Atmospheric Refraction of Light from Nearby Objects in Space

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Abstract

This note describes a minor issue in computing astronomical refraction, which is that refraction depends on the distance of the observed object, at least for objects very close to the Earth. In practice, the effect is important mainly for optical observations of satellites in low Earth orbit, but even for them, the difference in refraction amounts to only a few arcseconds. If the usual algorithms for astronomical refraction (for objects at infinity) are used, simple correction formulas are available for the nearby objects.

1 Introduction

This note describes an effect that is unfamiliar to most astronomers: atmospheric refraction of celestial objects is a function of their distance. The effect is important only for nearby objects, at distances out to a few hundred kilometers at most, such as artificial satellites in low Earth orbit. For natural objects, including the Moon or even very-close-approaching asteroids, the distance dependence can be neglected — which is undoubtedly the reason for its unfamiliarity. In many expositions on astronomical refraction, e.g., in the current *Explanatory Supplement* (Urban & Seidelmann 2012), Smart (1965), and Kovalevsky (2002), it is not discussed, even though the effect might be embedded in the mathematics. In the first *Explanatory Supplement* (1961), and in Woolard & Clemence (1966) and Green (1985), it is mentioned indirectly[1] but none of these provide formulas for computing its angular magnitude.

The effect is sometimes called "parallactic refraction" or "refraction parallax." It has been discussed at least a dozen times in the literature, although not always in the sources that astronomers are most familiar with. An ADS search on these terms results in about a half-dozen papers published over the last 50 years. A more complete list would include Herget (1959), Brown (1961), Jones (1961), Schmid (1963), Nugent & Condon (1966), Kabeláč (1976), Kakkuri & Ojanen (1979), Schildknecht (1994), and Tausworthe (2005). The effect is discussed in the books by Mueller (1964) and Murray (1983).

It is not intuitively obvious why refraction should depend on distance. It would seem that light from objects along the same line of sight, regardless of distance, should be bent the same amount. In fact, this statement is true, but only for one meaning of "line of sight." The previous accounts of the effect approach refraction using a stratified, spherical atmosphere, and the complexities of such a model can sometimes obscure some basic geometric features. In this note we explore a simplified

[1] It has to do with the height h in the first *Explanatory Supplement* (pp. 54-56) and Woolard & Clemence (pp. 90-91), and h_0 in Green (p. 90).

version of the geometry in enough detail to be able to derive our own formulas for the effect, which yield numerical results that closely match those produced by others based on the more complicated models.

It is important to note that the effect is entirely geometric; it has nothing to do with refraction by the Earth's tenuous upper atmosphere. As we shall see, the effect occurs even if we consider the atmosphere to abruptly end at some height. The effect also exists for objects high in the atmosphere, such as balloons or sounding rockets, but these cases are much more complicated and will not be discussed here.

2 The Geometry of Refraction, Greatly Simplified

Figure 1 shows the overall geometry of astronomical refraction; incoming light from a star is gradually bent by air of increasing density and therefore increasing index of refraction. The amount of bending shown is greatly exaggerated; at a zenith distance of 45° the amount of refraction is only about an arcminute, and even at the horizon it is only about $1/2\,^\circ$. A proper account of refraction requires an integral through a spherical model atmosphere, where the index of refraction, n, is a function of air density and temperature, i.e., a function of height. The observer measures the star to be in a direction tangent to that of the incoming light, as it enters his detector. That light has been bent by the atmosphere, so the star appears at a greater altitude — smaller zenith distance — than its geometric direction. Atmospheric refraction affects only the apparent zenith distance of the object observed, not its azimuth, so the geometry plays out entirely in a vertical plane.

However, a very simple model suffices quite well for heuristic purposes and even possibly for low-accuracy applications (such as traditional celestial navigation) carried out at low to moderate zenith distances: the atmosphere can be approximated as a uniform horizontal slab of constant n with a flat top surface orthogonal to the observer's zenith. See Figure 2. Snell's law at the air-space interface provides the bending: $n \sin z' = \sin z$, where z and z' are the unrefracted (geometric) and refracted (apparent) zenith distances, respectively, of the star, and n is the refractive index of air at sea level. We also have $z' < z$, and the angle of refraction, r, is $z - z'$. If we set $n = 1.000277$, the value for dry air at 1013.25 mb (1 atm) pressure, 15° C, and a light wavelength of 0.574 μm (Allen 1973), the results are shown in the table below.

Table 1 Total Refraction for Objects at Infinity

Zenith Dist.	Refraction (simple)	Refraction (US1976)	Difference
°	″	″	″
5	5.00	5.00	0.00
10	10.08	10.07	0.01
15	15.31	15.31	0.00
20	20.80	20.79	0.01
25	26.64	26.64	0.00
30	32.99	32.98	0.01
35	40.01	39.98	0.03
40	47.95	47.90	0.05

continued. . .

Zenith Dist.	Refraction (simple)	Refraction (US1976)	Difference
°	″	″	″
45	57.14	57.07	0.07
50	68.11	67.98	0.13
55	81.62	81.40	0.22
60	99.00	98.62	0.38
65	122.61	121.87	0.73
70	157.14	155.61	1.53

where the first column is the apparent zenith distance, the second column is the result from the simple atmospheric slab model, and the third column is from Table 2 of van der Werf (2003), which gives the refraction from the US1976 Standard Atmosphere for the sea-level conditions listed above. The index of refraction used for the simple model, from Allen, matches that given by the NIST online calculator (Stone & Zimmerman 2011).

If we assume that the angle of refraction, r, is small, then it is easy to show that the Snell's law relation between z and z' can be expressed as $r = (n-1)\tan z'$; see, e.g., Green (1985). The constant $(n-1)$, of order $3{\times}10^{-4}$, is the amount of refraction, in radians, at a zenith distance of 45°. As mentioned above, this is about 1 arcminute.

The point here is that the atmospheric slab model (uniform plane-parallel atmosphere) is good enough for conceptualizing the phenomenon and even for some calculations. The reason it works so well is that the height of the real atmosphere is small compared to the radius of the Earth, so its curvature is important only close to the horizon. If we can neglect the curvature of the atmosphere, then, according to Woolard & Clemence, "when a ray passes through a parallel-stratified medium the final direction is the same as if the entire medium had the density of the last stratum." This non-intuitive result is actually quite easy to prove; see Smart or Green.

3 Refraction Depends on Distance

Figures 3 and 4 show why refraction depends on distance. The figures are based on the simple plane-parallel atmospheric slab model with an abrupt cutoff of the atmosphere at some height below that of any object observed. In Figure 3, the observer sees a star (at infinity) and a satellite (nearby but outside the atmosphere) in the same place on the sky, i.e., the satellite is occulting the star. Light from the star and the satellite is along the same ray path and is refracted by the same amount, but the two objects are at different geometric zenith distances — i.e., different topocentric directions (indicated by the black dashed lines). It's easy to see that the light paths in this figure could also be drawn using a spherical atmosphere with gradually diminishing density, such as that in Figure 1, with the same general result.

If two objects, at different distances, can appear at the same place in the sky yet have different geometric directions, then the converse must also be true. Two objects in the same geometric direction, at different distances, must appear at different places in the sky. That is, refraction can serve to separate objects that would appear together if there were no atmosphere.[2] See Figure 4. Obviously this could affect the calculation of eclipse and occultation circumstances, but the Moon

[2]Effectively, refraction changes the viewing angle of the observer, adding to his height. This is the significance of the heights h and h_0 mentioned in footnote 1.

and other solar system objects are so distant (compared to the height of the atmosphere) that the effect is quite small except near the horizon.

4 Formulas for a Plane-Parallel Atmosphere

There are good existing algorithms for atmospheric refraction for objects at infinity (stars), so we only need to know the small quantity represented by the *difference* between the refraction for nearby objects and those at infinity.

Figures 5 and 6 show the quantities used to obtain the formulas we need. Zenith distances are represented by z or z', depending on whether they are geometric or apparent, respectively, with subscript "sat" or "star" added as appropriate. The symbol q in Figure 6 represents an auxiliary angle, which is the geometric zenith distance of the satellite at a specific point at the top of the atmosphere. The quantities s, h, and d (in blue) are distances; s is the height of the atmosphere and h is the height of the satellite. The quantity s is specifically the *height of the homogenous atmosphere*, approximately 8 km (if the air density decreases exponentially with height, s is the same as the scale height of the atmosphere). The length d is the sum of the lengths d_1 and d_2. In both figures we are working entirely with right triangles in the plane of the figure, so the geometry is relatively simple.

Case 1: Star and satellite with same apparent directions: Figure 5 illustrates the difference between the geometric directions of the star and satellite, which have the same apparent zenith distance z'. Given z', we want z_{star} and z_{sat}. Snell's law applied to the ray from the star is $n \sin z' = \sin z_{\text{star}}$ so we have

$$z_{\text{star}} = \arcsin(n \sin z') \tag{1}$$

We also have

$$d = d_1 + d_2 \qquad \text{which expands to} \qquad h \tan z_{\text{sat}} = s \tan z' + (h - s) \tan z_{\text{star}} \tag{2}$$

We can solve this equation for z_{sat}:

$$z_{\text{sat}} = \arctan\left(\frac{s}{h} \tan z' + \frac{h-s}{h} \tan z_{\text{star}}\right) \tag{3}$$

Then

$$\Delta z = z_{\text{sat}} - z_{\text{star}} \tag{4}$$

is the difference between the geometric directions of the satellite and star, a small negative angle. That is, the geometric direction of the satellite is above that of the star. Note that if we start out knowing z_{star}, we can obtain z' from Snell's law.

Case 2: Star and satellite with same geometric directions: Figure 6 presents a more complicated case, where the star and satellite have the same geometric zenith distance z. Given z, we want z'_{star} and z'_{sat}. Again applying Snell's law to the ray from the star gives

$$z'_{\text{star}} = \arcsin\left(\frac{\sin z}{n}\right) \tag{5}$$

We also have

$$d = d_1 + d_2 \qquad \text{which expands to} \qquad h \tan z = s \tan z'_{\text{sat}} + (h - s) \tan q \tag{6}$$

where q is the previously mentioned auxiliary angle, which is related to z'_{sat} by Snell's law: $n \sin z'_{\text{sat}} = \sin q$. Therefore eq. (6) becomes

$$h \tan z = s \tan z'_{\text{sat}} + (h - s) \tan\left(\arcsin(n \sin z'_{\text{sat}})\right) \tag{7}$$

which we want to solve for z'_{sat}. Since z'_{sat} occurs twice in this equation, the solution is not as straightforward as for the previous case. However, the first term on the right side is much smaller than the other two, because $s \ll h$. That provides the possibility that the equation can be solved iteratively. The algebra is simplified a bit if we make the angle q the variable that we wish to solve for:

$$h \tan z = s \tan\left(\arcsin\left(\frac{\sin q}{n}\right)\right) + (h - s) \tan q \tag{8}$$

from which we obtain

$$q = \arctan\left(\frac{h}{h-s} \tan z - \frac{s}{h-s} \tan\left(\arcsin\left(\frac{\sin q}{n}\right)\right)\right) \tag{9}$$

This equation can be solved iteratively, starting by setting $q = z$ in the second term on the right, then obtaining a new value for q, then using the new value on the right side in the next iteration, and so on. The process converges to 10^{-10} radians (20 μas) in 7 iterations or fewer for $s = 8$ km, $h \geq 100$ km, and $z \leq 70°$. Once we have the value of q from this process, then

$$z'_{\text{sat}} = \arcsin\left(\frac{\sin q}{n}\right) \qquad \text{and} \qquad \Delta z' = z'_{\text{sat}} - z'_{\text{star}} \tag{10}$$

where $\Delta z'$ is the difference between the apparent zenith distances of the satellite and star, a small positive angle. That is, the satellite has the greater apparent zenith distance (appears lower in the sky); refraction is a bit *less* for nearby objects than for more distant ones along the same geometric line of sight (see Figure 4 or 6). Equivalently, the quantity $\Delta z'$ is the amount to be *subtracted* from the refraction of the star to obtain the refraction of the satellite.

5 Results

In this section, results from previously published formulas are compared to the results from the equations obtained in this note. It will be shown that the uniform plane-parallel atmosphere model, developed above, works quite well, even out to large zenith distances.

Case 1 For Case 1, we want to know the difference in geometric zenith distances for a satellite and star in the same apparent direction. Equations (1), (3), and (4) apply, and there is a formula from the existing literature that can be used for comparison.

Nugent & Condon (1966) derive the following formula for the total atmospheric refraction of light originating at an object at a height h:

$$R_0 = (n_0 - 1) \tan Z_0 \left[h_0/h(e^{-h/h_0} - 1) + 1\right] \tag{11}$$

In our notation, this is

$$r_{\text{sat}} = (n - 1) \tan z' \left[\frac{s}{h}(e^{-h/s} - 1) + 1\right] \tag{12}$$

where r_{sat} denotes the total refraction for an object at height h above the Earth's surface. For stars, $h = \infty$ and we have

$$r_{\text{star}} = (n-1)\tan z' \tag{13}$$

In the above equations, the satellite and the star share the same apparent zenith distance, z', so by construction, this corresponds to Case 1. Therefore the parallactic refraction is

$$\Delta z = z_{\text{sat}} - z_{\text{star}} = r_{\text{sat}} - r_{\text{star}} = (n-1)\tan z' \left(\frac{s}{h}\right)\left(e^{-h/s} - 1\right) \tag{14}$$

Table 2 provides a complete table of parallactic refraction for Case 1, for objects out to just beyond the Moon's distance, computed according to the above Nugent & Condon formula.

Table 2 Case 1 Results

$z_{sat} - z_{star}$ for same z' (arcsec)

Object Height (km) \ Apparent Zenith Distance (°)	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
560,000	0.000	0.000	0.000	0.000	0.000	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.003	-0.005
320,000	0.000	0.000	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.003	-0.003	-0.004	-0.006	-0.008
180,000	0.000	0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002	-0.003	-0.003	-0.004	-0.005	-0.006	-0.007	-0.010	-0.015
100,000	0.000	-0.001	-0.001	-0.002	-0.002	-0.003	-0.003	-0.004	-0.005	-0.006	-0.007	-0.008	-0.010	-0.013	-0.018	-0.027
56,000	-0.001	-0.002	-0.002	-0.003	-0.004	-0.005	-0.006	-0.007	-0.009	-0.010	-0.012	-0.015	-0.018	-0.024	-0.032	-0.049
32,000	-0.001	-0.003	-0.004	-0.005	-0.007	-0.009	-0.011	-0.013	-0.015	-0.018	-0.021	-0.026	-0.032	-0.041	-0.056	-0.085
18,000	-0.002	-0.005	-0.007	-0.010	-0.012	-0.015	-0.019	-0.022	-0.027	-0.032	-0.038	-0.046	-0.057	-0.073	-0.100	-0.152
10,000	-0.004	-0.008	-0.013	-0.018	-0.022	-0.028	-0.034	-0.040	-0.048	-0.057	-0.069	-0.083	-0.103	-0.132	-0.180	-0.273
5600	-0.008	-0.015	-0.023	-0.031	-0.040	-0.050	-0.060	-0.072	-0.086	-0.103	-0.123	-0.149	-0.185	-0.236	-0.321	-0.488
3200	-0.013	-0.027	-0.040	-0.055	-0.070	-0.087	-0.105	-0.126	-0.151	-0.179	-0.215	-0.261	-0.323	-0.414	-0.562	-0.854
1800	-0.023	-0.047	-0.072	-0.097	-0.125	-0.155	-0.187	-0.225	-0.268	-0.319	-0.382	-0.464	-0.574	-0.735	-0.999	-1.518
1000	-0.042	-0.085	-0.129	-0.175	-0.225	-0.278	-0.337	-0.404	-0.482	-0.574	-0.688	-0.835	-1.033	-1.324	-1.798	-2.733
560	-0.075	-0.152	-0.231	-0.313	-0.401	-0.497	-0.602	-0.722	-0.860	-1.025	-1.229	-1.490	-1.845	-2.364	-3.211	-4.880
320	-0.132	-0.266	-0.403	-0.548	-0.702	-0.869	-1.054	-1.263	-1.506	-1.794	-2.150	-2.608	-3.229	-4.137	-5.619	-8.539
180	-0.234	-0.472	-0.717	-0.974	-1.248	-1.545	-1.874	-2.246	-2.677	-3.190	-3.823	-4.636	-5.741	-7.355	-9.990	-15.181
100	-0.422	-0.850	-1.291	-1.754	-2.247	-2.782	-3.374	-4.043	-4.818	-5.742	-6.881	-8.346	-10.333	-13.238	-17.982	-27.326

Both the heights and the color contours are on logarithmic scales. The colors separate values at 0.001, 0.01, 0.1, 1, and 10 arcseconds. Table 2 would be much the same if eqs. (1), (3), and (4) were used to generate it instead of the Nugent & Condon formula. The differences in the results are less than 0.001 arcscec for most of the table. The largest difference, at the lower right corner, is 0.109 arcsec, and there are only nine differences near that corner greater than 0.01 arcsec. For both algorithms, the index of refraction was set at $n = 1.000292$ to be comparable to the results by Murray discussed below for Case 2.

Table 3 shows the differences between the values obtained from eqs. (1), (3), and (4) and the corresponding Nugent & Condon values for $h = 100$ km. The third column in Table 3 corresponds to the bottom row in Table 2.

Table 3 Parallactic Refraction (Case 1) for Object at h=100 km

Zenith Dist.	$z_{\rm sat}$−$z_{\rm star}$ This Note	$z_{\rm sat}$−$z_{\rm star}$ N & C	Difference
°	″	″	″
5	-0.422	-0.422	-0.000
10	-0.850	-0.850	0.000
15	-1.291	-1.291	0.000
20	-1.754	-1.754	0.000
25	-2.247	-2.247	0.000
30	-2.782	-2.782	0.000
35	-3.374	-3.374	0.000
40	-4.043	-4.043	0.000
45	-4.818	-4.818	0.001
50	-5.741	-5.742	0.001
55	-6.880	-6.881	0.002
60	-8.343	-8.346	0.003
65	-10.327	-10.333	0.006
70	-13.226	-13.238	0.012
75	-17.951	-17.982	0.031
80	-27.217	-27.326	0.109

The first column is the apparent zenith distance of both the star and satellite, z'.

Case 2 For Case 2, we want to know the difference in apparent zenith distances for a satellite and star in the same geometric direction. Equations (5), (9), and (10) apply. If we have the refraction value for a star, Murray (1983), pp. 173–174, provides a correction for the satellite's refraction:

$$\Delta\gamma_0 \simeq -\frac{q}{h}\sin\theta_0\cos\theta_0 \qquad \text{where} \qquad q = 2.34\sec^2\theta_0 \quad \text{for optical observations} \tag{15}$$

and where q and h are in meters (Murray's q is different from the one used in this note). Murray's $\Delta\gamma_0$ corresponds to our $-\Delta z'$ and his θ_0 corresponds to our z'. Combining these two expressions, Murray's formula becomes

$$\Delta\gamma_0 \simeq -\frac{2.34}{h}\tan\theta_0 \qquad \text{or in our notation} \quad \Delta z' \simeq \frac{2.34}{h}\tan z' \tag{16}$$

where h is the height of the satellite in meters, and $\Delta z'$ is in radians. Murray's formula is repeated in Schildknecht (1994). It is not completely clear what atmospheric conditions this applies to. Murray seems to favor a temperature of 0° C and a wavelength of 0.5893 μm at the standard pressure of 1010.25 mb (1 atm). The index of refraction of air for these conditions is 1.000292.

Table 4 provides a complete table of parallactic refraction for objects out to just beyond the Moon's distance, computed according to Murray's formula.

Table 4 Case 2 Results

z'$_{sat}$ – z'$_{star}$ for same z (arcsec)

Object Height (km) \ Apparent Zenith Distance (°)	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
560,000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.003	0.005
320,000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.006	0.009
180,000	0.000	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.005	0.006	0.007	0.010	0.015
100,000	0.000	0.001	0.001	0.002	0.002	0.003	0.003	0.004	0.005	0.006	0.007	0.008	0.010	0.013	0.018	0.027
56,000	0.001	0.002	0.002	0.003	0.004	0.005	0.006	0.007	0.009	0.010	0.012	0.015	0.018	0.024	0.032	0.049
32,000	0.001	0.003	0.004	0.005	0.007	0.009	0.011	0.013	0.015	0.018	0.022	0.026	0.032	0.041	0.056	0.086
18,000	0.002	0.005	0.007	0.010	0.013	0.015	0.019	0.022	0.027	0.032	0.038	0.046	0.058	0.074	0.100	0.152
10,000	0.004	0.009	0.013	0.018	0.023	0.028	0.034	0.040	0.048	0.058	0.069	0.084	0.104	0.133	0.180	0.274
5600	0.008	0.015	0.023	0.031	0.040	0.050	0.060	0.072	0.086	0.103	0.123	0.149	0.185	0.237	0.322	0.489
3200	0.013	0.027	0.040	0.055	0.070	0.087	0.106	0.127	0.151	0.180	0.215	0.261	0.323	0.414	0.563	0.855
1800	0.023	0.047	0.072	0.098	0.125	0.155	0.188	0.225	0.268	0.320	0.383	0.464	0.575	0.737	1.001	1.521
1000	0.042	0.085	0.129	0.176	0.225	0.279	0.338	0.405	0.483	0.575	0.689	0.836	1.035	1.326	1.801	2.737
560	0.075	0.152	0.231	0.314	0.402	0.498	0.604	0.723	0.862	1.027	1.231	1.493	1.848	2.368	3.217	4.888
320	0.132	0.266	0.404	0.549	0.703	0.871	1.056	1.266	1.508	1.798	2.154	2.612	3.235	4.144	5.629	8.554
180	0.235	0.473	0.718	0.976	1.250	1.548	1.878	2.250	2.681	3.196	3.829	4.644	5.750	7.367	10.007	15.207
100	0.422	0.851	1.293	1.757	2.251	2.787	3.380	4.050	4.827	5.752	6.893	8.360	10.351	13.261	18.013	27.373

The format of the table and the color contours are the same as for Table 2. Table 4 would be much the same if eqs. (5), (9), and (10) were used to generate it instead of Murray's formula. The differences in the results are less than 0.001 arcscec for the entire upper left half of the table. The largest difference, at the lower right corner, is 0.400 arcsec, and there are only four differences greater than 0.1 arcsec.

Table 5 shows the differences between the values obtained from eqs. (5), (9), and (10) and the corresponding Murray values for h = 100 km. The third column in Table 5 corresponds to the bottom row in Table 4.

Table 5 Parallactic Refraction (Case 2) for Object at h=100 km

Zenith Dist.	$z'_{\mathrm{sat}} - z'_{\mathrm{star}}$ This Note	$z'_{\mathrm{sat}} - z'_{\mathrm{star}}$ Murray	Difference
°	″	″	″
5	0.421	0.422	-0.001
10	0.849	0.851	-0.002
15	1.291	1.293	-0.003
20	1.753	1.757	-0.004
25	2.246	2.251	-0.005
30	2.781	2.787	-0.006
35	3.372	3.380	-0.007
40	4.041	4.050	-0.009
45	4.815	4.827	-0.011
50	5.738	5.752	-0.015
55	6.874	6.893	-0.019
60	8.334	8.360	-0.026

continued. . .

Zenith Dist.	$z'_{\rm sat}-z'_{\rm star}$ This Note	$z'_{\rm sat}-z'_{\rm star}$ Murray	Difference
$\circ$	$''$	$''$	$''$
65	10.312	10.351	-0.039
70	13.196	13.261	-0.065
75	17.879	18.013	-0.134
80	26.973	27.373	-0.400

The first column is the apparent zenith distance of the star, $z'_{\rm star}$.

Note that aside from the sign reversal, the values in Tables 2 and 4 are nearly identical, and could be used interchangeably for all except the most precise measurements. These tables are also consistent, with some small differences, with Table IV given by Schmid (1963).

6 Discussion

Generally, parallactic refraction is too small to be included in telescope-pointing algorithms unless the field of view is extremely small. On the other hand, the Nugent & Condon formula for total atmospheric refraction — eq. (11) or (12) in this note — offers a convenient way to compute the amount of refraction for an object at any distance, at least for those observed up to moderate zenith distances. To apply the Nugent & Condon formula, the index of refraction of air at the observer, n_0 or n, would have to be computed for local conditions and the wavelength of the light being detected; formulas for this are given in Allen (1973), Murray (1983), and many other references.

Nugent & Condon indicate that their formula is accurate only for zenith distances up to about 70°. The factor within square brackets accounts for the distance of the object; it would be interesting to test whether the same factor can be successfully applied to more sophisticated closed-form formulas for the refraction of stars that work all the way down to the horizon. Note that for objects at infinity, the formula reduces to that for a plane-parallel atmosphere. Although the authors start with a stratified spherical atmosphere, it may be the case that the approximations they introduce effectively flatten out their model. That would certainly explain the good agreement of the results with those from the formulas developed in this note.

Let us consider a practical application in which parallactic refraction should be applied. Suppose a satellite is imaged against background stars in a relatively small field of view, to obtain the celestial coordinates of the satellite, with the goal of determining the geometric direction to the satellite. That is, we want to determine the straight line from the observer to the satellite's position when the observed light left it. One way of analyzing the image would be to initially assume that the refraction of the stars and satellite is the same and so does not need to be explicitly calculated (any differential refraction across the field would be taken out by the plate constants). We obtain the satellite's apparent coordinates based on those of the reference stars, in the usual way, without considering refraction. Essentially, this procedure gives us the position of an imaginary star coincident with the satellite. That done, we need only to make a small adjustment to the satellite's computed position, represented by parallactic refraction Case 1, as given in Table 2. The sense of the adjustment is that the satellite's corrected position should be higher in the sky (smaller zenith distance), as shown in Figure 3.

To be convenient, we want the parallactic refraction adjustment to be expressed as small increments to the satellite's right ascension and declination, i.e., values for $\Delta\alpha$ and $\Delta\delta$. Kakkuri & Ojanen (1979) provide the following formulas:

$$\begin{aligned}\Delta\alpha &= dz\left(\frac{\cos\phi\sin h}{\cos\delta\sin z'}\right)\\ \Delta\delta &= dz\left(\frac{\sin\phi\cos\delta-\cos\phi\sin\delta\cos h}{\sin z'}\right)\end{aligned}\tag{17}$$

where the uncorrected position of the satellite is (α,δ), h is its hour angle, z' its apparent zenith distance, ϕ is the latitude of the observer, and dz is the parallactic correction. Note that $\Delta\alpha$ represents the change in right ascension along the equator; it is not an "arc" measurement at the position of the satellite on the celestial sphere (Kakkuri and Ojanen include a factor 1/15). These formulas can be obtained from those given by Green (1985) for a general small displacement on the celestial sphere (not specific to refraction) in his Section 1.7 (pp. 16–20), although we end up with $\Delta\alpha$ and $\Delta\delta$ expressions with opposite signs. Since the sign of the parallactic refraction correction is arbitrary, the signs should be set such that in the case we are considering, for a satellite observed from a mid-northern latitude, the following should be true of the satellite's corrected coordinates:

- Satellite along southern meridian: δ should increase, α should not change.
- Satellite in western sky: δ should increase, α should increase.
- Satellite in eastern sky: δ should increase, α should decrease.

Another option is to use the vector algorithms for refraction's effect on celestial coordinates given in Kaplan (2008), AA Technical Note 2008-01. In applying that note to the current problem, r should be understood as the parallactic refraction, and the vector $\mathbf{p}$ should be understood as the direction vector toward the satellite, uncorrected for parallactic refraction, i.e., obtained directly from the stars in the field. Then the vector $\mathbf{p}'$ is the desired result. The above checks on the sign of the result should be made.

7 Conclusion

This note has provided several formulas for computing the "parallactic refraction," the difference between the atmospheric refraction of nearby objects in space and those at infinity, i.e., stars. The difference is small except for those objects in low Earth orbit, or higher ones observed at large zenith distances. The sense of the difference is that nearby objects appear at a higher zenith distance (closer to the horizon) than more distant ones along the same geometric line of sight (Case 2, Figure 4, Table 4). Equivalently, for near and far objects that appear at the same (refracted) zenith distance, the geometric zenith distance of the near object is less than that of the far one (Case 1, Figure 3, Table 2).

Section 2 demonstrated that a uniform plane-parallel model of the atmosphere works quite well as a basis for ordinary astronomical refraction — that is, for objects at infinity. Section 5 showed that the same model works well for parallactic refraction. It was mentioned earlier that the simple model works because the atmosphere is so thin compared to the radius of the Earth, so most astronomical lines of sight pass through atmospheric layers nearly parallel to those directly over the observer. A simple geometric construction can be used to quantify this statement, and it shows that the atmosphere near the visible horizon, at a height of 10 km, is tilted only 3.2° from a horizontal plane centered at the observer. At zenith distances of 70° and 80°, the tilt of the atmosphere at a height of 10 km is only a quarter and a half degree, respectively. It is possible to correct for this effect in the uniform plane-parallel model: the small tilt angle of the atmosphere at these zenith distances effectively decreases the total modeled refraction slightly, by about 2 and 17 arcseconds, respectively. These corrections improve the model's agreement with more sophisticated developments, but such refinements are hardly justified for such a simple model.

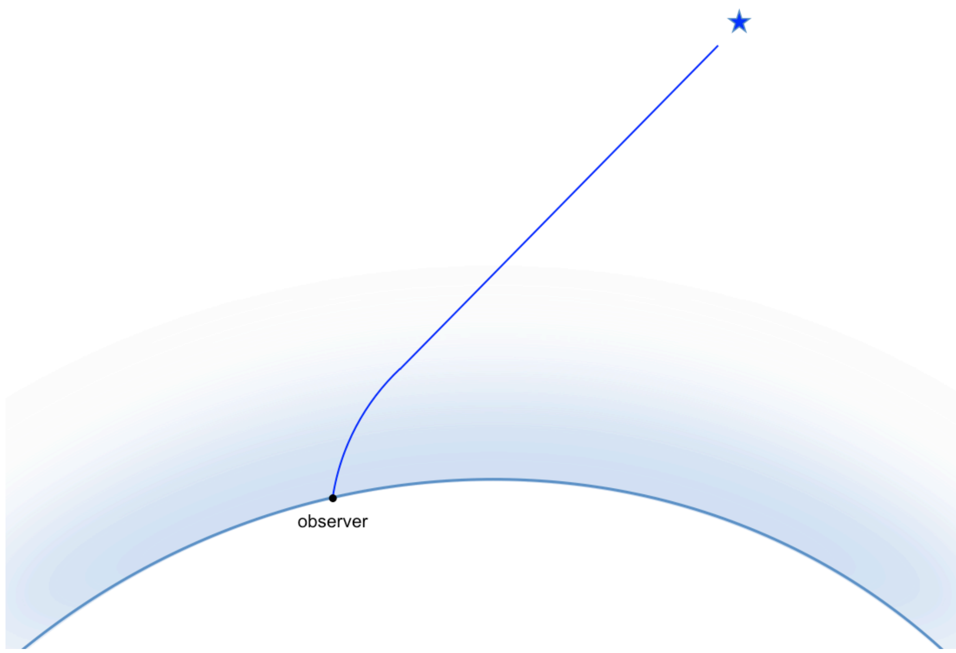

Figure 1 Atmospheric refraction of starlight — schematic.

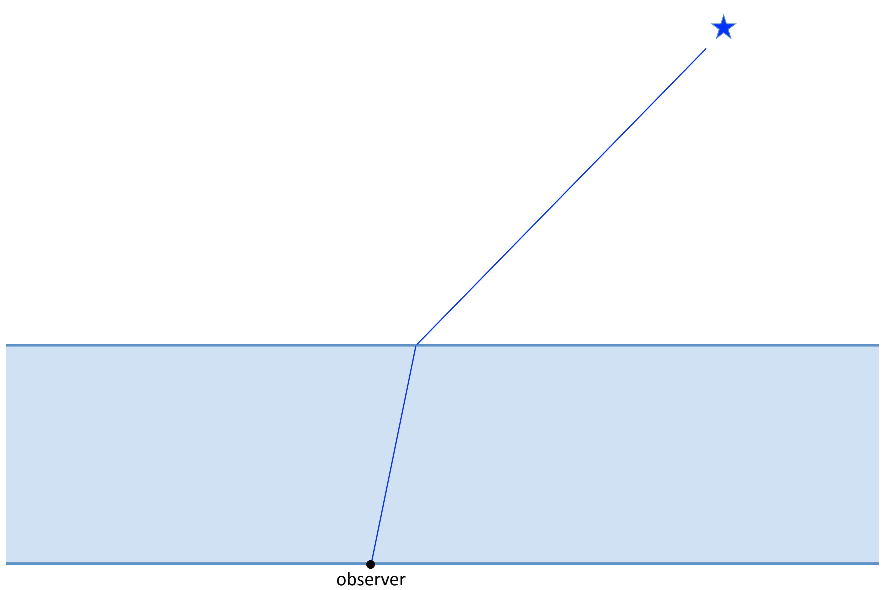

Figure 2 Atmospheric refraction of starlight — simple model using plane-parallel uniform atmospheric slab.

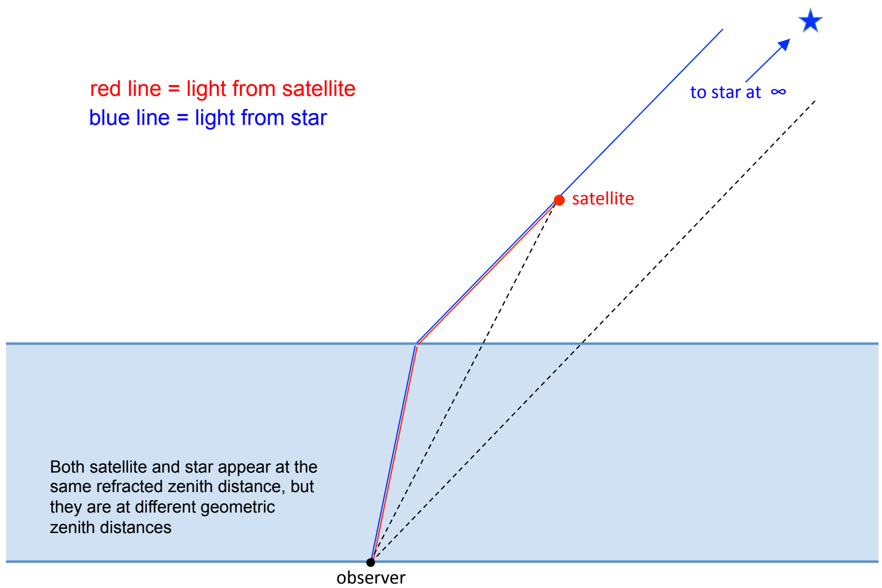

Figure 3 A star and satellite are seen at the same apparent zenith distance but are at different geometric zenith distances.

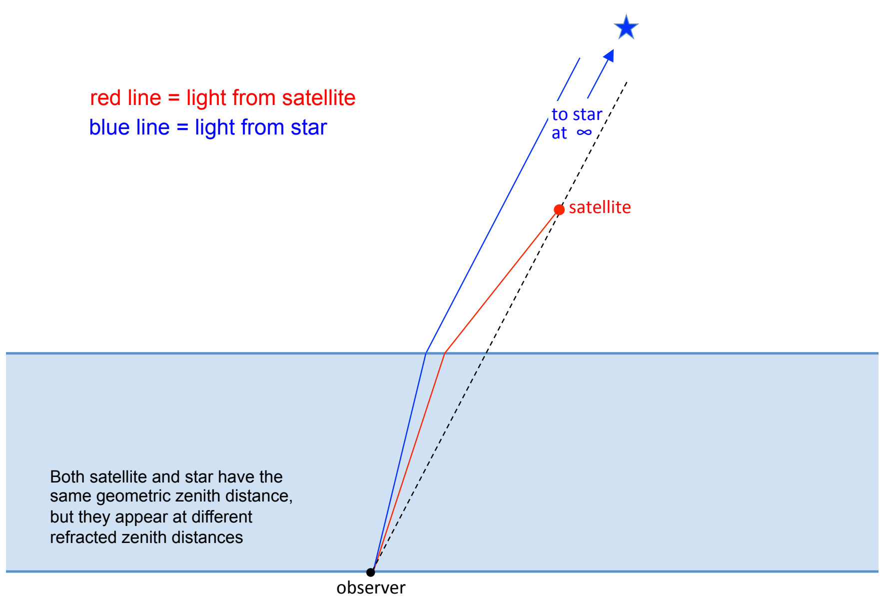

Figure 4 A star and satellite are at the same geometric zenith distance but are seen at different apparent zenith distances.

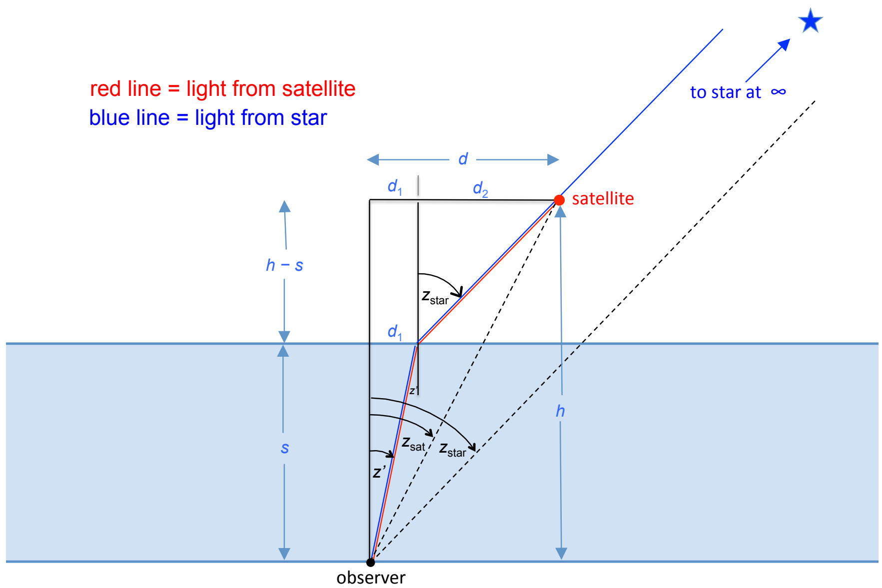

Figure 5 Quantities and triangles used to obtain formulas for Case 1, where a star and satellite are seen at the same apparent zenith distance but are at different geometric zenith distances.

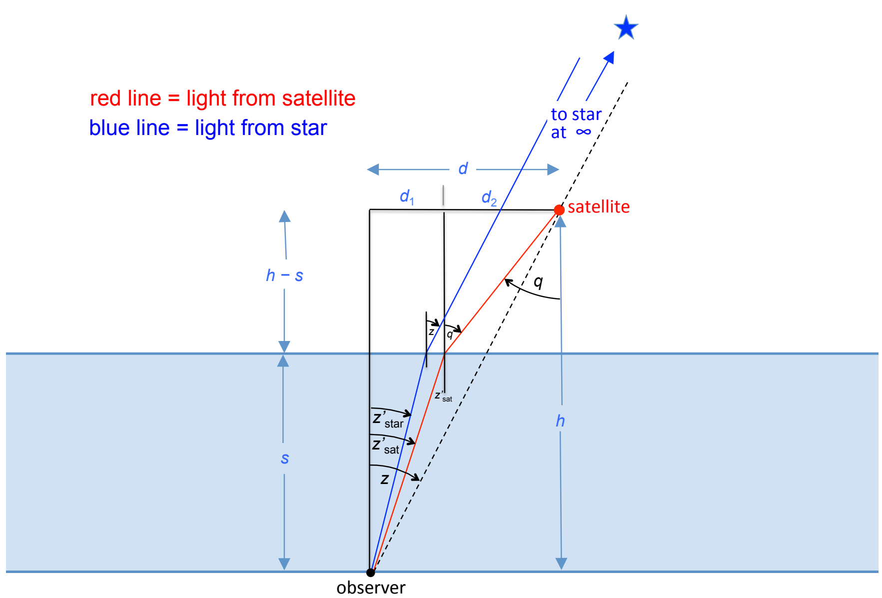

Figure 6 Quantities and triangles used to obtain formulas for Case 2, where a star and satellite are at the same geometric zenith distance but are seen at different apparent zenith distances.

References

Allen, C. W., 1973, *Astrophysical Quantities*, 3rd ed. (London: Athlone Press).

Brown, D. C., 1961, "Optical Refraction with Emphasis on Corrections for Points Outside the Atmosphere," *American Rocket Society Journal*, Vol. 31, pp. 549–550.

Green, R. M., 1985, *Spherical Astronomy* (Cambridge: University Press).

Herget, P., 1959, "Partial Astronomical Refraction," *Astronomical Journal*, Vol. 64, pp. 334–335.

Jones, B. L., 1961, "Photogrammetric Refraction Angle: Satellite Viewed from Earth," *Journal of Geophysical Research*, Vol. 66, pp. 1135–1138.

Kabeláč, J., 1976, "Parallactic Refraction Determined with the Aid of Meteorological Balloon Data," *Studia Geoph. et Geod.*, Vol. 20, pp. 1–9.

Kakkuri, J., & Ojanen, O., 1979, "Computing Parallactic Refraction for Stellar Triangulation," in *Refractional Influences in Astrometry and Geodesy*, ed. E. Tengström & G. Teleki, pp. 275–279.

Kaplan, G. H., 2008, "Refraction as a Vector," AA Technical Note 2008-01, Astronomical Applications Department, U.S. Naval Observatory (Washington: USNO).

Kaplan, G. H., Bangert, J., Bartlett, J., Puatua, W., & Monet., A., 2009, *User's Guide to NOVAS 3.0*, U.S. Naval Observatory Circular 180 (Washington: USNO). The latest version of the NOVAS library is at `http://aa.usno.navy.mil/software/novas/novas_info.php` .

Kovalevsky, J., 2002, *Modern Astrometry*, 2nd ed. (Berlin: Springer-Verlag).

Mueller, I. I., 1964, *Introduction to Satellite Geodesy* (New York: Ungar).

Murray, C. A., 1983, *Vectorial Astrometry* (Bristol, UK: Hilger).

Nautical Almanac Offices of the United Kingdom and the United States of America, 1961, *Explanatory Supplement to The Astronomical Ephemeris and The American Ephemeris and Nautical Almanac* (London: HM Stationery Office).

Nugent, L. J., & Condon, R. J., 1966, "Velocity Aberration and Atmospheric Refraction in Satellite Laser Communication Experiments," *Applied Optics*, Vol. 5, pp. 1832–1837.

Schildknecht, T., 1994, "Optical Astrometry of Fast Moving Objects Using CCD Detectors," Geodätisch-geophysikalische Arbeiten in der Schweiz, Vol. 49 (Zürich: Schweizerischen Geodätischen Kommission).

Schmid, H. H., 1963, "The Influence of Atmospheric Refraction on Directions Measured to and from a Satellite," GIMRADA Research Note No. 10 (Ft. Belvoir, VA: U.S. Army Corps of Engineers Geodesy, Intelligence and Mapping Research and Development Agency).

Smart, W. M., 1965, *Text-Book on Spherical Astronomy*, 5th ed. (Cambridge: University Press).

Stone, J. A., & Zimmerman, J. H., 2011, "Index of Refraction of Air," NIST online documentation and calculator, `http://emtoolbox.nist.gov/Wavelength/Documentation.asp` .

Tausworthe, R. C., 2005, "Computing the Apparent Elevation of a Near-Earth Spacecraft at Low Elevation Angles for an Arbitrary Refraction Model," IPN Progress Report 42–162 (Pasadena: Jet Propulsion Laboratory).

Urban, S. E., & Seidelmann, P. K., eds., 2012, *Explanatory Supplement to the Astronomical Almanac*, 3rd ed. (Mill Valley, CA: University Science Books).

van der Werf, S. Y., 2003, "Ray Tracing and Refraction in the Modified US1976 Atmosphere," *Applied Optics*, Vol. 42, No. 3, pp. 354–366.

Woolard, E. W., & Clemence, G. M., 1966, *Spherical Astronomy* (New York: Academic Press).