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Magneto-Solid Mechanics and the Aleph Tensor. 1. General Dynamic Master System and Specialized Models

by Michael Grinfeld and Pavel Grinfeld

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14. ABSTRACT

In a report published by Grinfeld and Grinfeld in 2019, we formulated a new master system applicable to the analysis of polarizable and/or magnetizable solids. The central distinction of the suggested approach consists in the systematic usage of the Cardinal Aleph tensor. By choosing different thermodynamic potentials, the suggested approach can be recommended for the analysis of a wide variety of static or dynamic engineering systems. Given the variety of possible applications, the system is relatively simple and can be analyzed not only computationally but also analytically. However, to make the analytical and computational results simpler and more transparent, it makes sense to adjust the general system for different applications. Using our approach, we formulate the exact nonlinear system for dynamics of magnetizable fluid and solids. Then, we apply our exact nonlinear general system to ferrofluids and its linearized version to piezo-magnetic solids.

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Contents

1. Introduction

Many dynamic applications of piezo-electric and piezo-magnetic substances require modeling that rigorously obeys the basic principles of Newtonian mechanics and thermodynamics laws. This sort of modeling is required, for instance, when dealing with high-rate phenomena and, especially, shock-waves analysis. In this report, we suggest such models for magnetoelastic substances. The most difficult in this analysis are not mathematical or computational difficulties, for which different models have been suggested over the 19th and 20th centuries; the most difficult are the obstacles implied by total misunderstanding of the fundamentals like energy, entropy, and stresses. John von Neumann justly claimed that "Nobody knows what entropy really is," whereas Richard Feynman wrote, "It is important to realize that in physics today, we have no knowledge what energy is." Because of that, we are destined to build our theories on the more or less intuitive basic concepts. Of course, these concepts are not arbitrary, but they in no way can be treated as ultimate truth. More often than not, we choose the fundamentals based on limited applied targets and deliberately sacrifice the universality for the sake of simplicity. Although, in our opinion it is important to present the basic assumptions in the compact, observable, and clear form, preferably in mathematical form. For the studies of polarizable or magnetizable fluids or solids, we suggest using the central concept of the Cardinal tensor, described in Grinfeld and Grinfeld (2019), where we did so for the static problems of polarizable solids. In this report, we generalize that approach for dynamic problems of magnetizable fluids and solids.

There are plenty of applications of magnomechanics in general engineering, mechanics, and physics, and in the defense-related applications in particular. Interested readers are referred to the world-known manuals of Stratton (1941), Landau and Lifshitz (1960), Vonsovsky (1974), Tamm (1979), as well as to the monographs of Moon (1984), Rosensweig (1985), Abele (1993), Kraus (1993), Visintin (1994), Bertotti (1998), Skomski and Coey (1999), Furlani (2001), among many others.

The classic medium-level manual for theoretical physicists is the one of Landau and Lifshitz (1960). It does not pursue the goal of giving the exhaustive presentation of the subject. Landau always emphasized that novices should not care too much about fundamentals. They mostly have to learn how to solve practical problems expecting that a really deep understanding should come at the later stages of combined learning and practical work. On the other hand, the book of Landau and Lifshitz (1960) is far deeper than the mathematically more rigorous books, not even speaking about the infinitely deeper understanding of the physical nature of the phenomena. One more important feature of Landau and Lifshitz's course of theoretical physics is the permanent emphasis on the crucial role of thermodynamics. The readers of this report are referred to Landau and Lifshitz (1960) if they are interested in the thermodynamics extensions.

In this report, we touch thermodynamics of magnetization or polarization quite superficially. We are talking about the internal energy density ψ per unit mass of the substance, which is treated as a function of the polarization density $\vec{P}$ or magnetization density $\vec{M}$, and of the so-called actual metrics X_{ab}. At the same time, we ignore the thermal arguments like entropy density η or the absolute temperature T. Nonetheless, our master systems are, more or less, directly applicable to the adiabatic and isothermal cases. In the adiabatic case (in the absence of shock waves), the entropy density remains fixed pointwise; in the isothermal case, the absolute temperature remains as a fixed constant. Therefore, in the first case, the function ψ is just the internal energy density $e(X_{ab}, P^a, \eta)$ or $e(X_{ab}, M^a, \eta)$ at fixed η, whereas in the second case, ψ is just the free energy density $\psi(X_{ab}, P^a, T)$ or $\psi(X_{ab}, M^a, T)$ at fixed T.

It is possible to simultaneously treat (with minor distinctions in notation) electric polarization and magnetization because we ignore any macroscopic currents and, in this situation, we can introduce the potential of the magnetic field.

2. Simplest Master Systems for Magnetizable and Polarizable Substances

In Grinfeld and Grinfeld (2019), we formulated a new master system applicable to the analysis of polarizable and/or magnetizable solids. The central distinction of the suggested approach consists in the systematic usage of the Cardinal Aleph tensor. By choosing different thermodynamic potentials, the suggested approach can be recommended for the analysis of a wide variety of static or dynamic engineering systems. Given the variety of possible applications, the system is relatively simple and can be analyzed not only computationally but also analytically. However, to make the analytical and computational results simpler and more transparent, it makes sense to adjust the general system for different applications. It can be done in different ways.

We present a slightly modified master system of Grinfeld and Grinfeld (2019). The modifications concern two aspects: 1) we use the magnetization rather than the polarization terminology, and 2) we add the inertia terms targeting applications to dynamic problems.

When dealing with anisotropic polarizable substances, it is convenient to use the mixed Eulerian–Lagrangian description of continuum media. Consider the immobile spatial coordinate system referred to by the coordinates z^i (the reference indexes from the middle of the Latin alphabet i,j,k run the values 1, 2, 3) and assume that our space is Euclidean. In this space, we consider a material body *B*, referred to the material coordinates x^a (the material indexes from the beginning of the Latin alphabet a,b,c run the values 1, 2, 3 as well). We accept the standard concepts of the covariant and contravariant indexes and accept the standard agreement regarding summation over the repeat covariant and contravariant indexes of the same type (i.e., of the reference or material type).

In addition to two different coordinates, we distinguish between two different configurations—the initial and current configurations of the body. Let the functions $z^i = z^i(x^a,t)$ be the Eulerian coordinates in the current configuration of the material point with the material coordinates x^a at the moment of time t. We use the notation $x^a = x^a(z^i,t)$ for the inverse of the function $z^i(x^a,t)$. Let us use the notation Z_{ij} for deformation-independent metrics of the reference spatial system, and the notation X_{ab} for the deformation-dependent metrics of the actual material configuration. These two metrics are connected by the relationships

$$X_{ab}(x,t) = Z_{ij} z^{i\cdot}_{.a} z^{i\cdot}_{.b}, \; Z_{ij} = X_{ab} x^{a\cdot}_{.i} x^{b\cdot}_{.j} \tag{1}$$

where the mixed shift-tensors $z^{i\cdot}_{.a}$ and $x^{a\cdot}_{.i}$ are defined as

$$z^{i\cdot}_{.a} \equiv \frac{\partial z^i(x,t)}{\partial x^a}, \;\; x^{a\cdot}_{.i} \equiv \frac{\partial x^a(z,t)}{\partial z^i} \tag{2}$$

The reference and the coordinate configurations are characterized by the current covariant bases $\boldsymbol{Z}_i(z)$ and contravariant bases and $\boldsymbol{X}_a(x,t)$, respectively.

We use the standard notation ∇_i and ∇_a for the reference and material contravariant differentiation in the metrics of the actual configuration.

Magnetization is a vector quantity. A distributed magnetization field is characterized by the density per unit mass $\boldsymbol{M}$ or per unit volume $\rho\boldsymbol{M}$, where ρ is the mass density. Vector $\boldsymbol{M}$ can be decomposed with respect to the material basis $\boldsymbol{M} = M^a \boldsymbol{X}_a$ or the spatial basis. By definition, in vacuum, the magnetization vector M is equal to zero. Thus, it experiences a discontinuity jump across the body's boundary Σ.

The bulk energy density ψ per unit mass is given as a function of the actual material metrics X_{ab}, the Lagrangian components M^a of the magnetization vector per unit mass, and fixed material constants or tensors, which we do not mention explicitly:

$$\psi = \psi(X_{ab}, M^a) \tag{3}$$

The magnetoelastic Aleph tensor $\aleph^{ij}$ is defined as follows

$$\aleph^{ij} \equiv 2\rho \frac{\partial \psi}{\partial X_{(cd)}} z^{i\cdot}_{.c} z^{j\cdot}_{.d} - \frac{1}{8\pi} H_k H^k Z^{ij} + \frac{1}{4\pi} H^i H^j \tag{4}$$

where H^i are the Eulerian component of the magnetic field.

The bulk dynamics equation reads

$$\rho\left(\frac{\partial V^i}{\partial t} + V^j \nabla_j V^i\right) = \nabla_j \aleph^{ij} \tag{5}$$

where V^i are the Eulerian components of the velocity field and ρ is the mass density.

The velocity field V^i is defined as

$$V^i(x,t) \equiv \frac{\partial z^i(x,t)}{\partial t} \tag{6}$$

We can also consider the velocity components as a function of the Eulerian coordinates z^i; we will use the notation $V^i(z,t)$ for this function. The functions $V^i(x,t)$ and $V^i(z,t)$ are different functions. This should not create any confusion even we do not show the arguments explicitly—which of the two functions is meant should be clear from the context; for instance, in Eq. 5 we mean $V^i = V^i(z,t)$.

The momentum condition at the boundary with vacuum reads

$$\left(2\rho\frac{\partial\psi}{\partial X_{(cd)}}z^{i\cdot}_{.c}z^{j\cdot}_{.d}+\frac{1}{4\pi}H^iH^j-\frac{1}{8\pi}H_kH^kZ^{ij}\right)_{sub}N_j=\left(\frac{1}{4\pi}H^iH^j-\frac{1}{8\pi}H_kH^kZ^{ij}\right)_{vac}N_j \tag{7}$$

The relationships in Eqs. 3–6 should be amended with the magnetostatics bulk equations and boundary conditions

$$H_i=-\frac{\partial\varphi(z,t)}{\partial z^i} \tag{8}$$

$$\nabla_i B^i=0 \tag{9}$$

with the magnetic induction B^i defined as

$$B^i\equiv H^i+4\pi\rho M^i \tag{10}$$

Equation 8 reflects the fact that in the absence of macroscopic electric current, the magnetic field is irrotational. Equation 9 reflects the fact that the magnetic induction is always divergence-free.

At the interfaces, the fields $\varphi(z,t)$, $H_i(z,t)$, and $B^i(z,t)$ and/or their derivatives experience finite jumps. Those jumps are not arbitrary but satisfy the boundary constraints of magnetostatics

$$\left[\varphi\right]_-^+=0 \tag{11}$$

and

$$\left[B^i\right]_-^+N_i=0 \tag{12}$$

The bulk equations (Eqs. 3–6 and 8–10) should be amended with the following thermodynamics-prompted relationship

$$H_a=\rho\frac{\partial\psi(X_{ab},M^a)}{\partial M^a} \tag{13}$$

To get the mathematically closed master system, the relationships in Eqs. 1–13 should be amended with the initial conditions and conditions at infinity.

The functions $\rho(z,t)$ and $V^i(z,t)$ satisfy the classical mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho V^i) = 0 \tag{14}$$

In hydrodynamics, Eq. 14 is used explicitly. However, in mechanics of solids there is no need to use Eq. 14.

Inserting Eq. 10 in Eq. 9 we get

$$\nabla_i(H^i + 4\pi\rho M^i) = 0 \tag{15}$$

Also, using Eq. 8 we can rewrite Eq. 11 as follows

$$\nabla_i(-\nabla^i\varphi + 4\pi\rho M^i) = 0 \tag{16}$$

Combining the thermodynamic identity (Eq. 13) with the magnetostatics relationship (Eq. 8), we get

$$\rho \frac{\partial \psi(X_{ab}, M^a)}{\partial M^a} x^a_{.i} = -\nabla_i \varphi \tag{17}$$

3. Model of Magnetizable Fluid

Fluid is a special case of solid deformable function. Often fluids are described with mathematical systems that are considerably simpler than the general master systems for solids. Sometimes, but not always, the simplifications are possible for polarizable fluids. Let us consider a special model of the substance described by the following energy density ψ

$$\psi(X_{ab}, M^a) = \Psi(\rho) + \frac{\alpha}{2} X_{ab} M^a M^b \ , \tag{18}$$

where α is a positive constant.

In the energy density (Eq. 18), the dependence upon the actual metrics X_{ab} enters not only explicitly through the term $X_{ab}M^aM^b$, but also implicitly through the density term $\Psi(\rho)$. Namely, we get

$$\frac{\partial \ln \rho^2}{\partial X_{ab}} = -X^{ab} \tag{19}$$

Using Eqs. 18 and 19, we get ψ

$$\psi(X_{ab}, M^a) = \Psi(\rho(X_{ab})) + \frac{\alpha}{2} X_{ab} M^a M^b \ , \tag{20}$$

and we arrive at the following formula:

$$\frac{\partial \psi}{\partial X_{(cd)}} z^{i\cdot}_{.c} z^{j\cdot}_{.d} = -\frac{1}{2}\rho \Psi_{\rho} Z^{ij} + \frac{\alpha}{2} M^i M^j, \tag{21}$$

as implied by the chain

$$\frac{\partial \psi}{\partial X_{(cd)}} z^{i\cdot}_{.c} z^{j\cdot}_{.d} = \left(-\frac{1}{2}\rho \Psi_{\rho} X^{cd} + \frac{\alpha}{2} M^c M^d\right) z^{i\cdot}_{.c} z^{j\cdot}_{.d} = -\frac{1}{2}\rho \Psi_{\rho} Z^{ij} + \frac{\alpha}{2} M^i M^j$$

Inserting Eq. 20 in the definition (Eq. 4) of the Aleph tensor, we get

$$\aleph^{ij} \equiv -\rho^2 \Psi_{\rho} Z^{ij} + \alpha \rho M^i M^j - \frac{1}{8\pi} H_k H^k Z^{ij} + \frac{1}{4\pi} H^i H^j \tag{22}$$

Using Eq. 21 we can rewrite the bulk momentum Eq. 5 as

$$\begin{gathered}\rho\left(\frac{\partial V^i}{\partial t} + V^j \nabla_j V^i\right) = \\ \nabla_j \left(-\rho^2 \Psi_{\rho} Z^{ij} + \alpha \rho M^i M^j - \frac{1}{8\pi} H_k H^k Z^{ij} + \frac{1}{4\pi} H^i H^j\right)\end{gathered} \tag{23}$$

The bulk thermodynamics Eq. 13 for the model (Eq. 20) implies

$$H_i = \alpha M_i \tag{24}$$

Using Eq. 24, we get the following relationship for the magnetic induction

$$B_i = \frac{\alpha + 4\pi}{\alpha} H_i \tag{25}$$

Combining the bulk magnetostatic Eq. 9 with Eq. 25 we get

$$\nabla_i H^i = 0 \tag{26}$$

Inserting Eq. 8 in Eq. 26, we arrive at the Laplace equation

$$\nabla_i \nabla^i \varphi = 0 \tag{27}$$

In the case of a magnetofluid media, the general momentum boundary condition (Eq. 7) at the boundary with vacuum reads

$$\begin{gathered}\left(-\rho^2 \Psi_\rho Z^{ij} + \alpha\rho M^i M^j + \frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij}\right)_{fluid} N_j = \\ \left(\frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij}\right)_{vac} N_j\end{gathered} \tag{28}$$

The magnetostatics boundary conditions imply

$$\left[\varphi\right]_-^+ = 0 \tag{29}$$

and

$$\frac{4\pi + \alpha}{\alpha} \nabla_i \varphi_{fluid} N^i = \nabla_i \varphi_{vac} N^i \tag{30}$$

The system (Eqs. 23–30) should be amended with the mass conservation equation (Eq. 14). Thus, in the case under study we eliminated the explicit use of the equations $x^a = x^a(z^i, t)$. This fact significantly simplifies the general master system for magnetizable solids.

4. Linearized Master System for Deformable Magnetizable Solids

We choose an affine reference coordinate system with the time-independent metrics Z_{ij}, Z^{ij}. Consider a uniform configuration with the uniform and time-independent shift tensors $x^{a}_{.i} = x^{a\circ}_{.i}$, $z^{i}_{.a} = z^{i\circ}_{.a}$, the uniform and time-independent metrics $X_{ab} = X^{\circ}_{ab}$, $X^{ab} = X^{ab\circ}$, and the vanishing fields $H^{\circ i} = M^{\circ a} = B^{\circ a} = V^{i\circ} = \varphi^{\circ} = 0$.

Let $\tilde{x}^{a}_{.i}$, $\tilde{z}^{i}_{.a}$, $\tilde{X}_{ab}$, $\tilde{X}^{ab}$, $\tilde{H}^{i}$, $\tilde{M}^{a}$, $\tilde{B}^{a}$, $\tilde{V}^{i}$, $\tilde{\varphi}$ be the small time- and coordinate-dependent perturbations of the equilibrium fields. Let us establish the linearized master system for the perturbations.

To within the first order terms, the relationship (Eq. 4) implies

$$\tilde{\aleph}^{ij} \equiv 2\rho^{\circ}\left(\left.\frac{\partial^2\psi^{\circ}}{\partial X_{(ab)}\partial X_{(cd)}}\right|_{\substack{X_{cd}=X^{\circ}_{cd}\\ M^{a}=0}}\tilde{X}_{(ab)} + \left.\frac{\partial^2\psi^{\circ}}{\partial X_{(cd)}\partial M^{a}}\right|_{\substack{X_{cd}=X^{\circ}_{cd}\\ M^{a}=0}}\tilde{M}^{a}\right) z^{i.\circ}_{.c} z^{j.\circ}_{.d} \tag{31}$$

Differentiating Eq. 31, we get

$$\frac{\partial\tilde{\aleph}^{ij}}{\partial t} \equiv 2\rho^{\circ}\left(\left.\frac{\partial^2\psi^{\circ}}{\partial X_{(ab)}\partial X_{(cd)}}\right|_{\substack{X_{cd}=X^{\circ}_{cd}\\ M^{a}=0}}(\nabla_a\tilde{V}_b + \nabla_b\tilde{V}_a) + \left.\frac{\partial^2\psi^{\circ}}{\partial X_{(cd)}\partial M^{a}}\right|_{\substack{X_{cd}=X^{\circ}_{cd}\\ M^{a}=0}}\frac{\partial\tilde{M}^{a}}{\partial t}\right) z^{i.\circ}_{.c} z^{j.\circ}_{.d} \tag{32}$$

where we used the relationship

$$\frac{\partial X_{ab}(x,t)}{\partial t} = \nabla_a V_b + \nabla_b V_a$$

We can now rewrite Eq. 32 as follows

$$\frac{\partial\,\tilde{\aleph}^{ij}}{\partial t} \equiv \left(C^{abcd} z^{k.\circ}_{.a} z^{l.\circ}_{.b} z^{i.\circ}_{.c} z^{j.\circ}_{.d} \nabla_{(k} \tilde{V}_{l)} + C^{cd}_{a} z^{i.\circ}_{.c} z^{j.\circ}_{.d} \frac{\partial \tilde{M}^a}{\partial t} \right) \tag{33}$$

In Eqs. 33 and following we use the following notation:

$$\begin{gathered} C^{abcd} \equiv 4\rho^\circ \left. \frac{\partial^2 \psi^\circ}{\partial X_{(ab)} \partial X_{(cd)}} \right|_{\substack{X_{cd} = X^\circ_{cd} \\ M^a = 0}}, \\ C^{cd}_{a} \equiv 2\rho^\circ \left. \frac{\partial^2 \psi^\circ}{\partial X_{(cd)} \partial M^a} \right|_{\substack{X_{cd} = X^\circ_{cd} \\ M^a = 0}},\ C_{ab} \equiv \rho^\circ \left. \frac{\partial^2 \psi\ \ X_{ab}, M^a}{\partial M^a \partial M^b} \right|_{\substack{X_{ab} = X^\circ_{ab} \\ M^a = 0}} \end{gathered} \tag{34}$$

The linearized bulk dynamics equation reads

$$\rho^\circ \frac{\partial \tilde{V}^i}{\partial t} = \nabla_j \tilde{\aleph}^{ij} \tag{35}$$

Differentiating Eq. 35 with respect to t and using Eq. 33, we arrive at the linearized momentum bulk equation

$$\rho^\circ \frac{\partial^2 \tilde{V}^i}{\partial t^2} = \nabla_j \left(C^{abcd} z^{k.\circ}_{.a} z^{l.\circ}_{.b} z^{i.\circ}_{.c} z^{j.\circ}_{.d} \nabla_{(k} \tilde{V}_{l)} + C^{cd}_{a} z^{i.\circ}_{.c} z^{j.\circ}_{.d} \frac{\partial \tilde{M}^a}{\partial t} \right) \tag{36}$$

Linearizing the magnetostatics Eq. 16, we get

$$\nabla_i \nabla^i \tilde{\varphi} = 4\pi \nabla_i (\rho^\circ z^{i.\circ}_{.a} \tilde{M}^a) \tag{37}$$

Linearizing Eq. 17, we get eventually

$$C_{ab} x^{a\circ}_{.i} \frac{\partial \tilde{M}^b}{\partial t} + C^{bc}_{a} x^{a\circ}_{.i} z^{k.\circ}_{.b} z^{j.\circ}_{.c} \nabla_k \tilde{V}_j = -\nabla_i \frac{\partial \tilde{\varphi}}{\partial t} \tag{38}$$

To establish Eq. 38 we first get, using Eq. 17

$$\rho^\circ \left(\frac{\partial^2 \psi(X_{ab}, M^a)}{\partial M^a \partial M^b} \frac{\partial M^b}{\partial t} + \frac{\partial^2 \psi(X_{ab}, M^a)}{\partial M^a \partial X_{(bc)}} (\nabla_b V_c + \nabla_c V_b) \right) x^{a\circ}_{.i} = -\nabla_i \frac{\partial \tilde{\varphi}}{\partial t} \tag{39}$$

and then, the symmetry, we rewrite Eq. 39 as

$$\rho^{\circ}\left(\frac{\partial^{2}\psi(X_{ab},M^{a})}{\partial M^{a}\partial M^{b}}\frac{\partial M^{b}}{\partial t}+2\frac{\partial^{2}\psi(X_{ab},M^{a})}{\partial M^{a}\partial X_{(bc)}}\nabla_{b}\tilde{V}_{c}\right)x^{a\circ}_{.i}=-\nabla_{i}\frac{\partial\tilde{\varphi}}{\partial t} \tag{40}$$

At last, using the definitions Eq. 34, we rewrite Eq. 40 as Eq. 38.

Under the assumptions regarding the ground configuration, the linearized momentum boundary condition (Eq. 7) implies

$$\left(C^{abcd}z^{k.\circ}_{.a}z^{l.\circ}_{.b}z^{i.\circ}_{.c}z^{j.\circ}_{.d}\nabla_{(k}\tilde{V}_{l)}+C^{cd}_{a}z^{i.\circ}_{.c}z^{j.\circ}_{.d}\frac{\partial\tilde{M}^{a}}{\partial t}\right)_{sub}N_{j}=0 \tag{41}$$

whereas the linearized boundary conditions (Eqs. 11 and 12) read

$$\left[\tilde{\varphi}\right]^{+}_{-}=0 \tag{42}$$

and

$$\left[\nabla^{i}\tilde{\varphi}-4\pi\rho^{\circ}z^{i.\circ}_{.a}\tilde{M}^{a}\right]^{+}_{-}N_{i}=0, \tag{43}$$

respectively.

5. Magnetoelastic Bulk Waves

Let us rewrite the system (Eqs. 36–38) as follows:

$$\rho^{\circ}\frac{\partial^{2}\tilde{V}^{i}}{\partial t^{2}}=\nabla_{j}\left(C^{ijkl}\nabla_{(k}\tilde{V}_{l)}+C^{ij}_{k}\frac{\partial\tilde{M}^{k}}{\partial t}\right) \tag{44}$$

$$\nabla_{i}\nabla^{i}\tilde{\varphi}=4\pi\nabla_{i}(\rho^{\circ}\tilde{M}^{i}) \tag{45}$$

$$C_{ki}\frac{\partial\tilde{M}^{i}}{\partial t}+C^{ij}_{k}\nabla_{i}\tilde{V}_{j}=-\nabla_{k}\frac{\partial\tilde{\varphi}}{\partial t} \tag{46}$$

where the magnetoelastic modules with the spatial indices are defined as follows:

$$C^{ijkl}\equiv C^{abcd}z^{k.\circ}_{.a}z^{l.\circ}_{.b}z^{i.\circ}_{.c}z^{j.\circ}_{.d},\ C^{ij}_{k}\equiv C^{cd}_{a}z^{i.\circ}_{.c}z^{j.\circ}_{.d}x^{a.\circ}_{.k},\ C_{ij}\equiv C_{ab}x^{a\circ}_{.i}x^{b\circ}_{.j} \tag{47}$$

Consider the following solutions of the bulk system (Eqs. 44–46):

$$\begin{bmatrix} \tilde{\varphi} \\ \tilde{V}_m \\ \tilde{M}^m \end{bmatrix} = \begin{bmatrix} \Phi \\ W_m \\ \mathrm{M}^m \end{bmatrix} e^{i(\omega t + k_i z^i)} \tag{48}$$

Inserting Eq. 48 in Eqs. 44–46, we arrive at the system of linear algebraic equations

$$(\rho^\circ \omega^2 z^{il} - C^{ijkl} k_j k_k) W_l = k_j \omega C_k^{ij} \mathrm{M}^k \tag{49}$$

$$-k_i k^i \Phi = 4\pi i k_i \rho^\circ \mathrm{M}^i \tag{50}$$

$$i\omega C_{ki} \mathrm{M}^i + i C_k^{ij} k_i W_j = k_k \omega \Phi \tag{51}$$

Excluding Φ between the Eqs. 50 and 51, we get

$$\omega\left(C_{ki} + 4\pi \frac{k_i k_k}{|\,k\,|^2} \rho^\circ\right) \mathrm{M}^i + C_k^{ij} k_i W_j = 0 \tag{52}$$

as implied by the chain:

$$i\omega C_{ki} \mathrm{M}^i + i C_k^{ij} k_i W_j = k_k \omega \Phi,\ \ \Phi = -4\pi i \frac{k_i}{|\,k\,|^2} \rho^\circ \mathrm{M}^i \ \rightarrow$$
$$\left(\omega C_{ki} + \omega 4\pi \frac{k_i k_k}{|\,k\,|^2} \rho^\circ\right) \mathrm{M}^i + C_k^{ij} k_i W_j = 0$$

Let us introduce the following vectors

$$m_i \equiv \frac{1}{\omega} k_i\,,\quad \Delta_i \equiv \frac{k_i}{\left|k\right|} \tag{53}$$

The vector m_i is not necessarily real. Obviously, the vector Δ_i is real and normalized:

$$\left|\Delta_i\right| = 1 \tag{54}$$

Then, we can rewrite Eqs. 49 and 51 as follows

$$(\rho^\circ z^{il} - C^{ijkl} m_j m_k) W_l = m_j C_k^{ij} \mathrm{M}^k \tag{55}$$

$$(C_{ki} + 4\pi \rho^\circ \Delta_i \Delta_k) \mathrm{M}^i + m_i C_k^{ij} W_j = 0 \tag{56}$$

Let us consider a special case, when the tensor C_{ki} has the form

$$C_{ki} = \gamma\rho^\circ z_{ki} \tag{57}$$

Now, we can rewrite Eq. 56 as follows

$$(\gamma\rho^\circ z_{ki} + 4\pi\rho^\circ \Delta_i \Delta_k)\mathrm{M}^i + m_i C_k^{ij} W_j = 0 \tag{58}$$

Resolving Eq. 58 with respect to M^i, we get

$$\mathrm{M}^k = \frac{1}{\gamma\rho^\circ}\left(\frac{4\pi}{\gamma + 4\pi}\Delta^k\Delta^l - z^{kl}\right) m_i C_l^{ij} W_j \tag{59}$$

Indeed, contracting Eq. 58 with Δ^k, we get

$$\Delta_i \mathrm{M}^i = -m_i \Delta^k \frac{C_k^{ij}}{(\gamma + 4\pi)\rho^\circ} W_j \tag{60}$$

Now, combining Eq. 59 with Eq. 57, we get Eq. 58.

Eliminating M^k between Eqs. 54 and 59, we get

$$\begin{aligned} &(\rho^\circ z^{in} - C^{ijkn} m_j m_k) W_n = \\ &m_j C_k^{ij} \frac{1}{\gamma\rho^\circ}\left(\frac{4\pi}{\gamma + 4\pi}\Delta^k\Delta^l - z^{kl}\right) m_m C_l^{mn} W_n \end{aligned} \tag{61}$$

or

$$\left\{\begin{aligned} &\rho^\circ z^{in} - \\ &C^{ijmn} m_j m_m - \frac{1}{\gamma\rho^\circ}\left(\frac{4\pi}{\gamma + 4\pi}\Delta^k\Delta^l - z^{kl}\right) C_k^{ij} C_l^{mn} m_j m_m \end{aligned}\right\} W_n = 0 \tag{62}$$

or else

$$\left\{ z^{in} - \left[c^{ijmn} + \frac{4\pi\Delta^k\Delta^l - (\gamma + 4\pi)z^{kl}}{\gamma(\gamma + 4\pi)} c_k^{ij} c_l^{mn} \right] m_j m_m \right\} W_n = 0 \tag{63}$$

where

$$c^{ijmn} \equiv \frac{1}{\rho^\circ} C^{ijmn},\; c_k^{ij} \equiv \frac{1}{\rho^\circ} C_k^{ij} \tag{64}$$

We can also rewrite Eq. 62 in the form

$$\left\{\omega^2 z^{in} - \left[c^{ijmn} + \frac{4\pi\Delta^k\Delta^l - (\gamma+4\pi)z^{kl}}{\gamma(\gamma+4\pi)} c_k^{ij} c_l^{mn}\right] k_j k_m\right\} W_n = 0 \quad (65)$$

6. Conclusion

Many dynamic applications of piezo-electric and piezo-magnetic substances require modeling that rigorously obeys the basic principles of Newtonian mechanics and thermodynamics laws. This sort of modeling is required, for instance, when dealing with high-rate phenomena and, especially, shock-waves analysis. In this report, we suggested such models for magnetoelastic substances.

In Section 2, we postulated the closed master system that allows one to model magnetoelastic or electroelastic systems without any assumptions of smallness of deformations and electromagnetic fields. The central element of our model is the Cardinal Aleph tensor, which has some common features with the stress tensor of the classical theory of elasticity and the Maxwell tensor of electromagnetic stresses. For the substances of any crystallographic symmetry, the Cardinal tensors appear to be symmetric. In Section 3, we specify and simplify our general master system for the case of ferrofluid. In Section 4, we specify our general system for the case of small magnetic fields and deformation, thus reducing the general nonlinear system to the linear one. At last, in Section 5, we provide a novel analysis of the linear piezomagnetic waves in solids.

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