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Finding Minimum Values of Distances and Distance Ratios in ATS Spectrum Management

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CAASD

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Abstract

Preventing interference among radio circuits used for air traffic services (ATS) often requires spectrum managers to observe channel-assignment rules based on minimum ratios of the great-circle distances (GCDs) traversed by desired and undesired (potentially interfering) signals. Each circuit operates within a service volume of airspace having a circular or polygonal "footprint" on the surface of a spherical Earth. Minimizing undesired-to-desired GCD ratios with respect to these footprints can be significantly facilitated in many cases through the use of stereographic projection. Instead of performing the minimization using the footprints on the surface of the sphere, a stereographic projection of the footprints to the complex plane is performed to transform the original minimization problem into a simpler problem of minimizing a ratio of distances in the complex plane. This ratio can be expressed in terms of a single real variable and then minimized using the Newton-Raphson method.

There are other assignment rules that require spectrum managers to observe channel assignment rules based on minimum distances between service volume footprints. Finding the minimum GCD distance between service volume footprints on the sphere can be done in an efficient manner using vector analysis, and as a result a closed form solution can be provided enabling the computation to be performed quickly.

Procedures described in this paper have been incorporated into Spectrum Prospector™, an automated tool developed by The MITRE Corporation's Center for Advanced Aviation System Development (CAASD) to perform spectrum analysis studies for the Federal Aviation Administration (FAA).

Acknowledgements

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1 Introduction

The Federal Aviation Administration (FAA) has approximately 530 very high frequency (VHF) radio channels available for assignment to an air traffic services (ATS) radio system comprising more than 7,000 air/ground (A/G) radio circuits. Each circuit operates within a service volume (SV) of airspace having a circular or polygonal "footprint" on the surface of a spherical Earth. Extensive channel reuse is essential to satisfying such a large demand with the resources available. However, preventing interference among the circuits often requires spectrum managers to observe channel-assignment rules based on minimum ratios of the great-circle distances (GCDs) traversed by desired and undesired (potentially interfering) signals. Such ratios are called undesired-to-desired (*u/d*) GCD ratios. In Figure 1-1, G_d and G_u are the GCDs (measured on the surface of the Earth) used as measures of the distances that the desired and undesired signals, respectively, must traverse.

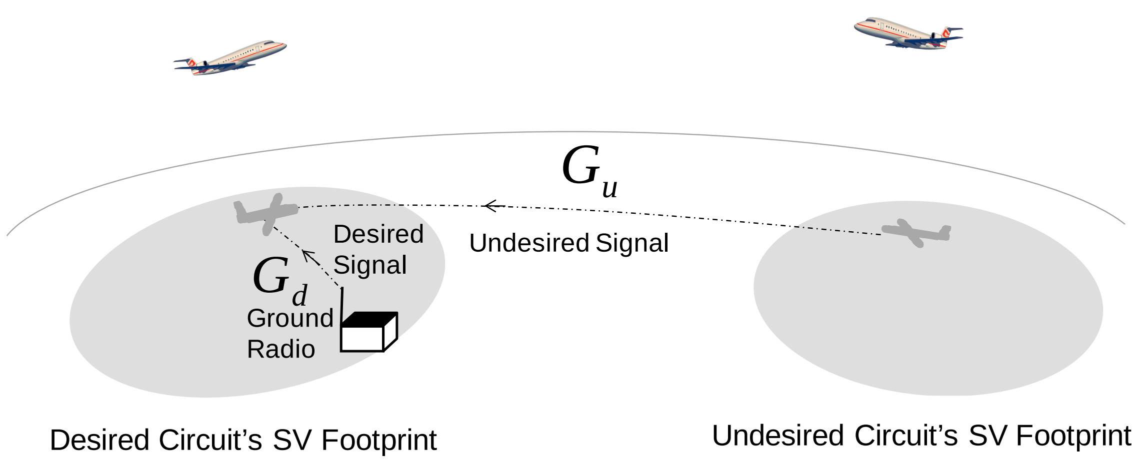

Figure 1-1. Components of *u/d* GCD Ratio

Minimizing *u/d* GCD ratios with respect to these footprints can be significantly facilitated through the use of stereographic projection. Instead of performing the minimization using the footprints on the surface of the sphere, a stereographic projection of the footprints to the complex plane is performed to transform the original minimization problem into a simpler problem of minimizing a ratio of distances in the complex plane. This ratio can be expressed in terms of a single real variable and then minimized using the Newton-Raphson method.

There are other assignment rules that require spectrum managers to consider assignments based on minimum GCD distances between service volume footprints. Finding the minimum GCD distance between service volume footprints on the sphere can be done in an efficient manner using vector analysis, and as a result a closed form solution can be provided enabling the computation to be performed quickly.

Procedures described in this report have been incorporated into Spectrum Prospector™, an automated tool developed by The MITRE Corporation's Center for Advanced Aviation System Development (CAASD) to perform spectrum analysis studies for the Federal Aviation Administration (FAA) [1,2,3,4]. The optimization procedures described in this report are designed to enable Spectrum Prospector to follow distance-based channel assignment rules as accurately as possible without excessive running time.

1.1 Background

In the ATS environment, pilots use airborne radios (ARs) to communicate with their designated ground-based radios (GRs), which are used by air traffic controllers. Each circuit operates within a SV, a three-dimensional volume of airspace whose horizontal cross section at any altitude has the same circular or polygonal "footprint" on the surface of a spherical Earth. Aircraft within the same SV communicate with the same GR on a given channel. The GR does not necessarily lie in the footprint of its SV. Because spectrum is a scarce resource, channels are reused, constrained by certain stringent rules that must be followed to prevent interference between the radios in one circuit and those in other circuits assigned the same channel. The problem of determining whether two circuits can be cochannel, i.e., assigned the same channel, can be posed as an optimization problem involving circles and polygons on the surface of a sphere.

1.2 Approach

Because the *u/d* ratio as it stands is a complicated function (i.e., an inverse trigonometric function) of four variables (the latitudes and longitudes of the two aircraft positions), attempting to find a global minimum in a straightforward manner leads to a very inefficient and cumbersome mathematical procedure. Much of the complication is due to the fact that the ratio is comprised of GCDs on the surface of the sphere. We realized that one possible simplifying step was to project the components involved in the ratio to the complex plane through stereographic projection. Stereographic projection is a viable approach because circles on the sphere are projected to circles in the complex plane, allowing us to retain the simple structure of the circle, which is crucial in simplifying the minimization process. Further, stereographic projection enables expressing the chordal length between points on the sphere in terms of distances between points in the complex plane. Thus, a ratio of chordal lengths can be expressed in this way, and it turns out, minimized in an efficient manner. But in this report we show that the solution to the problem of minimizing the ratio of chordal lengths is also the solution to the actual problem of minimizing the ratio of GCDs.

In terms of finding the minimum GCD between two SVs on the sphere, we were able to find an efficient procedure using vector analysis to obtain a closed-form solution to the problem of finding the minimum GCD from a point on the sphere to an arc of a great circle. Using this result and a feature of intersecting great circles, we were able to find an efficient procedure for finding the minimum GCD between any two SVs on the sphere.

2 Use of Stereographic Projection to Find Minimum Great-Circle Distance Ratios

This section first describes the problem of cochannel interference (CCI), which gives rise to the need for the distance-ratio calculations in this report. It then describes an efficient procedure for finding minimum *u/d* GCD ratios through the use of stereographic projection.

2.1 Cochannel Interference

Figure 2-1 shows an example of a situation where CCI could arise between two A/G radio circuits operating in neighboring SVs. In the example, the desired circuit's AR receives not only the desired signal from its own GR, but also a potentially interfering undesired signal transmitted by the AR of the undesired circuit. This constitutes a case of potential "air-to-air" interference. The desired circuit, of course, can also interfere with the undesired circuit, although interference in that direction is not shown in the figure. Air-to-ground, ground-to-air, or ground-to-ground interference might also occur between the two circuits but is not shown in the figure either. The altitudes of the aircraft above the earth are usually small enough, in relation to the horizontal extent of the SV, that the actual lengths of the signal paths are virtually identical to the associated GCDs.

Whenever dealing with SVs, only a "horizontal" slice at a given altitude is considered, and the boundary of the horizontal slice is projected to lie as a footprint on the sphere as shown in Figure 2-2. Any further reference to SVs will be to their footprints. Also, all distances between any two points in the horizontal slices, regardless of altitude, are measured along the surface of the earth using their footprints as shown in Figure 2-2. In Figure 2-2, G_d and G_u are the GCDs used as measures of the distances that the desired and undesired signals must traverse, respectively. The circuits of Figure 2-2 are close enough for ARs operating at their edges at the maximum allowable altitude to have an unobstructed mutual radio line of sight (RLOS), which in ATS spectrum management is generally regarded as an essential condition for interference.

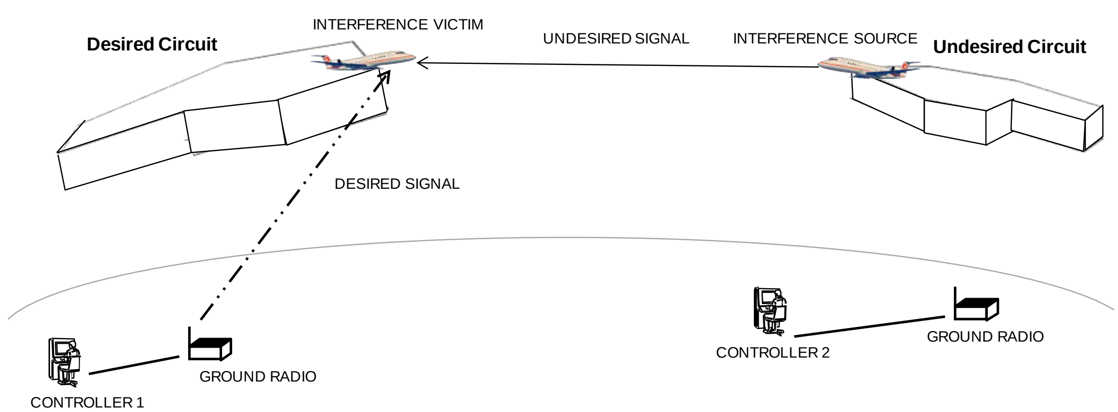

Figure 2-1. Cochannel Interference between A/G Radio Circuits

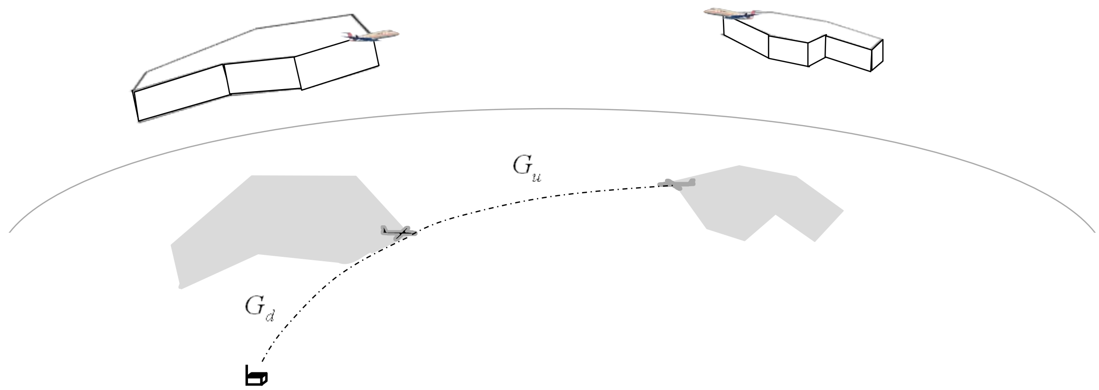

Figure 2-2. Sector Footprints and GCDs

For the problem we are considering, it is often assumed for simplicity that the desired and undesired radio signals have identical output powers, and that free-space propagation laws prevail over both the desired and the undesired signal paths. Under these conditions, the rules for avoiding CCI can be expressed as distance-based channel-assignment constraints.

Prior to the development of the procedure in this report, a worst-case *u/d* GCD ratio was used, because attempting to find the extremum solution was too complex and time-consuming. Figure 2-3 shows the values used to compute this worst-case *u/d* ratio. The denominator $G_{d,\max}$ is the GCD between the GR and the farthest point in the desired SV, and the numerator $G_{u,\min}$ is the minimum GCD between the desired and undesired SVs.

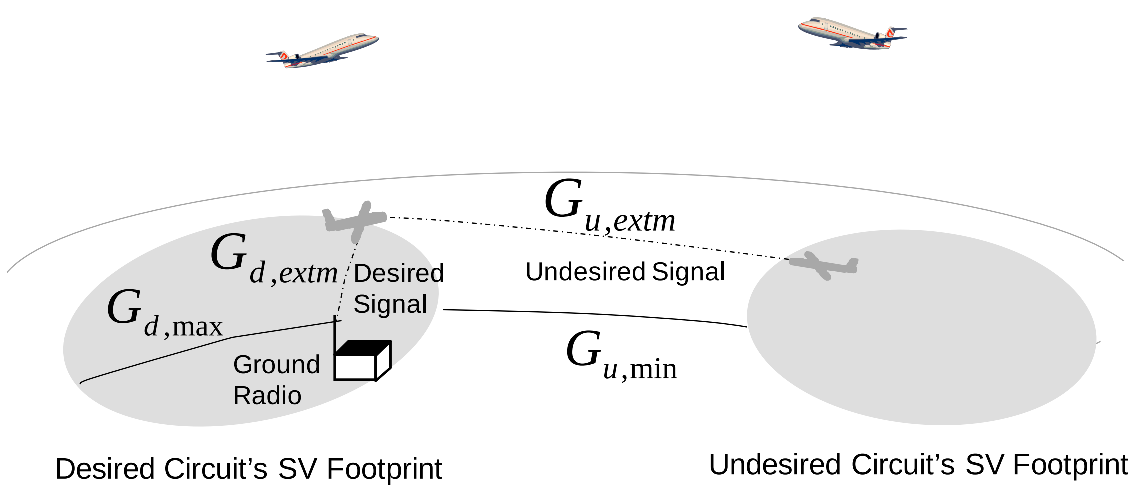

Figure 2-3. Components of Worst-Case *u/d* GCD Ratio

Let $\left(\frac{G_u}{G_d}\right)_{\min}$ be the minimum possible value of the *u/d* ratio, and let $G_{u,extm}$ and $G_{d,extm}$ be the values of G_u and G_d that produce that extremum *u/d* ratio. Then we would have

$$\left(\frac{G_u}{G_d}\right)_{\min} = \frac{G_{u,extm}}{G_{d,extm}} \geq \frac{G_{u,\min}}{G_{d,\max}}$$

Thus, using the worst-case ratio would be a lower bound on the extremum ratio and could result in a less efficient use of spectrum. In the special case when the GR is located at the center of a circular desired SV and the undesired SV is also circular, the minimum *u/d* ratio is equal to the worst-case ratio, and it is easy to calculate. For cases other than this special case, this report describes an efficient procedure for obtaining the true minimum *u/d* ratio.

In present FAA channel-planning practice, no pair of A/G circuits with an unobstructed mutual RLOS may operate on the same channel unless $\left(\frac{G_u}{G_d}\right)_{\min} \geq 5$ for that pair — i.e., unless the victim radio is at least 5.0 times as far from the undesired signal source as from the desired transmitter. Thus, $\left(\frac{G_u}{G_d}\right)_{\min}$ must be calculated to determine its value.

Calculating that ratio on the sphere for each of the millions of circuit pairs that exist in the U.S. ATS system can be a time-consuming process. This report describes a more efficient procedure that transforms the optimization problem on the sphere to an optimization problem in the complex plane through the use of stereographic projection.

Although the results we provide in this report are for the ratio of GCDs, Appendix A extends the results to the case where SV altitudes are incorporated with the GCDs to produce a ratio of "slant ranges."

2.2 Stereographic Projection

Figure 2-4 illustrates a stereographic projection, which is a projection from points on the sphere to points in the complex plane. The projection equations that transform a point (X, Y, Z) on the sphere with radius *R* to a point $z = x + iy$ in the complex plane are [6, 7]:

$$\begin{aligned} x &= \frac{2R}{2R - Z} X \\ y &= \frac{2R}{2R - Z} Y \end{aligned} \qquad (1)$$

The inverse equations from the complex plane to the sphere are:

$$X = \frac{4R^2}{|z|^2 + 4R^2} x$$
$$Y = \frac{4R^2}{|z|^2 + 4R^2} y \qquad (2)$$
$$Z = \frac{2R|z|^2}{|z|^2 + 4R^2}$$

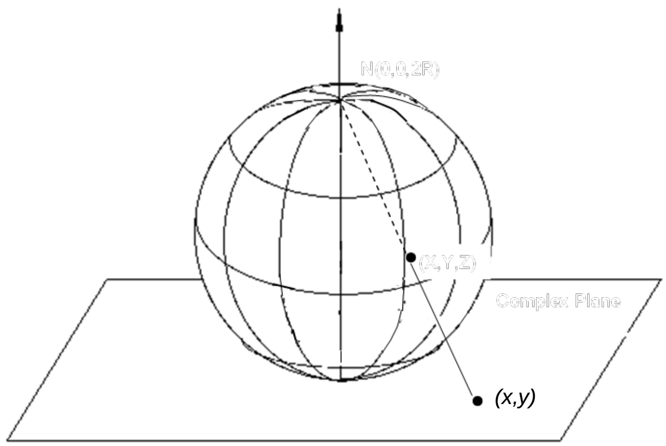

Figure 2-4. Stereographic Projection

2.2.1 Stereographic Projection of Circles and Polygons on the Sphere

It is known that the stereographic projection maps circles on the sphere (s-circles) to circles in the complex plane, and that the interior of an s-circle is mapped to the interior of its projected circle in the complex plane. Appendix B shows the derivation of the equation of a circle in the complex plane that is the stereographic projection of an s-circle on the sphere. These s-circles can be either great s-circles (i.e., those with center at [0,0,*R*]) or small s-circles (i.e., those with centers elsewhere). Projections of great circles contain the origin of the complex plane in their interiors.

We also need to show that the interiors of polygons on the sphere (s-polygons) are projected to the interiors of their projections in the complex plane. Since, in the cases we are considering now, each side of an s-polygon is an arc segment of a great circle on the sphere, the representation in the complex plane of a side is an arc segment of a circle. This is shown in Figure 2-5. An exception to the situation where the side of an s-polygon is projected to an arc of a circle in the complex plane is when the side lies along a meridian, i.e., when the two vertices of the side have the same longitude. In this case the projection of the s-polygon side is a line segment. In this report we deal with the more prevalent case where the side of an s-polygon does not lie along a meridian. The concepts and procedures developed here also have been adapted to the exceptional case.

A great circle partitions a sphere into two hemispheres. We define the interior of a great circle as the sphere's southern hemisphere, and the exterior as its northern hemisphere. Figure 2-5 also shows that any point in the interior (exterior) of a great circle is mapped under stereographic projection to the interior (exterior) of the projection of that great circle in the complex plane. Therefore, for any point on the sphere that is inside an s-polygon, its relationships (i.e., interior or exterior) with respect to the great circles on which the polygon sides lie are maintained with respect to the projected circles of those great circles in the complex plane. This means that the projection of a point on the sphere within an s-polygon will be within the "circular" polygon (a polygon whose sides are circular arc segments) that is the projection of the s-polygon (see Figure 2-6). That s-circles and s-polygons are mapped to circles and circular polygons and that interiors are mapped to interiors are results enabling the application of the Maximum Modulus Principle, which we will encounter later.

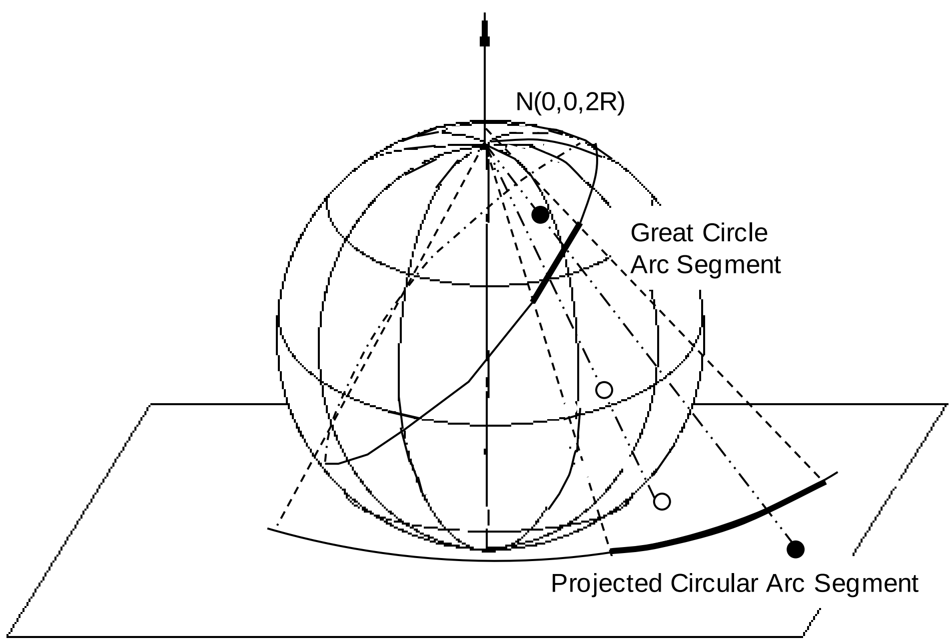

Figure 2-5. Stereographic Projection of a Great Circle Arc

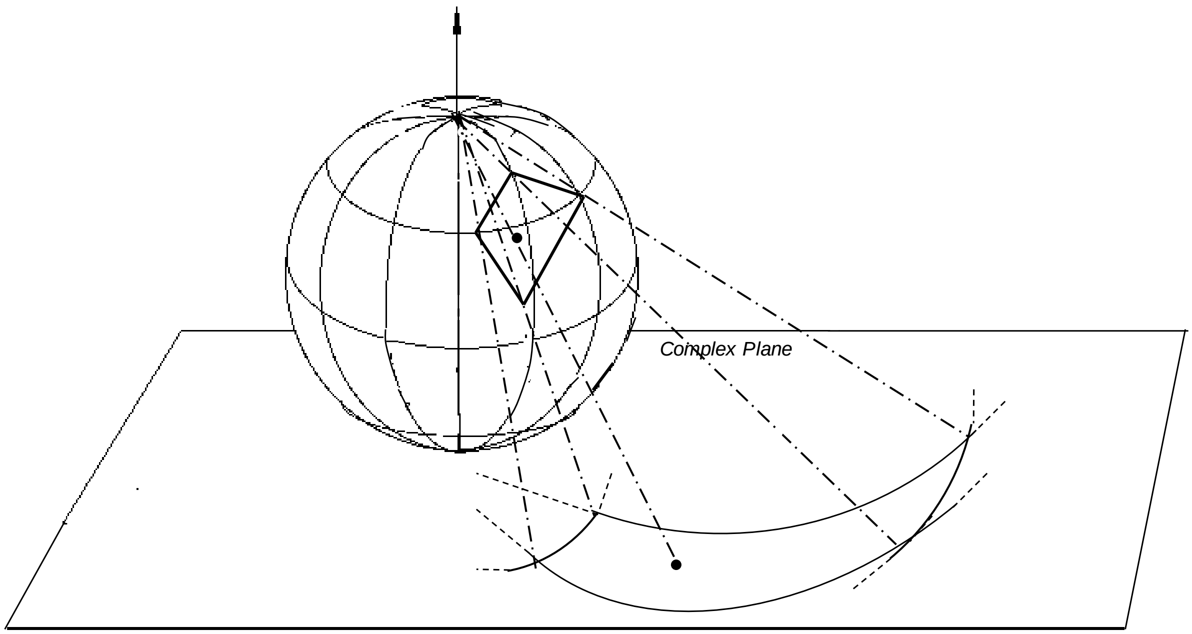

Figure 2-6. Representation of an s-Polygon in the Complex Plane

2.2.2 Computing Chordal Lengths

Useful to our optimization procedure is the computation of chordal length (D), which for any two points (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) on the sphere is defined as

$$D = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2} \qquad (3)$$

It can be shown that D can be computed in the complex plane, using the stereographic projections of the points (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2), as [6, 7]

$$D = \frac{4R^2 |z_1 - z_2|}{\sqrt{4R^2 + |z_1|^2}\sqrt{4R^2 + |z_2|^2}} \qquad (4)$$

This equation will be useful in minimizing the *u/d* ratio in an efficient manner.

2.3 Optimization of the *u/d* Ratio

Working with ratios of chordal lengths rather than ratios of GCDs enables us to develop a more efficient minimization procedure. The first step of our strategy is to show the equivalence of minimizing a *u/d* ratio of GCDs and its corresponding ratio of chordal lengths. The second step is to show that the minimum solution lies on the boundaries of the desired and undesired SVs.

2.3.1 Showing that GCD and Chordal *u/d* Ratios Have the Same Minimum Solution

In the applications we discuss later, the *u/d* ratio is required to be minimized over the entire boundary of a circle, or just over an arc of a circle. These two cases are dealt with separately to show that the *u/d* GCD ratio and the chordal length *u/d* ratio have the same minimum solution in each case.

2.3.1.1 Minimizing Over the Entire Boundary of a Circle

Let $G(P_x, P_y)$ be the GCD between two points P_x and P_y on the sphere. The problem we wish to solve is:

$$\underset{P_u, P_d}{Min} \frac{G(P_u, P_d)}{G(P_d, P_r)} \tag{5}$$

$$P_u = (X_u, Y_u, Z_u), P_d = (X_d, Y_d, Z_d), P_r = (X_r, Y_r, Z_r)$$

where P_u is on the boundary of the undesired SV, P_d is on the boundary of the desired SV, and P_r is the fixed location of the desired ground radio. That the solution of (5) occurs for P_d on the boundary of the desired SV and for P_u on the boundary of the undesired SV will be shown later.

Let $D(P_u, P_d)$ and $D(P_d, P_r)$ be the undesired and desired chordal lengths, respectively. We will show that the solution to the minimization problem:

$$\underset{P_u, P_d}{Min} \frac{D(P_u, P_d)}{D(P_d, P_r)} \tag{6}$$

is also the solution to the minimization problem (5). That (6) implies (5) means that if the ratio of chordal lengths is minimized at some point (P_{u_0}, P_{d_0}), then the ratio of corresponding GCDs is also minimized at (P_{u_0}, P_{d_0}).

We use the following relationship between chordal length and GCD. Let S be the length of the chord connecting the endpoints of an arc on the sphere of radius R with central angle α , so $S = 2R\sin\left(\frac{\alpha}{2}\right)$. The length of the corresponding arc, i.e., the GCD between the endpoints of the arc, is $R\alpha = 2R\sin^{-1}\left(\frac{S}{2R}\right)$. The ratio of GCDs can be expressed as:

$$\frac{G(P_u, P_d)}{G(P_d, P_r)} = \frac{2R\sin^{-1}\left(\frac{D(P_u, P_d)}{2R}\right)}{2R\sin^{-1}\left(\frac{D(P_d, P_r)}{2R}\right)} = \frac{\sin^{-1}\left(\frac{D(P_u, P_d)}{2R}\right)}{\sin^{-1}\left(\frac{D(P_d, P_r)}{2R}\right)} \tag{7}$$

Without loss of generality, we can assume that the sphere has unit radius and that the problem can be simplified by rotating the undesired s-circle so that its center on the sphere is at the South Pole, using the equations in Appendix D. Then its stereographic projection has its center at the origin of the complex plane. Using the same rotation equations, the desired SV and the desired GR location are rotated so that the undesired SV and the desired SV and its GR maintain the same geographical relationships as before the rotations and then are stereographically projected. The stereographic projections of the undesired and desired s-circles or s-polygons are referred to as the undesired and desired circles or polygons, respectively.

The strategy to show that the same pair of points minimizes both ratios is to show that for each fixed point in or on the desired projected s-circle or great circle, the point on the boundary of the undesired projected s-circle or great circle that minimizes the ratio of chordal lengths also minimizes the ratio of GCDs. Using (4) the ratio of chordal lengths can be written as

$$\frac{D(P_u, P_d)}{D(P_d, P_r)} = \frac{4|z_u - z_d|}{\sqrt{4+|z_u|^2}\sqrt{4+|z_d|^2}} \Bigg/ \frac{4|z_d - z_r|}{\sqrt{4+|z_d|^2}\sqrt{4+|z_r|^2}} = \frac{|z_u - z_d|}{|z_d - z_r|}\frac{\sqrt{4+|z_r|^2}}{\sqrt{4+|z_u|^2}} \tag{8}$$

Now we fix z_{d_0} at any point in or on the desired circle, but outside an arbitrarily small disc of radius ε centered at z_r. The reason for excluding this disc is that the ratio approaches infinity as z_{d_0} approaches z_r. Therefore, the minimum for the ratio would occur outside this disc. Assume that z_u is on an arbitrary circle c_u of radius r_u contained within the undesired circle, as shown in Figure 2-7. This arbitrary circle would be the stereographic projection of an arbitrary s-circle sc_u contained within the undesired SV on the sphere.

Then $z_u = |z_u|(\cos(\theta) + i\sin(\theta))$, $|z_u| = r_u \le R_u$, where R_u is the radius of the undesired circle. Since z_{d_0} and z_r are fixed, and since r_u is constant, the ratio simplifies a great deal to

$$\frac{D(P_u, P_{d_0})}{D(P_{d_0}, P_r)} = K_1 |z_u - z_{d_0}| \tag{9}$$

where K_1 is a positive constant equal to $\frac{1}{|z_{d_0} - z_r|} \frac{\sqrt{4 + |z_r|^2}}{\sqrt{4 + r_u^2}}$

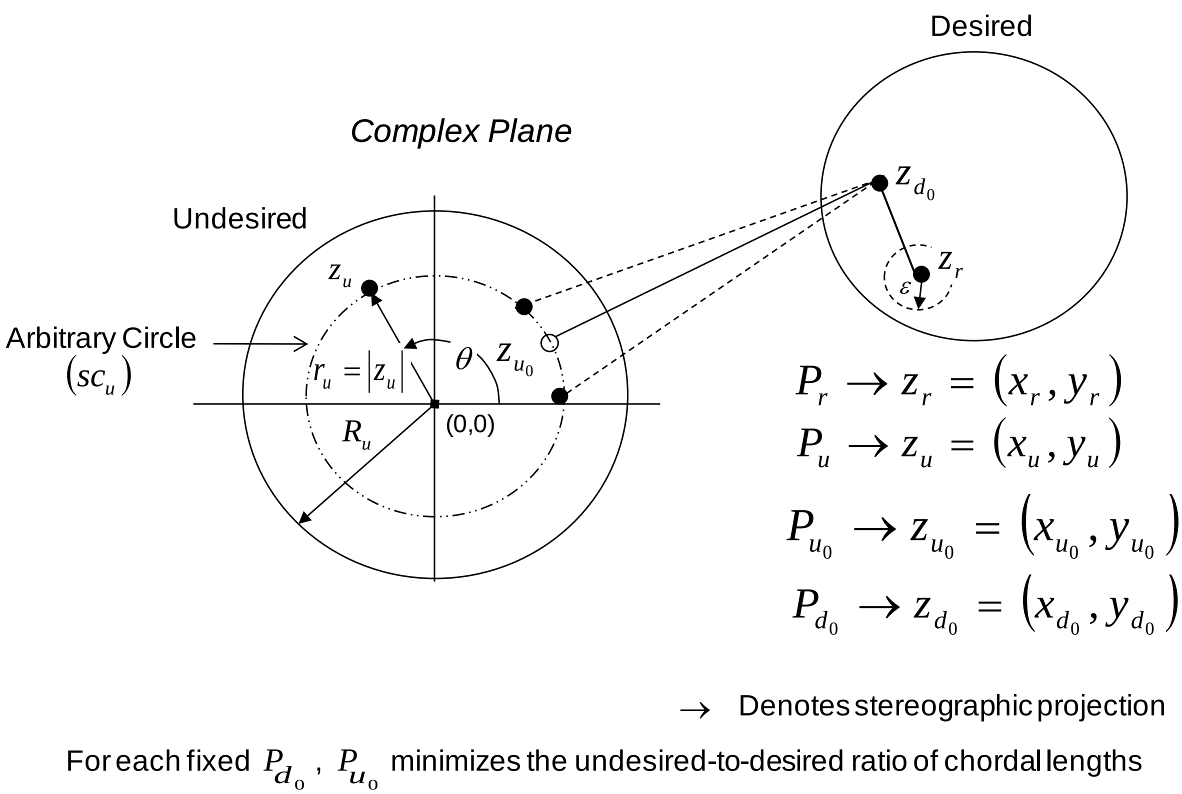

Figure 2-7. Minimizing Ratio of Chordal Lengths

The ratio of chordal lengths can then be expressed in terms of the single variable θ as

$$\frac{D\left(P_u(\theta), P_{d_0}\right)}{D\left(P_{d_0}, P_r\right)} = K_1 \left| z_u - z_{d_0} \right| = K_1 \sqrt{r_u^2 + x_{d_0}^2 + y_{d_0}^2 - 2 r_u x_{d_0} \cos(\theta) - 2 r_u y_{d_0} \sin(\theta)} \quad (10)$$

And because r_u is a constant, the ratio of chordal lengths is simply a constant times a function of θ for fixed P_{d_0} , i.e.,

$$\frac{D\left(P_u(\theta), P_{d_0}\right)}{D\left(P_{d_0}, P_r\right)} = K_1 \left| z_u - z_{d_0} \right| = K_1 f(\theta) \quad (11)$$

The ratio of GCDs can be expressed using (8) and applying (4) to the chordal length in the numerator as

$$\frac{\sin^{-1}\left(D\left(P_u(\theta), P_{d_0}\right) \big/ 2 \right)}{\sin^{-1}\left(D\left(P_{d_0}, P_r\right) \big/ 2 \right)} = K_2 \sin^{-1}\left(D\left(P_u(\theta), P_{d_0}\right) \big/ 2 \right) = K_2 \sin^{-1}\left(K_3 \left| z_u - z_{d_0} \right| \right)$$
$$= K_2 \sin^{-1}\left(K_3 \sqrt{r_u^2 + x_{d_0}^2 + y_{d_0}^2 - 2 r_u x_{d_0} \cos(\theta) - 2 r_u y_{d_0} \sin(\theta)} \right) \quad (12)$$

where K_2 and K_3 are positive constants equal to

$$K_2 = \left(\sin^{-1}\left(D\left(P_{d_0}, P_r\right) \big/ 2 \right) \right)^{-1}, \; K_3 = \left(\frac{1}{2} \sqrt{4 + r_u^2} \sqrt{4 + \left| z_{d_0} \right|^2} \right)^{-1} \quad (13)$$

This shows that if the ratio of chordal lengths is $\rho_{chdl}(\theta) = K_1 f(\theta)$, then the ratio of GCDs is $\rho_{GCD}(\theta) = K_2 \sin^{-1}\left(K_3 f(\theta) \right)$

Thus, $\frac{d}{d\theta}\rho_{chdl}(\theta) = K_1 \frac{d}{d\theta} f(\theta) = 0$, when $\frac{d}{d\theta} f(\theta) = 0$.

Also, $\frac{d}{d\theta}\rho_{GCD}(\theta) = \frac{K_2 K_3 \frac{d}{d\theta} f(\theta)}{\sqrt{1 - K_3^2 f^2(\theta)}} = 0$ when $\frac{d}{d\theta} f(\theta) = 0$.

If the extreme point of $\rho_{chdl}(\theta)$ is a minimum, then the extreme point of $\rho_{GCD}(\theta)$ is also a minimum. This is a result of applying the first derivative test, and the fact that K_1, K_2, and K_3 are positive constants. Therefore the value of θ that minimizes the *u/d* ratio of chordal lengths on an arbitrary circle of radius r_u within the undesired circle also minimizes the *u/d* ratio of GCDs on that arbitrary circle. This value of θ depends on P_{d_0} and on an arbitrary circle of radius r_u. We can write it as $\theta(P_{d_0})$, suppressing the notation for its dependence on the arbitrary s-circle. Since the same value of $\theta(P_{d_0})$ minimizes $\rho_{chdl}(\theta)$ and $\rho_{GCD}(\theta)$ and since P_{d_0} is an arbitrary point, we can say that $\underset{P_u, P_d}{Min} \frac{D(P_u, P_d)}{D(P_d, P_r)}$ and $\underset{P_u, P_d}{Min} \frac{G(P_u, P_d)}{G(P_d, P_r)}$ have the same solution for P_d in or on the desired s-circle and P_u on the undesired arbitrary s-circle sc_u. Because the circle within the undesired circle is arbitrary, we can say that $\underset{P_u, P_d}{Min} \frac{D(P_u, P_d)}{D(P_d, P_r)}$ and $\underset{P_u, P_d}{Min} \frac{G(P_u, P_d)}{G(P_d, P_r)}$ have the same solution for any points in or on the desired and undesired s-circles.

2.3.1.2 Minimizing Over an Arc

Now we find the minimum *u/d* ratio over an arc of a circle, instead of the whole circle as previously. This arises when the undesired SV is an s-polygon. Thus, let us consider that the desired SV is an s-circle as previously and that the undesired SV is now an s-polygon. Recall that each side of an s-polygon is an arc of a great circle on the sphere, and its projection is the arc of a circle in the complex plane. We perform the rotations described previously for an entire circle so the great circle is rotated to the equator, and its projection is a circle centered at the origin of the complex plane.

In this case the solution of $\frac{d}{d\theta} f(\theta) = 0$, as discussed previously for the entire circle, may fall outside of the undesired arc in the complex plane. Under this condition, we want to show that $\underset{P_u \,\varepsilon\, Arc, P_d}{Min} \frac{D(P_u, P_d)}{D(P_d, P_r)}$ and $\underset{P_u \,\varepsilon\, Arc, P_d}{Min} \frac{G(P_u, P_d)}{G(P_d, P_r)}$ have the same solution, since this would allow us to solve the simpler problem of minimizing the ratio of chordal lengths.

To prove this we use the following result, depicted in Figure 2-8, that from any point z_{d_0} external to a circle in the complex plane, the closest point on the boundary of that circle to the external point is the intersection with the circle of a line connecting the external point and the center of the circle. Thus, the result is that $\underset{\theta, |z_u(\theta)|=\eta}{Min} \left| z_u(\theta) - z_{d_0} \right| = \left| z_{u_0} - z_{d_0} \right|$ where $z_u(\theta_0) = z_{u_0}$ and $\left| z_{u_0} - z_{d_0} \right| = \left| z_{d_0} \right| - \left| z_{u_0} \right|$. This latter equality expresses the condition that z_{u_0}, z_{d_0}, and the center of the circle are collinear. The solution z_{u_0} that minimizes the distance will also minimize the ratio of chordal lengths $\rho_{chdl}(\theta) = K_1 \left| z_u(\theta) - z_{d_0} \right|$ for the same z_{d_0} since $\rho_{chdl}(\theta)$ is a positive constant multiple K_1 of the distance. In addition, z_{u_0} will also minimize the ratio of GCDs, $\rho_{GCD}(\theta) = K_2 \sin^{-1}\left(K_3 \left| z_u(\theta) - z_{d_0} \right|\right)$, for the same z_{d_0} where K_2 and K_3 are positive constants, due to the fact that $\rho_{GCD}(\theta)$ increases as $K_3 \left| z_u(\theta) - z_{d_0} \right|$ increases.

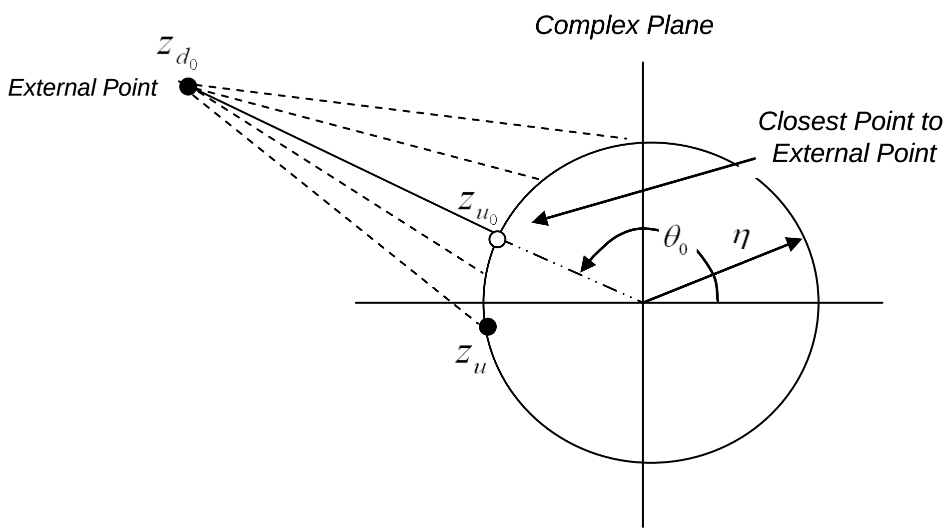

Figure 2-8. Closest Point on Circle to External Point

Another very useful result is that the distance from the circle to the external point monotonically increases as a point $z_u(\theta)$ on the circle moves away in either direction from $z_{u_0} = z_u(\theta_0)$, the closest point to the external point. This also implies that both *u/d* ratios, $\rho_{chdl}(\theta)$ and $\rho_{GCD}(\theta)$, monotonically increase as θ moves away from θ_0.

This result suffices to prove that if the arc on the undesired circle does not contain the location of the global minimum solution over all $\theta, 0 \le \theta \le 2\pi$ for a fixed P_{d_0}, then it must occur at one of the vertices, and it must be the vertex that is nearest the location of the global minimum solution. In Figure 2-9, the minimum occurs at $P_{vertex,1}$.

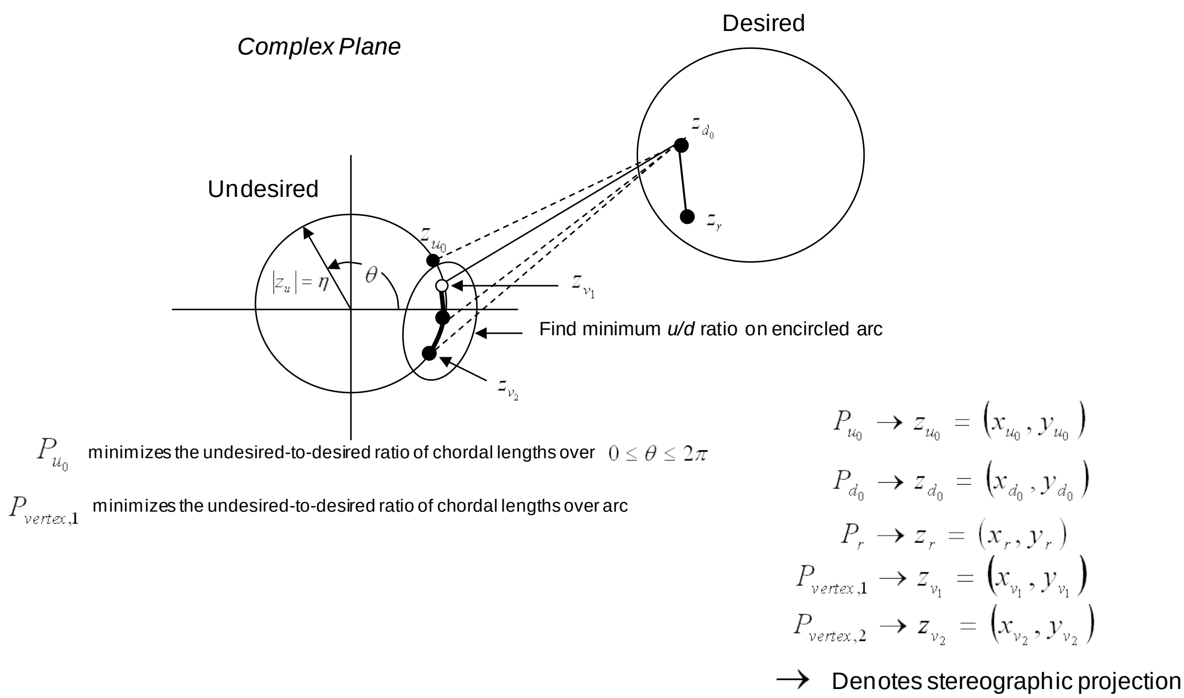

Figure 2-9. Minimum Distance Ratio Over an Arc

Note that the above proof shows that the same point optimizes the two ratios considered above *in* or on the boundary of the desired circle. This is important since, when we show that the optimal solution for the ratio of chordal lengths must occur on the boundary of the desired circle, it must also be true of the ratio of GCDs.

2.3.2 Proving that Minimum *u/d* Ratio of GCDs Occurs on the Boundary of the Desired SV

Now we prove that the optimal solution must occur on the boundary of the desired SV. In all of the proofs above we have assumed that P_{d_0} can occur anywhere in or on the boundary of the desired SV. We now prove that it must occur on the boundary of the desired SV for a fixed location on the boundary of the undesired SV. (It suffices to consider a point on the boundary of the undesired SV due to Section 2.3.3.) Because we showed in the previous section the equivalence of minimizing the *u/d* ratio of GCDs and chordal lengths, we will prove the result for a ratio of chordal lengths, which is a much easier task.

To begin, we stereographically project the SVs under consideration to the complex plane. Using the result of Section 2.3.1, we need only consider the following ratio of chordal lengths using (4):

$$\frac{D(P_u,P_d)}{D(P_d,P_r)}=\frac{\dfrac{4R^2|z_u-z_d|}{\sqrt{4R^2+|z_u|^2}\sqrt{4R^2+|z_d|^2}}}{\dfrac{4R^2|z_d-z_r|}{\sqrt{4R^2+|z_d|^2}\sqrt{4R^2+|z_r|^2}}} \tag{14}$$

$$=\frac{|z_u-z_d|\sqrt{4R^2+|z_r|^2}}{|z_d-z_r|\sqrt{4R^2+|z_u|^2}}$$

Since the term $\sqrt{4R^2+|z_r|^2}$ is a constant due to z_r being the fixed location of the desired radio, this implies that if we wish to minimize the *u/d* ratio over a region in the complex plane, we should solve the problem

$$\underset{z_u,z_d}{Min}\frac{|z_u-z_d|}{|z_d-z_r|\sqrt{4R^2+|z_u|^2}} \tag{15}$$

For a fixed point z_{u_0} of the projected undesired SV, we have to determine if

$$\underset{z_d}{Min}\frac{|z_{u_0}-z_d|}{|z_d-z_r|\sqrt{4R^2+|z_{u_0}|^2}} \tag{16}$$

occurs on the boundary of the projected desired SV in the complex plane, which would greatly simplify the problem. The variable z_d must be considered over the boundary and interior of the projected desired s-polygon or s-circle. It would be convenient to use the Minimum Modulus Principle [8] to show that the minimum occurs on the boundary and not in the interior, but the Minimum Modulus Principle applies only to analytic functions (i.e., differentiable functions of a complex variable). The function to be minimized is not analytic because the aircraft can be anywhere in the desired s-polygon or s-circle, and thus it can be directly over the desired radio site, which means that $|z_d - z_r|$, the denominator, can be zero if the radio site is within the SV. To resolve this problem, we can invert the expression and find

$$\underset{z_d}{Max} \frac{|z_d - z_r| \sqrt{4R^2 + |z_{u_0}|^2}}{|z_{u_0} - z_d|} \tag{17}$$

Since $\sqrt{4R^2 + |z_{u_0}|^2}$ is a positive constant, we denote it by *K* so that the maximization problem becomes

$$\underset{z_d}{Max}\, K \frac{|z_d - z_r|}{|z_d - z_{u_0}|} = K\, \underset{z_d}{Max} \frac{|z_d - z_r|}{|z_d - z_{u_0}|} \tag{18}$$

and thus *K* can be effectively ignored in the maximization. The function

$$g(z) = \frac{z - z_r}{z - z_{u_0}} \tag{19}$$

is an analytic function in and on the projection of the desired s-polygon or s-circle. Therefore, the Maximum Modulus Principle [8] can be used, which says that

$$\underset{z}{Max}|g(z)| = \underset{z}{Max} \frac{|z - z_r|}{|z - z_{u_0}|} \tag{20}$$

occurs on the boundary of the projection of the desired s-polygon or s-circle.

This proves that for a fixed z_{u_0} in the projection of an undesired s-polygon or s-circle, the maximum of $g(z)$ in and on the desired s-polygon or s-circle occurs on the boundary of the projection of the desired s-polygon or s-circle. Therefore, for fixed z_{u_0} the minimum of the *u/d* ratio of chordal lengths occurs on the boundary of the projection of the desired SV and hence on the boundary of the desired s-polygon or s-circle. In addition, for fixed z_{u_0} the *u/d* ratio of GCDs also takes its minimum on the boundary of the desired s-polygon or s-circle, since both ratios have the same minimum solution as shown in Section 2.3.1. Since z_{u_0} was arbitrary, we can state that the *u/d* ratio of GCDs takes its minimum on the boundary of the desired SV.

Appendix A shows that the results of this section also apply to ratios of "slant ranges," mentioned earlier.

2.3.3 Proof that the Minimum *u/d* Ratio of GCDs Occurs on the Boundary of the Undesired SV

We have shown that the minimum *u/d* ratio occurs on the boundary of the desired SV. Now we prove that it also occurs on the boundary of the undesired SV. As in the previous section, we will prove that the result for the *u/d* ratio of chordal lengths, and we will also work with the inverted *u/d* ratio or *d/u* ratio. We will prove the result first for an s-circle and then for an s-polygon.

2.3.3.1 Proof that the Minimum *u/d* (Maximum *d/u*) Distance Ratio of Chordal Lengths Occurs on the Boundary of the Undesired s-Circle

We begin by letting P_{d_0} be any fixed point of the desired SV. We use the equations in Appendix D to rotate the undesired s-circle so that its center is at the South Pole. To maintain the same geographical relationships with the undesired s-circle, P_{d_0} and P_r (the location of the desired GR) undergo the same rotation.

Then we stereographically project the undesired s-circle and desired SV to the complex plane. The *d/u* ratio of chordal lengths can be expressed using (4) and simplified as:

$$\frac{D(P_d, P_r)}{D(P_u, P_d)} = \frac{|z_d - z_r|}{|z_u - z_d|} \frac{\sqrt{4R^2 + |z_u|^2}}{\sqrt{4R^2 + |z_r|^2}} \qquad (21)$$

Let z_{d_0} be the projection of P_{d_0} and z_r be the projection of P_r. We note that z_{d_0} lies outside the projection of the rotated undesired s-circle, as shown in Figure 2-10.

Since z_r is fixed, $\sqrt{4R^2 + |z_r|^2}$ is a constant. We will show that the maximum of the following expression occurs on the boundary of the projection of the undesired s-circle, where z_u is any point in or on the projection of the undesired s-circle:

$$\underset{z_u}{Max} \frac{|z_{d_0} - z_r| \sqrt{4R^2 + |z_u|^2}}{|z_u - z_{d_0}|} \tag{22}$$

As Figure 2-10 shows, for any point $z_{u,\text{int}}$ in the interior of the projection of the undesired s-circle, there is always a point $z_{u,B}$ on its boundary that is closer to the external point z_{d_0}. Thus, we see that $|z_{u,B}| > |z_{u,\text{int}}|$ and $|z_{u,B} - z_{d_0}| < |z_{u,\text{int}} - z_{d_0}|$. It follows then that for any $z_{u,\text{int}}$ there exists a $z_{u,B}$ such that

$$\frac{\sqrt{4R^2 + |z_{u,B}|^2}}{|z_{u,B} - z_{d_0}|} > \frac{\sqrt{4R^2 + |z_{u,\text{int}}|^2}}{|z_{u,\text{int}} - z_{d_0}|}$$

Because z_{d_0} and z_r are fixed, this establishes that the solution of (22) occurs on the boundary of the projection of the undesired s-circle. Since P_{d_0} was picked as any point of the desired SV, thus making z_{d_0} an arbitrary point of the projection of the desired SV, we conclude that the maximum *d/u* ratio or the minimum *u/d* ratio occurs on the boundary of the undesired s-circle.

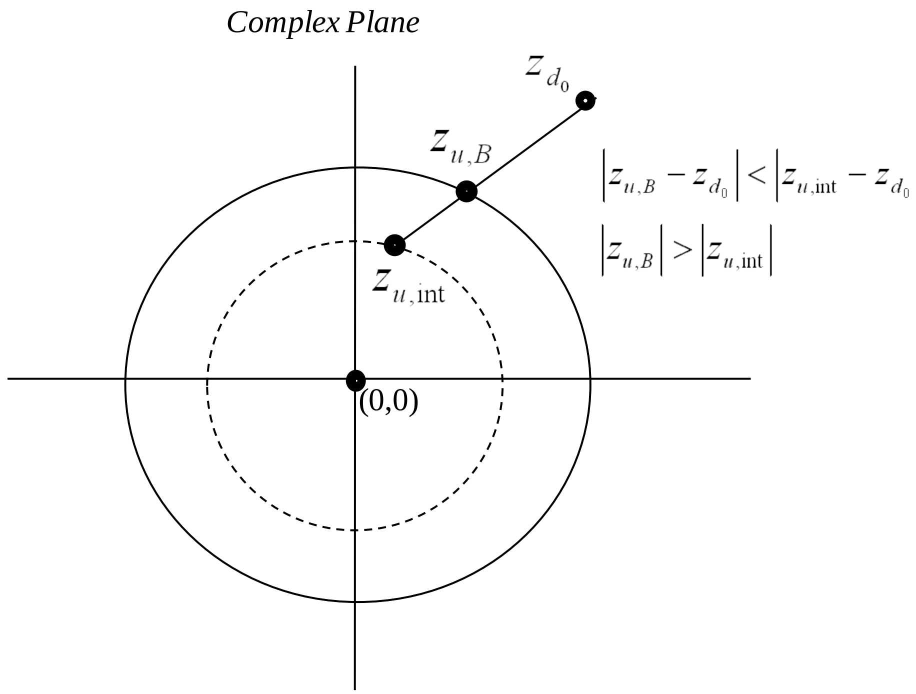

Figure 2-10. Illustration of Proof that Minimum Occurs on Undesired Circle, and Not in its Interior

2.3.3.2 Proof that the Minimum *u/d* (Maximum *d/u*) Distance Ratio of Chordal Lengths Occurs on the Boundary of the Undesired s-Polygon

Now assume that the undesired SV is an s-polygon and that P_{d_0} is the point of the desired SV and P_{u_0} is the point of the undesired SV that minimize the *u/d* ratio of chordal lengths on the sphere. Further assume that P_{u_0} is an interior point of the undesired s-polygon.

Then the great circle that connects P_{d_0} and P_{u_0} must intersect one of the sides of the s-polygon. Let the vertices of this side be P_{v_1} and P_{v_2}. Rotate the great circle on which the s-polygon side lies to the equator so that P_{d_0} is in the northern hemisphere. Apply the same rotation to P_{d_0}, P_{u_0}, and P_r. Then stereographically project the great circle and the points P_{d_0}, P_{u_0} and P_r to the complex plane. The projection of the great circle is a circle centered at the origin in the complex plane and z_{d_0} lies outside the circle. Thus, we obtain the configuration of Figure 2-11.

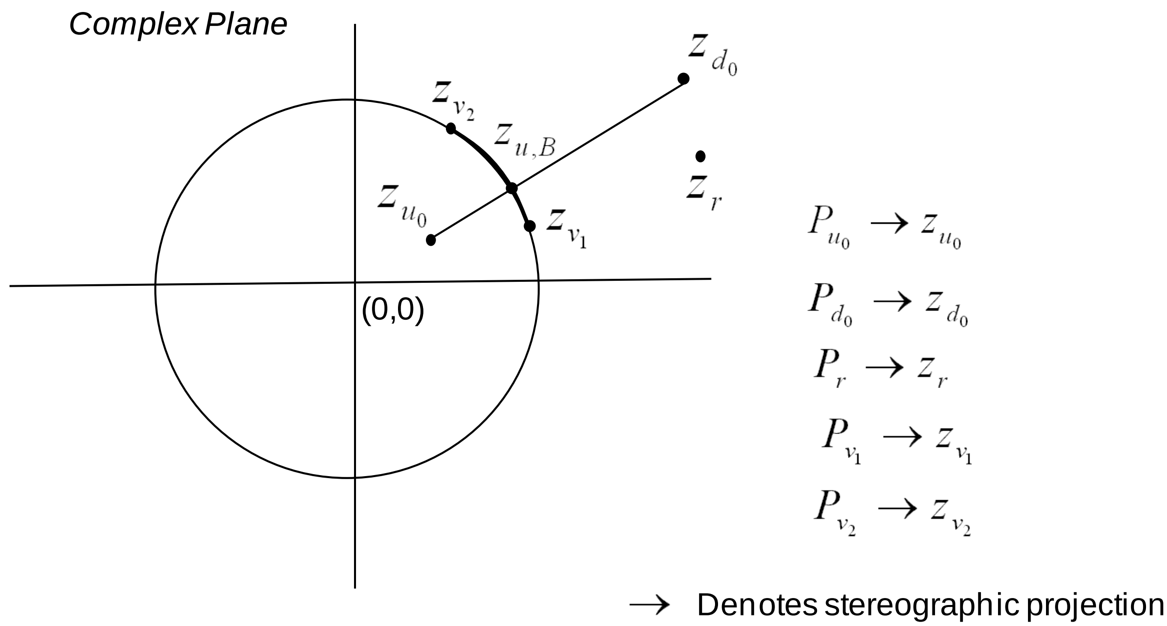

Figure 2-11. Illustration of Proof that Minimum Occurs on Undesired Polygon, and Not in its Interior

We work with the *d/u* ratio as in the previous section, starting with (21). Since z_r is fixed, $\sqrt{4R^2 + |z_r|^2}$ is a constant. Thus, the solution z_{d_0}, z_{u_0} that minimizes the *u/d* ratio would maximize the *d/u* ratio:

$$\frac{|z_{d_0} - z_r|\sqrt{4R^2 + |z_{u_0}|^2}}{|z_{u_0} - z_{d_0}|} \tag{23}$$

Now let $z_{u,B}$ be the intersection of the line connecting z_{u_0} and z_{d_0} with the arc z_{v_1}, z_{v_2}. Using the reasoning of the previous section, $|z_{u,B} - z_{d_0}| < |z_{u_0} - z_{d_0}|$ and $|z_{u,B}| > |z_{u_0}|$, so

$$\frac{|z_{d_0} - z_r|\sqrt{4R^2 + |z_{u,B}|^2}}{|z_{u,B} - z_{d_0}|} > \frac{|z_{d_0} - z_r|\sqrt{4R^2 + |z_{u_0}|^2}}{|z_{u_0} - z_{d_0}|}$$

This contradicts the assumption that z_{u_0} maximizes (23). Therefore, z_{u_0} cannot lie in the interior of the projection of the undesired s-polygon but must lie on its boundary. This implies that P_{u_0} cannot be in the interior of the undesired s-polygon but must be on its boundary. Thus, we conclude that the minimum *u/d* ratio of chordal lengths occurs on the boundary of the undesired s-polygon.

3 Minimizing *u/d* Ratios for Air-to-Air Communications

There are four cases to consider: air-to-air, ground-to-air, air-to-ground, and ground-to-ground. In this section we present only the air-to-air case, which is the most difficult to analyze because neither the interference source nor the interference victim has a fixed location. The concepts and procedures developed for the air-to-air case have been adapted to the other three cases involving the ground radio and are presented in Section 4.0.

Because there are two variable points, the location (latitude and longitude) of the aircraft in the desired service volume and the location of the one in the undesired service volume, the optimization problem can potentially become one of trying to minimize a function of four variables. However, we will show that the problem can always be reduced to minimizing a function of a single variable. In order to convert the four-dimensional minimization problem into a one-dimensional minimization problem, some useful results must first be developed.

The construction shown in Figure 3-1 in the complex plane is useful for our application. When the desired and undesired SVs are s-circles, the minimization is performed over the entire boundaries of both circles. In the complex plane, we define the Desired Containment Arc as the arc on the undesired circle subtended by the central angle of the undesired circle whose sides are tangent to the desired circle. The Desired Containment Arc is so named because the sides of its defining central angle "contain" the desired circle. Appendix C shows how to find the lines forming the central angle of the undesired circle that subtends the Desired Containment Arc. A feature of the Desired Containment Arc is that a line drawn from any selected point on the boundary of the desired circle to the center of the undesired circle will intersect it. The point of intersection is, in fact, the closest point on the undesired circle to the selected point on the desired circle. Any line drawn from anywhere on the boundary of the desired circle to any point on the undesired circle external to this arc will not pass through the center of the undesired circle.

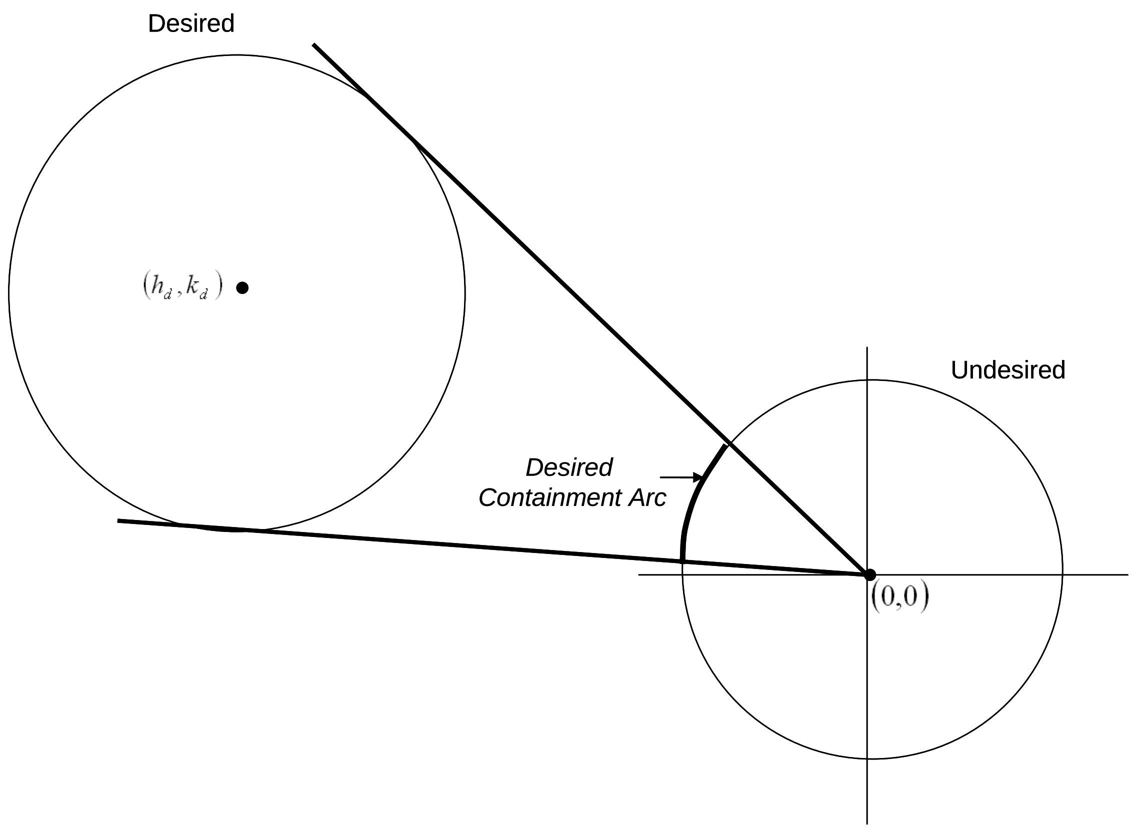

Figure 3-1. Desired Containment Arc

For the case where the desired and undesired service volumes are both s-circles, it was shown previously (see Figure 2-8) that for each fixed point z_{d_0} on the desired circle, the point z_{u_0} on the undesired circle that minimizes the ratio of chordal lengths, lies on the line connecting z_{d_0} with the center of the undesired circle. It is the intersection of this line with the undesired circle. Thus, because the center of the undesired circle is at the origin of the complex plane as shown in Figure 3-1, $|z_u|$ is a constant equal to the radius of the circle. Also, recall that because of collinearity $|z_d - z_u| = |z_d| - |z_u|$. Under these conditions and using (6) and (8), (6) can be simplified to the following minimization problem, on the right hand side, of one complex variable z_d:

$$\underset{z_d, z_u}{Min} \frac{|z_d - z_u|}{|z_d - z_r|} = \underset{z_d}{Min} \frac{|z_d| - |z_u|}{|z_d - z_r|} \tag{24}$$

(Note that using the result of Section 2.3.1, the original problem involving a ratio of GCDs has been reduced to one involving a ratio of distances in the complex plane.). Since z_d lies on a circle, it can be parametrized in terms of the angle formed by the radius of the circle of which it is an endpoint and the horizontal radius to the right of the center (see Figure 3-2).

Thus, minimization problem (24) can be further simplified so that the minimization can be done with respect to a single real variable. Therefore, solving problem (24) is just a matter of minimizing over the angle corresponding to points on the *desired* circle. Note that the angle is now associated with the desired circle rather than with the undesired circle as in our previous discussion. This is further discussed in the following section.

3.1 Two s-Circles

There are two cases to consider in this situation: the desired radio site is at the center of the desired s-circle, and the desired radio site is not at the center of the desired s-circle. This section discusses the latter case. The former case can be performed in a much simpler manner based on finding the minimum GCD between the two s-circles, which is described in Section 5.1.

For the latter case, the undesired s-circle is rotated to the South Pole (see Appendix D) and then projected to the complex plane to obtain the configuration of Figure 3-1. The desired s-circle and the desired radio site location undergo the same rotation, and then the procedure in Appendix B is used to find the center and radius of the desired circle's projection in the complex plane. The transformed desired radio site location is also projected to the complex plane. As just discussed, we need only find the solution to (24) to solve (6), and (5) and (6) have the same solution as shown in Section 2.3.1. Thus, solving (24), involving a ratio of distances, solves (5), involving a ratio of GCDs, which is the problem we set out to solve. Figure 3-2 shows the configuration used to minimize the *u/d* ratio.

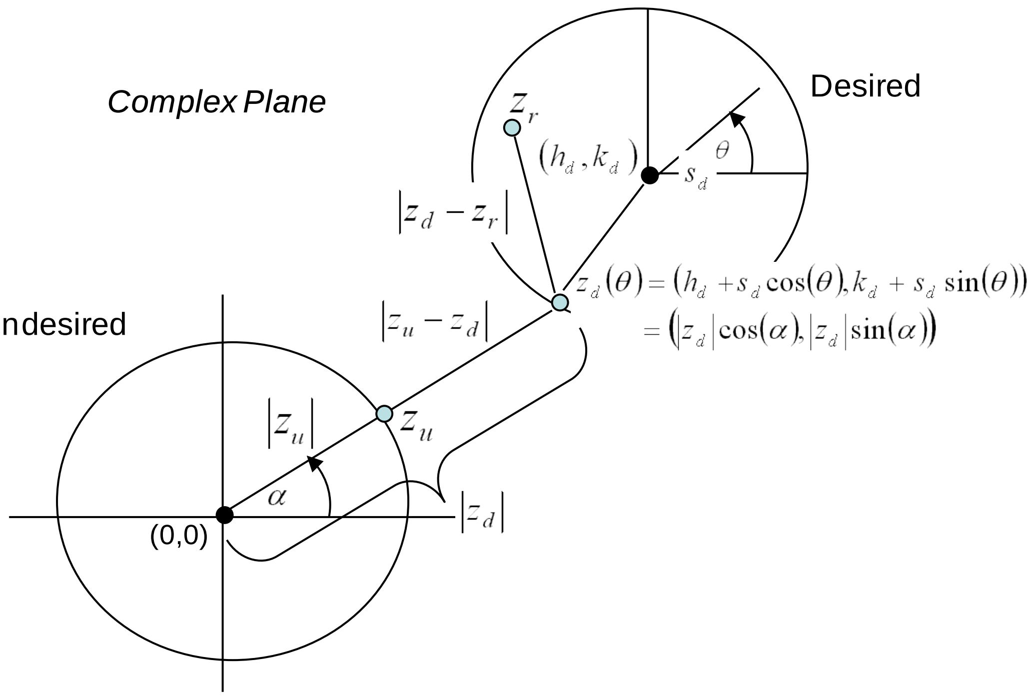

Figure 3-2. Configuration for Air-to-Air *u/d* Ratio Minimization (Two s-Circles)

The *u/d* ratio can then be expressed using (24) as follows:

$$\frac{|z_d - z_u|}{|z_d - z_r|} = \frac{|z_d| - |z_u|}{|z_d - z_r|} = \frac{\sqrt{(h_d + s_d\cos(\theta))^2 + (k_d + s_d\sin(\theta))^2} - |z_u|}{\sqrt{(h_d - x_r + s_d\cos(\theta))^2 + (k_d - y_r + s_d\sin(\theta))^2}}, \quad 0 \le \theta \le 2\pi \tag{25}$$

The single complex variable z_d has been parametrized in terms of the angle θ corresponding to points on the desired circle as shown in equation (25) and in Figure 3-2. As discussed in the previous section, finding the minimum *u/d* ratio over all (z_u, z_d) is just the problem of minimizing (25) with respect to θ. This can be done using the Newton-Raphson method [9]. Thus, the fact that the minimization can be done over a single real variable is a great simplification.

3.2 s-Circle and s-Polygon

In minimizing the *u/d* ratio when dealing with two different types of service volumes, namely an s-circle and an s-polygon, there are two scenarios to consider: (i) the s-polygon is the desired service volume; and (ii) the s-circle is the desired service volume. The procedure for finding the minimum *u/d* ratio for a side of the s-polygon is described below. The minimum ratio is found for each side and the minimum of those minima is the solution for the entire problem.

3.2.1 Desired s-Polygon and Undesired s-Circle

Minimizing the *u/d* ratio in this case is very similar to the two s-circle case covered in Section 3.1. The difference is that instead of having an entire circle for the desired SV, we have the arc of a circle, which is the projection of an s-polygon side, as shown in Figure 3-3. (The desired circle of Figure 3-3 is in actuality the projection of a great circle and thus would include the origin of the complex plane in its interior. Figure 3-3 does not show the desired circle containing the origin of the complex plane in order to more clearly show the details of the desired arc and its associated angles.) In order to determine the θ interval for the arc, the Cartesian coordinates (x_a, y_a) and (x_b, y_b) of the endpoints of the projected side of the s-polygon as shown in Figure 3-3 are used to find the angles $\theta_i, i = a, b$ by means of the following equation:

$$\theta_i = Cos^{-1}\left(\frac{x_i - h_d}{s_d}\right), i = a, b \qquad (26)$$

where s_d is the radius of the desired circle.

As in the two s-circle case, the problem is to minimize the expression shown in (27), which is the same as (25) since collinearity holds for all points on the arc, but now where the angle θ is restricted to lie between the angles θ_a and θ_b.

$$\frac{|z_d - z_u|}{|z_d - z_r|} = \frac{|z_d| - |z_u|}{|z_d - z_r|} = \frac{\sqrt{(h_d + s_d\cos(\theta))^2 + (k_d + s_d\sin(\theta))^2} - |z_u|}{\sqrt{(h_d - x_r + s_d\cos(\theta))^2 + (k_d - y_r + s_d\sin(\theta))^2}}, \quad \theta_a \le \theta \le \theta_b \qquad (27)$$

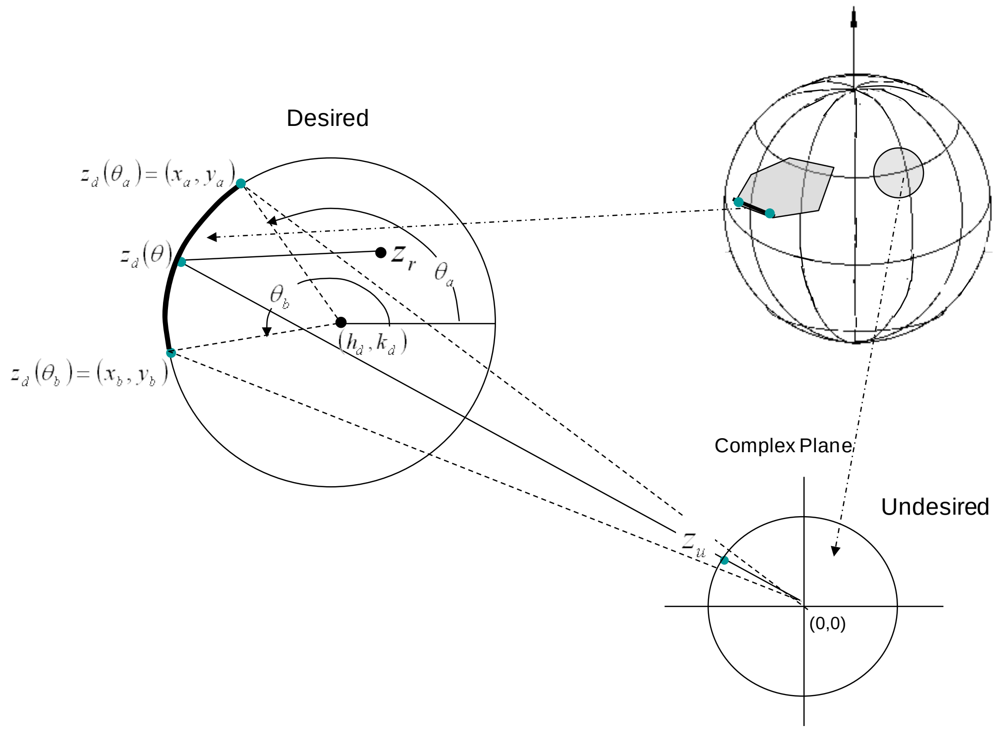

Figure 3-3. Configuration for Air-to-Air *u/d* Ratio Minimization (Desired s-Polygon, Undesired s-Circle)

If the optimal solution to (27) is found to lie outside of the interval $\theta_a \le \theta \le \theta_b$, then as shown previously using Figure 2-9, the optimal solution must be at one of the endpoints, corresponding to either θ_a or θ_b. Each one must be checked to determine which one provides the smaller value for the right hand side of (27).

3.2.2 Desired s-Circle and Undesired s-Polygon

There are two cases to consider in this situation: the desired radio site is at the center of the desired s-circle, and the desired radio site is not at the center of the desired s-circle. This section discusses the latter case. The former case can be performed in a much simpler manner based on finding the minimum GCD distance between the desired s-circle and the undesired s-polygon and is described in Section 5.2.

The case described now is more complicated than the cases discussed in Sections 3.1 and 3.2.1 because the collinearity condition ($|z_d - z_u| = |z_d| - |z_u|$) enabling a simplification of the minimization process is no longer guaranteed for every point on the desired circle. Figures 3-4 and 3-5 show the projections from the sphere to the complex plane of the desired s-circle and one side of the undesired s-polygon. The projection of a side of the undesired s-polygon will be referred to as an “undesired arc.” This undesired arc would be on a circle (a projection of the great circle upon which the s-polygonal side lies) in the complex plane. To center this circle at the origin of the complex plane, the spherical center (see Appendix D) of the great circle on the sphere is rotated to the South Pole, using the rotation equations in Appendix D. The center of the desired circle and the desired radio site location undergo the same rotation before being projected. There are four cases to consider: (i) the undesired arc completely overlays the Desired Containment Arc so that it is equal to it or overextends it on one or both sides; (ii) the entire undesired arc lies on the Desired Containment Arc as a subarc, i.e., neither endpoint of the undesired arc coincides with an endpoint of the Desired Containment Arc; (iii) no portion of the undesired arc lies on the Desired Containment Arc; (iv) the undesired arc lies partly on the Desired Containment Arc.

The first case is easily solved, since a line connecting any point z_d on the desired circle with the origin of the undesired circle would intersect the portion of the undesired arc overlaying the Desired Containment Arc. Thus, there would always be a point z_u on the undesired arc that is collinear with z_d, which together with z_d would minimize the *u/d* ratio. Therefore, the solution is found as in Section 3.1 by minimizing (25).

3.2.2.1 Undesired Arc Lies on the Desired Containment Arc

In order to determine the location of the undesired arc with respect to the Desired Containment Arc (see Figure 3-4), the intersections $(x_i, y_i), i = 1,2$ with the undesired circle of the (dashed) lines through the origin of the complex plane (the undesired circle’s center) and the endpoints of the undesired arc must be found. The method used to find these intersection points is provided in Appendix E. Figure 3-5 shows there are two arcs, highlighted by dashes, of the desired circle for which the collinearity condition can be used. The endpoints of these arcs are found as the intersection points with the desired circle of the sides of the central angle of the undesired circle that subtends the undesired arc. To each endpoint $(x_i, y_i), i = 1,2$ of the undesired arc, there correspond two intersection points on the desired circle $(x_{i,j}, y_{i,j}), j = 1,2$. The method to find these intersection points is provided in Appendix F. The corresponding $\theta_{i,j}, i = 1,2; j = 1,2$ are found using equation (26) with θ_i replaced by $\theta_{i,j}$ and x_i replaced by $x_{i,j}$.

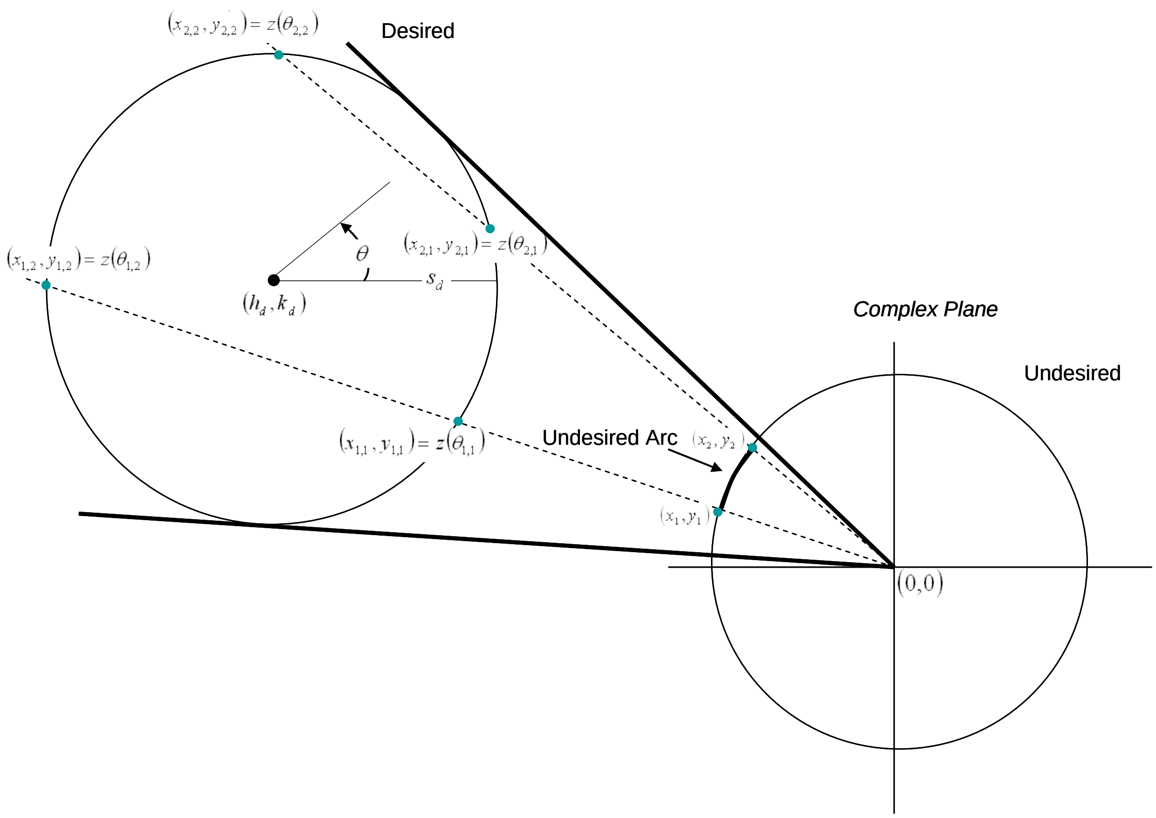

Figure 3-4. Configuration for Air-to-Air *u/d* Ratio Minimization (Desired s-Circle, Undesired s-Polygon) With Undesired Arc Intersections

Since the entire desired circle must be considered, points outside the intervals $(\theta_{1,1}, \theta_{2,1})$ and $(\theta_{2,2}, \theta_{1,2})$ shown in Figure 3-4 must also be considered as viable candidates for minimizing the *u/d* ratio. As shown in Figure 3-5, when the point z_{d_0} falls outside the arcs of the desired circle where the collinearity condition applies, the point on the undesired circle that is collinear with z_{d_0} and minimizes the *u/d* ratio for z_{d_0} is $z_{u_0} = z_u(z_{d_0})$, which is not on the undesired arc. As shown before, the *u/d* ratio for a given z_{d_0} monotonically increases as a point moves away from z_{u_0}, so its minimum value *on* the undesired arc would be attained at $z_{u,vertex,1}$, the closest endpoint of the undesired arc to z_{u_0}.

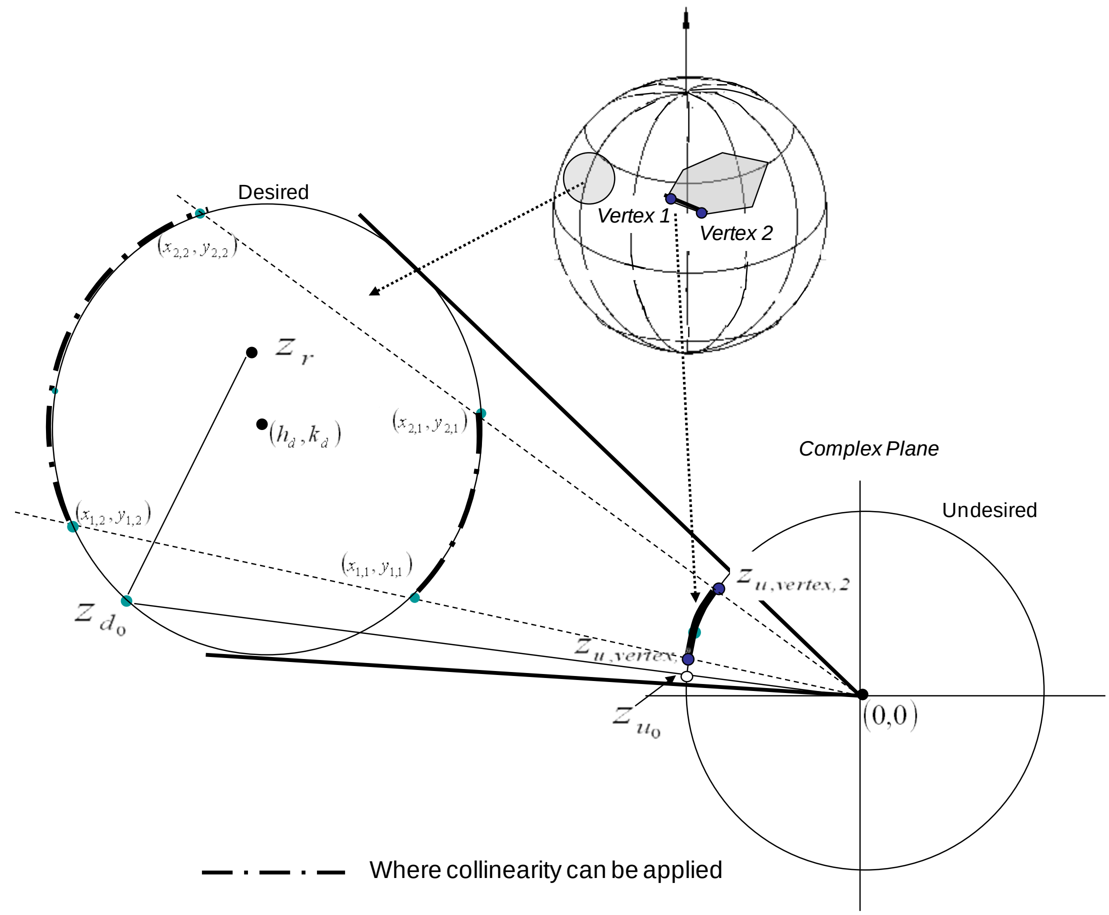

Figure 3-5. Configuration for Air-to-Air *u/d Ratio* Minimization (Desired s-Circle, Undesired s-Polygon) With Collinearity Condition Identification

This leads to the following procedure for finding the solution that minimizes the *u/d* ratio for points of the desired circle whose closest points on the undesired circle do not lie on the undesired arc (i.e., collinearity can't be applied). Referring to Figure 3-5, for points on the desired circle between $(x_{2,1}, y_{2,1})$ and $(x_{2,2}, y_{2,2})$ use $z_{endpt} = z_{u,vertex,2}$ as the endpoint, and for points between $(x_{1,2}, y_{1,2})$ and $(x_{1,1}, y_{1,1})$ use $z_{endpt} = z_{u,vertex,1}$. The optimization problem for these two instances is then formulated as

$$\underset{z_d, z_u \in arc}{Min} \frac{|z_d - z_u|}{|z_d - z_r|} = \underset{z_d}{Min} \frac{|z_d - z_{endpt}|}{|z_d - z_r|} = \underset{\theta}{Min} \frac{\sqrt{(h_d + s_d\cos(\theta) - x_{endpt})^2 + (k_d + s_d\sin(\theta) - y_{endpt})^2}}{\sqrt{(h_d + s_d\cos(\theta) - x_r)^2 + (k_d + s_d\sin(\theta) - y_r)^2}} \tag{28}$$

Thus, again, to find the solution requires minimizing a function of a single variable, which can be done using the Newton-Raphson method.

To summarize the procedure for the case where the entire undesired arc lies on the Desired Containment Arc as a subarc, there are four minimization sub-problems that must be solved: problem (28) over the arc counterclockwise from $(x_{2,1}, y_{2,1})$ to $(x_{2,2}, y_{2,2})$ using $z_{endpt} = z_{u,vertex2}$ and again over the arc counterclockwise from $(x_{1,2}, y_{1,2})$ to $(x_{1,1}, y_{1,1})$ using $z_{endpt} = z_{u,vertex1}$; problem (27) over the highlighted arc from $(x_{1,1}, y_{1,1})$ to $(x_{2,1}, y_{2,1})$ and again over the highlighted arc from $(x_{2,2}, y_{2,2})$ to $(x_{1,2}, y_{1,2})$. The minimum of the solutions from these four sub-problems is then selected as the solution for the polygon side under consideration.

3.2.2.2 Undesired Arc External to the Desired Containment Arc

Now consider the case where the undesired arc lies entirely outside the Desired Containment Arc, as shown in Figure 3-6.

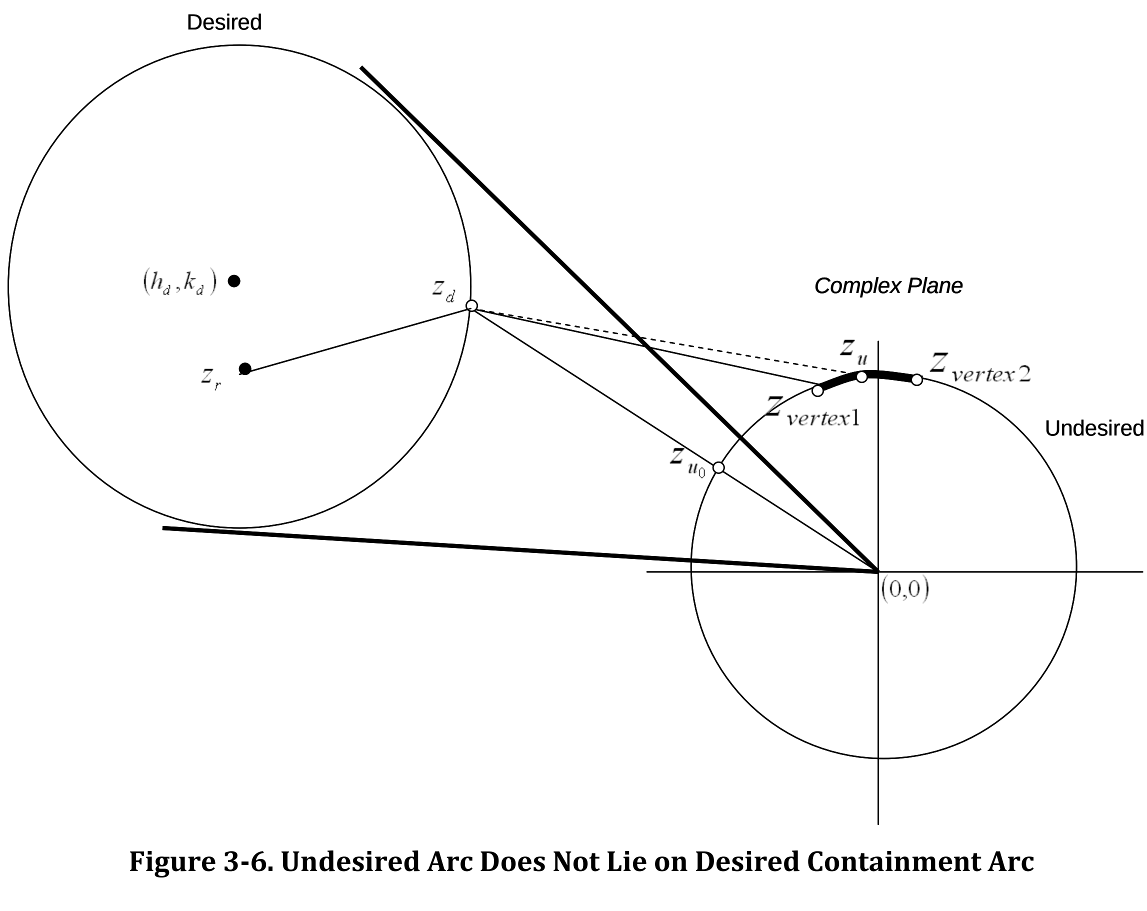

Figure 3-6. Undesired Arc Does Not Lie on Desired Containment Arc

Figure 3-6 illustrates that the same argument as above can be used to show that the optimal solution must be of the form (z_{endpt}, z_d). The endpoint closest to the Desired Containment Arc is selected and the optimization problem (28) above is solved with $0 \le \theta \le 2\pi$ to provide the solution for the polygon side under consideration.

3.2.2.3 Undesired Arc Lies Partly on the Desired Containment Arc

If the undesired arc partially overlays the Desired Containment Arc as shown in Figure 3-7, then the portion of the desired circle corresponding to that overlaying section of the undesired arc is where the collinearity condition holds. Therefore, for the highlighted portion of the desired circle between $z_d(\theta_1)$ and $z_d(\theta_2)$, the expression in (27) must be minimized over the interval $\theta_1 \le \theta \le \theta_2$.

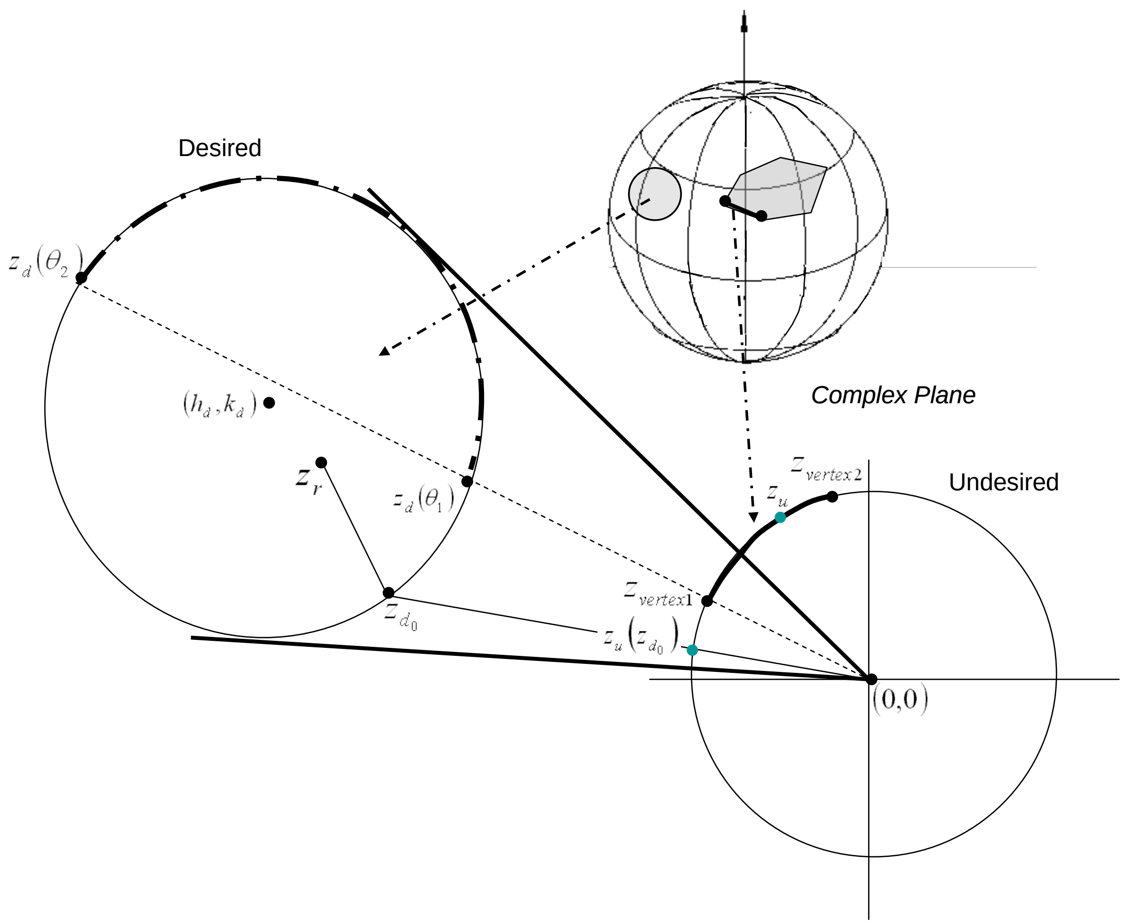

Figure 3-7. Undesired Arc Partially Lies on Desired Containment Arc and Partially Outside It

Because the entire desired circle must be considered, points outside the highlighted portion of the desired circle also must be considered. On the undesired arc, $z_{vertex1}$ being on the Desired Containment Arc, is the closest point to $z_u(z_{d_0})$ corresponding to any z_{d_0} on the non-highlighted portion of the desired circle. Thus the minimization problem (28) is solved using $z_{vertex1}$ as the endpoint over the interval $\theta_2 \le \theta \le \theta_1$. The minimum of the solutions to (27) and (28) is selected as the solution for the polygon side under consideration.

3.2.3 Desired s-Polygon and Undesired s-Polygon

For the case where both SVs are s-polygons, the optimization procedure must be applied to each pair of sides, one side from the desired s-polygon and the other side from the undesired s-polygon. Thus, we are always dealing with great circles. In all cases the center of the great circle containing the side under consideration of the undesired s-polygon is rotated to the South Pole. When these same rotation equations are applied to the great circle containing the side under consideration of the desired s-polygon, the two new vertices of that side may have the same longitudes. Thus, when that side is projected to the complex plane, we obtain a straight line instead of a circle. This case is covered in Section 3.3.

The case of two s-polygons is treated in a manner similar to the case of a desired s-circle and an undesired s-polygon discussed in Section 3.2.2. The difference is that the desired s-circle is replaced by the arc of a great circle as shown in Figure 3-8 after projection to the complex plane. The same four relationships described in Section 3.2.2 that can occur between an undesired arc and a desired circle can occur also between an undesired and a desired arc. Collinearity can be used where the central angle of the undesired circle that subtends the undesired arc (the undesired central angle) and the central angle of the undesired circle that subtends the desired arc (the desired central angle) overlap. Where collinearity cannot be used, the vertex of the undesired arc closest to the desired arc is used in the *u/d* ratio to be minimized. Figure 3-8 shows an example of the case that is similar to case (iv), identified in Section 3.2.2, and shown in Figure 3-7.

To find the solution, the *u/d* ratio (27) is minimized over the interval $\theta_a \le \theta \le \theta_{2,2}$, where collinearity holds, and the minimization problem (28) is solved over the interval $\theta_{2,2} \le \theta \le \theta_b$ using $z_{u,vertex,1}$, where collinearity does not hold. The angles θ_a, $\theta_{2,2}$, and θ_b are defined with respect to (h_d, k_d)(see Figure 3-3). For each minimization problem, the endpoints of the interval over which the minimization is performed must be checked whenever the derivative has no zero value in the interval. From the solutions obtained from minimization problems (27) and (28), the smaller one is selected as the solution for the side under consideration. The minimum *u/d* ratio is found for each pair of sides, and the minimum of these minima is the solution to the entire problem.

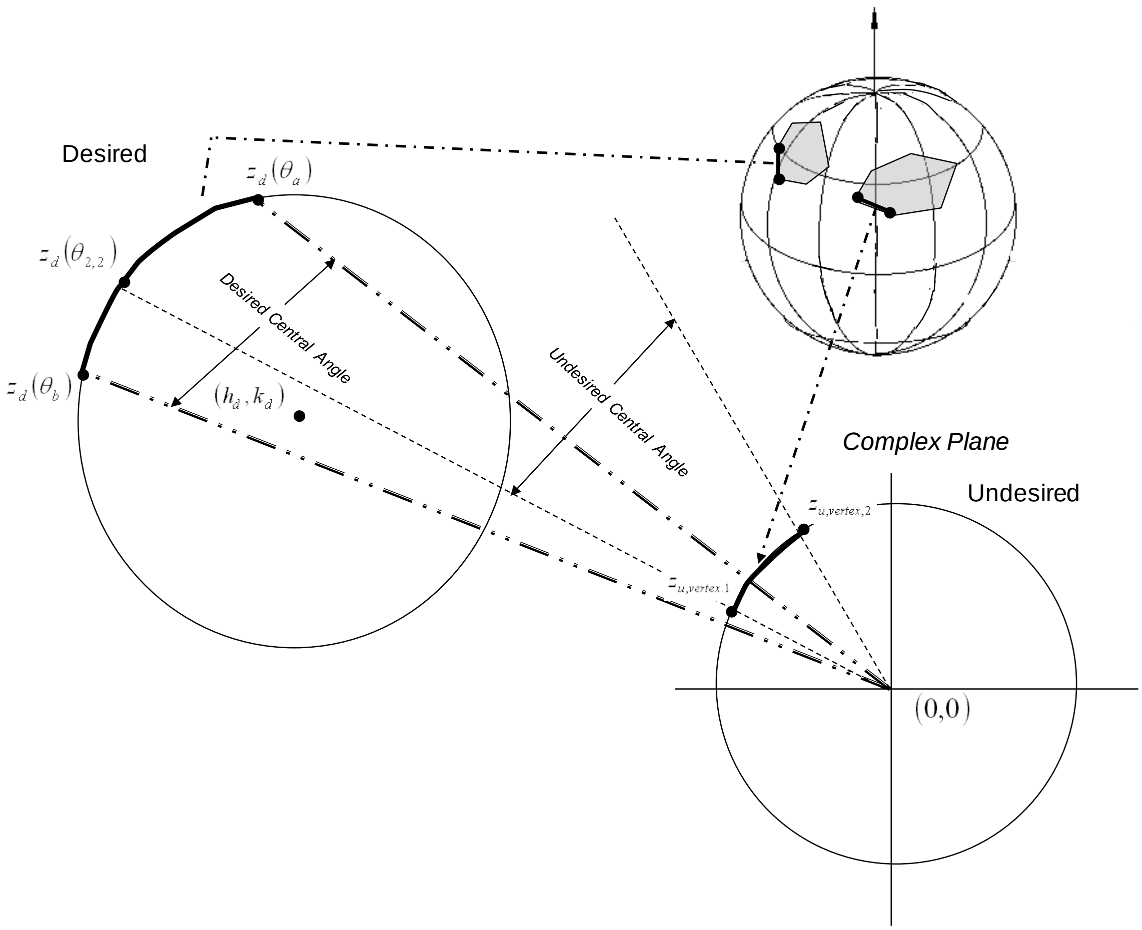

Figure 3-8. Desired Central Angle and Undesired Central Angle Overlap

3.3 Projection of a Side of Desired SV is a Straight Line

In this case, we assume that after the great circle on which the undesired arc lies has been rotated to the equator, the side under consideration of the desired s-polygon, after undergoing the same rotation, lies along a meridian. This means that the two vertices of the desired s-polygon side have the same longitudes, and the great circle containing this side passes through both Poles. A stereographic projection of this side would be a straight line segment in the complex plane passing through the origin of the complex plane as shown in Figure 3-9. Figure 3-9 shows the case where the desired line segment falls within the central angle of the undesired arc.

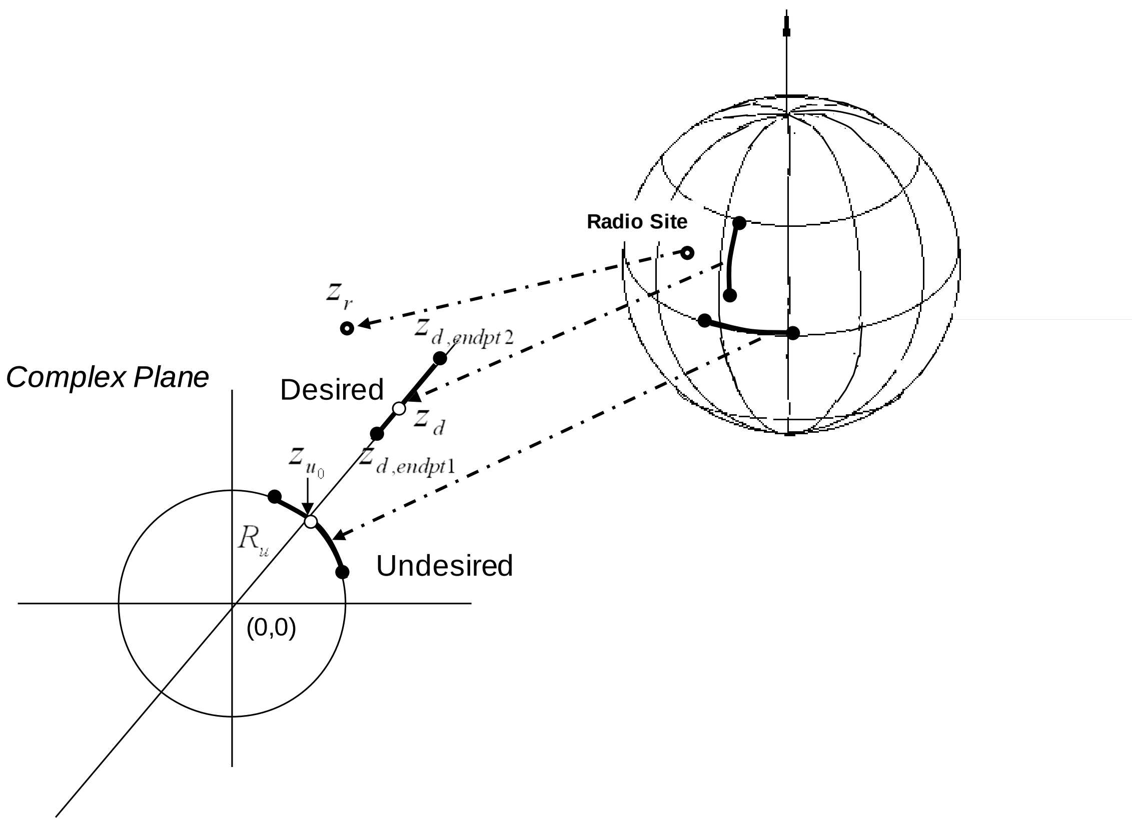

Figure 3-9. Side of Desired s-Polygon Projects as Straight Line Segment on Line Intersecting Undesired Arc

The endpoints of the desired line segment are easily obtained as the stereographic projection of the endpoints of the great circle arc segment shown in Figure 3-9, whose latitudes and longitudes are known. Since the line segment is on a line passing through the origin of the complex plane, any point on the line segment can be expressed as $z_d = s_d e^{i\theta_d}$ where s_d is variable and bounded by the magnitudes of the endpoints of the line segment, and θ_d is a constant since a vector from the origin to any point along the line segment makes the same angle with the real axis of the complex plane. For any point z_d on the line segment, the point on the undesired arc that minimizes the *u/d* ratio is the point z_{u_0} which is the intersection of the line through the origin containing the desired line segment with the undesired arc. Since the collinearity condition applies, the undesired distance can be expressed as $|z_d| - |z_{u_0}| = s_d - R_u$, where R_u is the radius of the undesired circle. The desired distance can be expressed as $\left|z_d - z_r\right| = \left|s_d e^{i\theta_d} - s_r e^{i\theta_r}\right| = \sqrt{s_d^2 + s_r^2 - 2s_d s_r \cos(\theta_d - \theta_r)}$. Thus, we must solve the problem

$$\underset{s_d}{Min} \frac{s_d - R_u}{\sqrt{s_d^2 + s_r^2 - 2s_d s_r \cos(\theta_d - \theta_r)}}, \left|z_{d,endpt1}\right| \le s_d \le \left|z_{d,endpt2}\right| \qquad (29)$$

The minimization would be performed over the single variable s_d, since all the other variables are constant, and can be implemented using the Newton-Raphson method.

All of the above also applies in the case when there is an undesired s-circle instead of an undesired s-polygon, the center of the undesired s-circle is rotated to the South Pole, and the side under consideration of the desired s-polygon undergoes the same rotation and lies along a meridian.

Figure 3-10 shows the case where the desired line segment falls outside the central angle of the undesired arc. In this case the endpoint on the undesired arc closest to z_{u_0} is selected to solve the minimization problem

$$\underset{s_d}{Min} \frac{\sqrt{s_d^2 + s_{endpt1}^2 - 2s_d s_{endpt1} \cos(\theta_d - \theta_{endpt1})}}{\sqrt{s_d^2 + s_r^2 - 2s_d s_r \cos(\theta_d - \theta_r)}}, \left|z_{d,endpt1}\right| \le s_d \le \left|z_{d,endpt2}\right| \qquad (30)$$

As before the minimization would be performed over the single variable s_d, since all the other variables are constant, and Newton-Raphson would be applied.

Note that in both problems (29) and (30), the minimization is applied to the ratio of chordal lengths; however, the solution obtained is also the solution to the problem where the ratio of chordal lengths is replaced with the ratio of GCDs, which is what we really desire to minimize. The proof provided in Section 3.0 can be shown to apply to this case where the projected desired side is a straight line segment.

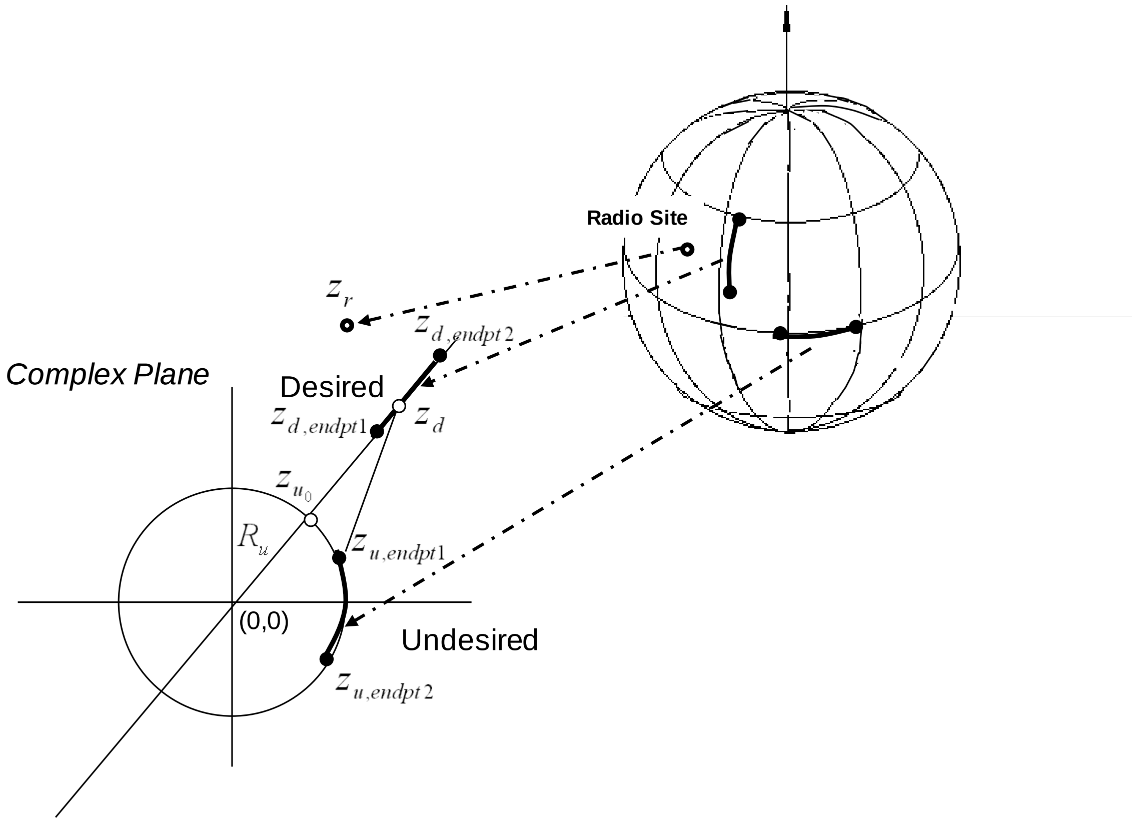

Figure 3-10. Side of Desired s-Polygon Projects as Straight Line Segment on Line Not Intersecting Undesired Arc

4 Minimizing *u/d* Ratios for Ground-to-Air, Ground-to-Ground, and Air-to-Ground

This section covers the *u/d* ratio for the cases where the ground radio either transmits the interfering signal or it is the victim of the interfering signal.

4.1 Ground-to-Air

The ground-to-air case shown in Figure 4-1 covers the situation where a radio at a ground site supporting a given SV A causes interference with an aircraft flying in a different SV B because SV A and SV B are cochannel. In this instance, the undesired signal is from the ground radio. Figure 4-2(a) shows the footprint associated with Figure 4-1 including the desired and undesired GCDs. This case falls under the case of an undesired s-circle and a desired s-circle, where the undesired s-circle has decreased to a circle of radius zero, i.e., it is just a point. Figure 4-2(b) shows the stereographic projection to the complex plane for this case, where it is assumed that the location of the undesired GR has been rotated to the South Pole. For the case of a desired s-polygon, the projection of a side would be the arc of a circle. In the case of a desired s-circle, the minimum *u/d* ratio can be found as a solution to (25) with $|z_u| = 0.$ For a desired s-polygon, the minimum *u/d* ratio for a given side is the solution to (27) with $|z_u| = 0.$ The minimum for all sides is then the solution.

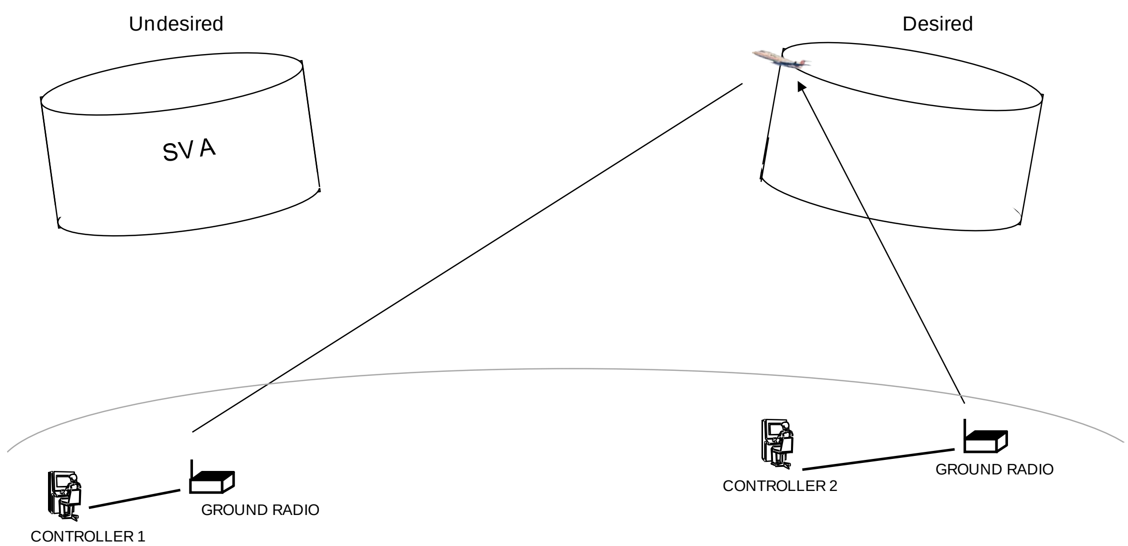

Figure 4-1. Ground-to-Air Case

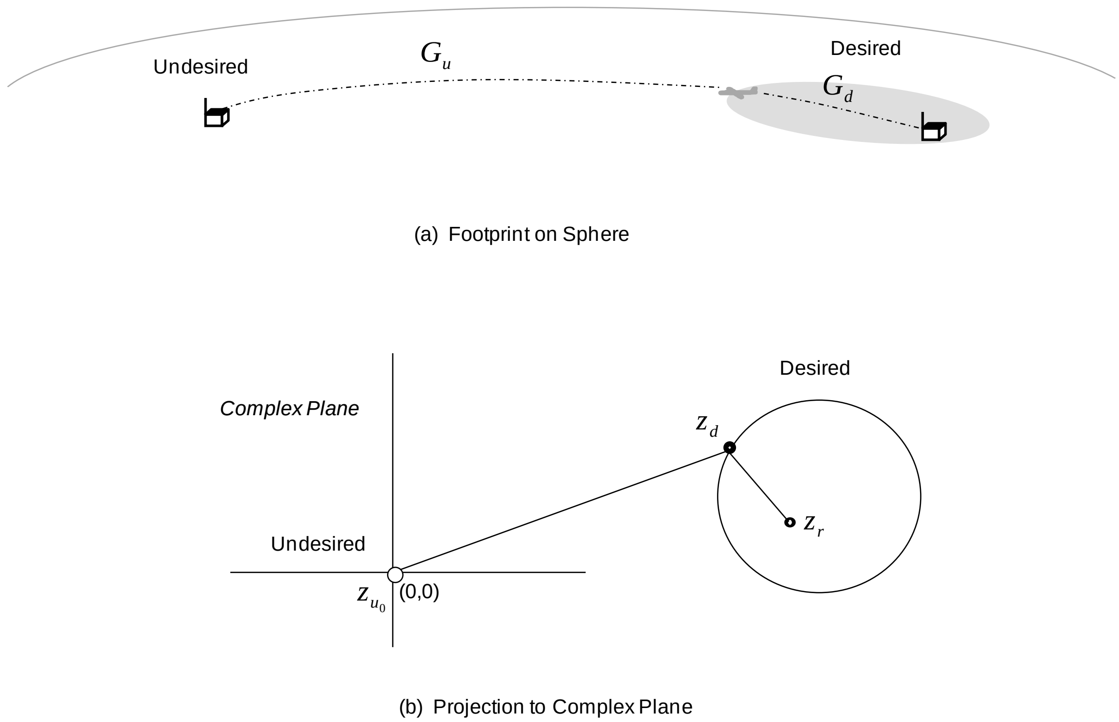

Figure 4-2. Footprint and Stereographic Projection for Ground-to-Air Case

4.2 Air-to-Ground

The air-to-ground case shown in Figure 4-3 covers the situation where an aircraft in SV B causes interference to a radio site supporting SV A because SV A and SV B are cochannel. Figure 4-4(a) shows the footprint associated with Figure 4-3 including the desired and undesired GCDs. Figure 4-4(b) shows the stereographic projection to the complex plane. In this case the ground radio for SV A is trying to receive a signal from an aircraft in SV A, but is being interfered with by a signal from an aircraft in SV B. In this case the *u/d* ratio always has the same numerator, which is the undesired chordal length between the point z_{u_0} on the undesired circle and z_r. This is because z_{u_0} is collinear with z_r and the center of the undesired circle, and therefore is the point where the undesired chordal length is the smallest (see Figure 2-9). It will never change because the radio site z_r is fixed. Notice that the center of the desired circle is at the origin of the complex plane (because we rotated the center of the desired s-circle to the South Pole), instead of the center of the undesired circle as in all the previous cases. This is done to simplify the minimization of the *u/d* ratio of chordal lengths, which can be expressed as

$$\frac{D(P_{u_0}, P_r)}{D(P_d, P_r)} = \frac{\dfrac{4R^2 |z_{u_0} - z_r|}{\sqrt{4R^2 + |z_{u_0}|^2}\sqrt{4R^2 + |z_r|^2}}}{\dfrac{4R^2 |z_d - z_r|}{\sqrt{4R^2 + |z_d|^2}\sqrt{4R^2 + |z_r|^2}}} = \frac{|z_{u_0} - z_r| \sqrt{4R^2 + |z_d|^2}}{|z_d - z_r| \sqrt{4R^2 + |z_{u_0}|^2}} \tag{31}$$

The terms $|z_{u_0} - z_r|$, $\sqrt{4R^2 + |z_{u_0}|^2}$, and $\sqrt{4R^2 + |z_d|^2}$ are all constants, and *R* is the radius of the sphere from which the projection is done. The latter term is a constant because z_d is a point on a circle with center at the origin of the complex plane. Therefore, the problem becomes one of minimizing the expression

$$\frac{D(P_{u_0}, P_r)}{D(P_d, P_r)} = \frac{K}{|z_d - z_r|}$$

This is equivalent to solving the problem

$$\underset{z_d}{Max} |z_d - z_r| = \underset{\theta}{Max} \sqrt{(s_d \cos(\theta) - x_r)^2 + (s_d \sin(\theta) - y_r)^2} \tag{32}$$

which has the solution $\theta = Tan^{-1}\left(\frac{y_r}{x_r}\right) + \pi$. This corresponds to the point on the desired circle which is farthest from z_r .

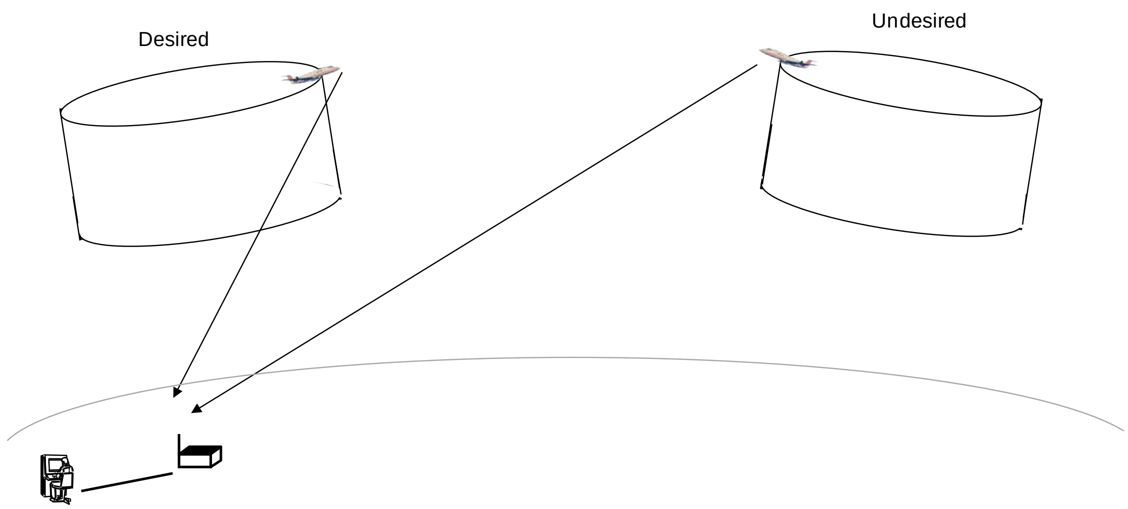

Figure 4-3. Air-to-Ground Case

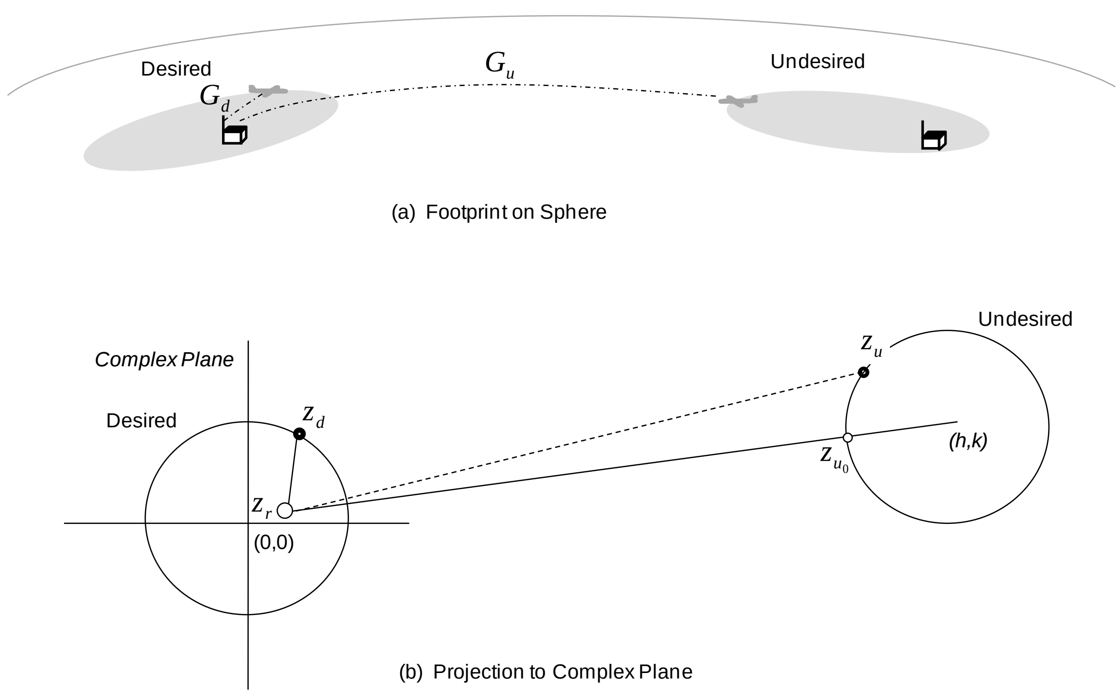

Figure 4-4. Footprint and Stereographic Projection for Air-to-Ground Case

In the above examples the SVs are s-circles. The same procedure can be applied to an s-circle and an s-polygon or to two s-polygons since the projection of an s-polygon side lies on a circle in the complex plane. The procedure must be applied to each of the sides of an s-polygon, and for each projected side the optimization would be restricted to an arc of the circle rather than to the entire circle. If the undesired projection is an arc, then z_{u_0} would be replaced by the closest point on the undesired arc to z_r to compute the minimum undesired chordal length.

4.3 Ground-to-Ground

The ground-to-ground case is shown in Figure 4-5. Figure 4-6(a) shows the footprint associated with Figure 4-5 including the desired and undesired GCDs. Figure 4-6(b) shows the stereographic projection to the complex plane. This case represents another instance where the center of the desired circle is at the origin of the complex plane because the center of the desired s-circle is rotated to the South Pole. The minimum *u/d* ratio in this case can be found by applying the formulation for the air-to-ground case and using (31) where $|z_{rA} - z_{rB}|$ is used in place of $|z_{u_0} - z_r|$ and $\sqrt{4R^2 + |z_{rB}|^2}$ is used in place of $\sqrt{4R^2 + |z_{u_0}|^2}$. Note that $|z_{rA} - z_{rB}|$ can be easily computed because the coordinates of the two radio sites are provided. If the desired SV is an s-polygon, then each side must be projected to obtain an arc in order to generate the solution, and the optimization of (32) is restricted to the arc. The maximum over all the solutions of (32) for each side is the solution.

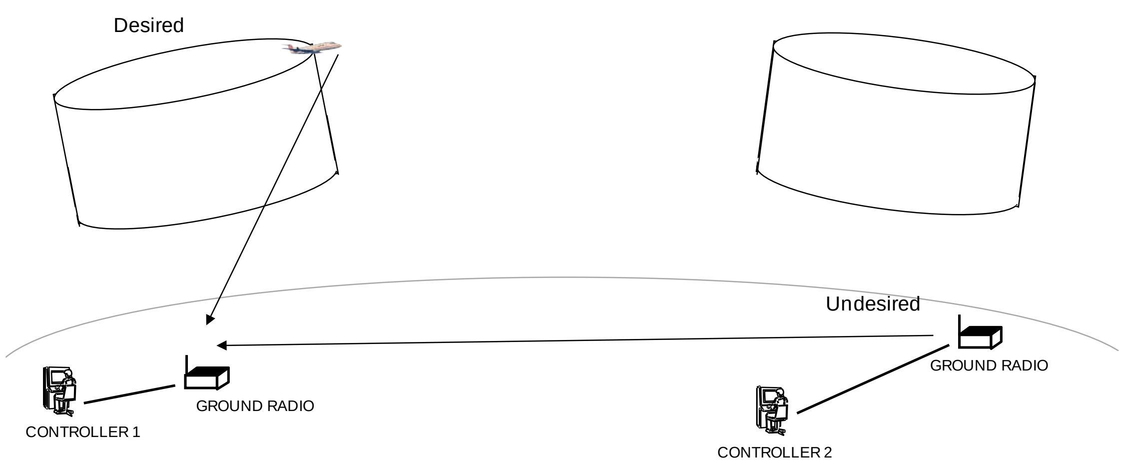

Figure 4-5. Ground-to-Ground Case

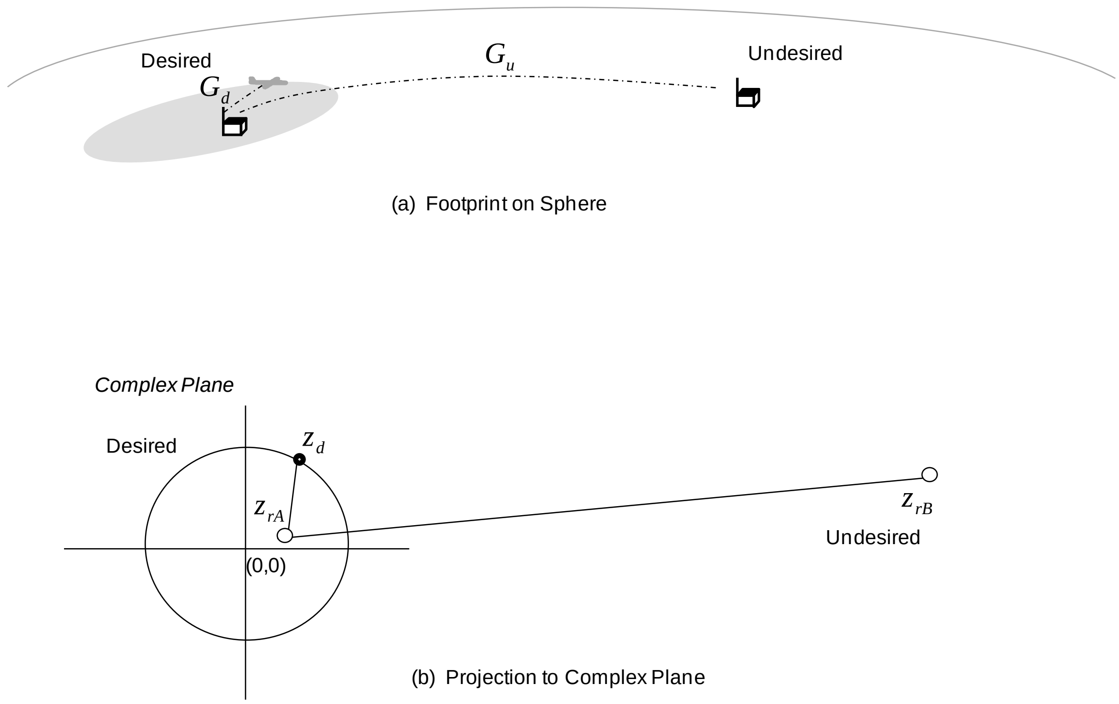

Figure 4-6. Footprint and Stereographic Projection for Ground-to-Ground Case

5 Minimizing Great-Circle Distance between s-Objects

Minimizing the GCD between the footprints on the sphere of two s-objects is a relatively easy task in comparison to minimizing the *u/d* ratio. We will discuss minimizing the GCD in the following sections between two s-circles, an s-circle and an s-polygon, and two s-polygons.

There are two basic applications that require finding the minimum GCD between s-objects. The first is in minimizing the *u/d* ratio in the case of a desired s-circle with the radio site at its center and an undesired s-polygon as mentioned in Section 3.2.2, and the second is to determine whether adjacent channels can be used in a pair of SVs based on the minimum GCD between them. In the current system, if the minimum GCD between two service volumes is less than 0.6 nmi, then their assignments must be separated by at least two channels. This rule, to prevent adjacent-channel interference (ACI), is referred to as the adjacent-channel assignment rule.

The reason why minimizing the GCD between service volume footprints on the sphere applies to the first application is the following. In the case where the radio site is at the center of the desired s-circle, the denominator of the *u/d* ratio is a constant equal to the radius of the s-circle, and thus minimizing the *u/d* ratio requires only minimizing the undesired GCD.

5.1 Minimizing Great-Circle Distance between Two s-Circles

Figure 5-1 shows finding the minimum GCD between two s-circles of a given radius. In this case, it is not necessary or helpful to perform a stereographic projection. The minimum GCD is easily obtained by computing the GCD between the centers of the two s-circles and subtracting the two radii. Thus the minimum GCD between the circles C_1 and C_2 would be computed as

$$\underset{BdryC1,BdryC2}{Min} G(C_1, C_2) = G((Lat1, Lon1), (Lat2, Lon2)) - r_1 - r_2 \qquad (33)$$

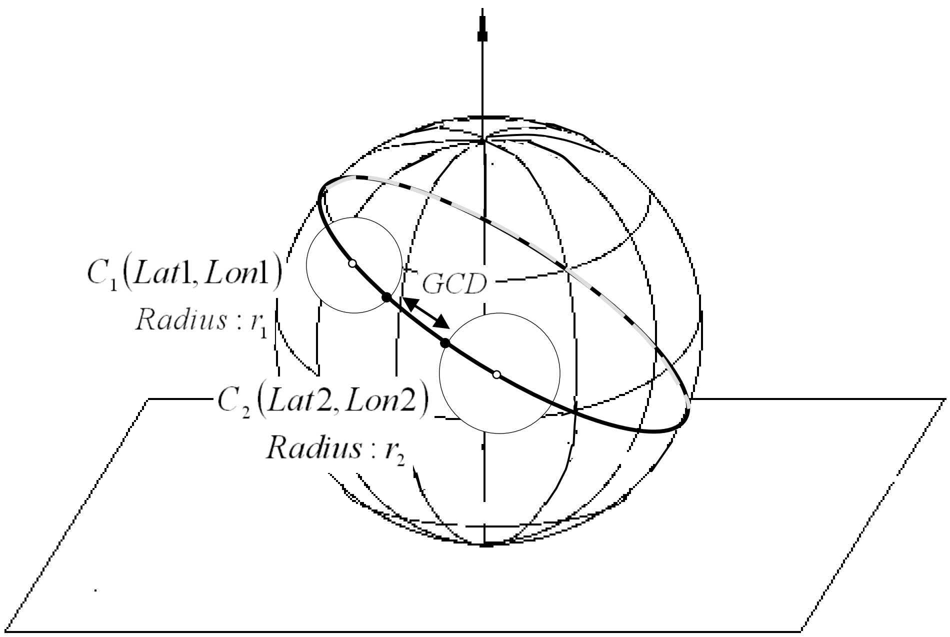

Figure 5-1. GCD Between Two s-Circles

5.2 Minimizing Great-Circle Distance Between Two s-Polygons and Between an s-Circle and an s-Polygon

The approach for minimizing the GCD between two s-polygons is based on the fact that any two great circles intersect as shown in Figure 5-2. It can be seen that the GCD between the two great circles is monotonic as one moves away from the point of intersection in either direction. For great-circle arcs, Figure 5-3 illustrates that the minimum occurs at an endpoint of one of the arcs, and would be the shortest distance between an endpoint of one of the arcs and the other arc. We have developed a very efficient point-to-arc algorithm, explained in the next section, to determine the minimum distance between a point external to an arc of a great circle and that arc. Given that procedure and given any two arcs of great circles, an approach to finding the minimum distance between the arcs would be the following. Using the point-to-arc algorithm, find the minimum distance between each endpoint of each arc and the other arc. The smallest of these four minimum distances would be the minimum GCD between the two arcs. If it is found that no shortest GCD arc from any endpoint of either arc intersects the other arc, then the shortest GCD would occur between a pair of endpoints from the two arcs as shown in Figure 5-4.

To find the minimum GCD between two s-polygons, the minimum GCD from each side (arc) of one of the s-polygons is found to the other s-polygon, and then the GCD from each side of the other s-polygon is found to the first s-polygon. The minimum of the GCDs over all pairs of sides of the two s-polygons is the solution.

The approach to find the minimum GCD between an s-circle and an s-polygon also relies on the point-to-arc algorithm. In this case the shortest GCD from the center of the circle to each side of the s-polygon is found, and the radius of the s-circle is subtracted. The smallest of these minimums is the solution.

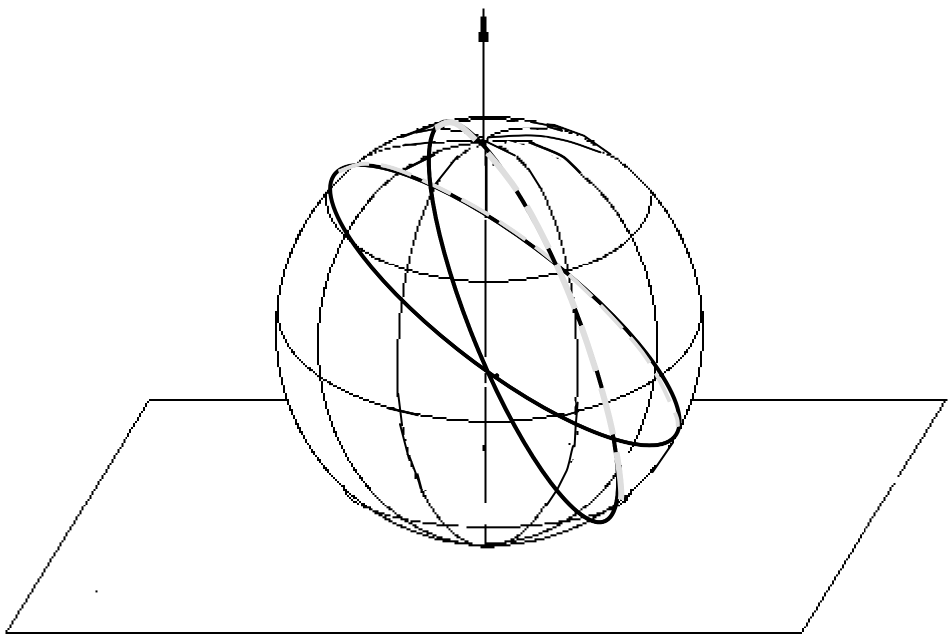

Figure 5-2. Intersecting Great Circles

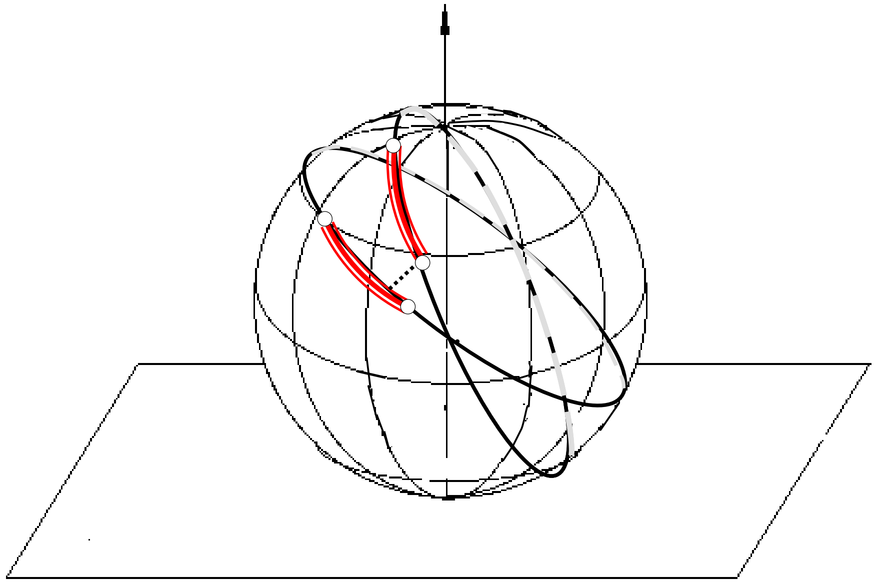

Figure 5-3. Shortest GCD between Arcs of Great Circles

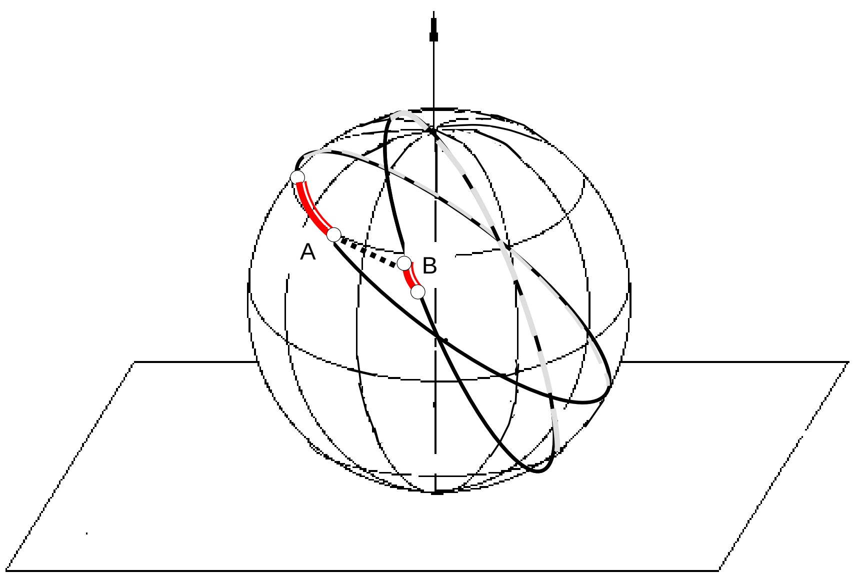

Figure 5-4. Minimum GCD Occurs at Endpoints

5.3 Point-to-Arc Algorithm

Figure 5-5 shows the constructs used to determine the minimum GCD from a fixed point (x_0, y_0, z_0) to the side of an s-polygon, where the side is a great circle arc between the points (x_1, y_1, z_1) and (x_2, y_2, z_2). There are two cases to consider that are described in this section.

The first case is when (x_0, y_0, z_0) lies on the great circle of the polygon side. Then either it lies on the polygon side or outside of it. In the first instance, the minimum GCD is zero, and in the second instance, the minimum GCD is the minimum of the GCDs from (x_0, y_0, z_0) to (x_1, y_1, z_1) and (x_2, y_2, z_2).

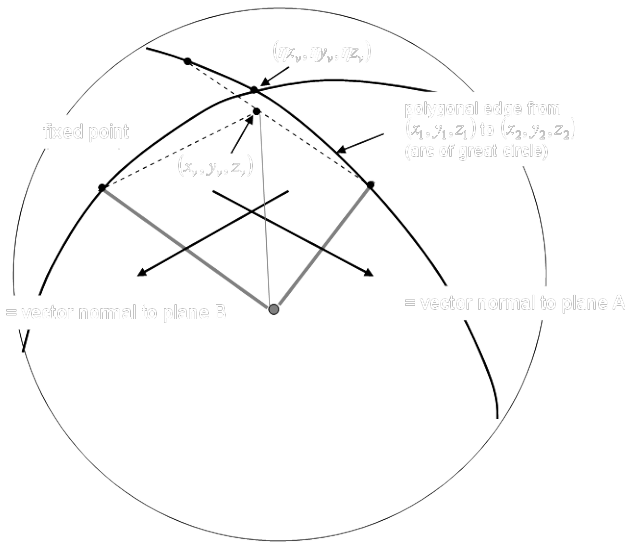

Figure 5-5. Construct for Finding Minimum GCD from Fixed Point to Side of s-Polygon

In the second case, the problem can be posed as the problem of finding the point (x_v, y_v, z_v) on the chord connecting (x_1, y_1, z_1) and (x_2, y_2, z_2) for which planes A and B are orthogonal. Once found, this point is projected along a vector from the origin of the sphere to the point $(\eta x_v, \eta y_v, \eta z_v)$ on the surface of the sphere where $\eta = \dfrac{1}{\sqrt{x_v^2 + y_v^2 + z_v^2}}$. The procedure for determining if two planes A and B are orthogonal is to find the normal vectors V_A and V_B to the planes and then require that these vectors be orthogonal. Two non-parallel vectors lying in plane A are the vectors from $(0,0,0)$ to (x_0, y_0, z_0) and from $(0,0,0)$ to (x_v, y_v, z_v); and two non-parallel vectors lying in plane B are the vectors from $(0,0,0)$ to (x_1, y_1, z_1) and from $(0,0,0)$ to (x_v, y_v, z_v). Thus, V_A can be determined using the cross product as:

$V_A = (y_0 z_v - y_v z_0,\ x_v z_0 - x_0 z_v,\ x_0 y_v - x_v y_0)$;

and V_B also can be determined using the cross product as:

$V_B = (y_1 z_2 - y_2 z_1,\ x_2 z_1 - z_2 x_1,\ x_1 y_2 - x_2 y_1)$

We want to determine (x_v, y_v, z_v) such that:

$$V_A \bullet V_B = 0 \tag{34}$$

where "$\bullet$" is the dot or scalar product.

Equation 34 can be written as:

$$(y_0 z_v - y_v z_0)K_1 + (x_v z_0 - x_0 z_v)K_2 + (x_0 y_v - x_v y_0)K_3 = 0 \tag{35}$$

where

$$\begin{aligned} K_1 &= y_1 z_2 - y_2 z_1 \\ K_2 &= x_2 z_1 - z_2 x_1 \\ K_3 &= x_1 y_2 - x_2 y_1 \end{aligned} \tag{36}$$

Note that $K_1, K_2,$ and K_3 are constants.

There are three variables in equation 35: x_v, y_v, and z_v. However, because these are coordinates of a point to be determined that lies on a straight line in 3-space, we can determine any two variables in terms of the third. We chose to determine x_v and z_v in terms of y_v. The relevant equations for doing this are

$$\frac{x_v - x_1}{y_v - y_1} = \frac{x_2 - x_1}{y_2 - y_1} \tag{37}$$

$$\frac{z_v - z_1}{y_v - y_1} = \frac{z_2 - z_1}{y_2 - y_1} \tag{38}$$

Note that $y_2 - y_1$ could be zero. In that case we would chose to solve for x_v and y_v in terms of z_v, or z_v and y_v in terms of x_v, whichever one does not result in a zero denominator.

From equations 37 and 38 we obtain

$$x_v = x_1 + m_{xy}(y_v - y_1) = m_{xy}y_v + C_1 \qquad (39)$$

$$z_v = z_1 + m_{zy}(y_v - y_1) = m_{zy}y_v + C_2 \qquad (40)$$

where

$m_{xy} = \dfrac{x_2 - x_1}{y_2 - y_1}$ and $C_1 = x_1 - m_{xy}y_1$

$m_{zy} = \dfrac{z_2 - z_1}{y_2 - y_1}$ and $C_2 = z_1 - m_{zy}y_1$

Equation 35 becomes

$$\left(y_0\left[m_{zy}y_v + C_2\right] - y_v z_0\right)K_1 + \left(z_0\left[m_{xy}y_v + C_1\right] - x_0\left[m_{zy}y_v + C_2\right]\right)K_2 + \left(x_0 y_v - \left[m_{xy}y_v + C_1\right]y_0\right)K_3 = 0 \qquad (41)$$

Equation 41 is an equation with the single unknown variable y_v. Solving for y_v, we obtain:

$$y_v = \frac{-y_0C_2K_1 - z_0C_1K_2 + x_0C_2K_2 + y_0C_1K_3}{m_{zy}y_0K_1 - z_0K_1 + m_{xy}z_0K_2 - m_{zy}x_0K_2 + x_0K_3 - m_{xy}y_0K_3} \qquad (42)$$

Once y_v is found using equation 42, equations 39 and 40 can be used to find x_v and z_v. If the point $(\eta x_v, \eta y_v, \eta z_v)$ is not found to be between (x_1, y_1, z_1) and (x_2, y_2, z_2), then the minimum GCD between (x_0, y_0, z_0) and the s-polygonal arc must occur at one of the endpoints of the arc and thus would be computed as $\min\left[G\left((x_0, y_0, z_0),(x_1, y_1, z_1)\right), G\left((x_0, y_0, z_0),(x_2, y_2, z_2)\right)\right]$. If the point $(\eta x_v, \eta y_v, \eta z_v)$ is found to be between (x_1, y_1, z_1) and (x_2, y_2, z_2), then the minimum GCD between (x_0, y_0, z_0) and the s-polygonal arc would be the GCD between $(\eta x_v, \eta y_v, \eta z_v)$ and (x_0, y_0, z_0) as shown in Figure 5-5.

6 Conclusions

This report has provided an efficient method, based on stereographic projection and complex analysis, for finding the minimum *u/d* ratio of GCDs between any two SVs on the sphere. A proof was provided showing that the same solution provides the minimum values for *u/d* ratios of both chordal lengths and GCDs. It was also proven, using the Maximum Modulus Principle, that the minimum value is achieved on the boundaries of both SVs and not in their interiors. Further it was shown that to minimize the ratio of chordal lengths, and therefore of GCDs, requires only minimizing a ratio of distances in the complex plane. Thus, it is only necessary to find a solution for the *u/d* ratio of planar distances in the complex plane in order to determine the solution for the *u/d* ratio of GCDs on the sphere.

In the complex plane the *u/d* ratio of distances can be expressed as a function of a single real variable to which the Newton-Raphson method can be applied in order to find the solution, requiring relatively few iterations. The result was found to be more accurate than the result produced by a method that performed the calculations directly on the sphere. In the case of s-circles in a large, complex problem, the computational time was reduced by two-thirds.

In addition this report has provided an efficient method for finding the minimum GCD between the footprints on the sphere of two s-objects.

The method described in this paper to solve the problem of *u/d* ratio minimization has been incorporated into a spectrum analysis tool developed for the FAA by MITRE CAASD. Accurately solving this problem enables a more efficient use of the FAA's spectral resources in the VHF A/G radio band.

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Appendix A Generalization of *u/d* Ratio to Consider Vertical Distances

Let $G(P_x, P_y)$ be the GCD between the two points P_x and P_y on the surface of the unit sphere. The solution that minimizes the ratio of GCDs can also be shown to minimize the following ratio

$$\frac{\sqrt{G(P_u, P_d)^2 + (\Delta h_u)^2}}{\sqrt{G(P_d, P_r)^2 + (\Delta h_d)^2}}$$

$$P_u = (X_u, Y_u, Z_u),\ P_d = (X_d, Y_d, Z_d),\ P_r = (X_r, Y_r, Z_r)$$

where P_u is on the boundary of the undesired SV, P_d is on the boundary of the desired SV, P_r is the fixed location of the ground radio, Δh_u is the vertical distance between the aircraft, and Δh_d is the vertical distance between the aircraft in the desired SV and the associated ground radio. We call the expression $\sqrt{G(P_u, P_d)^2 + (\Delta h_u)^2}$ the undesired slant range and the expression $\sqrt{G(P_d, P_r)^2 + (\Delta h_d)^2}$ the desired slant range. Figure A-1 shows the components involved in the slant range expressions.

There are two aspects to determining the minimum ratio of slant ranges. The first is to show that the ratio of slant ranges has its minimum at the same points on the boundaries of the footprints on the sphere of the desired and undesired SVs as the ratio of GCDs. The second aspect is to determine the values of Δh_u and Δh_d that produce the minimum value of the ratio of slant ranges.

To prove the first aspect select an arbitrary P_{d_0}, then $G(P_{d_0}, P_r)$ is the constant $\frac{2}{K_2}$ (see equations (7) and (13) in the main body of this report). Therefore, the ratio of GCDs can be written as:

$$\frac{G(P_u, P_{d_0})}{G(P_{d_0}, P_r)} = \frac{K_2}{2} G(P_u, P_{d_0}) \tag{A1}$$

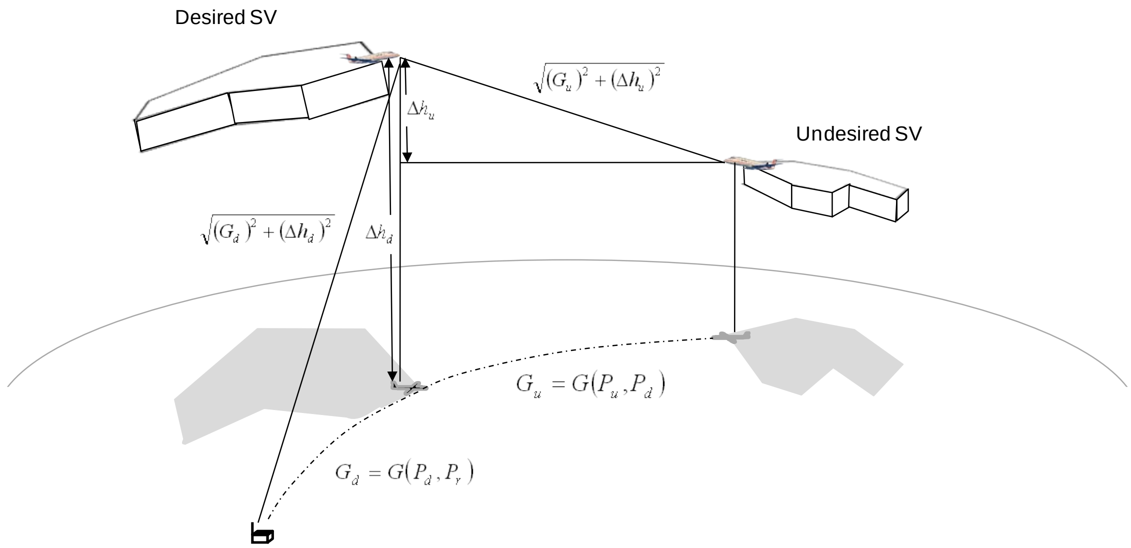

Figure A-1. Slant Ranges

Let P_{u_0} be the point on the boundary of the undesired SV that minimizes A1 for the given P_{d_0}. This means that

$$G\left(P_{u_0}, P_{d_0}\right) \leq G\left(P_u, P_{d_0}\right) \quad \forall \; P_u$$

This further implies that

$$\sqrt{G\left(P_{u_0}, P_{d_0}\right)^2 + \left(\Delta h_u\right)^2} \leq \sqrt{G\left(P_u, P_{d_0}\right)^2 + \left(\Delta h_u\right)^2} \quad \forall \; P_u$$

and

$$\frac{\sqrt{G\left(P_{u_0}, P_{d_0}\right)^2 + \left(\Delta h_u\right)^2}}{\sqrt{G\left(P_{d_0}, P_r\right)^2 + \left(\Delta h_d\right)^2}} \leq \frac{\sqrt{G\left(P_u, P_{d_0}\right)^2 + \left(\Delta h_u\right)^2}}{\sqrt{G\left(P_{d_0}, P_r\right)^2 + \left(\Delta h_d\right)^2}} \; \forall P_u$$

Since this result holds for arbitrary P_{d_0}, then we can claim that the points, P_u and P_d, on the SV boundaries where the minimum ratio of GCDs occurs are the same points where the minimum ratio of slant ranges occurs.

In determining the values of Δh_u and Δh_d that would minimize the value of slant ranges, we want to choose the smallest possible value of Δh_u since it is in the numerator, and the largest possible value of Δh_d since it is in the denominator. There are several cases to consider. Let CL_d be the ceiling of the desired SV and FL_d its floor. Let CL_u be the ceiling of the undesired SV and FL_u its floor.

The first case is when $CL_d > CL_u$ as in Figure A-1 except that we allow FL_d to go down to the ground. Let $x = \Delta h_d$ and $\Delta h_u = x - CL_u$. Note that Δh_u would not be negative since we are seeking to make *x* as large as possible and Δh_u as small as possible in absolute value. Let *c* and *b* be the values of G_u and G_d, respectively, that produce the minimum ratio of GCDs. Thus the ratio of slant ranges can be expressed as

$$\frac{\sqrt{c^2 + \left(x - CL_u\right)^2}}{\sqrt{b^2 + x^2}}$$

If this ratio is differentiated with respect to *x,* the value of *x* that minimizes the ratio is the root of a quadratic equation and can be expressed as

$$x = \frac{-(b^2 - c^2 - CL_u^2) + \sqrt{(b^2 - c^2 - CL_u^2)^2 + 4CL_u^2 b^2}}{2CL_u}$$

Note that if $FL_d > CL_u$ and the solution for *x* is less than FL_d, then *x* is set to FL_d so the desired aircraft remains in its SV.

A second case is when $CL_d \le CL_u$ and $FL_u \le CL_d$. In this case if the aircraft in the desired and undesired SVs are at altitude CL_d, then $\Delta h_u = 0$. Also, Δh_d assumes its maximum value of CL_d. These values of Δh_u and Δh_d provide the minimum value of the ratio of slant ranges.

A third case is when $CL_d < FL_u$. In this case if the aircraft in the desired SV is at altitude CL_d and the aircraft in the undesired SV is at altitude FL_u, then Δh_d assumes its maximum value of CL_d and Δh_u assumes its minimum value of $FL_u - CL_d$ for this case. These values of Δh_u and Δh_d provide the minimum values of the ratio of slant ranges.

Appendix B Equations of Circles in the Complex Plane that are Projections of Circles on the Sphere

Here we find the equation of the circle in the complex plane that is the stereographic projection of an s-circle. As shown in Figure B-1, the s-circle is the intersection of a plane with the sphere. Planes that pass through the center of the sphere (i.e., (0,0,*R*)) intersect the sphere in great circles.

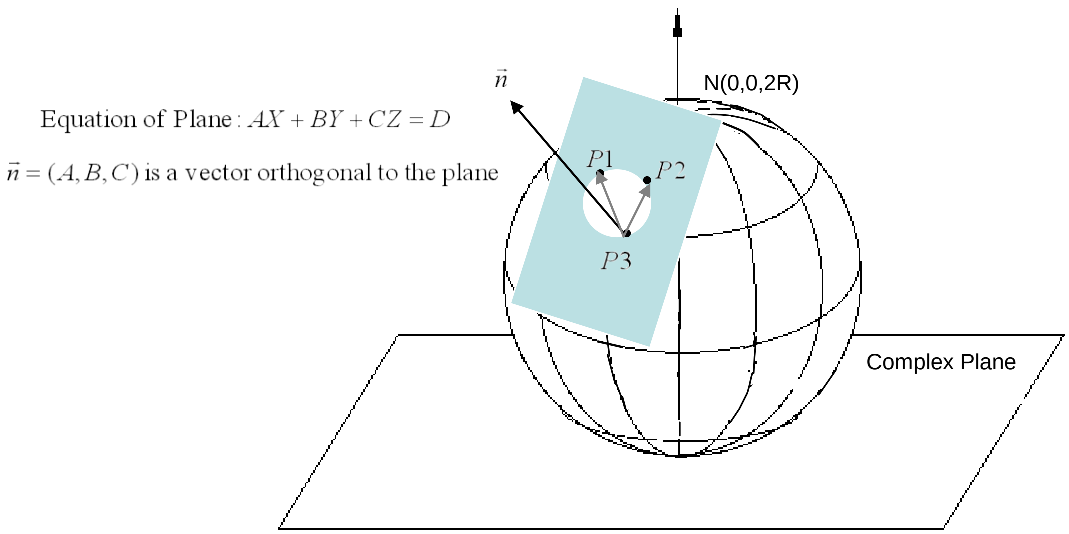

Figure B-1. s Circle Defined by Sphere and Intersecting Plane

Given an s-circle, the coefficients *A, B, C, D* of the intersecting plane must be found. In the case of finding the great circle upon which a side of an s-polygon lies, the two vertices of the side and the center of the sphere are sufficient to find the coefficients. When dealing with circular service volumes, the s-circles are "small" — i.e., are intersections of planes that do not pass through (0,0,*R*) with the sphere. In this case, three points on the s-circle must be found. We will discuss how this is done after we discuss how to find the coefficients for the equation of the plane. Given the three points $P1(X_1, Y_1, Z_1)$, $P2(X_2, Y_2, Z_2)$, $P3(X_3, Y_3, Z_3)$ on the sphere, compute the vectors

$$P1 - P3 = (X_1 - X_3,\ Y_1 - Y_3,\ Z_1 - Z_3) \text{ and } P2 - P3 = (X_2 - X_3,\ Y_2 - Y_3,\ Z_2 - Z_3)$$

and take the cross product

$$\vec{n} = (P1 - P3) \times (P2 - P3)$$
$$= \big((Y_1 - Y_3)(Z_2 - Z_3) - (Y_2 - Y_3)(Z_1 - Z_3), (X_2 - X_3)(Z_1 - Z_3) - (X_1 - X_3)(Z_2 - Z_3), (X_1 - X_3)(Y_2 - Y_3) - (X_2 - X_3)(Y_1 - Y_3)\big)$$

then

$$A = (Y_1 - Y_3)(Z_2 - Z_3) - (Y_2 - Y_3)(Z_1 - Z_3)$$
$$B = (X_2 - X_3)(Z_1 - Z_3) - (X_1 - X_3)(Z_2 - Z_3)$$
$$C = (X_1 - X_3)(Y_2 - Y_3) - (X_2 - X_3)(Y_1 - Y_3)$$

Since each of the points $P1(X_1, Y_1, Z_1)$, $P2(X_2, Y_2, Z_2)$, $P3(X_3, Y_3, Z_3)$ lies in the plane, *D* can be found by inserting any one of these points into the equation of the plane, where the coefficients *A, B, C* have been determined as components of the cross product.

To find three points on a small s-circle, the most convenient approach is to choose points that share longitudes with the center of the s-circle or are directly east or west of the center. Three choices are shown in Figure B-2 where points P1 and P3 share the same longitude and P2 is directly east of the center $(Latc, Lonc)$. For each point whose coordinates are to be found, the center of the s-circle and the radius *s* of the s-circle are then used in the following to find *P1, P2,* and *P3* (the calculations are in radians):

Finding coordinates of *P1*:

It is given that *LonP1=Lonc.* Also, P1 is directly north of the center, which has latitude *Latc* at a great circle distance of *s*. Therefore the incremental angle θ from the center would be such that $R_{earth}\theta = s$ or $\theta = \dfrac{s}{R_{earth}}$. Therefore,

$$LatP1 = Latc + \frac{s}{R_{earth}}$$

Finding coordinates of *P2*:

P2 is directly east of the center at a great circle distance of *s.* It can be shown by spherical trigonometry that

$$LatP2 = \sin^{-1}\left(\sin(Latc)\cos\left(\frac{s}{R_{earth}}\right)\right) \quad \text{and} \quad LonP2 = Lonc + \sin^{-1}\left(\frac{\sin\left(\frac{s}{R_{earth}}\right)}{\cos(LatP2)}\right)$$

Finding coordinates of *P3*:

It is given that *LonP3 = Lonc*. Also, P3 is directly South of the center, and so, using a similar argument used to find the latitude of P1, we find that

$$LatP3 = Latc - \frac{s}{R_{earth}}$$

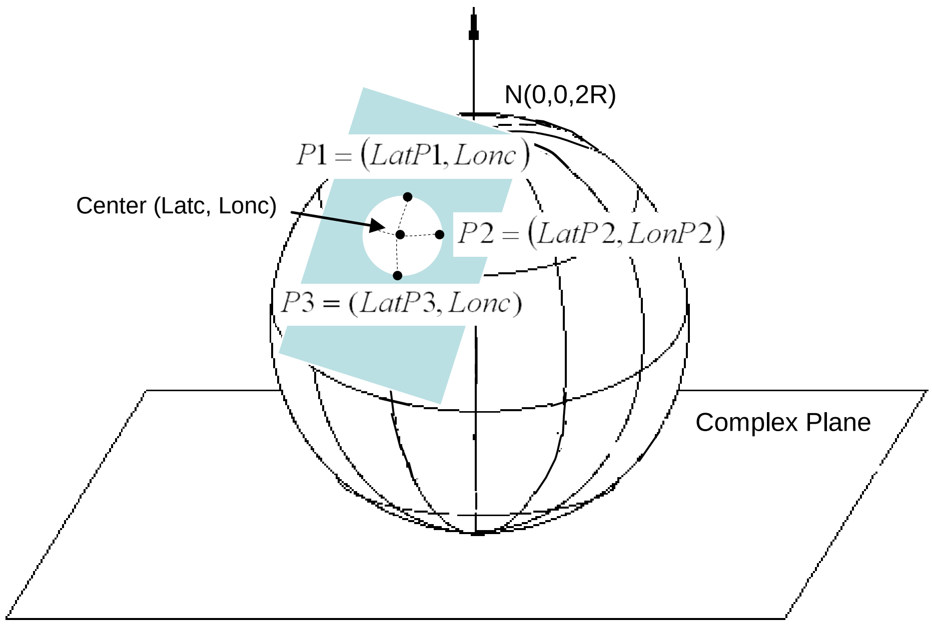

Figure B-2. Identifying Three Points on Small *s*-Circle

Replacing X, Y, Z in the equation of the plane with their inverse stereographic projections (see equations (2) in the main body of this paper), we obtain

$$AX + BY + CZ = A\frac{4R^2}{|z^2| + 4R^2}x + B\frac{4R^2}{|z^2| + 4R^2}y + C\frac{2R|z^2|}{|z^2| + 4R^2} = D$$

Through algebraic manipulation and using $|z^2| = x^2 + y^2$, the following equation for a circle is obtained

$$\left(x + \frac{2AR^2}{2CR - D}\right)^2 + \left(y + \frac{2BR^2}{2CR - D}\right)^2 = \frac{4R^2\left(2CRD - D^2 + A^2R^2 + B^2R^2\right)}{(2CR - D)^2}$$

with

$$\text{Center}: \left(-\frac{2AR^2}{2CR - D}, -\frac{2BR^2}{2CR - D}\right) \quad \text{Radius}: \frac{2R\sqrt{2CRD - D^2 + A^2R^2 + B^2R^2}}{2CR\text{-}D}$$

Appendix C Finding Tangent Points of Sides of Central Angle Forming Desired Containment Arc

In this appendix we show how to find the points of tangency on the desired circle of the sides of the central angle forming the Desired Containment Arc as shown in Figure C-1.

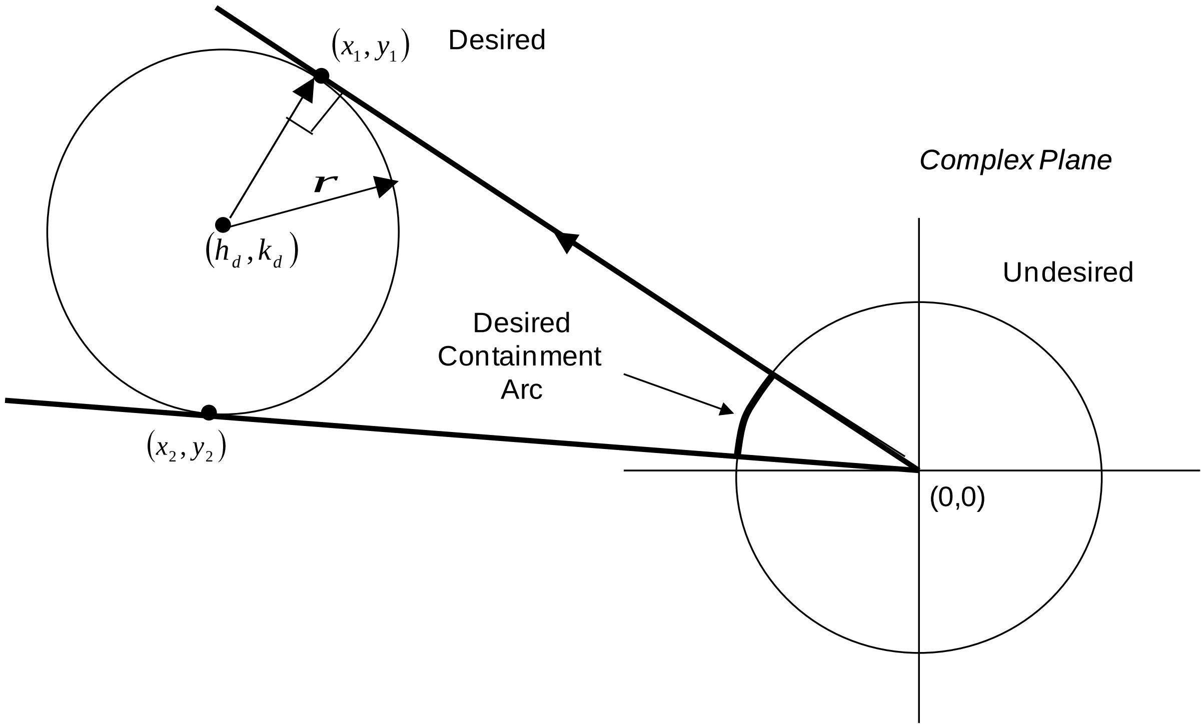

Figure C-1. Desired Containment Arc and Points of Tangency on Desired Circle

Find slope of tangent lines in terms of the derivative

$$(x-h_d)^2 + (y-k_d)^2 = r^2$$

$$2(x-h_d) + 2(y-k_d)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x-h_d}{y-k_d}$$

$$Note: \frac{y}{x} \text{ is also the slope, therefore}$$

$$\frac{y}{x} = -\frac{x-h_d}{y-k_d} \tag{C1}$$

$$y(y-k_d)=-x(x-h_d) \tag{C2}$$

Multiplying $(x-h_d)^2+(y-k_d)^2=r^2$ through by y^2 *gives*

$y^2(x-h_d)^2+y^2(y-k_d)^2=y^2r^2$

Using (C2) gives

$$y^2(x-h_d)^2+x^2(x-h_d)^2=y^2r^2$$

Solving gives

$$y^2=\frac{x^2(x-h_d)^2}{r^2-(x-h_d)^2}$$
$$y=\frac{|x(x-h_d)|}{\sqrt{r^2-(x-h_d)^2}}$$

Using (C1) and assuming $x(x-h_d)>0$

$$\begin{aligned}
&\frac{x-h_d}{\sqrt{r^2-(x-h_d)^2}}=\frac{-(x-h_d)}{\dfrac{x(x-h_d)}{\sqrt{r^2-(x-h_d)^2}}-k_d}\\
&\frac{1}{\sqrt{r^2-(x-h_d)^2}}=\frac{-\sqrt{r^2-(x-h_d)^2}}{x(x-h_d)-k_d\sqrt{r^2-(x-h_d)^2}}\\
&1=\frac{-\left(r^2-(x-h_d)^2\right)}{x(x-h_d)-k_d\sqrt{r^2-(x-h_d)^2}}\\
&x(x-h_d)-k_d\sqrt{r^2-(x-h_d)^2}=-\left(r^2-(x-h_d)^2\right)\\
&-k_d\sqrt{r^2-(x-h_d)^2}=-x(x-h_d)-r^2+(x-h_d)^2\\
&-k_d\sqrt{r^2-(x-h_d)^2}=(x-h_d)[-x+(x-h_d)]-r^2\\
&-k_d\sqrt{r^2-(x-h_d)^2}=(x-h_d)[-h_d]-r^2\\
&-k_d\sqrt{r^2-(x-h_d)^2}=-xh_d+h_d^2-r^2\\
&-k_d\sqrt{r^2-(x-h_d)^2}=-xh_d+D\\
&k_d^2\left(r^2-(x-h_d)^2\right)=x^2h_d^2-2xDh_d+D^2\\
&k_d^2\left(r^2-x^2+2xh_d-h_d^2\right)=x^2h_d^2-2xDh_d+D^2\\
&k_d^2r^2-k_d^2x^2+2xk_d^2h_d-k_d^2h_d^2=x^2h_d^2-2xDh_d+D^2\\
&Solve: x^2\left(-k_d^2-h_d^2\right)+x\left(2k_d^2h+2Dh_d\right)+k_d^2r^2-k_d^2h_d^2-D^2=0
\end{aligned}$$

where $D=h_d^2-r^2$. Solving the above quadratic gives the two solutions for the points of tangency.

Using (C1) and assuming $x(x-h_d)<0$

$$\frac{-(x-h_d)}{\sqrt{r^2-(x-h_d)^2}}=\frac{-(x-h_d)}{\frac{-x(x-h_d)}{\sqrt{r^2-(x-h_d)^2}}-k_d}$$
$$\frac{1}{\sqrt{r^2-(x-h_d)^2}}=\frac{\sqrt{r^2-(x-h_d)^2}}{-x(x-h_d)-k_d\sqrt{r^2-(x-h_d)^2}}$$
$$1=\frac{\left(r^2-(x-h_d)^2\right)}{-x(x-h_d)-k_d\sqrt{r^2-(x-h_d)^2}}$$
$$-x(x-h_d)-k_d\sqrt{r^2-(x-h_d)^2}=\left(r^2-(x-h_d)^2\right)$$
$$-k_d\sqrt{r^2-(x-h_d)^2}=r^2-(x-h_d)^2+x(x-h_d)$$
$$-k_d\sqrt{r^2-(x-h_d)^2}=r^2-(x-h_d)[x-h_d-x]$$
$$-k_d\sqrt{r^2-(x-h_d)^2}=r^2-(x-h_d)[-h_d]$$
$$-k_d\sqrt{r^2-(x-h_d)^2}=r^2+h_d x-{h_d}^2$$
$$-k_d\sqrt{r^2-(x-h_d)^2}=xh_d-D$$
$$k_d^2\left(r^2-(x-h_d)^2\right)=x^2{h_d}^2-2xDh_d+D^2$$
$$k_d^2\left(r^2-x^2+2xh_d-{h_d}^2\right)=x^2{h_d}^2-2xDh_d+D^2$$
$$k_d^2r^2-k_d^2x^2+2xk_d^2h_d-k_d^2{h_d}^2=x^2{h_d}^2-2xDh_d+D^2$$
$$Solve: x^2\left(-k_d^2-{h_d}^2\right)+x\left(2k_d^2h_d+2Dh_d\right)+k_d^2r^2-k_d^2{h_d}^2-D^2=0$$

Same solution as when $x(x\text{-}h_d\,)\;>0$.

Appendix D Rotation of Points on the Sphere to the South Pole

When performing the *u/d* ratio minimization procedure involving the side of an s-polygon, which is an arc of a great circle on the sphere, it may be necessary to find the spherical center of the great circle on which the arc lies. The definition and calculation are given below. It also is useful to be able to rotate the spherical center of a great circle or the center of a small s-circle to the South Pole. The equations to do this are presented below.

Computing the Spherical Center of a Great Circle

Figure D-1 illustrates the method to find the spherical center $\vec{C} = (C_1, C_2, C_3)$ of a great circle. It is the endpoint of a vector that is the cross product of the vectors $\vec{L}$ and $\vec{N}$. $\vec{L}$ and $\vec{N}$ are obtained using the endpoints of the arc under consideration of the great circle. Each one is a vector from the center of the sphere of radius *R* to an endpoint of the arc shown in Figure D-1. (The vectors $\vec{L}$ and $\vec{N}$ would typically be specified with respect to a sphere centered at (0,0,0) as shown. The calculations can be carried out in that manner since what we are interested in is the latitude and longitude of the spherical center to use in the rotation equations.) We denote the latitude, longitude point corresponding to $\vec{C}$ by *Latc*, *Lonc*, respectively. The spherical center is found using the following equations:

$$\vec{C} = \frac{\vec{L} \times \vec{N}}{\left\| \vec{L} \times \vec{N} \right\|} R$$

$$\vec{L} = (X_v, Y_v, Z_v)$$

$$\vec{N} = (X_r, Y_r, Z_r)$$

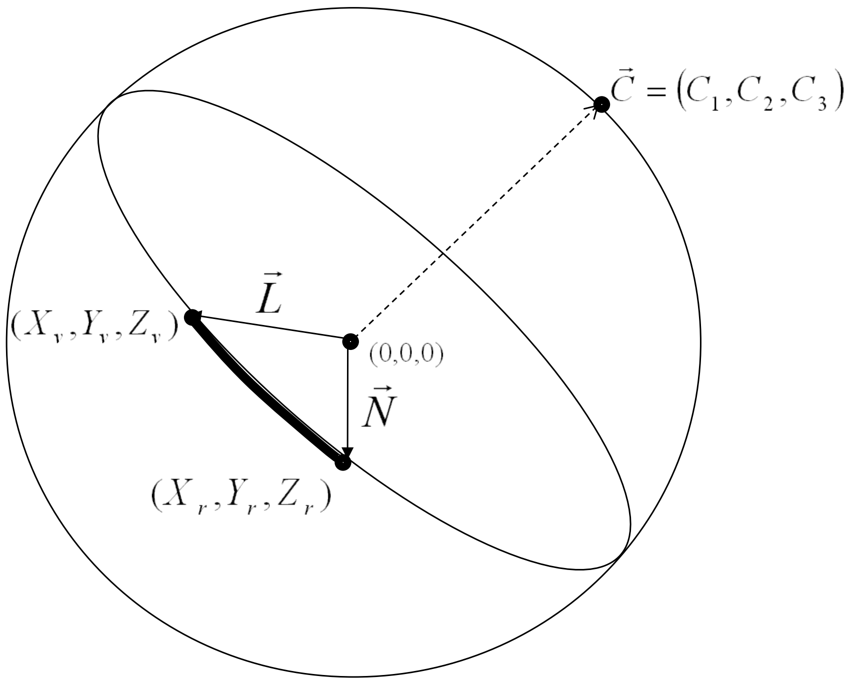

Figure D-1. Finding the Spherical Center of a Great Circle

Rotation to the South Pole

When performing the *u/d* ratio minimization procedure with two SVs and a desired ground radio site location, either the center of a small s-circle or the spherical center of a great circle associated with a side of an s-polygon is rotated to the South Pole. (In the case of a great circle, the rotation moves the great circle to the equator.) To maintain the same distance relationships among the two SVs and the desired ground radio site location, the equations used to rotate the center (*Latc*, *Lonc*) of the selected s-circle or the spherical center (*Latc*, *Lonc*) of a great circle to the South Pole are applied to the other SV and the radio site location. If the other SV is an s-circle, then the equations are applied to the center on the sphere of the s-circle. If the other SV is an s-polygon, then the equations are applied to the vertices of the s-polygon side.

The following equations are used to rotate a point of interest on a sphere of radius R as shown in Figure 2-4 with Cartesian coordinates (X, Y, Z) to point (X', Y', Z'):

$$\text{Define}: Lat\_nc = -\frac{\pi}{2} - Latc$$
$$Lon\_nc = Lonc$$
$$\text{Rotate } (X, Y, Z) \text{ about Z-axis by } Lon\_nc$$
$$W1\_1 = \cos(Lon\_nc)X + \sin(Lon\_nc)Y$$
$$W2\_1 = -\sin(Lon\_nc)X + \cos(Lon\_nc)Y$$
$$W3\_1 = Z$$
$$\text{Rotate } (W1\_1,\ W2\_1,\ W3\_1) \text{ about Y-axis by } Lat\_nc$$
$$X' = \cos(Lat\_nc)W1\_1 - \sin(Lat\_nc)W3\_1$$
$$Y' = W2\_1$$
$$Z' = \sin(Lat\_nc)W1\_1 + \cos(Lat\_nc)W3\_1 + R$$

Appendix E Finding Endpoints on Undesired Circle of Desired Containment Arc

In this appendix we find the points of intersection $(x_i, y_i), i = 1,2$ shown in Figure E-1 on the undesired circle of the sides of the central angle of the undesired circle that subtends the Desired Containment Arc. These intersection points would be the endpoints on the undesired circle of the Desired Containment Arc.

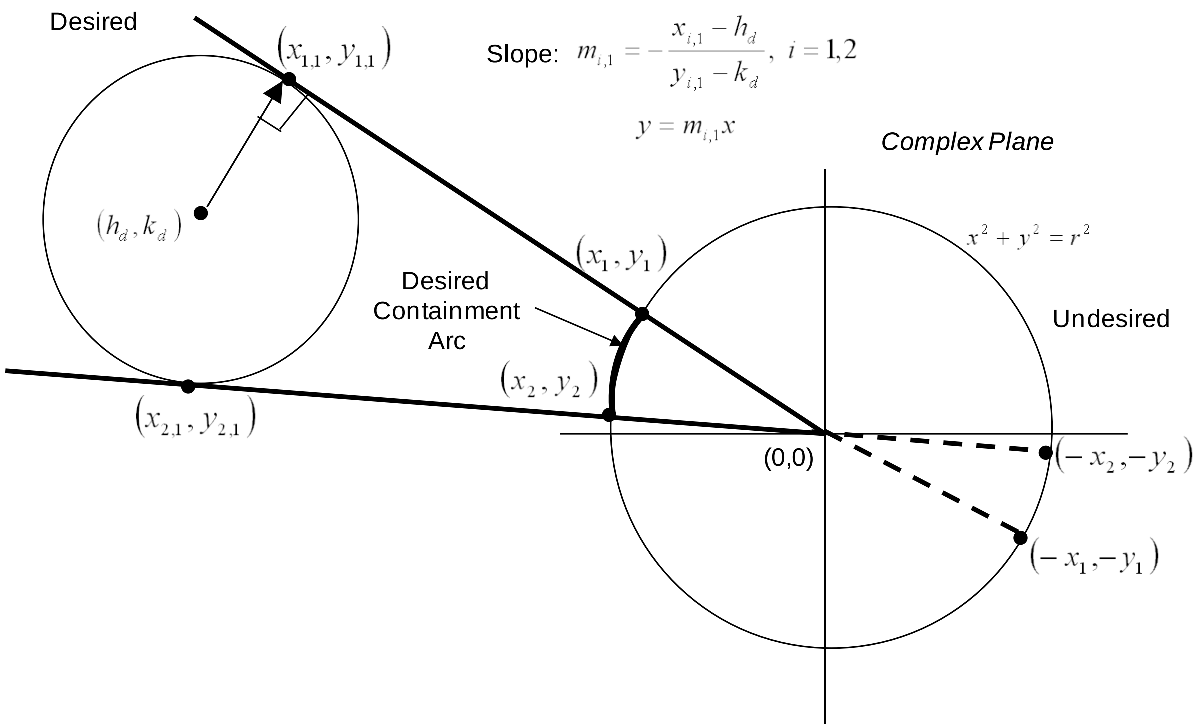

Figure E-1. Endpoints of Desired Containment Arc

The following equations show how to find these intersection points.

$$y^2 = r^2 - x^2$$

$$(m_{i,1}x)^2 = r^2 - x^2$$

$$(m_{i,1}^2 + 1)x^2 = r^2$$

$$x^2 = \frac{r^2}{m_{i,1}^2 + 1}$$

$$x = \pm\frac{r}{\sqrt{m_{i,1}^2 + 1}}, i = 1,2$$

As can be seen from these equations and Figure E-1, there are two solutions for each *i.* Only one of the solutions for each *i* is of interest. The intersection points of interest that correspond to the endpoints of the Desired Containment Arc are those closest to $(x_{i,1}, y_{i,1})$ for each *i*.

Appendix F Identifying Intersections of the Sides of the Undesired Arc's Central Angle with the Desired Circle

The endpoints (x_i, y_i), $i = 1, 2$ of the undesired arc are directly obtainable as the stereographic projection of the vertices of a side of the undesired s-polygon. Note that the great circle on which the side lies has been rotated to the equator. Using these endpoints, the intersection points of the sides of the central angle of the undesired arc with the desired circle shown in Figure F-1 can be found.

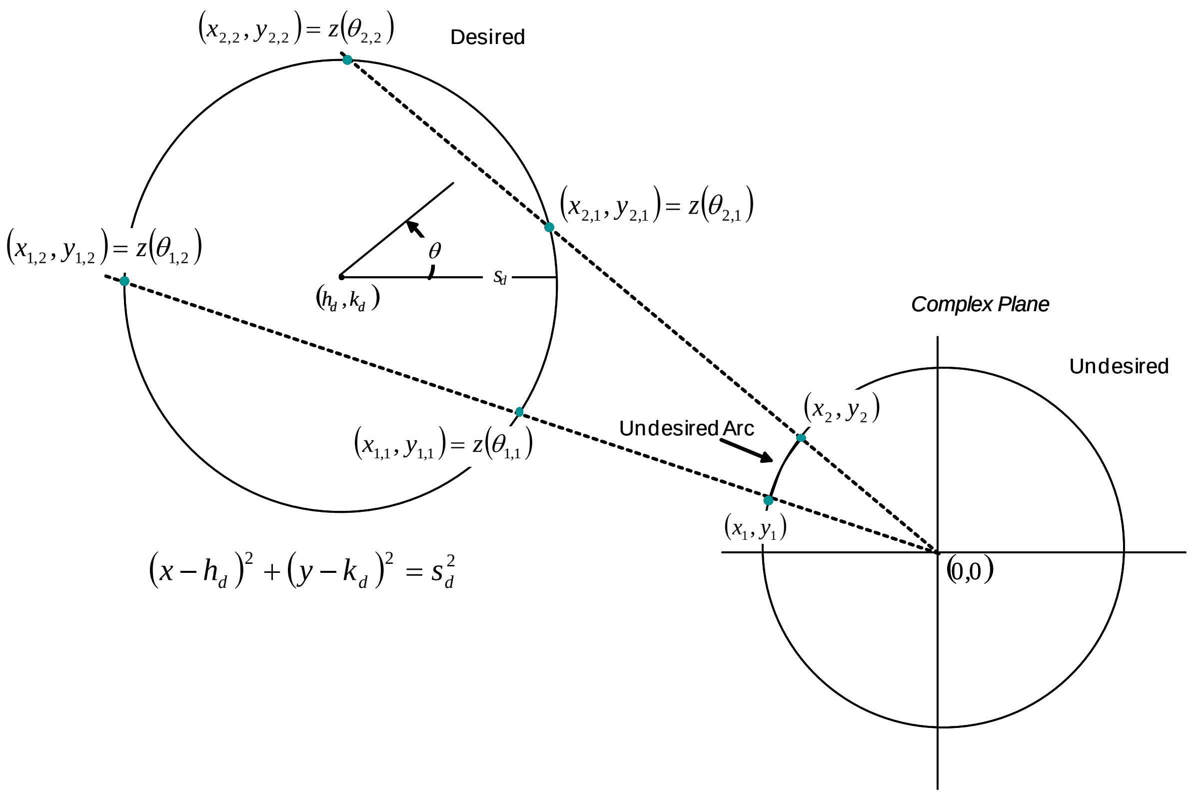

Figure F-1. Intersections on Desired Circle

To find these intersection points, we substitute $y = \frac{y_i}{x_i} x$ into the equation for the desired circle and thus solve the following

$$\left(\frac{y_i^2}{x_i^2}+1\right)x^2+\left(-2h_d-2k_d\frac{y_i}{x_i}\right)x+{h_d}^2+{k_d}^2-{s_d}^2=0 \text{ for } i=1,2$$

Two solutions $(x_{1,1}, y_{1,1})$ and $(x_{1,2}, y_{1,2})$ are obtained for i = 1 and two solutions $(x_{2,1}, y_{2,1})$ and $(x_{2,2}, y_{2,2})$ are obtained for i = 2. To find the corresponding angles $\theta_{i,j}$, the following equation is used

$$\theta_{i,j} = \cos^{-1}\left(\frac{x_{i,j}-h_d}{s_d}\right)$$

With this, then $z(\theta_{i,j}) = (h_d + s_d \cos(\theta_{i,j}), k_d + s_d \sin(\theta_{i,j}))$.

Appendix G Glossary

Acronyms

ACI	Adjacent-channel Interference
A/G	Air/Ground
AR	Airborne Radio
ATS	Air Traffic Services
CAASD	Center for Advanced Aviation System Development
CCI	Cochannel Interference
d/u	Desired-to-Undesired
DOT	Department of Transportation
FAA	Federal Aviation Administration
GCD	Great-Circle Distance
GR	Ground Radio
RLOS	Radio Line of Sight
SV	Service Volume
u/d	Undesired-to-Desired
VHF	Very High Frequency

Symbols

G_d	Great-circle distance of desired signal path
G_u	Great-circle distance of undesired signal path
$G(P_x, P_y)$	Great-circle distance between points P_x and P_y on the sphere
$D(P_x, P_y)$	Chordal length between points P_x and P_y on the sphere
P_u	A point on the sphere in the undesired SV
P_d	A point on the sphere in the desired SV
P_r	A point on the sphere representing the desired radio location
z_x	Stereographic projection in complex plane of P_x

ρ_{chdl}	Ratio of chordal lengths
ρ_{GCD}	Ratio of GCDs