$

$$t_1 = [(d_0^2 + d_2^2) - 2d_1^2]\beta, \tag{20b}$$

$$t_2 = [d_0^2\alpha - d_1^2(1-2\beta) - d_2^2\beta] \tag{20c}$$

Then the solution for density matrix is

$$\rho(t) = \begin{bmatrix} \omega_0' & -t_2\bar{B} & -t_1\bar{K} \\ -t_2B & \omega_1' & -t_0\bar{B} \\ -t_1K & -t_0B & \omega_2' \end{bmatrix} \tag{21}$$

leading to

$$Q_0' = \frac{1}{2}(q_0' + ia_0') = t_0\bar{B} \tag{22a}$$

$$Q_1' = \frac{1}{2}(q_1' + ia_1') = -t_1K \tag{22b}$$

$$Q_2' = \frac{1}{2}(q_2' + ia_2') = -t_2\bar{B} \tag{22c}$$

These definitions are similar to those given in Table 1. The mixed-state invariant vectors and their properties are given in the Table 3.

Table 3 IVR vectors for a mixed-state qutrit

IVR vectors	Cartesian components	Angles	Comments
$\vec{w}$	$w_1 = \sqrt{\omega_0'}$, $w_2 = \sqrt{\omega_1'}$, $w_3 = \sqrt{\omega_2'}$	Polar angle of $\vec{w}$: $\psi_1(t) = cos^{-1}\left(\sqrt{\omega_2'}\right)$; Azimuthal angle of $\vec{w}$: $\chi_1(t) = tan^{-1}\left(\sqrt{\frac{\omega_1'}{\omega_0'}}\right)$	Length of $w = 1$ Follows from unit trace relation of density matrix. It always has unit length.
$\vec{u}$	$u_1 = \sqrt{\omega_0'^2 + 4\alpha\beta t_0^2}$, $u_2 = \sqrt{\omega_1'^2 + 2\alpha^2 t_1^2}$, $u_3 = \sqrt{\omega_2'^2 + 4\alpha\beta t_2^2}$	Polar angle of $\vec{u}$: $\psi_2(t) = cos^{-1}(u_3/u)$; Azimuthal angle of $\vec{u}$: $\chi_2(t) = tan^{-1}(u_2/u_1)$	Length $u(t) = \sqrt{u_1^2 + u_2^2 + u_3^2}$
$\vec{v}$	$v_1 = \sqrt{\begin{matrix} 3\omega_0'^2 - 2\omega_0'^3 - \Delta' \\ +12\alpha\beta\omega_0' t_0^2 \end{matrix}}$, $u_2 = \sqrt{\begin{matrix} 3\omega_1'^2 - 2\omega_1'^3 - \Delta' \\ +6\alpha^2\omega_1' t_1^2 \end{matrix}}$, $u_3 = \sqrt{\begin{matrix} 3\omega_2'^2 - 2\omega_2'^3 - \Delta' \\ +12\alpha\beta\omega_2' t_2^2 \end{matrix}}$	Polar angle of $\vec{v}$: $\psi_3(t) = cos^{-1}(v_3/v)$; Azimuthal angle of $\vec{v}$: $\chi_3(t) = tan^{-1}(v_2/v_1)$	Length $v(t) = \sqrt{v_1^2 + v_2^2 + v_3^2}$ And $\Delta' = 8\alpha^2\beta t_0 t_0 t_1 t_2$

Plots of the IVR Vector Angles of Cascade Model: Mixed Qutrit

Here we study the time-dependence of three independent IVR vectors of the chosen mixed state (Figs. 5–7). The plots show the time-dependence of the polar angles (ψ_1, ψ_2, ψ_3), azimuth angles $(\chi_1, \chi_{2,}\chi_{3,})$, and lengths (u,v) of the IVR vectors $\vec{w}$, $\vec{u}$, and $\vec{v}$, respectively.

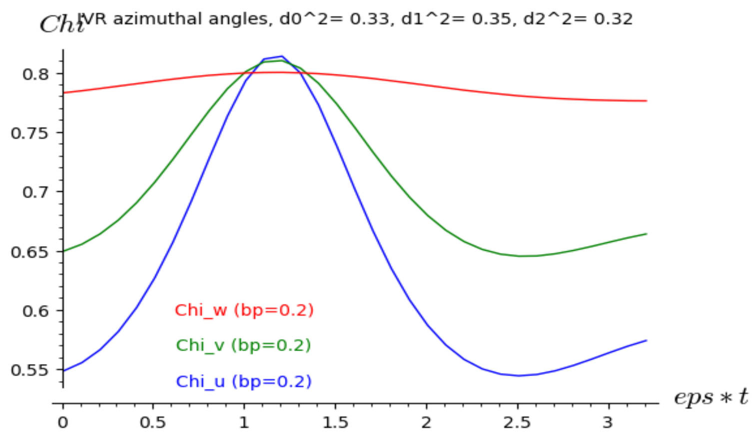

Fig. 5 Azimuthal angles of the invariant vectors $(\vec{u}, \vec{v}, \vec{w})$ as a function of (εt). The field-strength parameter is b*p = 0.2. The qutrit mixed-state parameters are $d_0^2 = 0.33$ and $d_1^2 = 0.35$.

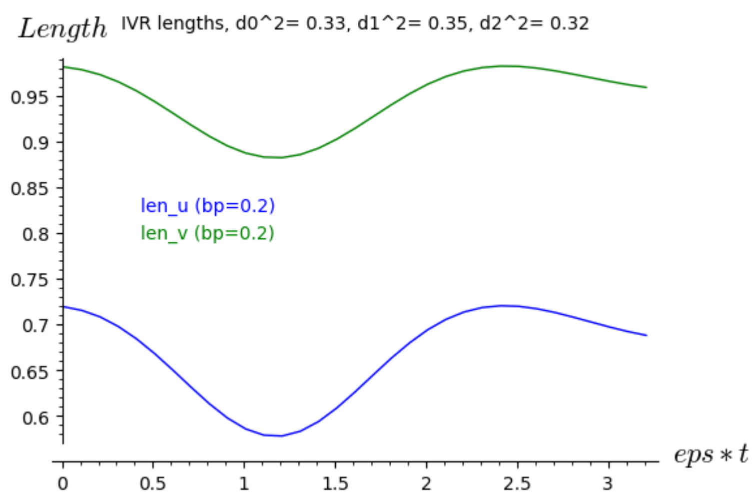

Fig. 6 Lengths of the invariant vectors $(\vec{u}, \vec{v}, \vec{w})$ as a function of (εt). The field strength parameter is b*p = 0.2. The qutrit mixed-state parameters are $d_0^2 = 0.33$ and $d_1^2 = 0.35$.

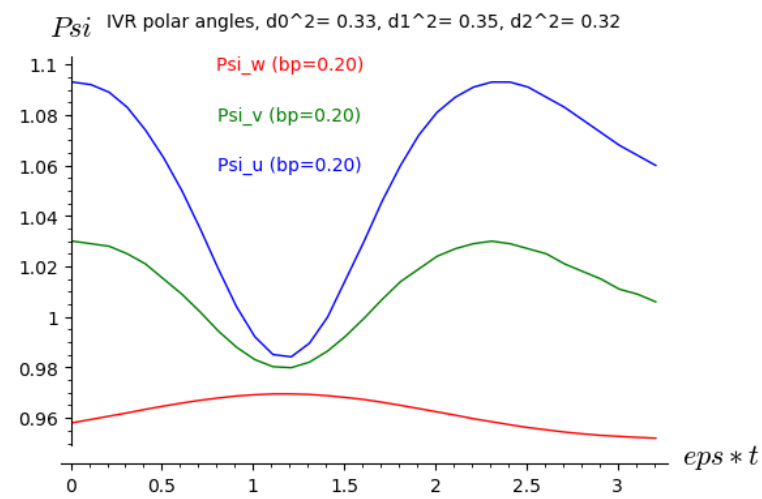

Fig. 7 Polar angles of the invariant vectors ($\vec{u}, \vec{v}, \vec{w}$) as a function of ($\varepsilon t$). The field-strength parameter is b*p = 0.2. The qutrit mixed-state parameters are $d_0^2 = 0.33$ and $d_1^2 = 0.35$.

The parameter space for mixed state is 3-D in (b*p, d_0^2, d_1^2). Only one combination has been presented here.The time-dependence of all eight density matrix parameters can be found independently using the current method. This ease of visualization should be contrasted with that of the traditional methods using the geometry of higher than 3-D spaces.

6. Conclusion and Next Steps

A method for visualizing the dynamics and geometry of both pure and mixed qutrit states has been presented. A simple Hamiltonian gives the temporal dynamics leading to time-dependent IVR vectors of the system. The state parameters have the following representations:

- The pure state has four parameters (four angles of two 3-D vectors).
- The mixed state has eight parameters (six angles and two lengths of three 3-D vectors).

The vectors represent invariant trace properties of the density matrix. This shows the versatility of this representation in which the 3-D IVR vectors can capture the temporal behavior of a qutrit. In the future, we would like to study the usefulness of this approach to two entangled qutrits and express the relevant metrics in terms of these vectors.

7. References

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List of Symbols, Abbreviations, and Acronyms

3-D	three-dimensional
8-D	eight-dimensional
IVR	invariant vector representation

1	DEFENSE TECHNICAL
(PDF)	INFORMATION CTR
	DTIC OCA
1	CCDC ARL
(PDF)	FCDD RLD DCI
	TECH LIB
1	CCDC ARL
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