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The Noise Figure for Multiple-Input RF Systems

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14. ABSTRACT
In this report we present an approach for calculating the noise figure for RF systems that comprise multiple inputs and a single output. We show that the proper definition of input signal-to-noise ratio depends on the coherence properties of the input signals.

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Contents

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1 EXECUTIVE SUMMARY

The radio-frequency (RF) noise figure is a performance parameter that is well understood for multiple devices arranged in series in a single channel, while the case of multiple inputs has been discussed less-frequently in the literature. In this report we present a method for calculating the noise figure for RF system containing multiple, parallel input signals. We consider both incoherent and coherent input signals. We derive an important result for typical beamformer configurations in which all the channels are identical and all the RF signals are in phase. Under these conditions, provided input signal-to-noise is properly defined, the noise factor of the entire system is independent of both the number of channels and the intrinsic loss of the combiner, and is equal to the noise factor of any one channel. We discuss the importance of a proper definition of input signal-to-noise ratio, a definition that depends on the coherence properties of the incoming RF signals. The results of this study may be of significance for beamforming and phased-array applications and for multiple-input/single-output (MISO) systems in general.

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2 INTRODUCTION

Noise figure *NF* is a well-established metric for characterizing the performance of an RF device or system in terms of how much the signal-to-noise ratio is degraded by the system itself. When multiple devices are connected serially to form a link between a single input and single output, the overall, end-to-end *NF* can be calculated easily using well-known formulas [See, for example Pozar [1]].

Certain systems comprise not just a single input but multiple, parallel input channels, each carrying an independent stream of information. Phased-array radars, for example, utilize multiple input RF channels all derived by power-division from a single source. Receive-beamforming systems use a spatial array of antennas to obtain multiple RF channels that are then coherently combined and measured to obtain angle-of-arrival information. Finally, unwanted multiple RF channel effects can occur when a presumed single path undergoes reflections leading to so-called multi-path interference effects.

Although discussions of noise figure for discrete devices and serial-cascaded systems are found in nearly every textbook on RF or microwaves, the literature on multiple, parallel-input configurations is relatively sparse. Lee (1993) [2] and Gatti (2004) [3] each calculated *NF* for an array of active antennas. Holzman (1996) [4] and Agrawal (1999) [5] compared the noise performance of RF beamformer architectures for phased-array applications. Prasad (2019) [6] recently investigated the noise figure for analog and digital beamforming systems. The advent of hybrid RF-photonic systems with sufficiently-good RF performance in the mid-late 1990's brought renewed interest in photonic phased-array and beamformer systems [7, 8]. An important aspect of fiber-optic implementations is the ability to provide the long-length, true-time delays necessary for phased-arrays but with extremely-low RF loss in the delay segments as shown recently by Mondich (2020) [9].

In this report we amplify and extend the work of Lee [2] and Gatti [3] as applied specifically to the noise figure for receive beamformer configurations. For completeness, we analyze the cases of both incoherent and coherent RF inputs and we do not restrict the analysis to the situation where all RF inputs are presumed to arrive in-phase. In addition, we calculate the effect of losing just one channel of the receive array as well as the effect of losing all but one channel in the array as these provide further insight into the behavior of NF in multiple-channel systems. Compared to archival publications, here we extend certain results and provide significantly more

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detail in the derivations. In contrast to much of the existing literature, we show that the proper definition of input SNR depends on the coherence properties of the RF input signals.

We always calculate noise in terms of power spectral density (PSD) $N(W/Hz)$ and we assume that, in the neighborhood of any RF frequency of interest, the noise PSD is relatively flat so conversion to noise power is accomplished simply by multiplication by the RF bandwidth B. Thus, signal-to-noise ratios are always expressed in units Hz^{-1}. Although the logarithmic quantity *noise figure* $NF(dB)$ is the parameter most-typically quoted to describe device and system performance, here we shall calculate instead the more natural linear quantity *noise factor* F where $NF = 10\log_{10} F$.

3 OVERVIEW

The three RF configurations to be analyzed in this report are shown in Fig. 1. Symbols shown in the figure will be defined in later sections and a complete list of symbols appears in Appendix C . In each configuration, the signals from each of the M individual parallel paths are summed together in an RF combiner. In this work, we model the combiner as an ideal Wilkinson device. We distinguish between incoherent Fig. 1(a) and coherent Figs. 1(b,c) cases since we have found that the appropriate definition of input signal-to-noise ratio depends on the coherence properties of the input signals. Beamformer systems rely on RF interference and, hence, must comprise coherent inputs. We include the incoherent case here for completeness and to illustrate the distinctions between the two cases. In configuration (c) of Fig. 1 we assume that the M coherent input signals are derived from a single RF source since this is likely how a beamformer system would be configured for laboratory testing using, for example, a network analyzer as source and receiver.

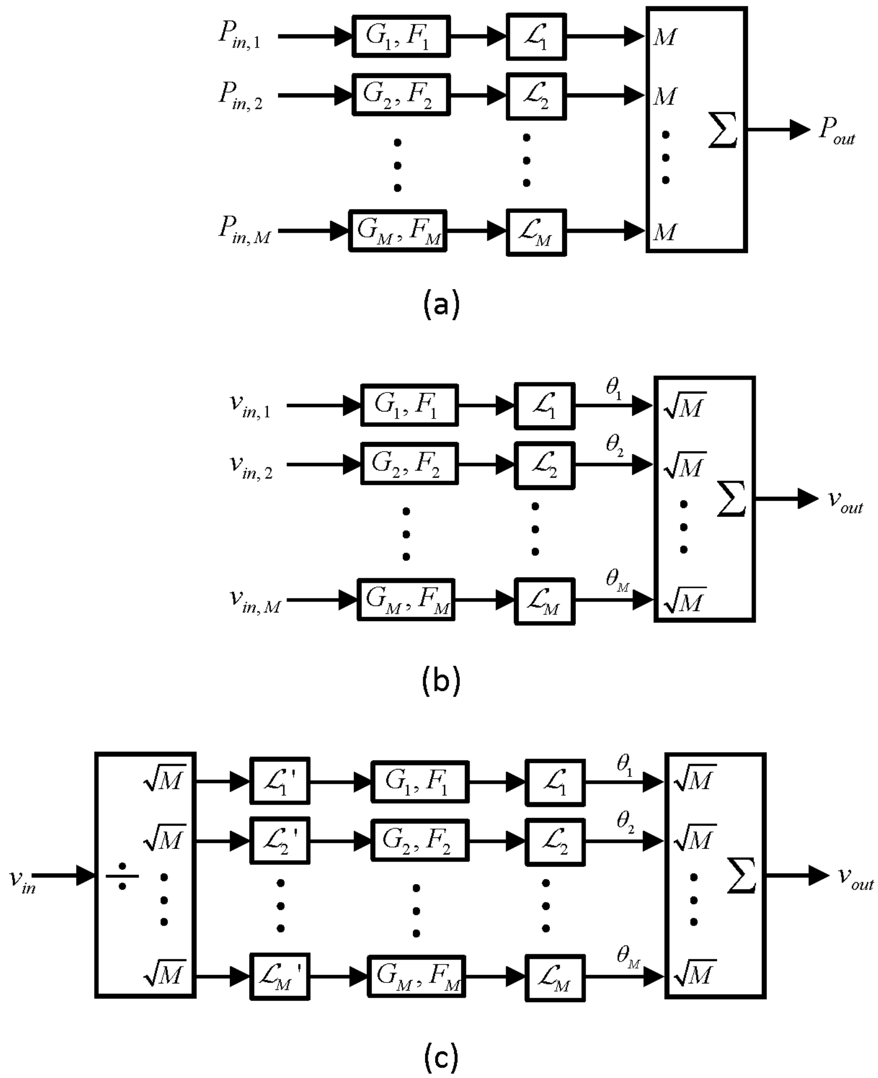


Figure 1: The three multi-channel RF configurations analyzed in this report, each comprising M input channels. (a) Incoherent input signals; (b) Coherent input signals; (c) Coherent input signals derived from a single RF source. Symbols are defined in the text and in Appendix C.

The report is organized as follows. Section 3 describes the overall goal and approach of the report. A brief review of noise factor for discrete gain and loss elements and serial cascades of discrete elements is provided in Section 4. Section 5 comprises the main results of the report in which we derive expressions for the noise factor F for all three configurations in Fig. 1 and discuss the behavior of F in various limiting cases. A brief summary and discussion is presented in Section 6. Three Appendices are included. Appendix A reviews and summarizes details of the SNR and noise factor for gain and loss elements and for some common serial cascade configurations. Appendix B reviews the general scattering matrix for an ideal Wilkinson device. A list of symbols is provided in Appendix C.

4 REVIEW OF NOISE FIGURE FOR SINGLE ELEMENTS AND SERIAL CASCADES OF ELEMENTS

In this section we review the noise figure for single, discrete elements and for serial cascades of discrete elements. These results are well-known to every RF engineer and are presented here only to establish nomenclature and to provide the background needed in subsequent calculations for multiple, parallel-input systems.

The *noise factor* $F \geq 1$ quantifies the degradation in signal-to-noise ratio (SNR) upon passage through a device or system.

$$F = \frac{SNR_{in}}{SNR_{out}} \tag{1}$$

under the assumption that the input SNR is due entirely to thermal noise at the standard temperature $T_0 = 290K$,

$$SNR_{in} = \frac{P_{in}}{k_B T_0}, \tag{2}$$

where P_{in} = input signal power and $k_B = 1.38 \times 10^{-23} J/K$ is the Boltzmann constant. The noise factor is obtained once the gain G and the output noise PSD N_{out} are known. That is, regardless of the actual operating temperature at the input, F is *defined* assuming the input noise is entirely thermal noise at the standard temperature. This in no way prevents us from calculating noise at the output when the input noise is something other than $k_B T_0$ as we shall see shortly. By definition, $SNR_{out} = GP_{in}/N_{out}$ and, hence, for a single, discrete element

$$F = \frac{1}{G} \frac{N_{out}}{k_B T_0}. \tag{3}$$

4.1 Noise Factor for Gain and Loss Elements

4.1.1 Active Gain Elements

An active element with RF gain G always does at least three things: 1) it amplifies the signal power $P_{in} \rightarrow GP_{in} = P_{out}$, 2) it amplifies the input noise $N_{in} \rightarrow GN_{in}$, and 3) it adds some noise of its own, N'_{added}. Then the total output noise is $N_{out} = GN_{in} + N'_{added}$. There are any number of ways to handle the added noise mathematically but the standard approach is to redefine $N'_{added} = GN_{added}$ and to write

$$N_{out} = GN_{in}\left(1 + \frac{N_{added}}{N_{in}}\right) \tag{4}$$

where the multiplicative factor $(1 + N_{added}/N_{in})$ quantifies how much larger the output noise is than the expected (and unavoidable) amplified-input noise GN_{in}. If we defined the input noise PSD as $k_B T_0$ then we can define a noise factor as

$$F \equiv 1 + \frac{N_{added}}{k_B T_0}. \tag{5}$$

In all cases, $F \geq 1$.

Next, whatever the value of N_{added}, we can always write $N_{added} = k_B T_e$ where $T_e = (F - 1)T_0$ is called the "equivalent noise temperature". Note that, although T_e is an entirely fictitious temperature that has nothing to do with any physical temperature in the device, it will turn out to be a very useful concept when we analyze cascaded systems. Hence, given input noise $k_B T_0$, the total noise at the output of a gain element with specified noise factor F or noise temperature T_e is

$$N_{out} = FGk_B T_0 = (Gk_B T_0)\left(1 + \frac{T_e}{T_0}\right). \tag{6}$$

The equivalent expression in terms of noise figure NF is the well-known

$$N_{out}(dBm/Hz) = -174dBm/Hz + G(dB) + NF(dB). \tag{7}$$

Therefore, for the purposes of noise factor analysis, any active gain element can be characterized by two parameters: either i) the RF power gain G and the noise factor F or, equivalently, ii) the RF power gain G and the equivalent noise temperature T_e. See Fig.2(a).

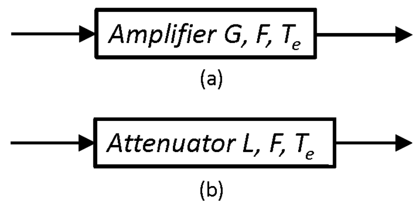


Figure 2: (a) Gain element characterized by RF power gain G and either noise factor F or equivalent noise temperature $T_e = (F-1)T_0$. (b) Loss element characterized by loss factor $L = 1/G$ and either noise factor F or equivalent noise temperature $T_e = (L-1)T_0$.

4.1.2 Passive Loss Elements

A passive loss element (attenuator) can be described as having either a power gain factor $G \leq 1$, or a power loss factor $L = 1/G \geq 1$. It is standard practice to use the latter so that L, just like F, is always greater than or equal to one. Given input signal power P_{in}, the output power $P_{out} = P_{in}/L$. Suppose now that there is no input signal and that the loss element is properly terminated so that the input noise is just $k_B T_0$. The output noise PSD in this case must also be just $k_B T_0$ - the minimum possible value of the output noise PSD for a passive loss element. Putting all these values together, the noise factor for a passive loss element is

$$F = \frac{SNR_{in}}{SNR_{out}} = \frac{P_{in}}{k_B T_0} \frac{k_B T_0}{(P_{in}/L)} = L. \tag{8}$$

It would be convenient if we could handle loss elements mathematically in exactly the same way as gain elements, especially in cascaded systems. To make this work we need to invent another fiction: We pretend that the device indeed attenuated the input noise $k_B T_0$ by $1/L$ but then added some noise back in to make the output noise come out right. Then $N_{out} = (1/L)(k_B T_0 + N_{added}) = k_B T_0$ which forces $N_{added} = (L-1)k_B T_0$. (The next section contains a more complete discussion of this concept.)

Hence, a passive loss element acts as if it had noise factor $F = L$ and equivalent noise temperature $T_e = N_{added}/k_B = (L-1)T_0$. Just as with gain elements, $F = (1 + T_e/T_0)$. See Fig. 2(b). Finally, given input noise $N_{in} \geq k_B T_0$, the output noise of a passive attenuator is

$$N_{out} = \frac{1}{L}(N_{in} + k_B T_e) = \frac{1}{L}(N_{in} + (L-1)k_B T_0). \tag{9}$$

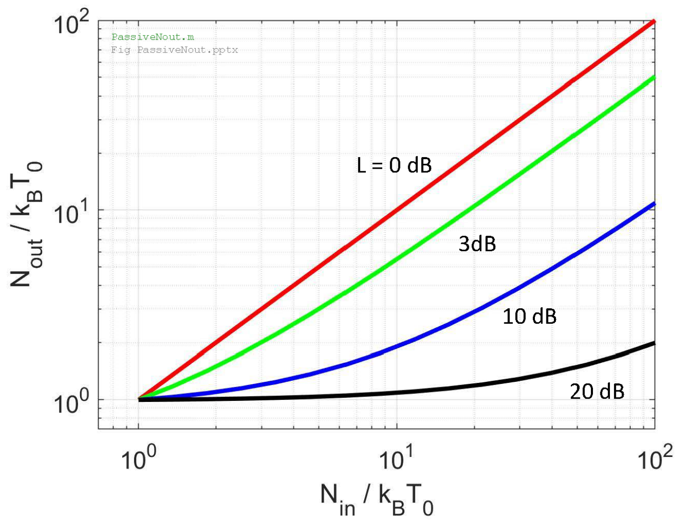


Figure 3: Output noise PSD vs input noise PSD for a passive loss element having various power loss factors L. All noise values are in units of $k_B T_0$.

This expression is plotted in Fig. 3 for a few different attenuation values. We see that, in the limit $N_{in} \to k_B T_0$, $N_{out} \to k_B T_0$ as required, independent of L and, in the limit $L \to 1$ (0 dB), $N_{out} \to N_{in}$ as expected.

4.1.3 Proper Use of Equivalent Noise Temperature

We have seen that noise added by an active gain element is given by $N'_{added} = G k_B T_e$ where T_e is the equivalent noise temperature. Now suppose the noise at the device input is not $k_B T_0$ but $N_{in} > k_B T_0$. Then the output noise is

$$N_{out} = G\left(N_{in} + k_B T_e\right) \tag{10}$$

where $T_e = (F-1)T_0$. So the equivalent noise temperature tells us how much "new" noise is added by the element. Note that the output noise is not given by $GN_{in} + Gk_B\left(T_0 + T_e\right)$ since this would incorrectly imply that there is an additional $k_B T_0$ noise source at the input. Similarly, for a loss element,

$$N_{out} = \frac{1}{L}\left(N_{in} + k_B T_e\right) \tag{11}$$

where $T_e = (L-1)T_0$.

For an active gain element, the gain factor and noise factor are typically not related while, for a passive loss element, the loss factor itself *defines* the

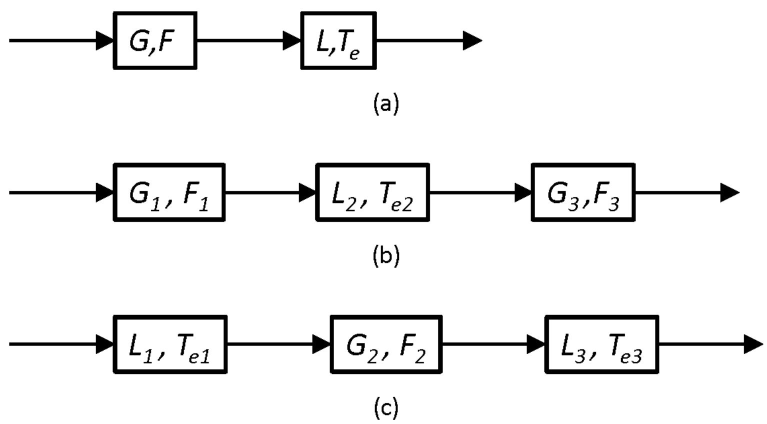


Figure 4: Three different series cascade gain/loss configurations. (a) Gain followed by loss. (b) Gain-loss-gain. (c) Loss-gain-loss.

noise factor. It is this fact that distinguishes an active gain element from a passive loss element, not simply whether to use G or $L = 1/G$.

4.2 Noise Factor for Cascaded Gain and Loss Elements

In Appendix A we tabulate expressions for the output noise and noise factor for a number of cascaded combinations of gain and loss elements. In each case, the noise factor for a series cascade of elements follows the well-known formula

$$F_{series} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \cdots \tag{12}$$

provided we use $G = 1/L$ for loss elements. For example, for the simple cascade of a gain element followed by a loss element shown in Fig.4(a), the output noise PSD and the noise factor are given by

$$N_{out} = \left(\frac{FG}{L} + \frac{L-1}{L}\right) k_B T_0 = \left(1 + \frac{FG-1}{L}\right) k_B T_0 \tag{13}$$

and

$$F_{series} = F + \frac{L-1}{G} = \left(\frac{L}{G}\right)\left(1 + \frac{FG-1}{L}\right). \tag{14}$$

The more cumbersome expression on the right in F_{series} is included here since it will turn out that comparisons between certain limiting cases for multiple-input systems will become more transparent using the second expression rather than the more-common first expression on the right in Eq. (14).

We now discuss two configurations that will be of utility in subsequent discussions. For the gain-loss-gain configuration of Fig. 4(b),

$$\begin{aligned} N_{out} &= G_3\left(\frac{F_1G_1}{L_2} + \frac{L_2 - 1}{L_2} + (F_3 - 1)\right)k_BT_0 \\ F_{series} &= F_1 + \frac{L_2 - 1}{G_1} + \frac{F_3 - 1}{G_1/L_2} \end{aligned} \tag{15}$$

and for the loss-gain-loss configuration of Fig.4(c), we find

$$\begin{aligned} N_{out} &= \left(\frac{F_2G_2}{L_3} + \frac{L_3 - 1}{L_3}\right)k_BT_0 \\ F_{series} &= F_2L_1 + \frac{L_3 - 1}{G_2/L_1}. \end{aligned} \tag{16}$$

5 NOISE FIGURE FOR MULTIPLE-INPUT RF CONFIGURATIONS

This section comprises the main results of this report in which we calculate the noise factors for RF parallel-receive architectures for both incoherent and coherent RF inputs. A critical parameter required in analyzing this type of system is the definition of overall input signal-to-noise ratio. Whereas in single-input systems the input SNR is usually unambiguous, its definition in a multiple-input system is not immediately evident. Furthermore, since noise factor is a manufactured characterization parameter for single-channel systems we could, in principle, choose any definition we like for a parallel-channel system. However in our analysis we sought expressions for F that a) conformed to well-accepted notions of noise factor for series-cascaded systems and b) always reduced to the series-cascade result in the appropriate limit. In this section, we provide detailed discussions of the background and the assumptions leading to various expressions and we test the relevant limiting cases. This development is in part an expansion and extension of the works by Lee [2] and Gatti [3].

5.1 Definition of the Input Signal-to-Noise Ratio

Consider a system comprising M RF inputs and a single RF output as shown in Fig. 5. We can easily define and measure a SNR for each input

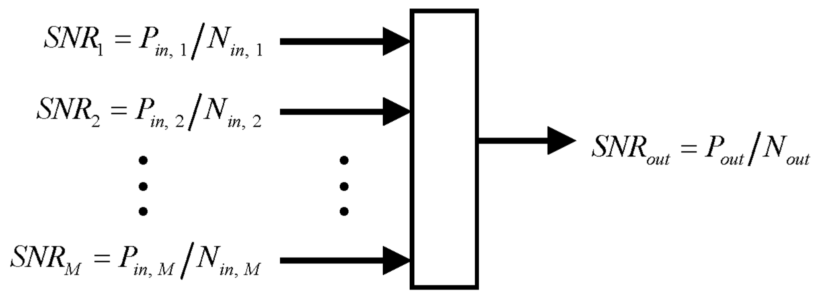


Figure 5: A generic multiple-input, single-output system.

channel, SNR_m, in the usual fashion but what do we mean by the input SNR, SNR_{in}, for the entire system? Some possibilities are

1) Arithmetic Mean (Average)

$$SNR_{in} = \frac{1}{M}\left(SNR_1 + SNR_2 + \cdots + SNR_M\right), \tag{17}$$

2) Total Power

$$SNR_{in} = \frac{1}{k_B T_0}\left(P_1 + P_2 + \cdots + P_M\right), \tag{18}$$

3) Geometric Mean

$$SNR_{in} = \left(SNR_1 \cdot SNR_2 \cdots SNR_M\right)^{1/M}, \tag{19}$$

4) Parallel Sum

$$\frac{1}{SNR_{in}} = \frac{1}{SNR_1} + \frac{1}{SNR_2} + \cdots + \frac{1}{SNR_M}. \tag{20}$$

We find that the proper choice of input SNR depends on whether the input signals are coherent or incoherent. For incoherent inputs, the simple arithmetic average is appropriate while, for coherent inputs, the ratio of total input power to $k_B T_0$ provides the most suitable definition of input SNR. In investigating various expressions we sought to always be consistent with common notions of F, namely, $F \geq 1$ and, in the limit where all gain and loss factors were unity and no noise was added by any of the elements, then $F \to 1$.

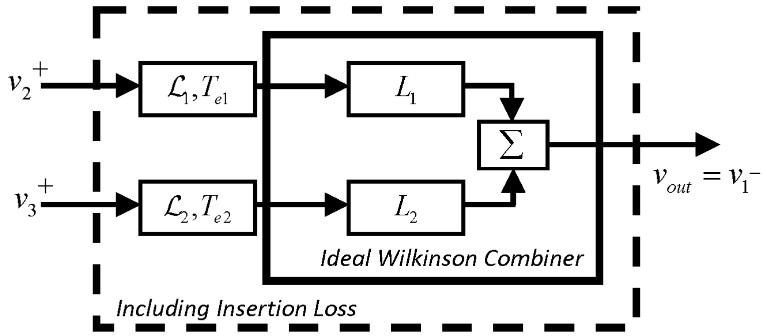


Figure 6: The model for a 2x1 RF combiner comprising an ideal Wilkinson combiner with intrinsic power loss factor $L = 2$ on each channel and insertion loss factors $\mathcal{L}_{1,2}$.

5.2 Model for the Power Combiner

Any beamformer system must contain some type of RF summing junction or signal combiner. In this report we model the RF combiner as an ideal Wilkinson device but we allow for insertion loss by explicitly adding an additional loss element at each input port. This is the loss factor $\mathcal{L}$ in Figs. 1 and 6. The scattering matrix for ideal Wilkinson power splitters and combiners is reviewed in Appendix B. To illustrate the approach, first consider a simple 2 x 1 combiner as shown in Fig. 6. For this ideal 3-port device, the scattering matrix equation, relating voltages entering a port v^+ and voltages exiting a port v^-, is given by

$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \frac{j}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix} . \tag{21}$$

Note that, even if there is no insertion loss so that $\mathcal{L}_{1,2} = 1$, each input signal suffers an intrinsic voltage attenuation $1/\sqrt{2}$ (power loss factor $L = 2$). In the general case, with M input ports, the intrinsic power loss factor is $L = M$. Later on, we will analyze a system that uses a Wilkinson device as a voltage splitter and, since the Wilkinson scattering matrix is its own transpose, the same matrix applies.

In the next two sections we analyze the noise factor for a multiple-input-channel system with either incoherent or coherent RF inputs.

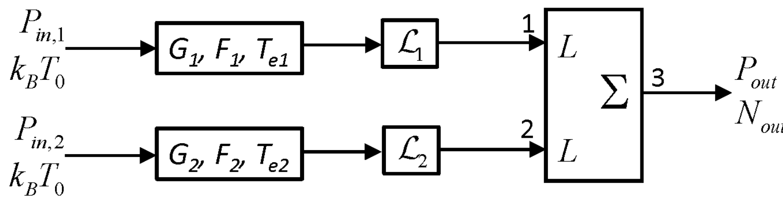


Figure 7: A 2x1 parallel power combination system with incoherent inputs.

5.3 Noise Figure for Incoherent Parallel Systems

In this section we calculate the noise factor for a system comprising M incoherent RF inputs that are summed in a device modeled as a Wilkinson combiner. We first consider an $M = 2$ system and then generalize the result to arbitrary M. The system we analyze is the power combiner of Fig. 6 with gain elements in each arm as shown in Fig. 7. Since the RF inputs are incoherent we can work directly with power rather than voltages. Each input is assumed to be thermal-noise limited. Then, from the discussion in Section 5.1, the input SNR is just the average

$$SNR_{in} = \frac{1}{2}\left(\frac{P_{in,1}}{k_B T_0} + \frac{P_{in,2}}{k_B T_0}\right) = \frac{P_{ave}}{k_B T_0} \tag{22}$$

where P_{ave} is the average input power.

The signal power at the input to each of the Wilkinson input ports is $P_m = (G_m/\mathcal{L}_m)\, P_{in,m}$ for $m = 1, 2$ while the corresponding noise was presented earlier in Eq. 13 in conjunction with Fig. 4, that is, $N_m = (1/\mathcal{L}_m)(F_m G_m + (\mathcal{L}_m - 1))k_B T_0$. In passing through the ideal Wilkinson, both the signal and noise suffer power loss L. Hence,

$$\begin{aligned} P_{out} &= \frac{1}{L}\left(\frac{G_1}{\mathcal{L}_1} P_{in,1} + \frac{G_2}{\mathcal{L}_2} P_{in,2}\right) \\ N_{out} &= \frac{1}{L}\left(N_1 + N_2\right) \\ &= \frac{1}{L}\left(\frac{F_1 G_1}{\mathcal{L}_1} + \frac{F_2 G_2}{\mathcal{L}_2} + \frac{\mathcal{L}_1 - 1}{\mathcal{L}_1} + \frac{\mathcal{L}_2 - 1}{\mathcal{L}_2}\right) k_B T_0. \end{aligned} \tag{23}$$

At this point it is worth pausing to test this noise expression in various limits.

i) Both channels equivalent: ($G1 = G2 = G$, $F1 = F2 = F$, and

$\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$.) Then, since $L = 2$

$$N_{out} \longrightarrow \left(1 + \frac{FG - 1}{\mathcal{L}}\right) k_B T_0. \quad (24)$$

This is the same expression as for a single-channel cascade gain and loss system that we found in Eq. 13. That is, the noise factor of two parallel, incoherent, identical channels is the same as the noise factor of a single channel, even though the intrinsic 3 dB loss per channel of the Wilkinson device is still included. The reason is that *both* signal and noise in each channel suffer a 3 dB loss in the combiner.

ii) No insertion loss ($\mathcal{L}_1 = \mathcal{L}_2 = 1$) and no gain elements ($G1 = G2 = F1 = F2 = 1$)

$$N_{out} \longrightarrow \frac{1}{2}(1 + 1)\, k_B T_0 = k_B T_0. \quad (25)$$

This must be the output noise for the ideal Wilkinson device with matched input loads.

iii) Infinite insertion loss in each arm ($\mathcal{L}_1, \mathcal{L}_2 \longrightarrow \infty$)

$$N_{out} \longrightarrow \frac{1}{2}(1 + 1)\, k_B T_0 = k_B T_0. \quad (26)$$

Again, this must be the output noise with extremely large attenuation in each input arm.

iv) Infinite insertion loss in just one arm ($\mathcal{L}_1 \longrightarrow \infty$)

$$N_{out} \longrightarrow \frac{1}{2}\left(2 + \frac{F_2 G_2 - 1}{\mathcal{L}_2}\right) k_B T_0. \quad (27)$$

This result is a bit more interesting and corresponds to "choking off" just one of the inputs to the combiner. We see that, for $(F_2 G_2/\mathcal{L}_2) >> 1$, the output noise is dominated initially by the input noise $(F_2 G_{2/}\mathcal{L}_2) k_B T_0$ attenuated by the intrinsic loss L of the combiner but then,as $\mathcal{L}_2$ becomes large or, equivalently, as $(F_2 G_2/\mathcal{L}_2)$ becomes small, the noise asymptotically approaches $k_B T_0$. A plot of $N_{out}/k_B T_0$ as a function of $\mathcal{L}_2$ for various values of $F_2 G_2$ is shown in Fig.8.

Finally, the noise factor for this configuration is given by

$$F_{2-||,\,incoh} = \frac{(P_{in,1} + P_{in,2})}{2\left(\frac{G_1 P_{in,1}}{\mathcal{L}_1} + \frac{G_2 P_{in,2}}{\mathcal{L}_2}\right)} \left(2 + \frac{F_1 G_1 - 1}{\mathcal{L}_1} + \frac{F_2 G_2 - 1}{\mathcal{L}_2}\right) \quad (28)$$

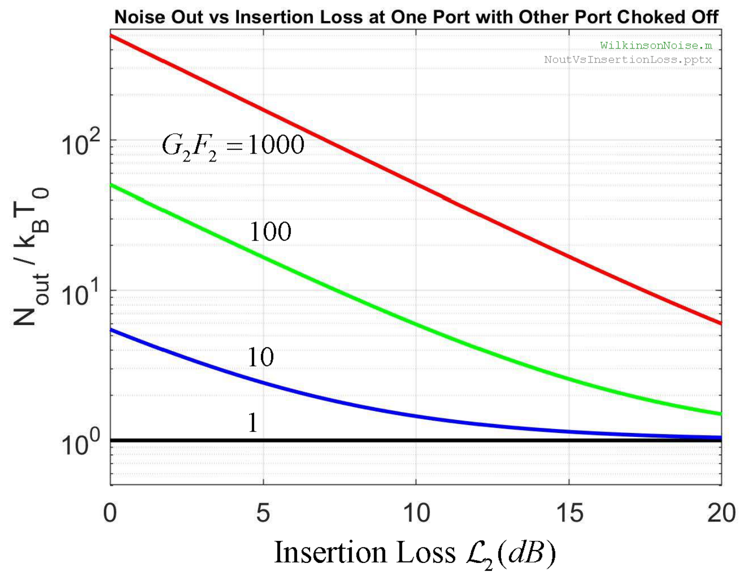


Figure 8: Output noise in units of k_BT_0 as a function of insertion loss $\mathcal{L}_2$ for various values of the product F_2G_2 under the assumption $\mathcal{L}_1 \longrightarrow \infty$.

where the notation "$2 - ||$" in the subscript indicates two parallel channels.

Generalization of these results to M incoherent parallel channels is straightforward.

$$SNR_{in} = \frac{1}{M} \sum_{m=1}^{M} \left(\frac{P_{in,m}}{k_BT_0} \right), \tag{29}$$

$$P_{out} = \sum_{m=1}^{M} \left(\frac{G_m P_{in,m}}{M\mathcal{L}_m} \right), \tag{30}$$

$$N_{out} = k_BT_0 \left[1 + \sum_{m=1}^{M} \left(\frac{F_mG_m - 1}{M\mathcal{L}_m} \right) \right], \tag{31}$$

and

$$F_{M-||,\, incoh} = \frac{\sum_{m=1}^{M} P_{in,m}}{\sum_{m=1}^{M} \left(\frac{G_m P_{in,m}}{\mathcal{L}_m} \right)} \left[1 + \sum_{m=1}^{M} \left(\frac{F_mG_m - 1}{M\mathcal{L}_m} \right) \right]. \tag{32}$$

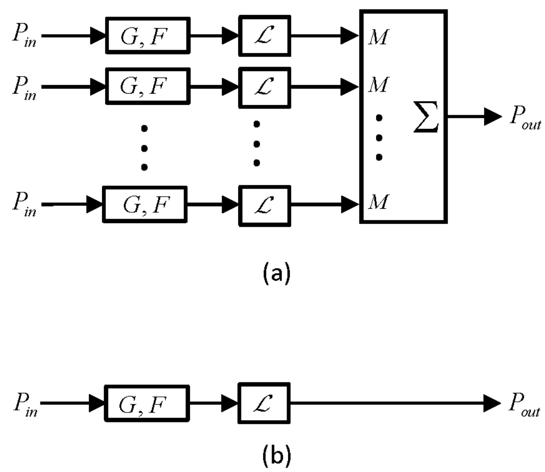


Figure 9: When the corresponding parameters in each channel are equal, so the channels are identical, then the noise factor of the full $M-$channel incoherent system shown in (a) is independent of M and equal to the noise factor of the single-channel system in (b).

We now examine some limiting cases.

i) All channels equivalent. That is, for all $1 \leq m \leq M$, $P_{in,m} = P, F_m = F, G_m = G$ and $\mathcal{L}_m = \mathcal{L}$. Then

$$F_{M-||, incoh} \longrightarrow \left(\frac{\mathcal{L}}{G}\right)\left[1 + \left(\frac{FG-1}{\mathcal{L}}\right)\right]. \tag{33}$$

Again, this is exactly the same noise factor as if there were just one channel. And again it arises because both the signal and the noise powers arriving at each input port of the combiner are attenuated by the same power loss factor M of the combiner.

ii) One channel blocked, all other channels equivalent. Then

$$F_{M-||, incoh} \longrightarrow \left(\frac{M}{M-1}\right)\left(\frac{\mathcal{L}}{G}\right)\left[1 + \left(\frac{M-1}{M}\right)\left(\frac{FG-1}{\mathcal{L}}\right)\right] \tag{34}$$

This result is slightly larger than the single channel result due to eliminating the signal power, but not the noise power, from one input port. For $M >> 1$, the penalty for "losing" one channel is relatively small, as expected.

iii) Only one channel (the $i - th$ channel) passes, all other channels equivalent. Then

$$F_{M-||,\,incoh} \longrightarrow M\left(\frac{\mathcal{L}}{G}\right)\left[1+\left(\frac{1}{M}\right)\left(\frac{FG-1}{\mathcal{L}}\right)\right]. \tag{35}$$

This is again similar to the noise factor for one channel except now the penalty term has the form $M\mathcal{L}/G$ which will be severe for $M >> 1$.

iv) No gain, gain-element noise factor, or insertion loss ($G_m = 1, F_m = 1, \mathcal{L}_m = 1$) for all $1 \leq m \leq M$ but the intrinsic Wilkinson power loss factor M remains for each port. Then

$$F_{M-||,\,incoh} \longrightarrow \left(\frac{M}{M}\right)[1+0] \longrightarrow 1. \tag{36}$$

With no gains or added noise, the noise factor must be unity.

From these considerations, we believe that Eq.(32) is the proper expression for the noise factor of a system comprising M incoherent input signals that are subsequently summed in a Wilkinson-type combiner to produce a single output.

5.4 Noise Figure for Coherent Parallel Systems

5.4.1 Multiple Coherent Inputs

We now consider a system with M coherent signal inputs. Here we must keep track, not of power, but of signal voltages and their RF phases. Voltages will be treated as phasors at some fixed RF frequency $f = \Omega/2\pi$. That is, voltage $v_0 \cos(\Omega t + \theta)$ will be represented by the phasor $v_0 \exp j\theta$ and the corresponding time-averaged rms electrical power delivered to load impedance Z is $P = |v_0 \exp j\theta|^2/2Z$. Noise will still act incoherently and, in fact, will be handled in exactly the same way as in the incoherent case analyzed earlier.

Given two voltages of the same amplitude but different phases, $v_0 e^{j\theta_1}$ and $v_0 e^{j\theta_2}$, the rms power in the sum of these voltages is $P_{sum} = \left(v_0^2/2Z\right)\left|e^{j\theta_1} + e^{j\theta_2}\right|^2$. Now $\left|e^{j\theta_1} + e^{j\theta_2}\right|^2 = 2\left(1 + cos(\theta_2 - \theta_1)\right)$. By definition, the time-average of the quantity $cos(\theta_2 - \theta_1)$ is zero for incoherent signals in which case the power in the sum is just the sum of the powers $P_{sum} = 2\left(v_0^2/2Z\right)$. But for coherent signals, by definition, there exists a

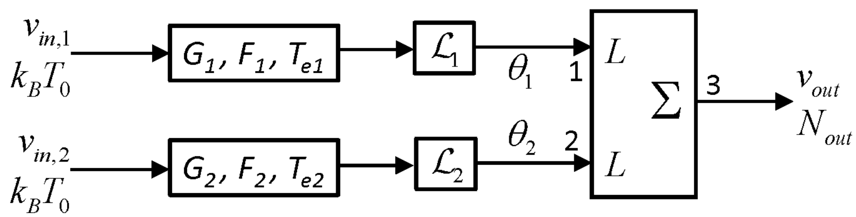


Figure 10: A 2-channel coherent system. $\theta_{1,2}$ denotes the RF phase accumulated by each signal in propagating from some reference plane to the inputs of the Wilkinson combiner.

fixed, time-invariant difference in the relative phase and, hence, $cos(\theta_2 - \theta_1)$ does not *average* to zero although it can be zero when $(\theta_2 - \theta_1)$ is an odd multiple of $\pi/2$. On the other hand, when $(\theta_2 - \theta_1)$ is an even multiple of π, then $cos(\theta_2 - \theta_1) = 1$ and the power in the sum is twice the sum of the individual powers, $P_{sum} = 4\left(v_0^2/2Z\right)$. Such is the nature of interferometric addition!

Consider the 2-channel coherent system shown in Fig. 10 and let $\theta_{1,2}$ denote the RF phases accumulated by the signals in propagating from their source to ports 1 and 2, respectively. Then, using the scattering matrix for a 3-port Wilkinson device (Eq. 21), we have

$$v_{out} = \frac{j}{\sqrt{2}}\left(\sqrt{\frac{G_1}{\mathcal{L}_1}}e^{j\theta_1}v_{in,1} + \sqrt{\frac{G_2}{\mathcal{L}_2}}e^{j\theta_2}v_{in,2}\right). \tag{37}$$

So the output power is

$$\begin{aligned} P_{out} = \frac{|v_{out}|^2}{2Z} &= \frac{1}{2Z}\frac{1}{2}\left|\sqrt{\frac{G_1}{\mathcal{L}_1}}v_{in,1}e^{j\theta_1} + \sqrt{\frac{G_2}{\mathcal{L}_2}}v_{in,2}e^{j\theta_2}\right|^2 \\ &= \frac{1}{2Z}\frac{1}{2}\left[\frac{G_1}{\mathcal{L}_1}v_{in,1}^2 + \frac{G_2}{\mathcal{L}_2}v_{in,2}^2 + 2\sqrt{\frac{G_1 G_2}{\mathcal{L}1\mathcal{L}_2}}v_{in,1}v_{in,2}\cos(\theta_2 - \theta_1)\right]. \end{aligned} \tag{38}$$

To understand the situation more clearly it will be helpful to assume, for the moment, that both channels are identical, except for phase accumulation, in which case $P_{in} = 2(v_{in}^2/2Z)$ and

$$P_{out} = \left(\frac{G}{\mathcal{L}}\right)\frac{v_{in}^2}{2Z}\left[1 + \cos\left(\theta_2 - \theta_1\right)\right]. \tag{39}$$

Depending on the phase difference $(\theta_2 - \theta_1)$ the output power can vary from a maximum of $(G/\mathcal{L})P_{in}$ to zero when $(\theta_2 - \theta_1)$ is an odd multiple of π so that $\cos(\theta_2 - \theta_1) = -1$. Since noise inputs reaching the Wilkinson combiner are mutually incoherent, the output noise is the same as it was in the incoherent case Eq.(24). We now have almost everything needed to calculate noise factor for this case.

We still need to define the input SNR. Clearly, the input thermal noises on each channel are incoherent while the input RF signals are coherent. Are we still justified in defining input SNR as the average of the individual SNR's? We believe the answer is no. If we expect the noise factor to be unity in the limiting case where all gain, loss, and noise factors are one, then we must compare the output SNR to an input SNR defined by the ratio of the *total* input power to $k_B T_0$. If we had used the average, and if the phases of all the incoming signals were equal, then it would appear as if the SNR improved upon passage through the combiner, which cannot be the case.

Returning to the system in Fig. 10, the output SNR is

$$SNR_{out,\,coh} = \frac{\frac{1}{2Z}\left|\sqrt{\frac{G_1}{2\mathcal{L}_1}}e^{j\theta_1 v_{in,1}} + \sqrt{\frac{G_2}{2\mathcal{L}_2}}e^{j\theta_2 v_{in,2}}\right|^2}{\left[1 + \left(\frac{F_1 G_1 - 1}{2\mathcal{L}_1} + \frac{F_2 G_2 - 1}{2\mathcal{L}_2}\right)\right] k_B T_0}, \tag{40}$$

so the noise factor

$$F_{2-||,\,coh} = \frac{(P_{in,1} + P_{in,2})\left[1 + \left(\frac{F_1 G_1 - 1}{2\mathcal{L}_1} + \frac{F_2 G_2 - 1}{2\mathcal{L}_2}\right)\right]}{\frac{1}{2Z}\left|\sqrt{\frac{G_1}{2\mathcal{L}_1}}e^{j\theta_1} v_{in,1} + \sqrt{\frac{G_2}{2\mathcal{L}_2}}e^{j\theta_2} v_{in,2}\right|^2}. \tag{41}$$

By direct extension from the 2-input case, we can write the expressions for SNRs and noise factor for the general case of M inputs.

Based on the above argument, the input SNR in the general case is

$$SNR_{in,\,coh} = \frac{1}{k_B T_0}\sum_{m=1}^{M}\frac{v_{in,m}^2}{2Z} \tag{42}$$

where Z is the load impedance. The output SNR is

$$SNR_{out,\,coh} = \frac{\dfrac{1}{2Z}\left|\displaystyle\sum_{m=1}^{M}\sqrt{\frac{G_m}{M\mathcal{L}_m}}e^{j\theta_m}v_{in,m}\right|^2}{\left[1+\displaystyle\sum_{m=1}^{M}\left(\frac{F_mG_m-1}{M\mathcal{L}_m}\right)\right]k_BT_0}, \tag{43}$$

and the noise factor for M coherent parallel input channels is

$$F_{M-||,\,coh} = \frac{\left(\displaystyle\sum_{m=1}^{M}v_{in,m}^2\right)\left[1+\displaystyle\sum_{m=1}^{M}\left(\frac{F_mG_m-1}{M\mathcal{L}_m}\right)\right]}{\left|\displaystyle\sum_{m=1}^{M}\sqrt{\frac{G_m}{M\mathcal{L}_m}}e^{j\theta_m}v_{in,m}\right|^2}. \tag{44}$$

These results should be compared to the corresponding results for incoherent input signals in Eqs.(29)-(32). The main differences are a) the definition of input SNR, of course, and b) the appearance of the phase terms in the denominator of $F_{M-||,coh}$ for which there are no corresponding terms at all in the incoherent case. That is, if the RF phases are not all properly aligned, the denominator in Eq.(44) can approach zero and the noise factor can approach infinity - an undesirable value for F.

Just as we did for the incoherent case, we can test this noise factor expression in some limiting cases.

i) All channels equivalent. That is, for all $1 \leq m \leq M$, $P_{in,m} = P, F_m = F, G_m = G$ and $\mathcal{L}_m = \mathcal{L}$. Then

$$F_{M-||,\,coh} \longrightarrow \frac{M^2}{|Q|^2}\left(\frac{\mathcal{L}}{G}\right)\left[1+\left(\frac{FG-1}{\mathcal{L}}\right)\right] \tag{45}$$

where

$$Q = \sum_{m=1}^{M} e^{j\theta_m} \tag{46}$$

and note that $0 \leq |Q| \leq M$. Equation (45) is an important result. It shows that when all channels are equivalent and when all the RF signals are in phase, $|Q| = M$, then the noise factor for the entire system is independent of both the number of channels and the intrinsic loss of the combiner, and is equal to the noise factor for any one channel (Fig. 11). Recall that we obtained a similar result for the case of equivalent channels in an incoherent

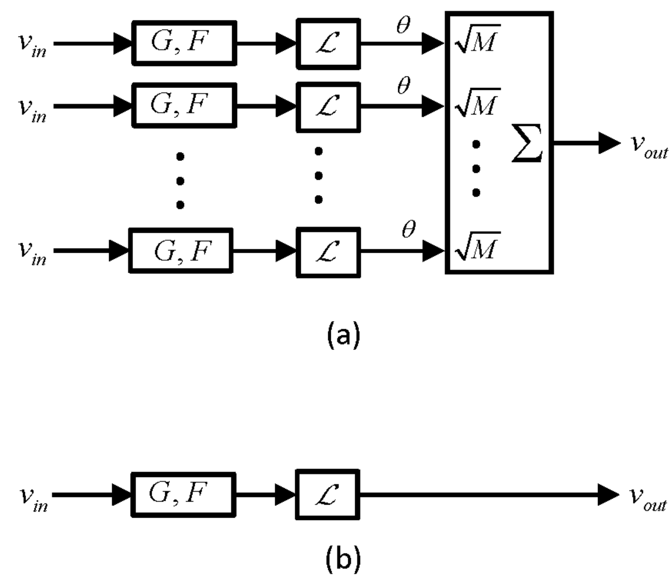


Figure 11: When the corresponding parameters in each channel are equal, so the channels are identical, then the noise factor of the full coherent $M-$channel system (a) is independent of M and equal to the noise factor of the single-channel (b).

system but we emphasize that this result came about with very different definition of input SNR and should not be construed to mean that coherent- and incoherent-input systems are equivalent when all the channels in each case behave equivalently.

To be sure, F does indeed depend on the gain G and insertion loss $\mathcal{L}$ in each channel, but not on the number M of channels or on the intrinsic combiner loss factor $L = M$. *Hence, under conditions that are typical for a beamformer system, increasing the number of channels improves beamformer spatial selectivity but does not improve the system noise factor.*

ii) One channel blocked (say, the i-th channel), all other channels equivalent. Then

$$F_{M-||, coh} \longrightarrow \frac{M^2}{|Q(m \neq i)|^2} \left(\frac{\mathcal{L}}{G}\right) \left[1 + \left(\frac{M-1}{M}\right) \left(\frac{FG-1}{\mathcal{L}}\right)\right] \qquad (47)$$

where $Q(m \neq i)$ indicates to take the sum in Eq.(46) over all m values except $m = i$. Since the largest possible value of $|Q(m \neq i)|$ is $(M-1)$

the prefactor $M^2/\left|Q(m \neq i)\right|^2 \geq 1$ and blocking one channel degrades F in spite of the fact that the multiplier of the second term in brackets is less than one. As in the incoherent case, F suffers a penalty here because the signal, but not the noise, has been removed from one channel.

iii) Only one channel (say, the i-th channel) passes, all other channels equivalent. Then

$$F_{M-||,\,coh} \longrightarrow M^2 \left(\frac{\mathcal{L}}{G}\right) \left[1 + \left(\frac{1}{M}\right)\left(\frac{FG-1}{\mathcal{L}}\right)\right]. \tag{48}$$

Here, as expected, the penalty is severe being proportional to M^2.

iv) No gain, gain-element noise factor, or insertion loss ($G_m = 1, F_m = 1, \mathcal{L}_m = 1,$ for all$1 \leq m \leq M$) but the intrinsic Wilkinson power loss factor M remains for each port. Then

$$F_{M-||,\,coh} \longrightarrow \frac{M^2}{|Q|^2} \tag{49}$$

which approaches unity as $|Q|$ approaches M.

From these considerations, we believe that Eq.(44) is the proper expression for the noise factor of a system comprising M coherent input signals that are subsequently summed in a Wilkinson-type combiner to form a single output.

At this point, the alert reader may ask, how does one produce the M coherent signals in a configuration such as the one shown in Fig. 1(b)? In almost all cases, multiple coherent signals arise from the division of a signal from a single source. This is exactly how a beamformer, which relies on multiple, coherent input signals, would be tested in the laboratory. A system comprising M coherent input signals derived from a single RF source is analyzed in the next section.

5.4.2 Multiple Coherent Inputs derived from a Single RF Source

In this section we analyze the configuration in Fig. 1 (c) comprising a single RF source that is first divided in amplitude along M separate paths and then recombined. The difference between configurations (b) and (c) in Fig. 1 is a practical one, namely, do we wish to specify the noise factor when we have physical access to the single source providing the multiple coherent signals (system (c)), or when we do not have such access (system (b)). Unlike the coherent and incoherent cases discussed above, here there is absolutely no ambiguity about the input SNR. It is simply $SNR_{in} = (v_{in}^2/2Z)/(k_B T_0)$. Also, this configuration employs a Wilkinson device on the front end to divide the source signal and so both the intrinsic divider loss factors M and any insertion loss factors $\mathcal{L}'$ of the front-end splitter must be taken into account. But just like the previous two cases, noise at each of the inputs to the combiner is mutually incoherent and will be treated as we did previously.

We will dispense with a preliminary analysis using a 2-path configuration and simply present the expressions for the general $M-$path configuration in Fig. 1(c).

The output signal power is

$$P_{out} = \frac{|v_{out}|^2}{2Z} = \frac{\left|\sum_{m=1}^{M} \sqrt{\frac{G_m}{M^2 \mathcal{L}'_m \mathcal{L}_m}} e^{j\theta_m}\right|^2 v_{in}^2}{2Z}. \tag{50}$$

Compared to the expression for the previous coherent system, here there is another factor of M in the denominator of the term inside the square root and the appearance of the new insertion loss factors $\mathcal{L}'$, both arising from the signal splitter on the system's front end.

The output noise PSD is

$$N_{out} = \left[1 + \sum_{m=1}^{M} \left(\frac{F_m G_m - 1}{M \mathcal{L}_m}\right)\right] k_B T_0, \tag{51}$$

and thus the noise factor is

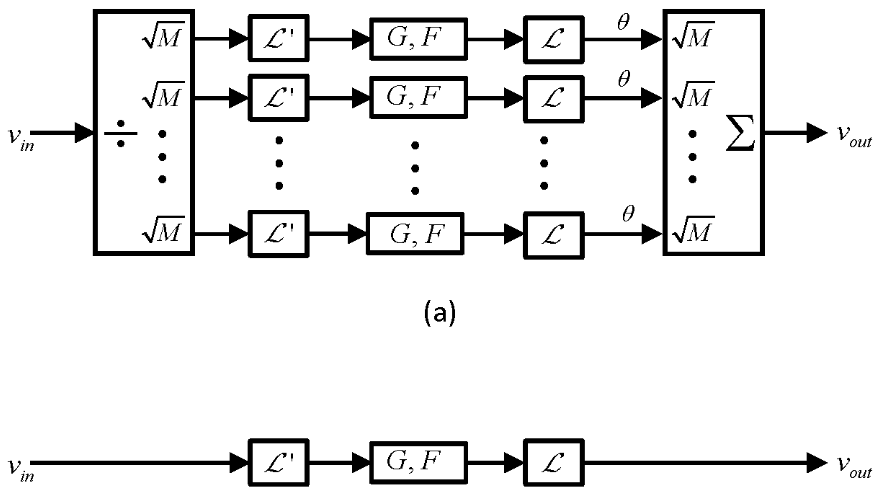
$$F_{M-||,\,cohS} = \frac{\left[1 + \sum_{m=1}^{M} \left(\frac{F_m G_m - 1}{M \mathcal{L}_m}\right)\right]}{\left|\sum_{m=1}^{M} \sqrt{\frac{G_m}{M^2 \mathcal{L}'_m \mathcal{L}_m}} e^{j\theta_m}\right|^2} \tag{52}$$

where the notation $M-||$, $cohS$ indicates M parallel coherent channels all derived from a single RF source.

As we have done previously, we now test this expression in various limits.

i) All channels equivalent. That is, for all $1 \leq m \leq M$, $F_m = F, G_m = G$ and setting $\mathcal{L}_m = \mathcal{L}$ and $\mathcal{L}'_m = \mathcal{L}'$,

$$F_{M-||,\,coh} \longrightarrow \frac{M^2}{|Q|^2} \left(\frac{\mathcal{L}'\mathcal{L}}{G}\right) \left[1 + \left(\frac{FG-1}{\mathcal{L}}\right)\right] \tag{53}$$



(a)

(b)

Figure 12: As with the coherent system in Fig. 11, when the corresponding parameters in each channel are equal, so the channels are identical, then the noise factor of the full coherent $M-$channel system (a) is independent of M and equal to the noise factor of the single-channel (b).

where, as before,

$$Q = \sum_{m=1}^{M} e^{j\theta_m} \tag{54}$$

and note that $0 \leq |Q| \leq M$ This is the expression analogous to the multiple-input coherent case Eq.(45) but here includes the effect of both the intrinsic loss M and the insertion loss $\mathcal{L}'$ of the front-end power divider. But just as in Eq. (45), it yields the same important result that, when all channels are equivalent and when all the RF signals are in phase, $|Q| = M$, then the noise factor for the entire system is independent of both the number of channels and the intrinsic loss of the combiner, and is equal to the noise factor for any one channel. That is, under the stated conditions, the noise factor of the systems (a) and (b) shown in Fig. 12 are equal.

ii) One channel blocked (say, the i-th channel), all other channels equivalent. Then

$$F_{M-||,\,coh} \longrightarrow \frac{M^2}{|Q(m \neq i)|^2}\left(\frac{\mathcal{L}'\mathcal{L}}{G}\right)\left[1+\left(\frac{M-1}{M}\right)\left(\frac{FG-1}{\mathcal{L}}\right)\right] \tag{55}$$

where, as before, $Q(m \neq i)$ indicates to take the sum in Eq.(46) over all m values except $m = i$. Again this is the expression analogous to Eq.(47).

iii) Only one channel (say, the i-th channel) passes, all other channels equivalent. Then

$$F_{M-||,\,coh} \longrightarrow M^2\left(\frac{\mathcal{L}'\mathcal{L}}{G}\right)\left[1+\left(\frac{1}{M}\right)\left(\frac{FG-1}{\mathcal{L}}\right)\right] \tag{56}$$

is the expression analogous to Eq.(48).

iv) No gain, gain-element noise factor, or insertion loss ($G_m = 1, F_m = 1, \mathcal{L}_m = 1, \mathcal{L}'_m = 1$ for all $(1 \leq m \leq M)$ but the intrinsic Wilkinson power loss factor M remains for each port. Then

$$F_{M-||,\,coh} \longrightarrow \frac{M^2}{|Q|^2} \tag{57}$$

which approaches unity as Q approaches M.

From these considerations, we believe that Eq.(52) is the proper expression for the noise factor of a system comprising M coherent input signals derived from a single RF source, split into M parallel channels by a Wilkinson-type divider, and subsequently summed in a Wilkinson-type combiner to produce a single output.

6 SUMMARY AND DISCUSSION

Expressions for the input and output SNRs and the noise factors for the three configurations are summarized in Fig. 13. Fig. 14 is a summary of the noise factors in various limiting cases. In Fig. 15 we plot the noise factor as a function of the number M of channels for three different gain factors $G = 1$, 2, 1000 and for the limiting cases of one channel blocked and all-but-one channel blocked. We assumed fixed values $F = 1.3\ dB$, $\mathcal{L} = \mathcal{L}' = 1.5\ dB$ for all channels. At any G value, the noise factor is practically independent of M for $M > 10$ or so.

In summary, we have calculated the noise factor for three major system types having multiple, parallel input channels: a)multiple incoherent inputs;

b) multiple coherent inputs; and c) multiple coherent inputs derived from a single, accessible source. System types b) and c) correspond to beamformer systems while the incoherent case a) was included for completeness. The practical distinction between c) and b) lies mainly in whether or not the single RF source giving rise to multiple, coherent signals is accessible experimentally.

We showed that, for the purposes of calculating noise factor, the definition of input SNR depends on whether the RF input signals are coherent or incoherent. For incoherent input signals, $SNR_{in} =$ (average input power)/$k_B T_0$ while for coherent input signals, $SNR_{in} =$ (total input power)/$k_B T_0$. For a system utilizing coherence of the input signals, the output signal and, hence the noise factor, depend strongly on the relative phases of the signals at the point of summation. In the worst case, the signals interfere destructively, the output signal power goes to zero, and the noise figure becomes infinite. In the optimum case, the signals interfere constructively

We also showed that for beamformer-like configurations, which are clearly of the coherent-input type, increasing the number of receive-antenna elements typically does not improve the system noise figure although it does improve spatial selectivity of the array.

	SNRin	SNR out	Noise Factor F
Incoherent	$\frac{1}{M}\sum_{m=1}^{M}\frac{P_{in,m}}{k_B T_0}$	$\frac{\sum_{m=1}^{M}\frac{G_m}{M\mathcal{L}_m}P_{in,m}}{\left[1+\sum_{m=1}^{M}\left(\frac{F_m G_m-1}{M\mathcal{L}_m}\right)\right]k_B T_0}$	$\frac{\left(\sum_{m=1}^{M}P_{in,m}\right)\left[1+\frac{1}{M}\sum_{m=1}^{M}\left(\frac{F_m G_m-1}{\mathcal{L}_m}\right)\right]}{\left(\sum_{m=1}^{M}\frac{G_m}{\mathcal{L}_m}P_{in,m}\right)}$
Coherent	$\frac{1}{k_B T_0}\sum_{m=1}^{M}\frac{v_{in,m}^2}{2Z}$	$\frac{\frac{1}{2Z}\left\lvert\sum_{m=1}^{M}\sqrt{\frac{G_m}{M\mathcal{L}_m}}e^{j\theta_m}v_{in,m}\right\rvert^2}{\left[1+\sum_{m=1}^{M}\left(\frac{F_m G_m-1}{M\mathcal{L}_m}\right)\right]k_B T_0}$	$\frac{\left(\sum_{m=1}^{M}v_{in,m}^2\right)\left(1+\sum_{m=1}^{M}\left(\frac{F_m G_m-1}{M\mathcal{L}_m}\right)\right)}{\left\lvert\sum_{m=1}^{M}\sqrt{\frac{G_m}{M\mathcal{L}_m}}e^{j\theta_m}v_{in,m}\right\rvert^2}$
Coherent Single – Source	$\frac{1}{k_B T_0}\frac{v_{in}^2}{2Z}$	$\frac{\frac{1}{2Z}\left\lvert\sum_{m=1}^{M}\sqrt{\frac{G_m}{M^2\mathcal{L}'_m\mathcal{L}_m}}e^{j\theta_m}\right\rvert^2 v_{in}^2}{\left[1+\sum_{m=1}^{M}\left(\frac{F_m G_m-1}{M\mathcal{L}_m}\right)\right]k_B T_0}$	$\frac{\left[1+\sum_{m=1}^{M}\left(\frac{F_m G_m-1}{M\mathcal{L}_m}\right)\right]}{\left\lvert\sum_{m=1}^{M}\sqrt{\frac{G_m}{M^2\mathcal{L}'_m\mathcal{L}_m}}e^{j\theta_m}\right\rvert^2}$

Figure 13: General expressions for performance metrics for each of the three configurations shown in Fig.1 assuming M channels.

	Identical Channels	One Channel Blocked	All-but-one Blocked	No G, L, or Noise
Incoherent	$\left(\frac{\mathcal{L}}{G}\right)\left[1+\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	$\left(\frac{\mathcal{L}}{G}\right)\left[\frac{M}{(M-1)}+\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	$\left(\frac{\mathcal{L}}{G}\right)\left[M+\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	1
Coherent	$\frac{M^2}{\lvert Q\rvert^2}\left(\frac{\mathcal{L}}{G}\right)\left[1+\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	$\frac{M^2}{\lvert Q_{m\neq i}\rvert^2}\left(\frac{\mathcal{L}}{G}\right)\left(1+\frac{(M-1)}{M}\left(\frac{FG-1}{\mathcal{L}}\right)\right)$	$\left(\frac{\mathcal{L}}{G}\right)\left(M^2+M\left(\frac{FG-1}{\mathcal{L}}\right)\right)$	$\frac{M^2}{\lvert Q\rvert^2}\geq 1$
Coherent Single Source	$\frac{M^2}{\lvert Q\rvert^2}\left(\frac{\mathcal{L}'\mathcal{L}}{G}\right)\left[1+\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	$\frac{M^2}{\lvert Q_{m\neq i}\rvert^2}\left(\frac{\mathcal{L}'\mathcal{L}}{G}\right)\left[1+\frac{(M-1)}{M}\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	$\left(\frac{\mathcal{L}'\mathcal{L}}{G}\right)\left[M^2+M\left(\frac{FG-1}{\mathcal{L}}\right)\right]$	$\frac{M^2}{\lvert Q\rvert^2}\geq 1$

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Figure 14: Summary of the noise factors in certain limiting cases.

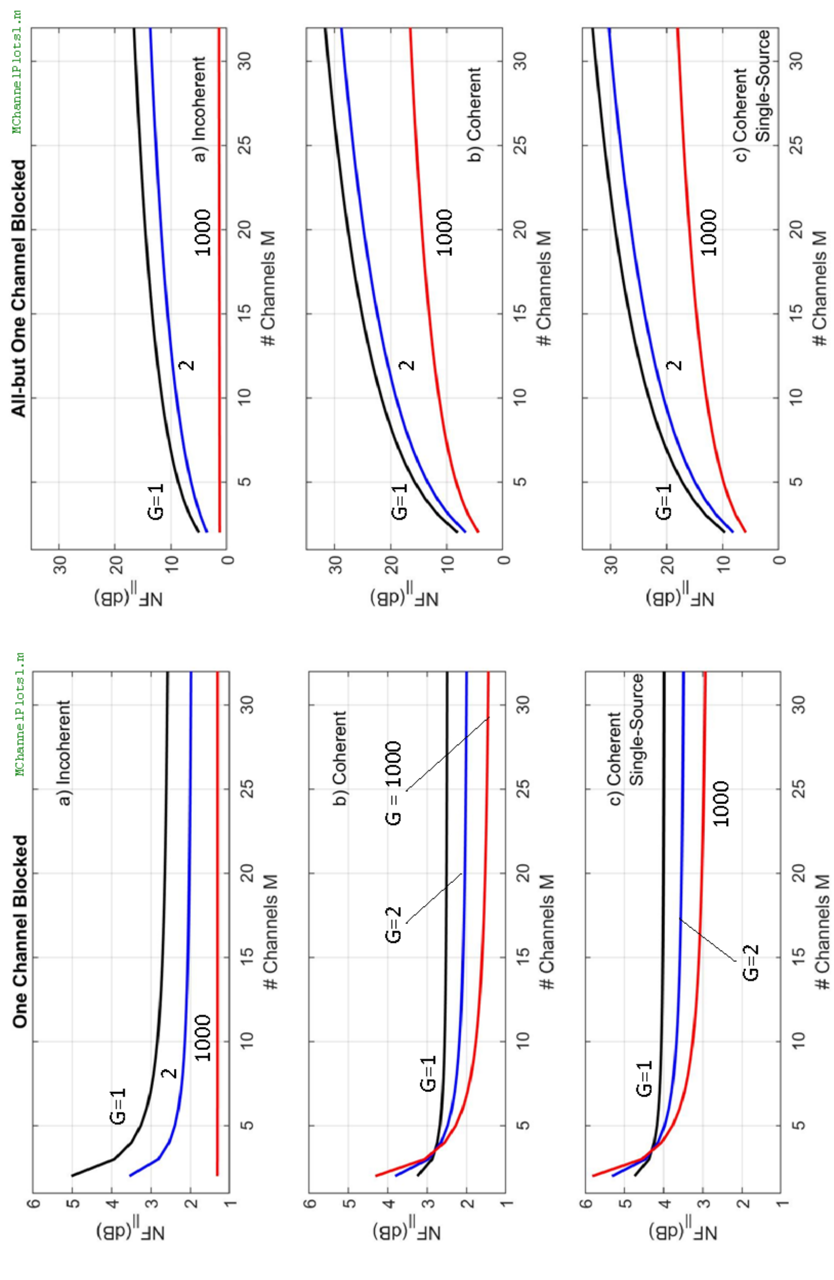


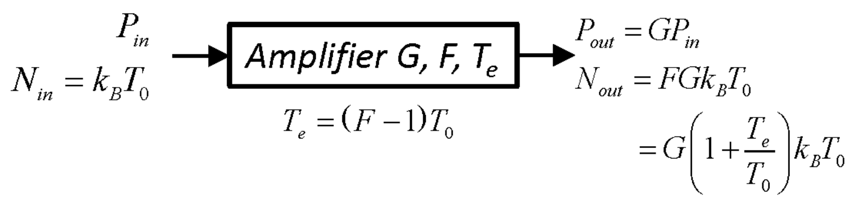
Figure 15: Parallel noise figure $F_{||}$ (dB) versus number of channels (M) for various gain values for the three configurations in Fig. 1. All plots assume the channels are otherwise identical with $F = 1.3\ dB$, $\mathcal{L} = \mathcal{L}' = 1.5\ dB$ and, for the coherent cases, that the signals are all completely in phase.

A APPENDIX A: SERIAL CASCADE NOISE FACTORS

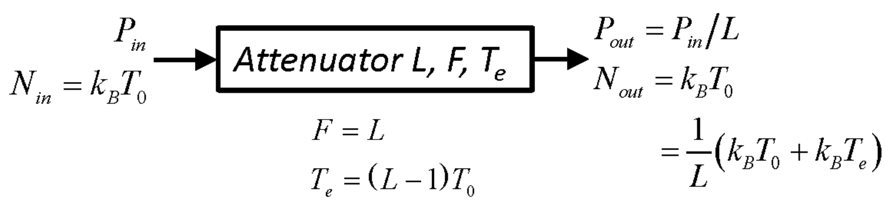
In this appendix we summarize well-known expressions for SNR and noise factor for a few different configurations. We also provide some details of the derivations leading to the expressions. The configurations are:

A) Gain Element,
B) Loss Element,
C) Gain Elements in Series,
D) Loss Elements in Series,
E) Loss + Gain in Series,
F) Gain + Loss in Series,
G) Gain + Loss + Gain in Series,
H) Loss + Gain + Loss in Series.

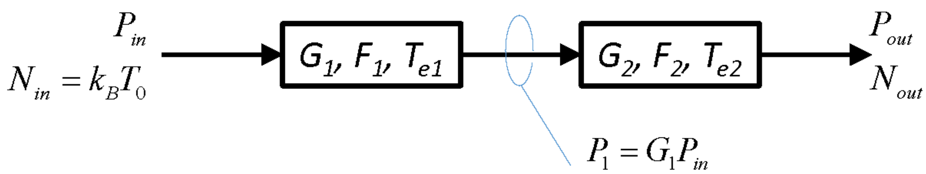
A) Gain Element *G = power gain factor*



B) Loss Element *L = power loss factor*



C) Gain Elements in Series



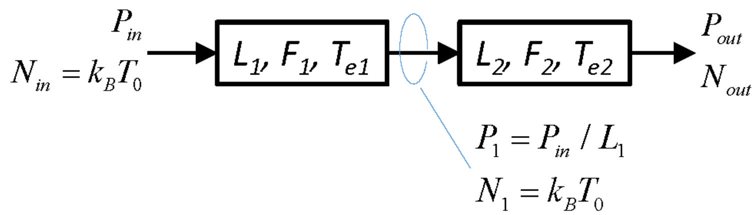
$$P_1 = G_1 P_{in}$$
$$N_1 = G_1 k_B \left(T_0 + T_{e1}\right) = F_1 G_1 k_B T_0$$

$$P_{out} = G_2 G_1 P_{in}$$
$$N_{out} = G_2 N_1 + G_2 k_B T_{e2} = \left(G_2 F_1 G_1 + G_2 \left(F_2 - 1\right)\right) k_B T_0$$

$$F_{series} = \frac{P_{in}}{k_B T_0} \frac{\left(G_2 F_1 G_1 + G_2 \left(F_2 - 1\right)\right) k_B T_0}{G_2 G_1 P_{in}} = F_1 + \frac{F_2 - 1}{G_1}$$

For multiple gain elements in series ... $F_{series} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_2 G_1} + \frac{F_4 - 1}{G_3 G_2 G_1} + \cdots$

D) Loss Elements in Series L_m = *power loss factors*



$$P_{out} = P_{in} / L_1 L_2$$
$$N_{out} = k_B T_0$$

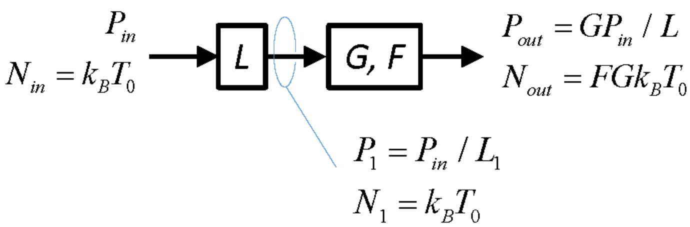
$$F_{series} = \frac{P_{in}}{k_B T_0} \frac{k_B T_0}{P_{in} / L_1 L_2} = L_1 L_2$$

Obtain the same result using the cascade equation for noise factors and setting $G = 1 / L$:

$$F_{series} = F_1 + \frac{F_2 - 1}{G_1} = L_1 + L_1 \left(L_2 - 1 \right) = L_1 L_2 .$$

For multiple loss elements in series ... $F_{series} = L_1 L_2 L_3 \cdots$

E) Loss + Gain in Series



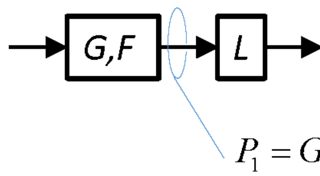
$$F_{series} = \frac{P_{in}}{k_B T_0} \frac{FGk_B T_0}{GP_{in} / L} = LF$$

Same result using the cascade equation for noise factors :

$$F_{series} = L + \frac{F-1}{1/L} = LF.$$

F) Gain + Loss in Series

$$P_{in}$$
$$N_{in} = k_B T_0$$



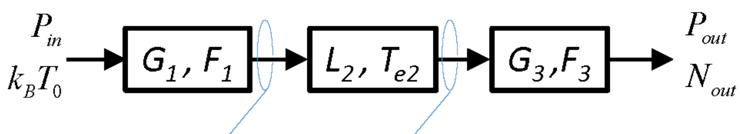
$$P_{out} = GP_{in} / L$$
$$N_{out} = \frac{FG}{L} k_B T_0 + \frac{L-1}{L} k_B T_0$$

$$P_1 = GP_{in}$$
$$N_1 = FGk_B T_0$$

$$F_{series} = \frac{P_{in}}{k_B T_0} \frac{\left(\frac{FG}{L} + \frac{L-1}{L} \right) k_B T_0}{GP_{in} / L} = F + \frac{L-1}{G}$$

Exactly the same result as using the cascade equation for noise factors.

G) Gain + Loss + Gain in Series



$$P_1 = G_1 P_{in}$$
$$N_1 = F_1 G_1 k_B T_0$$

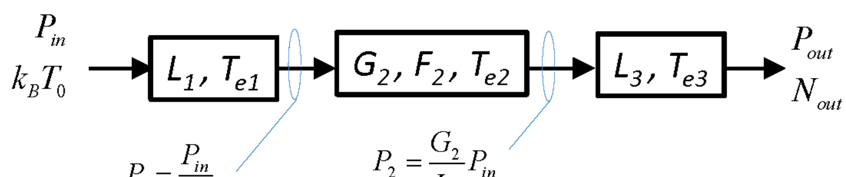
$$P_2 = \frac{G_1}{L_2} P_{in}$$
$$N_2 = \frac{F_1 G_1}{L_2} k_B T_0 + \frac{L_2 - 1}{L_2} k_B T_0$$

$$P_{out} = \frac{G_3 G_1}{L_2} P_{in}$$
$$\begin{aligned} N_{out} &= G_3 N_2 + G_3 k_B T_{e3} \\ &= G_3 \left(\frac{F_1 G_1}{L_2} + \frac{L_2 - 1}{L_2} + (F_3 - 1) \right) k_B T_0 \end{aligned}$$

$$\begin{aligned} F_{series} &= \frac{P_{in}}{\frac{G_3 G_1}{L_2} P_{in}} \frac{G_3 \left(\frac{F_1 G_1}{L_2} + \frac{L_2 - 1}{L_2} + (F_3 - 1) \right) k_B T_0}{k_B T_0} \\ &= \frac{L_2}{G_1} \left(\frac{F_1 G_1}{L_2} + \frac{L_2 - 1}{L_2} + (F_3 - 1) \right) \\ &= F_1 + \frac{L_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 / L_2} \end{aligned}$$

exactly the cascade formula result.

H) Loss + Gain + Loss in Series



$$P_1 = \frac{P_{in}}{L_1} \qquad P_2 = \frac{G_2}{L_2} P_{in}$$

$$N_1 = k_B T_0 \qquad N_2 = F_2 G_2 k_B T_0$$

$$P_{out} = \frac{G_2}{L_3 L_1} P_{in}$$

$$N_{out} = \frac{N_2}{L_3} + \frac{L_3 - 1}{L_3} k_B T_0 = \left(\frac{F_2 G_2}{L_3} + \frac{L_3 - 1}{L_3} \right) k_B T_0$$

$$F_{series} = \frac{P_{in}}{\frac{G_2}{L_3 L_1} P_{in}} \frac{\frac{1}{L_3}\left(F_2 G_2 + (L_3 - 1)\right) k_B T_0}{k_B T_0}$$

$$= L_1 \left(F_2 + \frac{L_3 - 1}{L_3} \right) \qquad = F_2 L_1 + \frac{L_3 - 1}{G_2 / L_1}$$

And the cascade formula gives $F_{series} = L_1 + \frac{F_2 - 1}{1/L_1} + \frac{L_3 - 1}{G_2 / L_1}$

$$= F_2 L_1 + \frac{L_3 - 1}{G_2 / L_1} \ .$$

B APPENDIX B: GENERAL SCATTERING MATRIX FOR A WILKINSON DIVIDER/COMBINER

This Appendix contains the general scattering matrix representation for an ideal Wilkinson power splitter/combiner for which there are no reflections from any port [10]. For an $N-$port device, the scattering matrix is the square $N \times N$ matrix

$$S = \frac{j}{\sqrt{N-1}} \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{58}$$

The transpose $S^T = S$ so the same matrix is used whether the device is configured as a combiner or as a splitter.

A 2×1 Wilkinson combiner is a 3-port device and, therefore, its scattering matrix is

$$S = \frac{j}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{59}$$

C APPENDIX C: LIST OF SYMBOLS

$F =$ Noise factor

$G =$ Gain factor

$k_B =$ Boltzmann's constant

$L =$ Loss factor

$M =$ Number of channels

$MISO =$ Multiple in / single out (systems)

$N =$ Noise power spectral density

$NF =$ Noise figure

$P =$ RF power

$PSD =$ Power spectral density

$Q = \sum_{m=1}^{M} \exp\left(j\theta_m\right)$

$RF =$ Radio-frequency

$SNR =$ Signal-to-noise ratio

$T_e =$ Noise-equivalent temperarture

$T_0 =$ Standard temperature (290K)

$v =$ RF voltage

$Z =$ Impedance

$\mathcal{L}, \mathcal{L}' =$ Insertion loss factor

$\theta_m =$ RF phase

$\Omega =$ RF (radian) frequency

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