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Optimal Sensor Tasking for Space Situational Awareness

Tarunraj Singh
RESEARCH FOUNDATION OF STATE UNIVERSITY OF NEW YORK THE

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Air Force Research Laboratory
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Tarunraj Singh, Manoranjan Majji, Puneet Singla

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14. ABSTRACT

This report documents the major project findings during the duration of the AFOSR Award titled Optimal Sensor Tasking for Enhanced Space Situational Awareness. Continuing work on Gaussian mixture model (GMM), we have developed an adaptive mechanism to automatically select the architecture of the Gaussian mixture model. The developed method is the only method in the literature which constrained the GMM approximation to satisfy the Fokker-Planck-Kolmogorov equation. We have developed computationally efficient semi-analytical approaches for uncertainty propagation while making use of tools from convex optimization. The CUT algorithm has been used to derive information theoretic sensor tasking framework. Our work clearly shows the benefit of using mutual information as a tasking metric as opposed to Fisher information generally
used in the literature. Also with the Conjugate Unscented Transform for uncertainty propagation, these approaches provide a computationally efficient framework to simultaneously task sensors and track multiple targets.

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1 FINAL REPORT

This report documents the major project findings during the duration of the AFOSR Award titled *Optimal Sensor Tasking for Enhanced Space Situational Awareness.* The main accomplishments of our work during the duration of the project are summarized as follows:

- **Uncertainty Quantification:** Continuing on our work on Gaussian mixture model (GMM), we have developed an adaptive mechanism to automatically select the architecture of the Gaussian mixture model. Unlike other methods in literature, we used the Kolmogorov equation error as a feedback to automatically select the Gaussian kernels in the mixture model. Furthermore, we have exploited our prior work in *how nonlinear is it?* to develop a mechanism to find the direction in which uncertainty growth is maximum. The developed method is the only method in the literature which constrained the GMM approximation to satisfy the Fokker-Planck-Kolmogorov equation.

  In a parallel effort, we have developed computationally efficient semi-analytical approaches for uncertainty propagation while making use of tools from convex optimization. In particular, non-product cubature method known as Conjugate Unscented Transformation (CUT) is developed to compute desired order statistical moments in a computationally efficient manner. CUT method is also exploited to develop sparse collocation methods to compute higher order state transition matrices and state density function with guaranteed convergence. It should be noted that the CUT method provide the *minimum number of simulation points* in the literature to accurately compute the desired order moments. The CUT method was successfully used to compute the probability of collision between two space objects objects with only 1490 simulations while 10 million Monte Carlo simulations were required to achieve the same accuracy.

  Following is the list of publications in this effect:

  1. M. Mercurio, "Sparse Collocation Methods for Solving High Dimension PDEs in Estimation and Control of Dynamical Systems," Ph.D. Dissertation, Department of Mechanical & Aerospace Engineering, University at Buffalo, January, 2017.
  2. K. Vishwajeet and P. Singla, "Adaptive Split-Merge based Gaussian Mixture Model Approach for Uncertainty Propagation," AIAA Journal of Guidance, Control and Dynamics, Vol. 41, No. 3, 2018.
  3. N. Adurthi, P. Singla, and T. Singh, "Conjugate Unscented Transformation: Applications to Estimation and Control," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 140, No. 3, 2018.
  4. M. Mercurio, and P. Singla, "A Tree-Based Approach for Efficient and Accurate Conjunction Analysis," Computer Modeling in Engineering & Sciences, Special Issue on Computational Methods in Celestial Mechanics, Vol. 111, Issue 3, pp. 229?256, Jan. 2016.
  5. N. Adhurthi and P. Singla, "A Conjugate Unscented Transformation Based Approach for Accurate Conjunction Analysis," AIAA Journal of Guidance, Control and Dynamics, Vol. 38, Issue 9, pp. 1642?1658, Sep. 2015 .
  6. M. Mercurio, M. Majji and P. Singla, "How Non-Gaussian Is it?: Applications to Astrodynamics," Special Issue of Journal of Astronautical Sciences, In Review.
  7. D. Ciliberto, M. Majji and P. Singla, "Extended Kalman Filtering in Regularized Co- ordinates: Applications to Astrodynamics ," Special Issue of Journal of Astronautical Sciences, To be Submitted.
  8. N. Adurthi, and M. Majji, "Uncertain Lambert Problem," Special Issue of Journal of Astronautical Sciences, To be Submitted.

9. Z. Hall, T. Lee and P. Singla, “Higher Order Polynomial Series Expansion for Uncertain Lambert Problem,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
10. D. Gueho, P. Singla and R. Melton, “Learning Capabilities of Neural Networks and Keplerian Dynamics,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
11. T. Lee, M. Majji and P. Singla, “A High Order Filter For Estimation of Nonlinear Dy- namic Systems,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19- 23 August 2018.
12. M. Mercurio, M. Majji and P. Singla, “How Non-Gaussian Is it?: Applications to Astrodynamics,” John L. Junkins Dynamic Systems Symposium, College Station, TX, 20-21 May 2018.
13. N.Adurthi,and M.Majji, “Method of Characteristics based Nonlinear Filter: Applications to Space Object Tracking,” 2018 AIAA/AASAstrodynamicsSpecialist Conference, Snowbird, UT, 19-23 August 2018
14. Lee, T.W., Singla, P., Majji, M., “Conjugate Unscented Transform Approach to Compute High Order State Transition Matrices: Applications to Uncertainty Propagation” presented at AAS/AIAA Astrodynamics Specialist Conference, Stevenson, WA, 2017.
15. M. Mercurio and P. Singla, “A Tree-Based Approach for Efficient and Accurate Conjunction Analysis,” 2015 International Conference on Computational & Experimental Engineering & Sciences (ICCES), Reno, NV, July 20?24, 2015.
16. P. Singla, and Manoranjan Majji, “How Non-Gaussian Is It?,” 2015 International Conference on Computational & Experimental Engineering & Sciences (ICCES), Reno, NV, July 20?24, 2015.
17. N. Adurthi and P. Singla, “Conjugate Unscented Transform Based Approach for Accurate Conjunction Analysis,” 2015 International Conference on Computational & Experimental Engineering & Sciences (ICCES), Reno, NV, July 20?24, 2015, Keynote Paper.

It should be mentioned that that our prior work in this area has garnered a lot of attention and is among the highest cited paper in this area.

- G. Terejanu, P. Singla, T. Singh, and P. D. Scott, “Uncertainty Propagation for Nonlinear Dynamic Systems Using Gaussian Mixture Models,” Journal of Guidance, Control, and Dynamics, Vol. 31, No. 6 (2008), pp. 1623-1633. **(citations: 128)** *(paper from earlier YIP grant)*
- G. Terejanu, P. Singla, T. Singh, and P. D. Scott, “Adaptive Gaussian sum filter for nonlinear Bayesian estimation,” IEEE Transactions on Automatic Control 56.9 (2011): 2151-2156. **(citations: 96)** *(paper from earlier YIP grant)*

- **Sensor Tasking:** The CUT algorithm has been used to derive information theoretic sensor tasking framework. Our work clearly shows the benefit of using mutual information as a taskign metric as opposed to Fisher information generally used in the literature. As optimization of sensor modalities leads to a mixed-integer programming problem which is combinatorial in nature, greedy algorithms are developed to recursively optimize sub problems. Particularly, methods that are greedy in time, greedy in sensors and greedy in objects are developed in a moving horizon approach for sensor tasking problem. Coupled with the Conjugate Unscented Transform for uncertainty propagation, these approaches provide a computationally efficient framework to simultaneously task sensors and track multiple targets.

  Following is the list of publications in this respect:

1. N. Adurthi, "Conjugate Unscented Transformation based Framework for Uncertainty Quantification, Nonlinear Filtering, Optimal Control and Dynamic Sensing," Ph.D. Dissertation, Department of Mechanical & Aerospace Engineering, University at Buffalo, January, 2016.
2. N. Adurthi, P. Singla, and M. Majji, "Sparse Approximation Based Collocation Scheme for Nonlinear Optimal Feedback Control Design," AIAA Journal of Guidance, Control, and Dynamics, vol. 40, no. 2, pp. 248–264, Feb. 2017.
3. M. Mercurio, M. Majji and P. Singla, "A Conjugate Unscented Transform-Based Scheme for Optimal Control with Terminal State Constraints," 2018 American Control Conference, Milwaukee, WI, June 27?29, 2018.
4. N. Adurthi, P. Singla and M. Majji, "Conjugate Unscented Transform Based Approach for Dynamic Sensor Tasking and Space Situational Awareness," 2015 American Control Conference, Chicago, IL, July 1–3, 2015.
5. N. Adurthi, P. Singla and M. Majji, "Conjugate unscented transformation based orbital state estimation and sensor tasking for efficient space surveillance." In AIAA/AAS Astrodynamics Specialist Conference, p. 4168. 2014.
6. M. Mercurio, N. Adurthi, P. Singla, and M. Majji, "A Collocation-Based Approach to Solve the Finite Horizon Hamilton-Jacobi-Bellman Equation," 2016 American Control Conference, Boston, MA, July 6–8, 2016.

In addition to aforementioned papers, two more journal papers are in the editing stage and are expected to be submitted by end of this year.

- **Data Association:** PIs have incorporated the CUT algorithms with well-known Joint Probability Data Association (JPDA) and Multiple Hypothesis Tracking (MHT) framework to develop computationally efficient framework for accurate data association. These new frameworks are based upon the premises that accurate uncertainty quantification leads to accurate data association. Furthermore, the solution of uncertain Lambert problem is being incorporated in the data association framework to consider only the physically viable hypotheses for data association. Our preliminary studies shows that the incorporation of uncertain Lambert problem in DA framework leads to significant reduction of hypotheses to be consider.

  Following is the list of publications in this effect:

  1. N. Adurthi, M. Majji, Utkarsh R. Mishra and and P. Singla, "Multiple Hypothesis Tracking and Joint Probabilistic Data Association Filters for Multiple Space Object Tracking," 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
  2. N. Adurthi, M. Majji, U. R. Mishra, and P. Singla, "Conjugate Unscented Transform Based Joint Probability Data Association," presented at the 2017 AIAA/AAS Astrodynamics Specialist Conference, Stevenson, WA, 2017.

  In addition to aforementioned papers, two more journal papers are in the editing stage and are expected to be submitted by end of this year.

- **Transition Activities:** PIs have established active technical interchanges with researchers from AFRL-Kirtland (Dr. Ryan Weisman) and AFRL-Rome (Dr. Joseph Raquepas). In this respect, PIs have conducted two days summer workshop at AFRL, Kirtland, NM to transition developed algorithms to AFRL researchers. PIs have also established contacts with industry partners (Applied Defense Services (ADS), Inc. & ExoAnalytic Inc.) to transition their research work to industry. PI's former student, Michael Mercurio (supported through this grant) is currently working full time at ADS. One of the student, Zachary Hall, supported through this effort has received the 2018 SMART fellowship. PIs have also exploited many existing mechanisms such as AFRL Space Scholar Program and summer internships at industry to help with transition activities.

High Fidelity Uncertainty Quantification and its Applications in SSA.

PI: Puneet Singla
Co-PI: Manoranjan Majji

Department of Aerospace Engineering
The Pennsylvania State University
Texas A&M University

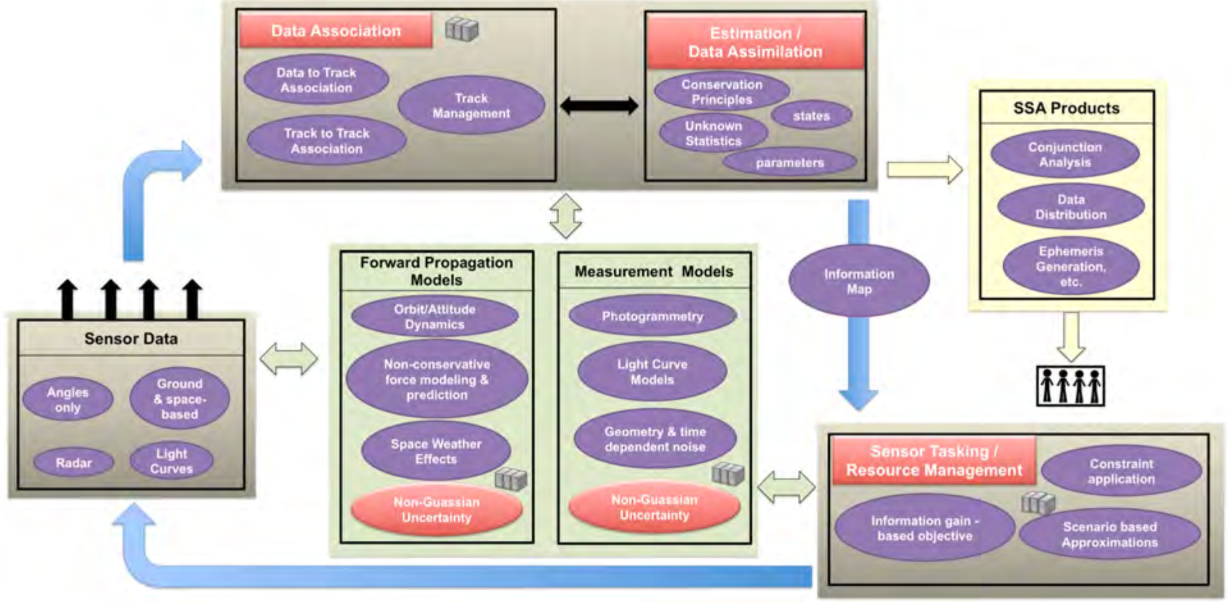
AFOSR Grant No.: AFOSR FA9550-15-1-0313
Start Date: 15 July, 2015.
PM: Dr. Stacie Williams

2018 Remote Sensing and Imaging Physics Program Review
Albuquerque, New Mexico, 4th – 6th September, 2018.

Space Object Tracking

Where are we going?

***Research Philosophy**: A unique combination of data association, tracking and resource management algorithms for effective and efficient SSA may not exist.*



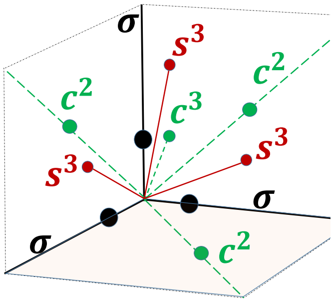
Semi Analytical Means for Uncertainty Propagation

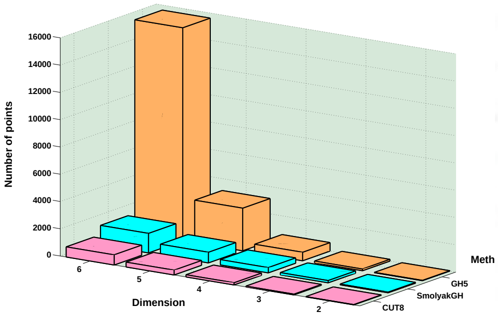
- ✓ **We have developed: Sparse Collocation, Adaptive Gaussian Mixture Models (AGMM), Method of Characteristics Methods for uncertainty propagation by solving the Fokker Planck Kolmogorov (FPK) and Liouville equations for large scale systems.**
- ✓ **Our tools employ convex optimization tools for guaranteed convergence.**
- ✓ **Non-product cubature method known as Conjugate Unscented Transformation (CUT) has been developed to compute desired order statistical moments in a computationally efficient manner.**
- ✓ ***Higher-order State Transition Tensors (STTs) are evaluated with the help of CUT.***

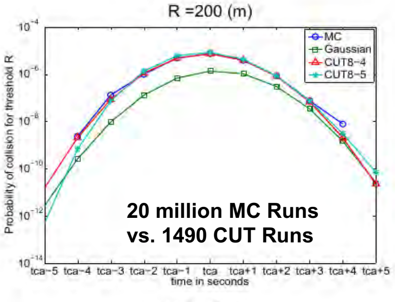
- **G. Terejanu, P. Singla, T. Singh, and P. D. Scott. "Uncertainty Propagation for Nonlinear Dynamic Systems Using Gaussian Mixture Models", Journal of Guidance, Control, and Dynamics, Vol. 31, No. 6 (2008), pp. 1623-1633. (128) (paper from earlier YIP grant)**
- **G. Terejanu, P. Singla, T. Singh, and P. D. Scott, "Adaptive Gaussian sum filter for nonlinear Bayesian estimation." *IEEE Transactions on Automatic Control* 56.9 (2011): 2151-2156. (96) (paper from earlier YIP grant)**

Conjugate Unscented Transformation

“Optimal” Quadrature Approach







20 million MC Runs vs. 1490 CUT Runs

(c) $R = 200m$

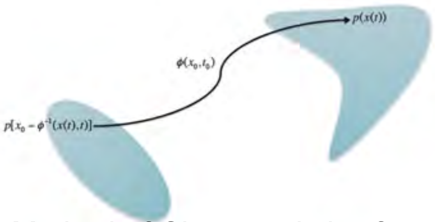
Adurthi, Singla, 2015

- ***CUT:*** An ***efficient quadrature scheme*** for the determination of high dimension expectation integrals involving symmetric pdfs.
  - –*non-product cubature rule.*
  - –extends *unscented transformation rules* to compute higher order moments.
  - *–20 millions MC Runs vs. 1490 CUT Runs*.
- *CUT* provides a computationally efficient tool for ***accurate uncertainty propagation.***
  - – *allows one to trade-off between accuracy and computational resources!!*

Sparse Collocation Methods

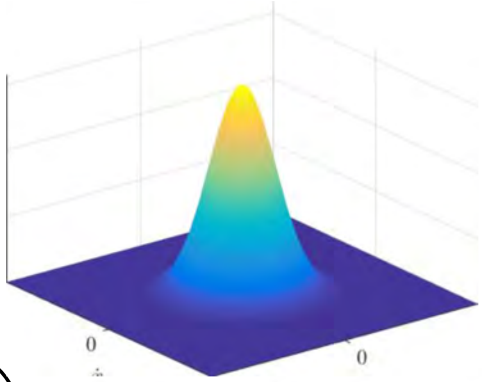
State PDF Approximation

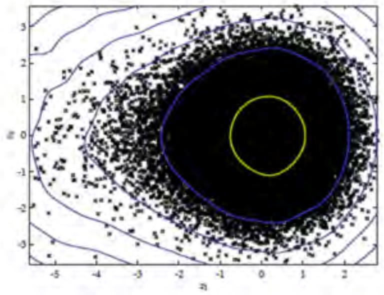
Semi Analytical Machine Learning Tools for Uncertainty Propagation



Method of Characteristics for Domain Tracking

1. Collocation points generated using Conjugate Unscented Transformation (CUT)
2. Sparsity enhancing l_1 optimization tools to ensure parsimonious basis functions
3. No assumptions made on structure of log-PDF





Higher Order Filter
Accurate Uncertainty Propagation

- **JMEKF Moment Propagation**

$$\delta x_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \Phi_{i,k_1,k} \quad \cdot \quad \delta x_{k_1}^0 \ldots \delta x_{k_p}^0$$

$$\left. \frac{\partial^p x_i}{\partial x_{k_p}^0} \right|_{x=x^*}$$

Even for someone who enjoys taking partials, this is way too much!!!

STTs Differential I

$$\dot{\Phi}_{i,j_1} = f_{i,k_1}^* \Phi_{k_1,j_1}$$

$$\dot{\Phi}_{i,j_1 j_2} = f_{i,k_1}^* \Phi_{k_1,j_1 j_2}$$

$$\dot{\Phi}_{i,j_1 j_2 j_3} = f_{i,k_1}^* \Phi_{k_1,j_1 j_2 j_3} + f_{i,k_1 k_2}^*$$
$$+ f_{i,k_1 k_2 k_3}^* \Phi_{k_1,j_1} \Phi_{k_2,j_2} \Phi_{k_3,j_3}$$

$$\dot{\Phi}_{i,j_1 j_2 j_3 j_4} = f_{i,k_1}^* \Phi_{k_1,j_1 j_2 j_3 j_4}$$
$$+ f_{i,k_1 k_2}^* (\Phi_{k_1,j_1 j_2 j_3} \Phi_{k_2,j_4} + \Phi_{k_1,j_1 j_2 j_4} \Phi_{k_2,j_3} + \Phi_{k_2,j_2 j_3,j_4} \Phi_{k_1,j_1} +$$
$$+ f_{i,k_1 k_2}^* (\Phi_{k_1,j_1 j_2} \Phi_{k_2,j_3 j_4} + \Phi_{k_1,j_1 j_3} \Phi_{k_2,j_2 j_4} + \Phi_{k_2,j_2 j_3} \Phi_{k_1,j_1 j_4})$$
$$+ f_{i,k_1 k_2 k_3}^* (\Phi_{k_1,j_1 j_2} \Phi_{k_2,j_3} \Phi_{k_3,j_4} + \Phi_{k_1,j_1 j_3} \Phi_{k_2,j_2} \Phi_{k_3,j_4} + \Phi_{k_2,j_2 j_3} \Phi_{k_1,j_1} \Phi_{k_3,j_4})$$
$$+ f_{i,k_1 k_2 k_3}^* (\Phi_{k_1,j_1 j_4} \Phi_{k_2,j_3} \Phi_{k_3,j_3} + \Phi_{k_1,j_1} \Phi_{k_2,j_2 j_4} \Phi_{k_3,j_3} + \Phi_{k_2,j_2} \Phi_{k_1,j_1} \Phi_{k_3,j_3 j_4})$$
$$+ f_{i,k_1 k_2 k_3 k_4}^* \Phi_{k_1,j_1} \Phi_{k_2,j_2} \Phi_{k_3,j_3} \Phi_{k_4,j_4}$$

with initial conditions

ODEs up to 4th order

$$\Phi_{i,j}(t_0) = \delta_{i,j}, \quad \Phi_{i,j_1 j_2 \cdots j_p}(t_0) = 0, \forall p > 1$$

Moment Propagation (Vector-Matrix models)

$$\psi_P^{(2)t} = [\Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)}] \psi_P^{(2)+} + \frac{1}{2!} \begin{bmatrix} \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} \\ +\Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \end{bmatrix} \psi_P^{(3)+}$$

$$\ldots \Theta_\Phi^{(4)} \otimes \Theta_\Phi^{(2)} \ldots \Bigg) \psi_P^{(4)+} + \left(\frac{1}{4!} \begin{bmatrix} \Theta_\Phi^{(5)} \otimes \Theta_\Phi^{(2)} \\ +\Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(5)} \end{bmatrix} + \frac{1}{2!2!} \begin{bmatrix} \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(4)} \\ +\Theta_\Phi^{(4)} \otimes \Theta_\Phi^{(3)} \end{bmatrix} \right) \phi_P^{(5)+}$$

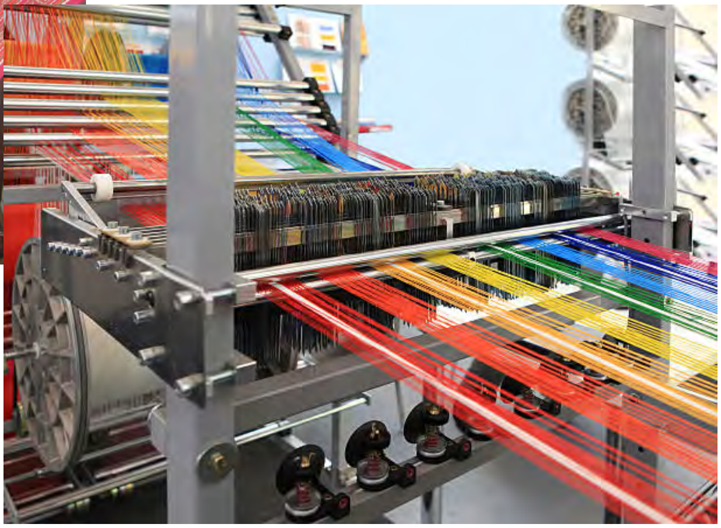
$$\ldots \begin{bmatrix} \ldots \otimes \Theta_\Phi^{(2)} \\ \ldots \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \\ \ldots \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} \end{bmatrix} \phi_P^{(4)+}$$

$$+ \left(\frac{1}{3!} \begin{bmatrix} \Theta_\Phi \ldots \\ +\Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \otimes \ldots \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \end{bmatrix} + \frac{1}{2!2!} \begin{bmatrix} \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \\ +\Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} + \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(3)} \end{bmatrix} \right) \phi_P^{(5)+} + \cdots$$

Majji, Junkins & Turner 2008

Out with the old …
in with the new !!!!

Can we supplant the high order partial generation process ???



Higher Order State Transition Tensors

Non-Intrusive Approach

Dynamical System

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t))$$

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t f(\tau, \mathbf{x}(\tau))d\tau = \psi(t, \mathbf{x}_0)$$

Departure motion dynamics

$$\delta\mathbf{x}(t) = \psi(t, \mathbf{x}_0 + \delta\mathbf{x}_0) - \psi(t, \mathbf{x}_0) \approx \mathbf{\Phi}(t, t_0)\delta\mathbf{x}_0$$

$$\delta\mathbf{x}(t) \approx \sum_{N_1, N_2, \ldots, N_n} \frac{\delta x_{0_1}^{N_1} \delta x_{0_2}^{N_2} \cdots \delta x_{0_n}^{N_n}}{N_1! \, N_2! \cdots N_n!} \frac{\partial^{N_1+N_2+\cdots+N_n}}{\partial x_{0_1}^{N_1} \partial x_{0_2}^{N_2} \cdots \partial x_{0_n}^{N_n}} \psi(t, \mathbf{x}_0)$$

this motivates

Polynomial representations of the flow:

$$\delta\mathbf{x}(t) \approx \sum_{i=1}^{m} c_i(t) p_i(\delta\mathbf{x}_0)$$

Weighted norm Minimization:

$$\min_{c_i(t)} J = \frac{1}{2} \int \left(\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)\right)^T \left(\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)\right) \rho(\mathbf{x}_0) d\delta\mathbf{x}_0$$

$$= \frac{1}{2} \langle (\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)), (\delta\mathbf{x}(t) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)) \rangle$$

$$\mathbf{M}(t)\mathbf{c}(t) = \mathbf{b}(t)$$

$$M_{ij}(t) = \langle p_i(\delta\mathbf{x}_0), p_j(\delta\mathbf{x}_0) \rangle$$

$$b_i(t) = \langle \delta\mathbf{x}(t), p_i(\delta\mathbf{x}_0) \rangle$$

$$= \sum_{i=1}^{N} w_i p_i(\boldsymbol{\xi}_i) \delta\mathbf{x}(t, \boldsymbol{\xi}_i)$$

use quadratures for inner product evaluation

- State transition tensors are equivalent to the coefficients of this expansion, i.e., $c_i(t)$
- Derivative free approach to evaluate sensitivities over a domain of interest.
  - Domain of interest is represented by the state PDF.
- ***Tensors are evaluated using minimal number of model evaluations.***

Lee, Majji, & Singla, 2018

Higher Order State Transition Tensors

Non-Intrusive Approach

Step 5. Compute prior statistical moments

$$\mu^-(t) = E[x(t)] = \sum_{i=1}^{q} w_i X_i^-$$

$$\Sigma^{(2)-}(t) = E[(x(t)-\mu^-(t))^2] = \sum_{i=1}^{q} w_i (X_i^- - \mu^-)^2 \ : 2nd \text{ moment}$$

$$\Sigma^{(3)-}(t) = E[(x(t)-\mu^-(t))^3] = \sum_{i=1}^{q} w_i (X_i^- - \mu^-)^3 \ : 3rd \text{ moment}$$

$$\Sigma^{(4)-}(t) = E[(x(t)-\mu^-(t))^4] = \sum_{i=1}^{q} w_i (X_i^- - \mu^-)^4 \ : 4th \text{ moment}$$

$$\vdots$$

<=>

STTs

$$\begin{aligned}
P^{(2)}(t) &= E[\delta x^2(t)] \\
&= \phi_{(1)}^2 P^{(2)+}(t_k) + \phi_{(1)}\phi_{(2)} P^{(3)+}(t_k) \\
&+ \left[\frac{1}{2!\,2!}\phi_{(2)}\phi_{(2)} + \frac{2}{3!}\phi_{(3)}\phi_{(1)}\right] P^{(4)+}(t_k) + HOT \\
P^{(3)}(t) &= E[\delta x^3(t)] \\
&= \phi_{(1)}^3 P^{(3)+}(t_k) + \frac{3}{2!}\phi_{(2)}\phi_{(1)}^2 P^{(4)+}(t_k) + HOT \\
P^{(4)}(t) &= E[\delta x^4(t)] \\
&= \phi_{(1)}^4 P^{(4)+}(t_k) + HOT
\end{aligned}$$

- **Step 8. Compute the high order moments**

$$\Sigma^{(2)+} = \sum_{i=1}^{q} w_i \delta Z_i^{+2} + K^2 R$$

$$\Sigma^{(3)+} = \sum_{i=1}^{q} w_i \delta Z_i^{+3} + K^3 E[v^3]$$

$$\Sigma^{(4)+} = \sum_{i=1}^{q} w_i \delta Z_i^{+4} + 6K^2\Sigma^{(2)-}R + K^4 E[v^4]$$

$$\vdots$$

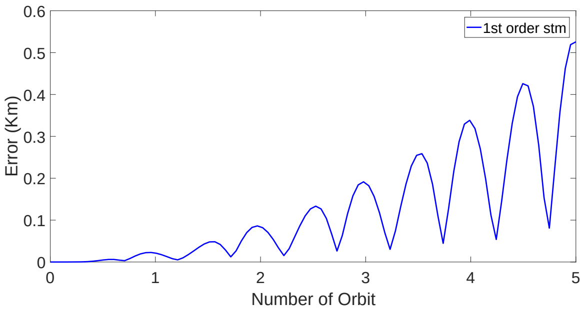
<=>

$$\begin{aligned}
\psi_P^{(3)t} = [\Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)}]\psi_P^{(3)+} + \frac{1}{2!}\begin{bmatrix} \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \\ +\Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \\ +\Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} \end{bmatrix} \phi_P^{(4)+} \\
+ \left(\begin{array}{l} \frac{1}{3!}\begin{bmatrix} \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(4)} \otimes \Theta_\Phi^{(2)} \\ +\Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(4)} + \Theta_\Phi^{(4)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(2)} \end{bmatrix} \\ +\frac{1}{2!2!}\begin{bmatrix} \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \\ +\Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} + \Theta_\Phi^{(2)} \otimes \Theta_\Phi^{(3)} \otimes \Theta_\Phi^{(3)} \end{bmatrix} \end{array} \right) \phi_P^{(5)+} + \cdots
\end{aligned}$$

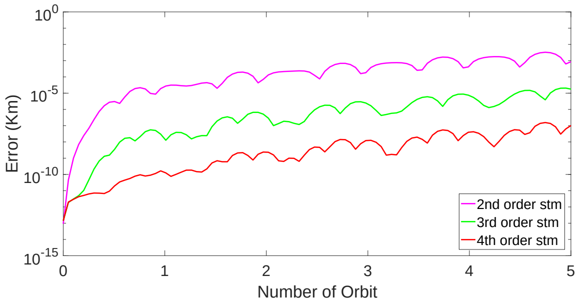
Lee, Majji, & Singla, 2018

Higher Order State Transition Tensors

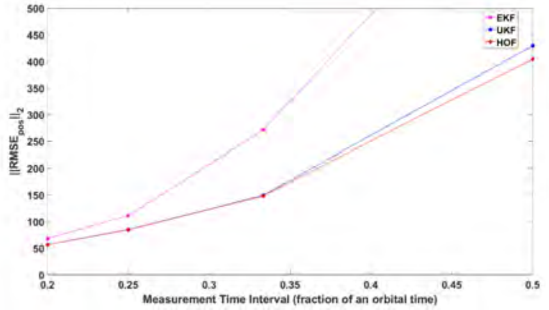
Non-Intrusive Approach

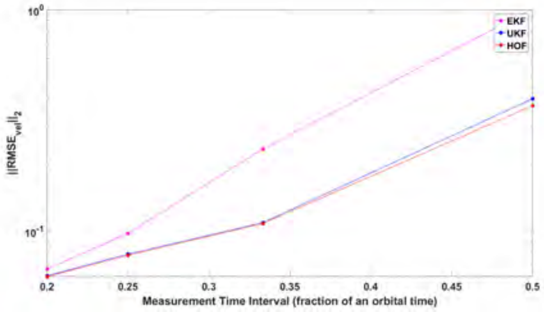


(a) 2-norm Error by 1st order STM



(b) 2-norm Error by higher order STM

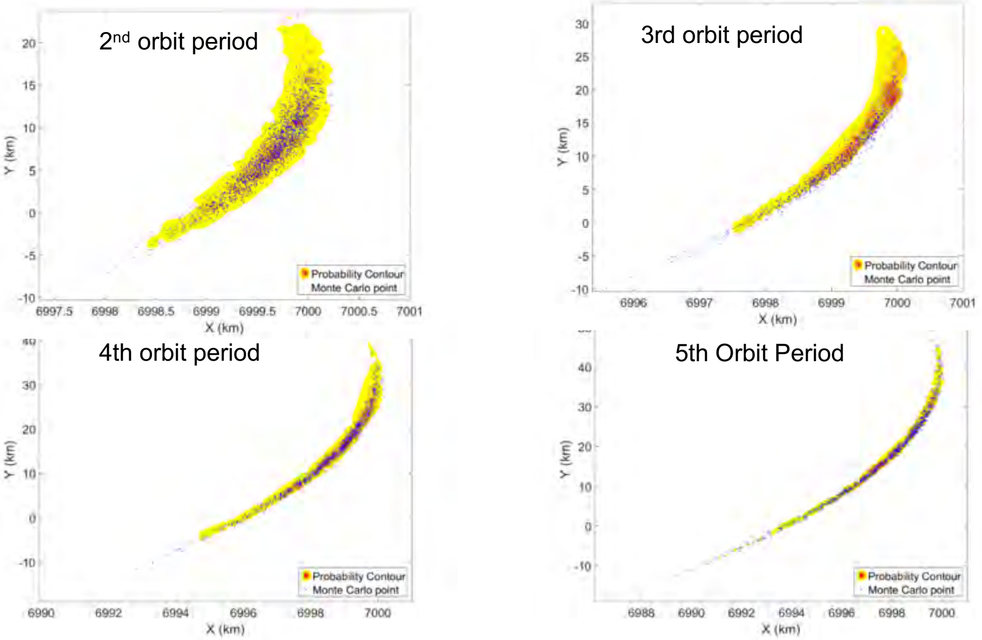




Higher Order State Transition Matrices

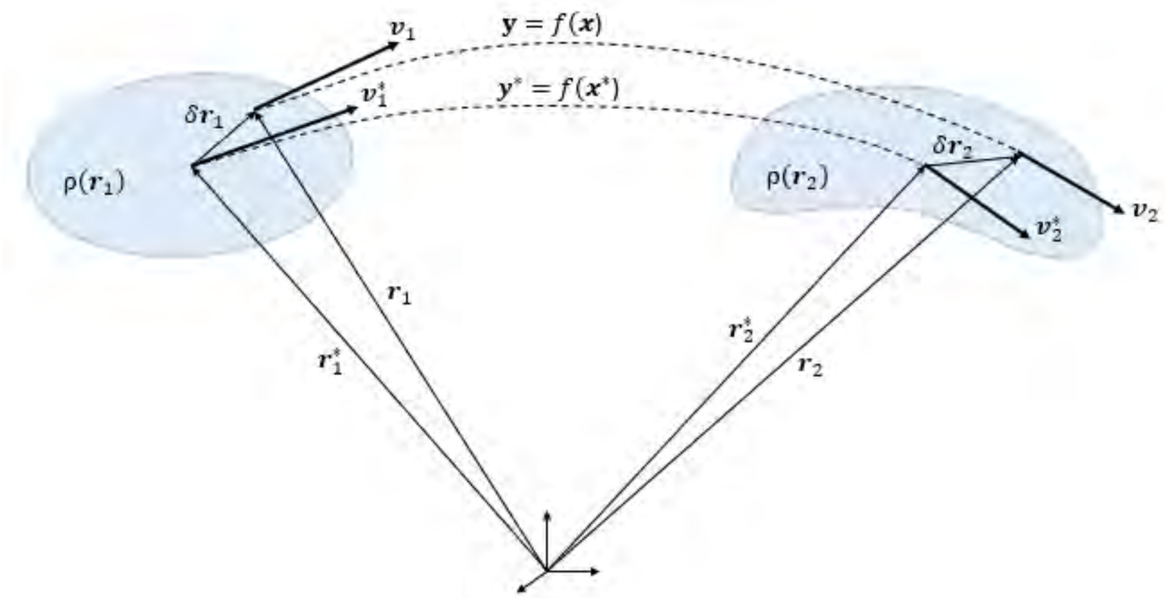
Non-Intrusive Approach

Method of Characteristics: $$p(\mathbf{x}(t)) = \rho\left(\underbrace{\psi^{-1}(\mathbf{x}(t))}_{\psi_0(x_t)}\right)\left\|\frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_0}\right\|^{-1}_{\mathbf{x}_0=\psi^{-1}(\mathbf{x}(t))}$$



Uncertain Lambert Problem

Higher Order Sensitivities



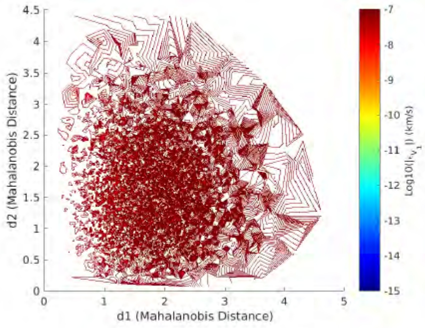
- *Higher order STTs are computed through 745 CUT points.*
    - *resulting distribution is validated against 100,000MC runs.*

Hall, Singla, 2018

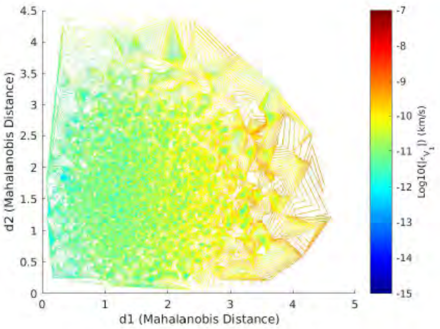
Uncertain Lambert Problem

Higher Order Sensitivities

Test Case : GTO

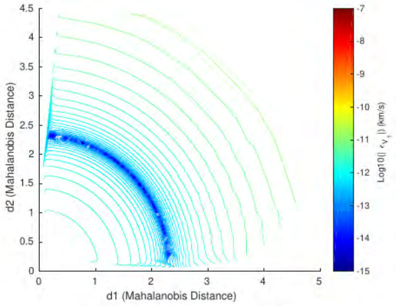


(b) 1st Order Approximation Error



(c) 2nd Order Approximation Error

Mahalanobis Distance Representation



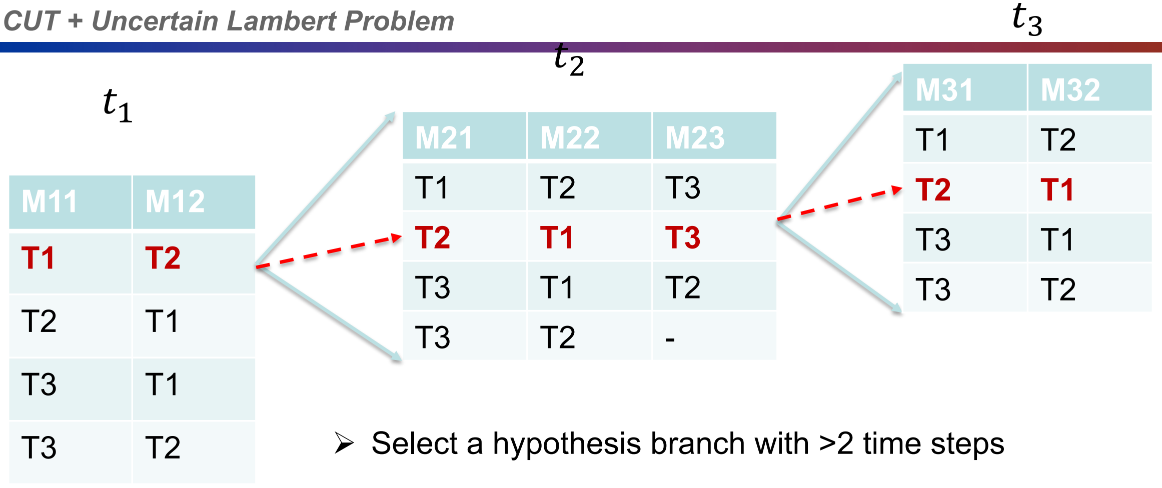
(d) 3rd Order Approximation Error

Data Association:

- ✓ Handshake of CUT with Joint Probability Data Association (JPDA) & Multiple Hypothesis Tracking (MHT).
- ✓ Accurate bearing only data association.
- ✓ Gating based upon uncertain Lambert problem.

Data Association

CUT + Uncertain Lambert Problem



t_1

M11	M12
T1	**T2**
T2	T1
T3	T1
T3	T2

t_2

M21	M22	M23
T1	T2	T3
T2	**T1**	**T3**
T3	T1	T2
T3	T2	-

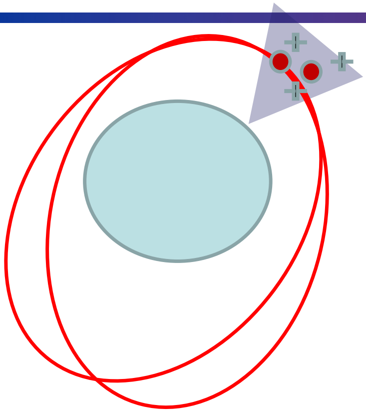
t_3

M31	M32
T1	T2
T2	**T1**
T3	T1
T3	T2

- Select a hypothesis branch with >2 time steps
- Select the track of measurements associated with target T1
- Measurement Track → M11 , M22, M32
- IOD methods to check for feasibility → prune branch
- For example : Solve Lambert Problem between pairs of measurements → consistent orbital elements

Data Association

Covariance Realism

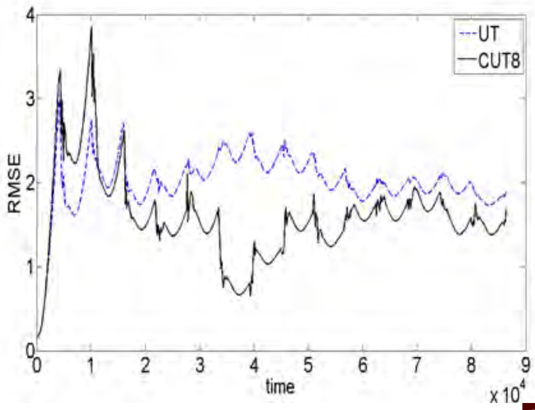


- Single sensor with limited Field of View

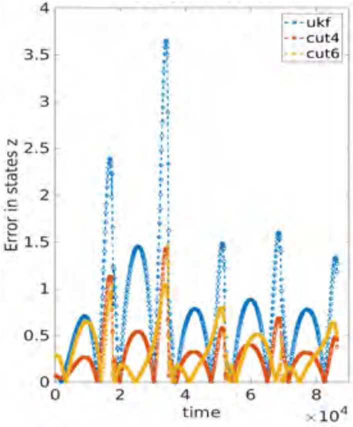
Better UQ ➔ Better Data Association ➔ Better State estimation

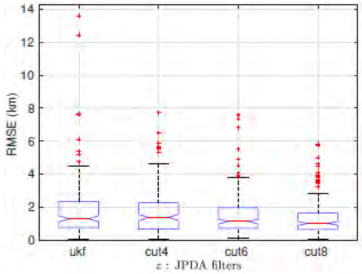
Satellite 1	Satellite 2
$a = 7003.025$	$a = 7018.6397$
$e = 0.181$	$e = 0.11480$
$E = 0.189$	$E = 0.25602$
$w = 0.785$	$w = 0.7853$
$i = 0.207$	$i = 0.18471$
$\Omega = 0.523$	$\Omega = 0.523$

Use high order moments



Data Association Results





(c) State z

General observations
- ➢ High order CUT filters lead to improved association
- ➢ Improved association leads to improved state estimation
- ➢ Estimation errors decrease …

- ➢ Covariance realism is also improved by increasing quadrature order.
- ➢ Gating based upon uncertain Lambert problem decreases the number of hypotheses.

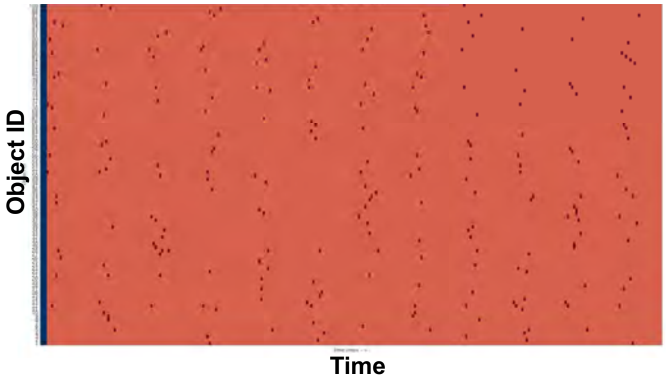
Sensor Tasking:

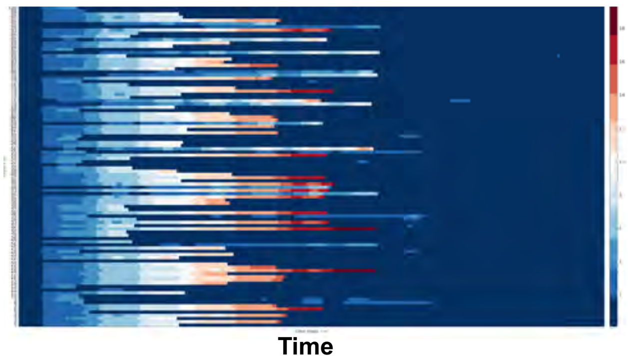
- ✓ **CUT + Information Theory + Mixed Integer Programming (MIP).**
- ✓ **Appropriate Simplifications for tractable numerical solution for MI optimization.**
  - ➢ *Greedy in Time AND/OR Targets AND/OR Sensors approximations to the cost function*

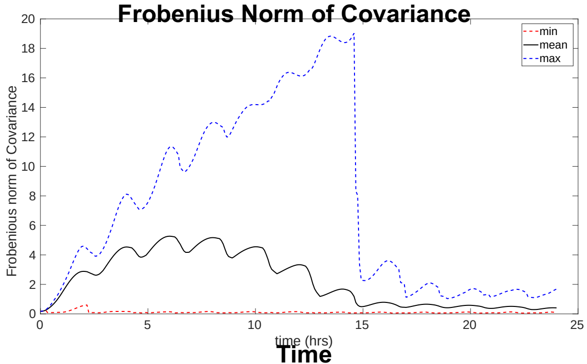
Optimal Information Collection

3D Satellite Tracking scenario➔ 100 satellites and 3 sensors

- Greedy in target.
- It took around 107 seconds on laptop computer to do tasking over next 24 hrs. !!!




Trace of Covariance


Frobenius Norm of Covariance

Conference Publications:

1. Adurthi, N., Majji, M., Mishra U. R., Singla, P., “Conjugate Unscented Transform Based Joint Probability Data Association,” presented at AAS/AIAA Astrodynamics Specialist Conference, Stevenson, WA, 2017.
2. Mirzaei, M., Singla, P., Majji, M., “A Sparse Collocation Approach for Optimal Feedback Control of Spacecraft Attitude Maneuvers,” presented at AAS/AIAA Astrodynamics Specialist Conference, Stevenson, WA, 2017.
3. Lee, T.W., Singla, P., Majji, M., “Conjugate Unscented Transform Approach to Compute High Order State Transition Matrices: Applications to Uncertainty Propagation” presented at AAS/AIAA Astrodynamics Specialist Conference, Stevenson, WA, 2017.
4. N. Adurthi, P. Singla and M. Majji, “Conjugate Unscented Transform Based Approach for Dynamic Sensor Tasking and Space Situational Awareness,” 2015 American Control Conference, Chicago, IL, July 1–3, 2015.
5. M. Mercurio and P. Singla, “A Tree-Based Approach for Efficient and Accurate Con- junction Analysis,” 2015 International Conference on Computational & Experimental En- gineering & Sciences (ICCES), Reno, NV, July 20–24, 2015.
6. P. Singla, and Manoranjan Majji, “How Non-Gaussian Is It?,” 2015 International Conference on Computational & Experimental Engineering & Sciences (ICCES), Reno, NV, July 20–24, 2015.
7. N. Adurthi and P. Singla, “Conjugate Unscented Transform Based Approach for Accurate Conjunction Analysis,” 2015 International Conference on Computational & Experimental Engineering & Sciences (ICCES), Reno, NV, July 20–24, 2015, Keynote Paper.

Journal Publications:

1. Vishwajeet, K. and Singla, P., “Adaptive Split-Merge based Gaussian Mixture Model Approach for Uncertainty Propagation,” AIAA JGCD, Vol. 41, No. 3, 2018.
2. Wong, X., Majji, M., “Extended Kalman Filter for Stereo Vision Based Localization and Mapping Applications,” ASME Journal of Dynamic Systems, Measurement and Control, Vol. 140, No. 3, March 2018.
3. Adurthi, N, Singla, P., Singh, T., “Conjugate Unscented Transformation: Applications to Estimation and Control,” ASME Journal of Dynamic Systems, Measurement and Control, Vol. 140, No. 3, March 2018.
4. K. Vishwajeet, P. Singla, and M. Majji, “Random Matrix Based Approach for Uncertainty Analysis of the Eigenvalue Realization Algorithm,” AIAA Journal of Guidance, Control & Dynamics, Vol. 40, No. 8, pp. 1877-1891, August 2017.
5. M. Mercurio, and P. Singla, “A Tree-Based Approach for Efficient and Accurate Conjunction Analysis,” Computer Modeling in Engineering & Sciences, Special Issue on Computational Methods in Celestial Mechanics, Vol. 111, Issue 3, pp. 229–256, Jan. 2016.
6. N. Adurthi, P. Singla and M. Majji, “Sparse Approximation based Collocation Scheme for Nonlinear Optimal Feedback Control Design,” AIAA Journal of Guidance, Control and Dynamics, Vol. 40, Special Issue on Computational Guidance and Control, No. 2, Feb. 2017. **(3)**
7. N. Adhurthi and P. Singla, “A Conjugate Unscented Transformation Based Approach for Accurate Conjunction Analysis,” AIAA Journal of Guidance, Control and Dynamics, Vol. 38, Issue 9, pp. 1642–1658, Sep. 2015 . **(16)**
8. K. Vishwajeet, P. Singla and M. Jah, “Nonlinear Uncertainty Propagation for Perturbed Two-Body Orbits,” AIAA Journal of Guidance, Control and Dynamics, Vol. 37, Issue 5, pp. 1415-1425, Sep. 2014. **(29)**

Conference Publications: (Continued)

8. M. Mercurio, M. Majji and P. Singla, “A Conjugate Unscented Transform-Based Scheme for Optimal Control with Terminal State Constraints,” 2018 American Control Conference, Milwaukee, WI, June 27–29, 2018.
9. D. Ciliberto, M. Majji and P. Singla, “Extended Kalman Filtering in Regularized Coordinates: Applications to Astrodynamics ,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
10. N. Adurthi, M. Majji, Utkarsh R. Mishra and and P. Singla, “Multiple Hypothesis Tracking and Joint Probabilistic Data Association Filters for Multiple Space Object Tracking,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
11. Z. Hall, T. Lee and P. Singla, “Higher Order Polynomial Series Expansion for Uncertain Lambert Problem,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
12. D. Gueho, P. Singla and R. Melton, “Learning Capabilities of Neural Networks and Keplerian Dynamics,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018.
13. T. Lee, M. Majji and P. Singla, “A High Order Filter For Estimation of Nonlinear Dy- namic Systems,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19- 23 August 2018.
14. M. Mercurio, M. Majji and P. Singla, “How Non-Gaussian Is it?: Applications to Astrodynamics,” John L. Junkins Dynamic Systems Symposium, College Station, TX, 20-21 May 2018.
15. N. Adurthi, and M. Majji, “Method of Characteristics based Nonlinear Filter: Applications to Space Object Tracking,” 2018 AIAA/AAS Astrodynamics Specialist Conference, Snowbird, UT, 19-23 August 2018

Journal Publications:

9. M. Mercurio, M. Majji and P. Singla, “How Non-Gaussian Is it?: Applications to Astrodynamics,” Special Issue of Journal of Astronautical Sciences, In Review.
10. N. Adurthi, M. Majji and P. Singla, “Information Theoretic Static Sensor Tasking: Applications to Space Situational Awareness,” AIAA Journal of Guidance, Control & Dynamics, To be Submitted.
11. N. Adurthi, M. Majji and P. Singla, “Information Theoretic Optimal Sensor Path Planning,” AIAA Journal of Guidance, Control & Dynamics, To be Submitted.
12. D. Ciliberto, M. Majji and P. Singla, “Extended Kalman Filtering in Regularized Co- ordinates: Applications to Astrodynamics ,” Special Issue of Journal of Astronautical Sciences, To be Submitted.
13. N. Adurthi, and M. Majji, “Uncertain Lambert Problem,” Special Issue of Journal of Astronautical Sciences, To be Submitted.
14. G. Terejanu, P. Singla, T. Singh, and P. D. Scott. "Uncertainty Propagation for Nonlinear Dynamic Systems Using Gaussian Mixture Models", Journal of Guidance, Control, and Dynamics, Vol. 31, No. 6 (2008), pp. 1623-1633. **(128) (paper from earlier YIP grant)**
15. G. Terejanu, P. Singla, T. Singh, and P. D. Scott, "Adaptive Gaussian sum filter for nonlinear Bayesian estimation." *IEEE Transactions on Automatic Control* 56.9 (2011): 2151-2156. **(96) (paper from earlier YIP grant)**

+ 3 Ph.D. Dissertations

Students Supported

- Dr. Nagavenkat Adurthi (Texas A&M)
- Dr. Michael Mercurio (ADS Inc.),
- Dr. Xue Iuan Wong (Ford Motor Company.)
- Dr. Kumar Vishwajeet (Delphi Advanced Engineering Center)
- Dr. Taewook Lee (Hitachi Research Center)

- ***Current Students***:
- *Zachary Hall (SMART 2018)*
- *Damien Gueho*
- *David Ciliberto*
- *David Schwab*
- *Utkarsh Ranjan Mishra*
- *Roshan Suresh Kumar.*

Future Research:

- **Data Association:**
  - **Exploiting the solution of uncertain Lambert problem.**
  - **Exploit machine learning tools to learn feasible hypothesis based upon numerical simulations.**

- **Reachability Set Calculations:**
  - **Exploit CUT algorithm to compute reachability sets for continuous as well as impulsive maneuver.**
  - **Exploit reachability calculations for maneuver detection & reconstruction.**
  - **Develop search strategies for optical sensors based upon reachability set calculations to find lost targets.**

- **Uncertainty Propagation**
  - **How to exploit dynamic system properties to reduce the effective dimension of sampling space.**
  - **Exploit regularized variables for uncertainty propagation.**