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Multi-scale Higher Order Methods for Underresolved Simulations Useful in Turbulence Modelling

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Mathematically rigorous techniques that are tailored for and integrated into specific application areas are necessary from both a mathematical and engineering perspective. One of these application areas is that of simulating turbulence. Simulating turbulence requires resolution of a significant range of scales, such as fine dissipative scales and scales containing most of the kinetic energy. These problems are found in atmospheric modelling as well as traditional aerodynamic applications. One of the key ingredients in simulating turbulence is that of passing numerical information from fine scales to coarse scales. That is, multi-scale/multi-resolution information. This proposal focuses on isolating the numerical errors in passing information between scales. This one-year initial phase established how the errors are improved when combining accuracy-enhancing post-processing with multi-resolution analysis.

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Abstract
Mathematically rigorous techniques that are tailored for and integrated into specific application areas are necessary from both a mathematical and engineering perspective. One of these application areas is that of simulating turbulence. Simulating turbulence requires resolution of a significant range of scales, such as fine dissipative scales and scales containing most of the kinetic energy. These problems are found in atmospheric modelling as well as traditional aerodynamic applications. One of the key ingredients in simulating turbulence is that of passing numerical information from fine scales to coarse scales. That is, multi-scale/multi-resolution information. This proposal focuses on isolating the numerical errors in passing information between scales. This one-year initial phase established how the errors are improved when combining accuracy-enhancing post-processing with multi-resolution analysis.

Status/Progress
A foundation phases of AFOSR funding for one-year was obtained to support this research was obtained in September 2019. This funding was used to support two months salary for the PI. Additionally, this funding was used to support travel and collaborative visits related to this grant.

This funding has allowed the PI to make the following contributions:

- Constructing reliable and accurate post-processors play a fundamental role in providing efficient error control through *a posteriori* error analysis.
- Construct reliable approximation refinement techniques that are able to *move information from a coarse mesh to a fine mesh with reduced errors*.

Key Findings

- The PI and her collaborators have shown that constructing accurate post-processors plays a fundamental role in a posteriori error analysis in order to provide efficient error control. In order to show this, they enforced Galerkin orthogonality of the post-processed solution. This orthogonal post-processor has an increased order in the L^2 norm. This finding was supported both theoretically and numerically. The numerical support utilized two types of post-processors – that of SIAC and SPR – for smooth and non-smooth initial data.
- Passing numerical information from fine scales to coarse scales and vice-versa is a key ingredient in multi-scale/multi-resolution information. It is desirable to isolate the numerical errors in passing information between scales. One way to do this is to exploit the multi-scale structure of the underlying numerical approximation while taking

advantage of the hidden accuracy of the approximation, specifically exploiting superconvergence to obtain more accurate multi-resolution analysis. One can enhance the quality of passing of information between scales by implementing the Smoothness-Increasing Accuracy-Conserving (SIAC) Filtering combined with multi-wavelets. This includes the ability to use a one-dimensional filter to extract information for multi-dimensional data.

Most Significant Accomplishments

Mesh Adaptivity using convolution filters
In [4], Ryan and her collaborators extended the use of post-processing convolution filters to *a posteriori* analysis for approximations of partial differential equations (PDEs). The goal of an *a posteriori* error bound is to computationally control the error committed in approximating the solution to a PDE.
A key observation is that the post-processed solution must be at least as good of an approximation of the solution as the given approximation. In fact, for some classes of PDE, hyperbolic ones for example, it must actually be a better approximation. In fact, accurate evaluation of the post-processed approximation is more important than the approximation, hence it is natural to estimate the error in the post-processor.

The conditions under which the post-processed solution is a superconvergent approximation from the *a posteriori* viewpoint are given in [4]. Understanding this allows for providing reliable and efficient error control for the post-processed solution, which is used in refining the grid near discontinuities and complex phenomena. The effectiveness of these ideas using the Smoothness-Increasing Accuracy-Conserving (SIAC) filters and Superconvergent Patch Recovery (SPR) techniques. However, the analysis is quite general and makes only very mild assumptions on the postprocessing operator. Specifically, it is only required that:

1. The postprocessed solution belongs to a finite dimensional space that contains piecewise polynomials, although it does not necessarily need to be piecewise polynomial itself.
2. The postprocessed solution should be piecewise smooth over the same triangulation, or a subtriangulation, of the finite element approximation.

Given a postprocessor that satisfies these rather mild assumptions, it is perturbed slightly to ensure it satisfies an orthogonality condition which then allows us to show various desirable properties including:

1. The orthogonal postprocessor satisfies better approximation than the original postprocessor in the energy norm.
2. The orthogonal postprocessor has an increased order in the L^2 norm. Practically, this is not always the case for the original postprocessor.
3. Efficient and reliable a posteriori bounds are available for the error committed by the orthogonal postprocessor.

In Figure 1 a demonstration of these results for a smooth solution to Laplace's equation is given and the general expected improvement given in Table 1.

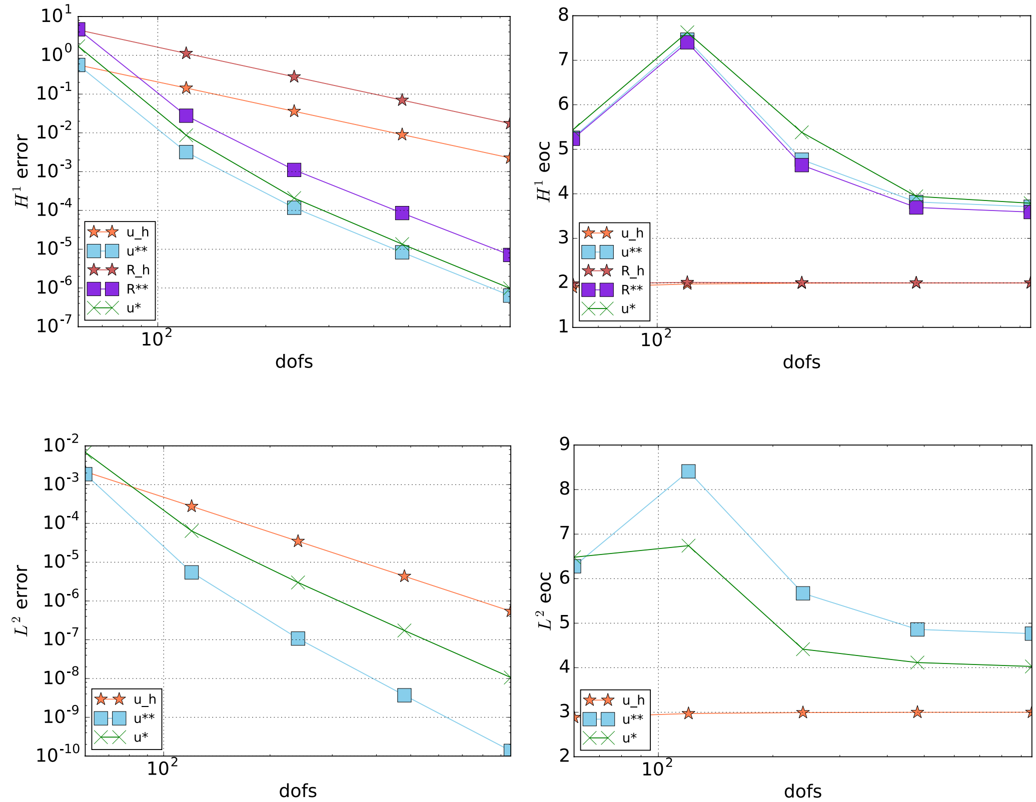

Figure 1: Errors (Left column) and convergence rates (Right column) for H^1 (top) and L^2 (bottom) for a quadratic polynomial approximation to Laplace's equation with a smooth solution.

	$p=1$			$p=2$			$p=3$		
	EOC(u_h)	EOC(u^*)	EOC(u^{**})	EOC(u_h)	EOC(u^*)	EOC(u^{**})	EOC(u_h)	EOC(u^*)	EOC(u^{**})
L^2-error	2	2	4	3	4	5	4	6	6
H^1-error	1	2	3	2	4	4	3	5	5

Table 1: Experimental rates of convergence for a smooth solution to Laplace's equation using different polynomial degree p. The convergence rates are shown for the three approximations, u_h, u^*, u^{**}.

SIACMRA

During the initial one-year foundation phase of this project, the PI established the effectiveness of combining the underlying superconvergent information contained in the approximation with MRA techniques for uniformly spaced data [6]. This allows for accurate transfer of information when moving data from a course mesh to a fine mesh, and hence more accurate multi-scale information. When beginning with a piecewise polynomial of degree *p*, this can be done for *p* levels of refinement. In Figure 2, a log plot showing the errors for the usual MRA against SIAC MRA is presented. It clearly demonstrates that the errors can not only be maintained, but improved when transferring information from a coarse mesh to a fine mesh.

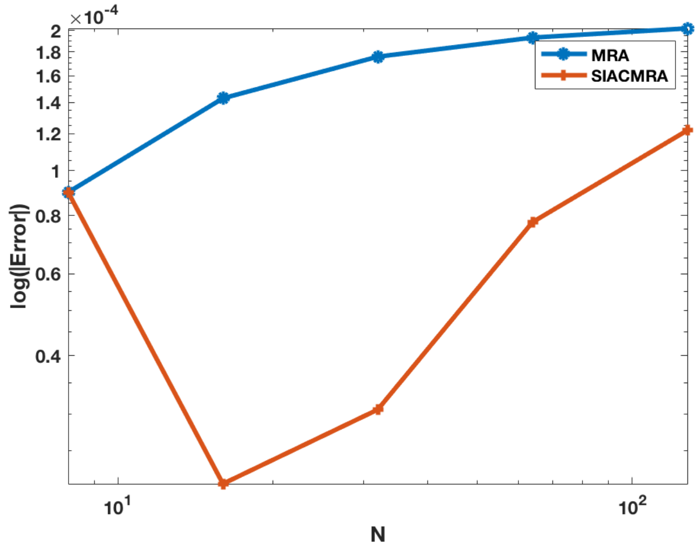

Figure 2: Convergence rate plots for a piecewise cubic approximation when converting the information from a coarse mesh to a fine mesh for a smooth function. Information is initially given on a mesh of 8 elements. The final mesh is a mesh consisting of 128 elements. The blue line represents an L^2–projection onto the finer mesh and the red line represents the combination of the SIAC filter with the Multi-Resolution reconstruction.

Additionally, these ideas are able to translate to finite difference data. In Figure 3, information obtained using a third order finite difference scheme consisting of 256 points was given to the SIACMRA algorithm. This translates into a $p=3$ discontinuous Galerkin solution with 64 elements. This was then refined four times using the SIACMRA algorithm ($p=3$, *1024* elements). The results are then compared with a finite difference scheme consisting of the same number of elements (1024). The plot shows that we are able to reproduce the solution reasonably well.

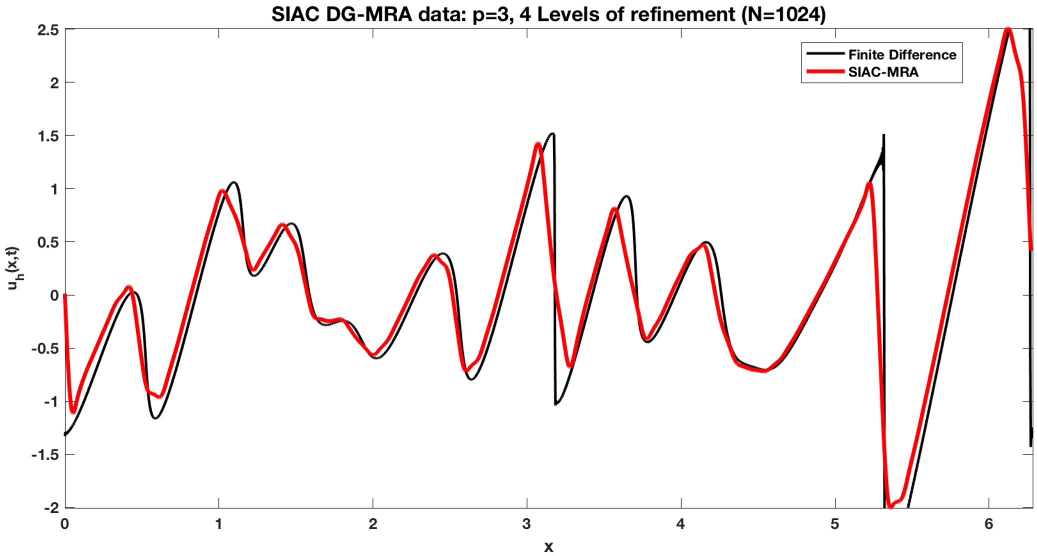

Figure 3: This plot shows the ability of the SIACMRA algorithm to translate to finite difference methods. In this plot, the SIACMRA algorithm was fed in a finite difference solution consisting of 256 points. The algorithm was applied four times and compared with the finite difference solution solved over a finer grid consisting of 1024 points.

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References (Presentations and Publications)

Presentations

1. Plenary Speaker, *The Mathematics of Finite Elements and Applications (MAFELAP),* Brunel University. Uxbridge, UK. June 17-21, 2019.
2. Invited Workshop Speaker, *Wavelets and Computational Fluid Dynamics*, ONERA – The French Aerospace Lab. Paris, France. June 11, 2019.
3. Colloquium Speaker,
    a. Scientific Computing Seminar, *Uppsala University*. Uppsala, Sweden. May 23, 2019.
    b. Mathematics Institute, *University of Darmstadt*. Darmstadt, Germany. February 6, 2019.
    c. Mathematics Institute, *University of Köln*. Köln, Germany. January 11, 2019.
    d. Mathematics Institute, *University of Würzburg.* Würzburg, Germany. September 24, 2018.

e. Mathematics Institute, *Hamburg University of Technology*. Hamburg-Harburg, Germany. September 6, 2018.

Publications (Preprints and Submissions)

4. Andreas Dedner, Jan Giesselmann, Tristan Pryer, Jennifer K Ryan, ”Residual estimates for post-processors in elliptic problems .” Submitted to *SIAM Journal on Scientific Computing*. (https://arxiv.org/abs/1906.04658)
5. Julia Docampo, Gustaaf B. Jacobs, Xiaozhou Li, and Jennifer K. Ryan, “Enhancing Accuracy with a Convolution Filter: What Works and Why!”. Submitted to *Computers and Fluids*.
6. Jennifer K. Ryan, “Capitalizing on superconvergence for more accurate multi-resolution discontinuous Galerkin approximation”, preprint, 2019.

Personnel Supported During Duration of Grant

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