AA Tech Note 2018-1

Depression of a Foreground Reference Line

George Kaplan

In the Appendix on the following pages is a simple geometric development that provides a result that might be of use for celestial navigation. It assumes an observer at some height above the sea surface (height << radius of the Earth) looking toward the horizon. The development provides a formula for the depression angle (the angle below a horizontal plane at zenith distance 90°) of some reference line or point that lies on the sea surface, short of the horizon. The waterline of a ship or a distant shoreline would be examples. It assumes that the observer's height h and the line-of-sight distance l to the reference line or point are known. The distance l does not have to be known very precisely. Such a line could be used in place of the true horizon for celestial navigation in the case where the true horizon was not visible — if, for example, it was obscured by low-lying haze or fog or blocked by a land mass. However, to use this kind of foreground reference line for celestial navigation, its depression angle must be known, and used in place of the depression of the horizon. Obviously the depression of such a line will be greater than that of the true horizon. This development provides a formula for the depression angle.

The formula derived is

$$d = 0.57 \;\; (h\,/\,l\,) + 0.41 \;\; l$$

where d is in arcminutes, h is in feet, and l is in nautical miles. The same formula is given in the 1977 Bowditch, but not the 1995 edition.

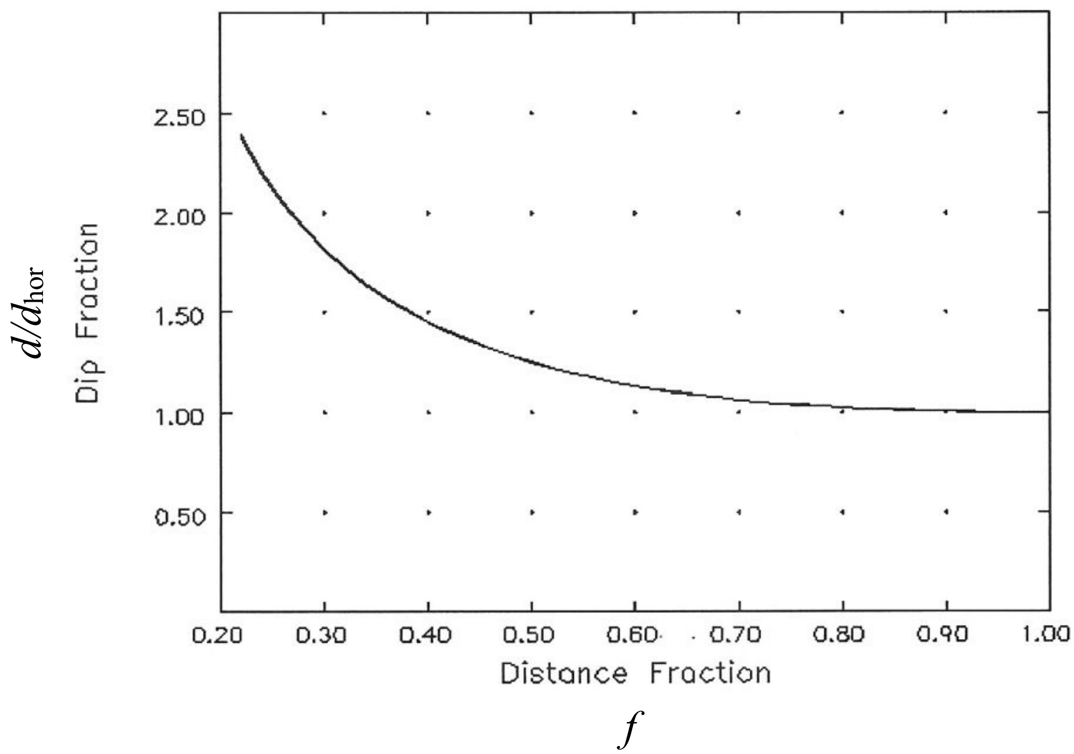
There is perhaps a more convenient form of this equation, where everything is expressed as a ratio:

$$d/d_{\mathrm{hor}} = 0.502\ (1/f) + 0.495\, f \;\approx \tfrac{1}{2}\,(1/f + f)$$

Here, the depression has been replaced by the ratio d/d_{hor} of the depression of the reference line to that of the true horizon, and the distance to the reference line has been replaced by the ratio f of the distance of the reference line to that of the horizon ($d/d_{\mathrm{hor}} \geq 1$ and $f \leq 1$). The height of the observer does not explicitly appear. It may not be obvious that this form is simpler to use in practice, but it relies on the fact that most navigators know their height of eye, which is usually a constant for a given ship, and therefore they know both the depression of the horizon and the distance to the horizon.

A graph of this form of the equation is given on the next page. It shows that the correction for using a foreground reference line instead of using the true horizon is small when the foreground line is at least halfway to the horizon. For a reference line halfway to the horizon, the depression is only 1.25 times that of the horizon. In such a case, if the height of eye is 10 feet, the difference in the depressions (3.75 arcminutes vs. 3 arcminutes) is just about within the error of measurement by sextant. It is also clear that only a crude estimate of the foreground line's distance is needed, as the slope of the curve is quite shallow for distances beyond halfway to the horizon.

It should be mentioned that the coefficients in the above equations have been adjusted for refraction, but the adjustment scheme, given on page A5 of the Appendix, uses refraction at the horizon as the boundary condition. It is possible that this overestimates refraction for shorter distances and that the dip predicted by these formulas is therefore somewhat less than the true value. This is undoubtedly a small effect, however.

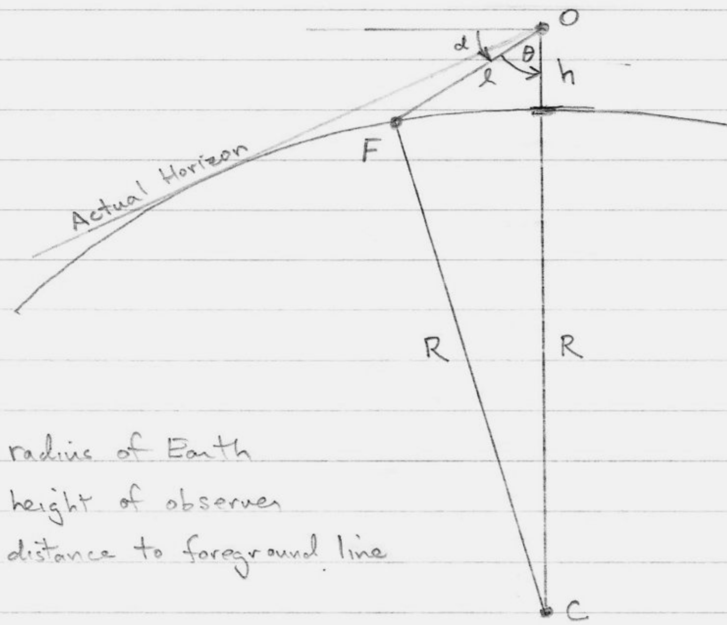


2.50
2.00
1.50
1.00
0.50
d/d_hor
Dip Fraction
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
Distance Fraction
f

Appendix

A1

Depression of Foreground Reference Line



R = radius of Earth

h = height of observer

ℓ = distance to foreground line

From triangle COF:

$$R^2 = (R+h)^2 + \ell^2 - 2\ell(R+h)\cos\theta$$
$$= R^2 + 2Rh + h^2 + \ell^2 - 2\ell(R+h)\cos\theta$$

$$2\ell(R+h)\cos\theta = 2Rh + h^2 + \ell^2$$
$$\cos\theta = \frac{2Rh + h^2 + \ell^2}{2\ell(R+h)}$$
$$= \frac{2Rh}{2\ell(R+h)} + \frac{h^2}{2\ell(R+h)} + \frac{\ell^2}{2\ell(R+h)}$$

Since $R+h \approx R$,

$$\cos\theta = \frac{h}{\ell} + \frac{h^2}{2\ell R} + \frac{\ell}{2R}$$

$$\cos\theta = \frac{h}{\ell}\left(1+\frac{h}{2R}\right)+\frac{\ell}{2R} \qquad \text{but } h \ll R$$

so $$\cos\theta = \frac{h}{\ell}+\frac{\ell}{D} \qquad \text{where } D = 2R$$

$$\theta = \arccos(\cos\theta) = \frac{\pi}{2} - \cos\theta - \cdots$$

$$\theta = \frac{\pi}{2} - \frac{h}{\ell} - \frac{\ell}{D}$$

Depression $d = \frac{\pi}{2} - \theta = \frac{h}{\ell} + \frac{\ell}{D}$

where $D = 12{,}756$ km $= 7926$ miles $= 6888$ nmi

Note: Here, D is twice the equatorial radius of the Earth. Actually, the local radius of curvature of the adopted Earth ellipsoid, which varies with latitude and azimuth, should be used. But those radii differ from the equatorial radius by less than 1%, and the latter could be considered to be a crude average of all the possibilities.

Suppose h is in feet, ℓ in nmi, d in arcmin

$$d = \left(\frac{h}{6076\,\ell} + \frac{\ell}{6888}\right) \times 57.296 \times 60$$

$$d = 0.5658\,\frac{h}{\ell} + 0.4991\,\ell$$

Does this formula work in the limit of the actual geometric horizon, where $d = \ell = 1.063\sqrt{h}$ (unrefracted)

$$d = 0.5658\,\frac{h}{1.063\sqrt{h}} + 0.4991 \times 1.063\sqrt{h}$$

$$= 0.5323\sqrt{h} + 0.5305\sqrt{h}$$

$$= 1.063\sqrt{h} \quad \checkmark$$

How about using the refracted formulas for dip and distance?
$(d = 0.97\sqrt{h},\quad \ell = 1.17\sqrt{h})$

$$d = 0.5658 \frac{h}{1.17\sqrt{h}} + 0.4991 \times 1.17\sqrt{h}$$

$$= 0.4836\sqrt{h} + 0.5839\sqrt{h}$$

$$= 1.068\sqrt{h} \neq 0.97\sqrt{h}$$

(predicts dip too high by 10%)

Could compensate by adjusting the constants:

$$d = 0.57\frac{h}{\ell} + 0.41\,\ell$$

See page A5 of the development for how the adjustment is made.

Then, at the horizon we have

$$d = 0.57\frac{h}{1.17\sqrt{h}} + 0.41 \times 1.17\sqrt{h}$$

$$= 0.49\sqrt{h} + 0.48\sqrt{h}$$

$$= 0.97\sqrt{h} \quad \checkmark$$

Although this works for all horizons, it may overcompensate for refraction for nearby points ($\ell \ll$ horizon distance).

Fraction of horizon dip as a function of fraction of distance to horizon:

$$d = 0.57\frac{h}{\ell} + 0.41\,\ell$$

$$\frac{d}{d_{hor}} = \frac{0.57\frac{h}{\ell} + 0.41\,\ell}{0.97\sqrt{h}}$$

$$= \frac{0.57}{0.97}\frac{h}{\ell\sqrt{h}} + \frac{0.41}{0.97}\frac{\ell}{\sqrt{h}}$$

$$= \frac{0.57}{0.97}\frac{\sqrt{h}}{\ell} + \frac{0.41}{0.97}\frac{\ell}{\sqrt{h}}$$

But if we let $\ell = f\,\ell_{hor} = f \times 1.17\sqrt{h}$ then

$$\frac{d}{d_{hor}} = \frac{0.57}{0.97}\frac{\sqrt{h}}{1.17f\sqrt{h}} + \frac{0.41}{0.97}\frac{1.17f\sqrt{h}}{\sqrt{h}}$$

$$= 0.502\,\frac{1}{f} + 0.495\,f \approx \frac{1}{2}\left(\frac{1}{f}+f\right)$$

Equivalent increase in height for use in usual horizon depression formula:

Usual formula is $d_{hor} = 0.97\sqrt{h}$. What is h' such that $d = 0.97\sqrt{h'}$? Let $d/d_{hor} = q$. Then

$$d = q\,d_{hor} = q \times 0.97\sqrt{h} = 0.97\sqrt{q^2 h}$$

So $h' = q^2 h$ where $q^2 = \frac{1}{4}\left(\frac{1}{f}+f\right)^2 = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{f^2}+f^2\right)$

Adjusting Constants for Refraction

The geometric (unrefracted) formula is $d = 0.5658\frac{h}{\ell} + 0.4991\,\ell$, i.e., of the form $d = A\frac{h}{\ell} + B\ell$. What should A and B be for the refracted case? The refracted dip of the horizon, d_{hor} is $0.97\sqrt{h}$ and the refracted distance to the horizon, ℓ_{hor} is $1.17\sqrt{h}$. We need $d = d_{hor}$ when $\ell = \ell_{hor}$:

$$d = d_{hor} = 0.97\sqrt{h} = A\frac{h}{1.17\sqrt{h}} + B \times 1.17\sqrt{h}$$

$$0.97 = \frac{A}{1.17} + 1.17B \qquad (1)$$

We also want the function $d(\ell)$ to have a negative slope for $\ell < \ell_{hor}$. Let $\ell = f \times 1.17\sqrt{h}$, with $0 < f \leq 1$, then

$$\frac{d}{d\ell}d = -A\frac{h}{(1.17f\sqrt{h})^2} + B \leq 0$$

$$-\frac{0.73}{f^2}A + B \leq 0$$

$$\frac{0.73}{f^2}A \geq B \qquad (2)$$

In fact, intuitively, we expect d to asymptotically approach d_{hor} as ℓ approaches ℓ_{hor} $(f \to 1)$. So we require $\frac{d}{d\ell}d = 0$ for $f = 1$, leading to

$$0.73A = B \qquad (3)$$

Substituting back into (1), we get $A = 0.57$ and (3) immediately gives $B = 0.41$. Then (2) also holds for $0 < f \leq 1$.

Bowditch (1995) p. 340 gives

$$d = 60 \arctan \left(\frac{h}{6076.1\, l} + \frac{l}{8268} \right)$$

Bowditch (1977) p. 1255 gives

$$d = 0.4156\, l + 0.5658 \frac{h}{l}$$

virtually identical to what has been derived on the previous pages.

GHK