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A GEOMETRICAL APERTURE–WIDTH RELATIONSHIP FOR ROCK FRACTURES

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Abstract

The relationship between fracture aperture (maximum opening; $d_{\max}$) and fracture width (w) has been the subject of debate over the past several decades. An empirical power law has been commonly applied to relate these two parameters. Its exponent (n) is generally determined by fitting the power-law function to experimental observations measured at various scales. Invoking concepts from fractal geometry we theoretically show, as a first-order approximation, that the fracture aperture should be a linear function of its width, meaning that $n = 1$. This finding is in agreement with the result of linear elastic fracture mechanics (LEFM) theory. We compare the model predictions with experimental observations available in the literature. This comparison generally supports a linear relationship between fracture aperture and fracture width, although there exists considerable scatter in the data. We also discuss the limitations of the proposed model, and its potential application to the prediction of flow and transport in fractures. Based on more than 170 experimental observations from the literature, we show that

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such a linear relationship, in combination with the cubic law, is able to scale flow rate with fracture aperture over $\sim$14 orders of magnitude for variations in flow rate and $\sim$5 orders of magnitude for variations in fracture width.

Keywords: Boundary Fractal Dimension; Fracture Aperture; Fracture Width; Surface Roughness; Cubic Law.

1. INTRODUCTION

Modeling flow and transport within fracture networks require knowledge of fracture attributes and their scaling properties. A fracture is typically characterized by its aperture, width, displacement, and surface roughness. It is well documented in the literature that the surfaces of natural rock fractures are rough and follow self-affine scaling from fractal geometry.[1–4] The self-affinity of the surfaces of fractures is best understood in a statistical sense.[5] According to self-affinity, if one considers the height difference between two points separated by distance $|x_i - x_j|$ on a self-affine surface, then

$$\langle h(x_i) - h(x_j) \rangle \propto |x_i - x_j|^H, \qquad (1)$$

where $h(x_i)$ and $h(x_j)$ are the heights at points x_i and x_j, respectively, and H is the Hurst exponent characterizing the surface roughness. Generally speaking, the larger the Hurst exponent, the smoother the fracture surface. A value of $H \sim 0.8$ was reported for granite,[6] while $H \sim 0.5$ for sandstones.[7] However, a wider range of the Hurst exponent e.g. $0 < H < 0.9$ has been reported for a variety of rock joints.[5]

One of the most important characteristics of an individual fracture is its aperture, defined herein as the maximum opening ($d_{\max}$). The maximum opening is related to the average opening via $d_{\max} = 4/\pi d_{\mathrm{ave}}$.[8] Although aperture has been typically characterized as either the maximum or the average opening, due to the rough surface of fractures, there exists a distribution of aperture openings, rather than a single unique value for a given fracture. It is worth mentioning that the aperture opening distribution can only be inferred once the fracture surfaces have been characterized, which requires accurate imaging of the surface roughness.

Fracture aperture has been shown to control fluid flow and solute transport processes. For example, the well-known cubic law[9,10] relates flow in a single fracture to the product of fracture width w and fracture aperture raised to the power three (i.e.

$Q \propto w d_{\mathrm{ave}}^3 \nabla h \propto w d_{\max}^3 \nabla h$ in which Q is flow rate and ∇h is hydraulic gradient). Accordingly, the scaling relationship between fracture aperture and its width has been the subject of active research and debate over the past several decades. We should point out that the terms width and length have been interchangeably used in the literature to describe the straight line from one tip of a rock fracture to another, which can be a source of confusion.

A commonly applied model linking fracture aperture to fracture width is the following empirical power law[11,12]:

$$d_{\max} = cw^n \qquad (2)$$

in which c is a constant coefficient, $d_{\max}$ is the fracture aperture as previously defined, w is the fracture width, and n is an empirical exponent. Different values of n have been reported in the literature based on experimental observations, stochastic reasoning, and theoretical models. In Table 1, we summarize the various values of n (and if appropriate the corresponding coefficients of determination, R^2, determined by directly fitting Eq. (2) to measurements), that have been obtained using these three different approaches. For a comprehensive review, see Ref. 11.

From a theoretical perspective, linear elastic fracture mechanics (LEFM) theory, which has been successfully applied to rock fractures, predicts that the fracture aperture should scale linearly with fracture width,[13] meaning that the value of n in Eq. (2) should be equal to 1. A relationship similar to Eq. (2) was suggested by Oron and Berkowitz[14] between mean aperture and mean width for varying length sections within a self-affine fractal fracture. According to Oron and Berkowitz,[14] $n = H$, the Hurst exponent that characterizes the roughness of the fracture surface (see their Eq. (24)).

It is possible to stochastically relate fracture aperture to fracture width.[15] Based on this approach, it is well documented in the literature (see e.g. Ref. 11) that fracture width- and aperture-size distributions both conform

Table 1 **Various n Values Determined by Directly Fitting Eq. (2) to Measurements, and their Corresponding Coefficients of Determination (R^2) for Several Datasets Reported in the Literature, Including Theoretical and Stochastic Approaches.**

Reference	Remarks	No. of Samples	Exponent n^*	R^2
13	LEFM theory	—	1	—
15	Stochastic model for faults in a variety of tectonic environments	NA	1	NA
20	Kelduhverfi, Iceland, Fracture width <3 m	79	2.2 ± 0.24	0.41
	Kelduhverfi, Iceland, Fracture width >3 m		0.89 ± 0.08	0.76
	Myvatn, Iceland, Fracture width <3 m	42	1.78 ± 0.47	0.22
	Myvatn, Iceland, Fracture width >3 m		0.63 ± 0.08	0.81
62	Bonticou Crag	NA	1	0.96
	Outcrop 1, Forillon Park, Gaspe Peninsula	NA	1	0.86
	Outcrop 2, Forillon Park, Gaspe Peninsula	NA	1	0.94
	Anse a Mercier, Gaspe Peninsula	NA	1	0.73
	Les Petite Anse, gaspe Peninsula	NA	1	0.66
	Petite Vallee, Gaspe Peninsula	NA	1	0.68
	White-hall Dike	NA	1	0.66
	Lake Champlain	NA	1	0.24
23	Fractures in clay	$\sim$3000	0.47 ± 0.03	NA
63	Croagh Patrick quartz veins	431	1.20	0.73
	Section parallel to the vein array, Croagh Patrick	139	1.47	0.64
	Loughshinny	158	0.99	0.64
	Loughshinny, section perpendicular to shear zone and transport direction	180	1.08	0.32
	Bridges of Ross quartz veins in sandstone	300	1.30	0.68
	Bridges of Ross, section parallel to zone	96	0.95	0.26
	Skelpoonagh Bay	305	1.08	NA
	Quartz veins in granite	16	1.49	0.44
64	Veins in the damage zone of the Húsavík–Flatey fault	384	1	0.66
8	Ship Rock Dike Segments from Delaney and Pollard (1981)	NA	0.4	0.55
	LEFM theory incorporating subcritical and critical fracture propagation criteria	—	0.5	—
46	14 datasets from various studies (details given in their Fig. 1)	—	1	NA
65	Open fractures in tight gas sandstones	110	1	0.99
	Calcite-filled fractures in tight gas sandstones	300	1	0.95

Note: * n represents the exponent in the power-law aperture–width relationship (Eq. (2)); NA means not available.

to power-law probability density functions (i.e. $f(d_{\max}) = C_d d_{\max}^{-\alpha}$ and $f(w) = C_w w^{-\beta}$ in which C_d and C_w are constant coefficients). Assuming that $d_{\max}$ and w are related via the mathematical power-law function, Eq. (2), one can write $f(d_{\max})d(d_{\max}) = f(w)dw$. Given that $d(d_{\max})/dw = cnw^{n-1}$ from Eq. (2), then $n = (\beta-1)/(\alpha-1)$. Scholz and Cowie[15] reported $\alpha = 2.2$ and $\beta = 2.1$ and found $n \approx 1$ for a variety of tectonic environments. Those authors also showed that $n = 1$ could accurately represent their measured data. Besides power law scaling,[16,17] the log-normal probability density function has also been used to

describe distributions of fracture aperture and/or width.[18,19]

In addition to theoretical and stochastic values for n, there are numerous empirical estimates of n in the literature. For example, Hatton *et al.*[20] experimentally analyzed two datasets (i.e. Kelduhverfi and Myvatn) from the Krafla fissure swarm, Iceland, and found a break in slope in the aperture–width data when plotted on a log–log scale. For the Kelduhverfi dataset, they reported $n = 2.2 \pm 0.24$ with $R^2 = 0.41$ for $w < 3$ m and $n = 0.89 \pm 0.08$ with $R^2 = 0.76$ for $w > 3$ m, while for the Myvatn dataset, $n = 1.78 \pm 0.47$ with $R^2 = 0.22$

fractures,[38] and soil and rock pores.[39,40] Furthermore, it has been also used in combination with the model of Patzek and Silin[41] to describe fluid flow in tubular pores with rough surfaces, and to accurately estimate water relative permeability.[42]

We can compute the width of the fractal perimeter of a rough-boundary ellipse shown in Fig. 1, by invoking the Mandelbrot[25] approach, followed by Wheatcraft and Tyler[43] and many others. Based on this approach, a fractal length L_f is a function of some measurement scale ε and the straight-line distance L_s between the two ends of the fractal path as follows:

$$L_f = \varepsilon^{1-D_l} L_s^{D_l} \qquad (4)$$

in which D_l is the fractal dimension of the fractal length. Equation (4) is valid for both self-similar and self-affine fractal curves. For a self-similar fractal curve, divider method, box counting, and mass scaling estimates of D_l will all be the same. However, for a self-affine fractal curve, different values of D_l will be obtained using different evaluation methods.[44] We should also note that the concept underlying fractals is self-similarity or self-affinity, that is, invariance against variations in scale or size (scale-invariance). Accordingly, the fractal dimension is theoretically scale-invariant. However, in the nature, objects are only approximately fractal and scale invariant, and one may expect the fractal dimension to vary from one scale to another. A notable example is fractal dimension determination from images. For example, Baveye et al.[45] demonstrated the effects of image resolution, thresholding, and algorithm used to generate binary images on the estimation of fractal dimension.

Assuming that the rough boundary shown in Fig. 1 is a fractal length, setting $D_l = D_b$ and $L_s = w$ in Eq. (4) in combination with $P \propto L_f$ yields:

$$P \propto \varepsilon^{1-D_b} w^{D_b}. \qquad (5)$$

Equation (5) necessarily means that perimeter is measurement scale-dependent. Given that $1 \leq D_b < 2$, P is directly proportional to the fracture width w, while inversely proportional to the measurement scale ε. As a consequence, when ε tends to zero, P approaches infinity.

The area of an ellipse with a smooth boundary (represented by the black dashed line in Fig. 1) is a function of the product of its semi-minor and semi-major axes ($w d_{\max}$). Accordingly, for the rough-boundary ellipse (shown in red) one can

approximately set

$$A \propto w d_{\max}. \qquad (6)$$

Note that Walmann et al.[23] also used a relationship similar to Eq. (6) to relate the area of a fracture to its aperture and width.

Substituting Eqs. (5) and (6) into Eq. (3) gives

$$d_{\max} = C w, \qquad (7)$$

where C is a numerical prefactor whose value depends on the measurement scale ε (i.e. $C \propto \varepsilon^{\frac{2-2D_b}{D_b}}$).

Equation (7) represents a geometrical scaling relationship linearly relating fracture aperture to fracture width. It is in agreement with the LEFM approach, which predicts that the fracture aperture should scale linearly with fracture width.[13] Within the LEFM theory framework, the coefficient C would be equal to $\sigma(1-\nu)/\mu$ in which σ is the effective driving stress (remote tension plus the internal fluid pressure), ν is Poisson's ratio, and μ is the shear modulus.[13]

As the derivation of Eq. (7) shows the roughness exponent cancels meaning that it is mainly the fracture geometry governing the linear $d_{\max}$–w relationship rather than the roughness *per se*. It is interesting that the rough-boundary ellipse produces the same result as the smooth-boundary ellipse applied by LEFM. Although our geometrical terminology is different from that of LEFM, the obtained results indicate that perturbing the boundary of a smooth-boundary ellipse, used in LEFM theory, with self-affinity duplicates the linear w–$d_{\max}$ relationship. In Appendix A, using the same geometrical terminology, we, however, demonstrate that there exists no simple linear relationship between fracture aperture and its width for a smooth-boundary ellipse, unless w is significantly greater than $d_{\max}$. In Appendix B, we assume a rectangular fracture embroidered with a fractal rough boundary (see Fig. B.1) and demonstrate that when $w \gg d_{\text{ave}}$ the average fracture aperture (d_{ave}) should scale linearly with its width (w).

Equation (7) is also in agreement with the stochastic approach of Scholz and Cowie[15] and the average of the experimentally determined n values presented in Table 1. However, there are some exponents reported in the literature (e.g. those from Ref. 20) that apparently do not match the theoretical prediction of $n = 1$. In what follows, we compare the geometrical model with two datasets from Ref. 20 and demonstrate that Eq. (7), with

different numerical prefactor C values, is able to accurately match the experimental measurements by Hatton *et al.*[20]

3. COMPARISON WITH HATTON *et al.* EXPERIMENTS

Hatton *et al.*[20] collected two datasets, namely Kelduhverfi and Myvatn, from the Krafla fissure swarm, one of five volcano tectonic systems in the active rift zone of northeast Iceland. It is probably worth pointing out that the Kelduhverfi and Myvatn exposures have heavy joint sets at a characteristic scale, that control the fracturing (see Fig. 2 of Ref. 20). However, this is not always the case (I. Main, personal communication (2018)). Hatton *et al.*[20] measured fracture width, the straight line from tip to tip, and fracture aperture, the maximum opening displacement along the width of a fracture, using a tape measure. A total of 72 and 42 fractures were digitized for the Kelduhverfi and Myvatn areas, respectively, from Fig. 3 in Ref. 20. The yield of 72 for Kelduhverfi is less than the 79 stated in Ref. 20. The difference between the two might be due to low figure quality or duplications in the measurements.

Fracture aperture as a function of fracture width is shown on a log–log scale for the two areas in Fig. 2. As can be observed in Fig. 2a, the data points are highly scattered, not only at short length scales but also over the entire range of length scales. For

example, at short length scales a fracture aperture of 0.001 m corresponds to a wide range of fracture widths from near 0.1 to 1 m, almost one order of magnitude. Such scatter in the data causes substantial uncertainties in the optimization of any mathematical function's parameters through a direct fitting process.

In the Kelduhverfi dataset (Fig. 2a) both fracture aperture and width span nearly four orders of magnitude. However, in the Myvatn dataset, fracture width spans nearly three and half orders of magnitude while aperture only varies over approximately two orders of magnitude. Nonetheless, as we show in Fig. 2, Eq. (7) with various numerical prefactors can accurately capture the trends in the observations. We plotted Eq. (7) with $C = 0.001$, 0.01, and 0.1 in Fig. 2; however, one can clearly see that other values of C might be relevant to some data points. In both plots shown in Fig. 2, the envelope of estimated fracture aperture values via $C = 0.001$, 0.01, and 0.1 coincides remarkably with results from the two datasets from Ref. 20.

Our results presented in Fig. 2 are in agreement with those of Schultz *et al.*[46] who experimentally demonstrated for 14 datasets that fracture aperture should linearly scale with fracture width (see their Fig. 1), similar to Eq. (7). In their Fig. 2, however, they discuss that for some experiments the value $n = 0.5$ might be more relevant than $n = 1$. Interestingly, Schultz *et al.*[46] found that the same numerical prefactors $C = 0.001$, 0.01, and 0.1 could
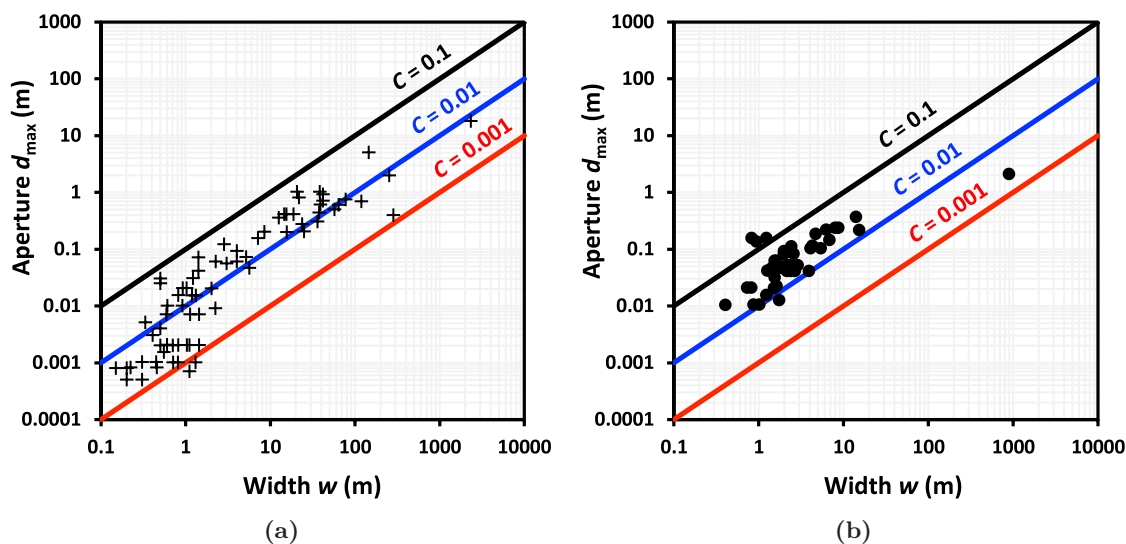

Fig. 2 Fracture aperture versus fracture width for the **(a)** Kelduhverfi and **(b)** Myvatn datasets from Ref. 20. The solid lines represent Eq. (7) with various numerical prefactor values (i.e. $C = 0.001$, 0.01, and 0.1). To be consistent with Figs. 3a and 3b of Ref. 20, same symbols have been used.
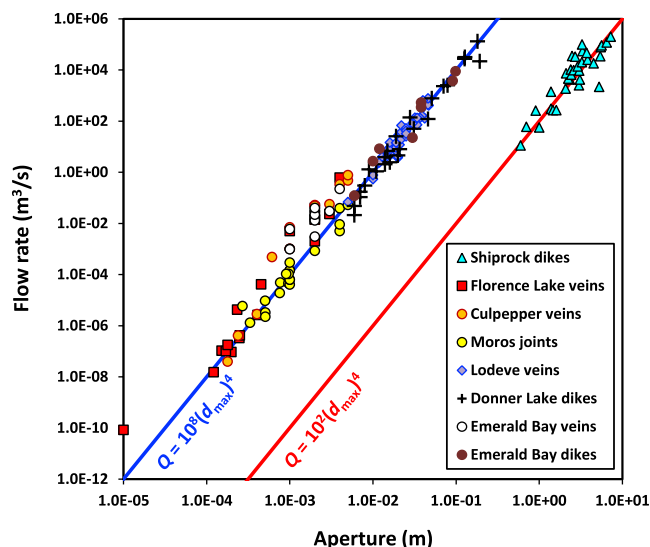
Fig. 3 Measured flow rate Q as a function of measured aperture for eight datasets and 173 fractures digitized from Ref. 48. The red and blue solid lines respectively represent Eq. (9) with $n = 1$ (from Eq. (7)) and numerical prefactors 10^2 and 10^8 (i.e. $Q = 10^2 d_{\max}^4$ and $Q = 10^8 d_{\max}^4$).

accurately capture aperture–width trends for more than 170 fractures from 14 datasets including normal faults, strike-slip faults, and thrust faults.

4. COMPARISON WITH FLOW RATE EXPERIMENTS

The correlation between aperture and width has an impact on flow rate Q in a single fracture, which can be approximated by the following cubic law[9,10]

$$Q \propto w d_{\text{ave}}^3 \nabla h \propto w d_{\max}^3 \nabla h. \qquad (8)$$

For the validity of the cubic law in smooth and rough fractures, see pp. 156 and 416–417 of Ref. 5 for a review as well as Ref. 47. Equation (8) clearly shows that flow rate Q is mainly controlled by fracture aperture $d_{\max}$ rather than fracture width w. This is because in Eq. (8) aperture is raised to the power three, while width is only raised to the power of unity. However, the effect of fracture width might not be negligible, as we discuss in the following paragraphs.

If the fracture aperture and the fracture width are uncorrelated, Eq. (8) in its present form ($Q \propto d_{\max}^3$) should accurately scale flow rate with aperture in a single fracture. However, if $d_{\max}$ and w are correlated via Eq. (2), then w in Eq. (8) can be replaced with $d_{\max}^{\frac{1}{n}}$, and accordingly Eq. (8)

changes to

$$Q \propto d_{\max}^{3+\frac{1}{n}}, \qquad (9)$$

and the values $n = 0.5$ (Refs. 8, 46, 48) and $n = 1$ (see Table 1 and Eq. (7)) result in $Q \propto d_{\max}^5$ and $Q \propto d_{(\max)}^4$, respectively. Equation (9) clearly shows deviation from the cubic law that should be attributed to the aperture-width relationship (Eq. (2)). However, as we discuss later, there is evidence in the literature indicating that such a deviation can be attributed to surface roughness. Accordingly, Eq. (9) will likely only provide accurate results when the aperture is wide enough so that the effect of surface roughness is minimized. In what follows, we compare $Q \propto d_{\max}^4$ with data from flow rate experiments.

Klimczak *et al.*[48] collected eight datasets from various studies in the literature including Shiprock dikes, Florence Lake veins, Culpeper veins, Moros joints, Lodeve veins, Donner Lake dikes, Emerald Bay veins, and Emerald Bay dikes (see their Fig. 6). Measured flow rate (m³/s) versus measured fracture aperture (m) are shown in Fig. 3 on a log–log scale. As can be observed, flow rate and aperture span near 16 and six orders of magnitude on the vertical and horizontal axes, respectively. Generally speaking, the measurements can be grouped into two classes: (1) Shiprock dikes (represented by triangles in Fig. 3, and (2) all others. Although both classes have similar slopes on a log–log scale, the Shiprock dikes require a larger aperture than other types of fractures shown in Fig. 3 to return the same flow rate value.

In Fig. 3, we also show Eq. (9) with $n = 1$ (from Eq. (7)) and two different numerical prefactors i.e. 10^2 and 10^8. As can be seen, $Q = 10^2 d_{\max}^4$ and $Q = 10^8 d_{\max}^4$ precisely scale the measurements over 14 orders of magnitude variations in flow rate and five orders of magnitude of fracture widths. This result is a definite improvement on the $Q \propto d_{\max}^5$ scaling propounded by Klimczak *et al.*[48] (see their Fig. 6).

It is worth pointing out that one should expect Eq. (9) to be valid only under the laminar flow (low Reynolds number) condition. However, some of the high discharge experiments of Klimczak *et al.*[48] presented in Fig. 3 would likely violate this condition for water. Those experiments apparently correspond to magma flow through dikes with wide apertures. Given that magma's viscosity is remarkably high and significantly greater than that of water,

the Reynolds numbers associated with those experiments should not be too large. However, magmas are typically quasi- or non-Newtonian, for which applying Darcy's law causes uncertainties in calculations. Given the lack of detailed information on the flow rate experiments of Klimczak *et al.*,[48] care should be taken in interpreting the close correspondence between Eq. (9) and the data in Fig. 3 as model validation. Clearly, additional experimental work on water flow rates in fractures is needed in order to fully evaluate our proposed model.

Although we have shown that a simple geometrical scaling relationship, Eq. (7), in combination with the cubic law can accurately represent the experimental data from Klimczak *et al.*,[48] there is experimental and numerical evidence in the literature that surface roughness can remarkably affect fluid flow.[49] Accordingly, an exponent different than 4 might better scale flow rate Q with fracture aperture $d_{\max}$. For example, lattice-gas simulations in single self-affine (anisotropic) fractures by Zhang *et al.*[34] revealed exponents greater than 4. For three-dimensional fractures with relatively rough ($H = 0.8$) and rough ($H = 0.3$) surfaces, those authors found exponents 2.67 and 3.15, respectively. With further decreases in H they found significant increases in the exponent scaled permeability with mean aperture. Zhang *et al.*[34] stated that "Our study shows *quantitatively* that the experimentally observed deviation from the cubic law can be attributed to surface roughness". Sahimi[5] also stated that in fractures with rough surfaces, an exponent as large as 6 might be expected. Eker and Akin[50] simulated fluid flow through two-dimensional fractures with rough surfaces via the lattice-Boltzmann technique. Those authors reported that as the mean aperture-fractal dimension ratio increased, permeability increased. They stated that, "The resulting permeability values were less than the ones obtained with the cubic law estimates". Eker and Akin[50] also found that the exponent scaled permeability with mean aperture was linearly correlated to fractal dimension characterized fracture surface roughness for isotropic fractures (see their Eq. (12)). More specifically, their simulations resulted in exponents ranging from 4.27 to 5.66, in accord with the results of Zhang *et al.*[34] Recently, Liu *et al.*[51] combined concepts from fractal curves, Eq. 4, with the cubic law, Eq. 8, and proposed $Q \propto d_{\mathrm{ave}}^{6-D_T}$ in which D_{T} represents the tortuosity fractal dimension. Given that theoretically $1 \leq D_T < 3$, the exponent $6 - D_T$ can be

expected to range between 3 and 5, with $D_T = 2$ corresponding to our model prediction. Further research on comparing these different approaches could be valuable.

5. DISCUSSION

One of the main assumptions in our theoretical derivation is that the boundary of the fracture surface is fractal. Many studies in the literature indicate that fracture surfaces are rough and obey fractal geometry.[1,2,52–55] However, not every rock fracture surface is necessarily fractal at all scales. In fact, natural fractures that exhibit self-similarity or self-affinity might lose their fractal properties above and below upper and lower cutoff scales. The aperture–width relationship for fractures whose surface is non-fractal may well deviate from the theoretical linear function given in Eq. (7), as we show in Appendix A.

In addition, we assumed that fractures are elliptical with a rough boundary. Although natural fractures have been frequently represented by ellipses,[56–59] in reality they typically have irregular cross-sections with converging–diverging opening geometry. Furthermore, rather than a unique aperture for a given fracture, there exists a distribution of gap sizes due to the rough surfaces of fractures. Thus, we should caution that any significant deviation from elliptical shape may cause uncertainties in our proposed scaling relationship.

Madadi *et al.*[60] studied fluid flow in two-dimensional fractures with rough self-affine surfaces, quantified with the Hurst exponent H, using the lattice-Boltzmann method. They used five values of H ranging from less than 0.5 (which generated rough surfaces) to near 1 (which provided relatively smooth surfaces). In their study, the mean aperture of the fractures was varied to generate narrow to wide fractures. They stated that, "Using a simple mean aperture for representing a fracture with rough internal surfaces will always result in gross errors, unless, of course, the fractures are wide, in which case the surface roughness does not really matter".

A rough profile (i.e. the fractal path shown in Fig. 1) can be characterized by two parameters: (1) the boundary fractal dimension (or Hurst exponent), and (2) the root-mean-square of the roughness height. The former characterizes the boundary roughness, while the latter controls the roughness thickness. The higher the boundary

fractal dimension, the rougher the fractal path. Likewise, the higher the root-mean-square of the roughness height, the thicker the fractal profile. In our theoretical framework, described in Sec. 2, only the boundary fractal dimension was employed. The effect of the root-mean-square of the roughness height was not incorporated. Further investigation is required to understand how the root-mean-square of the roughness height might impact the predictions of our theoretical linear relationship, Eq. (7).

6. CONCLUSIONS

In this study, we developed a first-order linear approximation of the relationship relating fracture aperture (d_{max}) to fracture width (w). Linear relationships have been previously proposed either experimentally or theoretically in the literature. However, for the first time, we invoked concepts from fractal geometry to show that d_{max} should scale linearly with w. We reanalyzed the Hatton et al.[20] database, for which nonlinear power-law equations, with breaks in slope, have been reported, and demonstrated that a linear relationship with various numerical prefactors could accurately capture the trends in the data. Comparison with previously published flow rate experiments also showed that our linear scaling relationship, in combination with the cubic law, could accurately scale flow rate with fracture aperture for more than 170 experimental observations.

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APPENDIX A

In this section, we consider an elliptical fracture with a smooth boundary presented in Fig. A.1. We show that there exists no simple relationship between fracture aperture and its width, unless the latter is significantly greater than the former.

Following Eq. (3), one may relate fracture perimeter to its area for a smooth-boundary object ($D_b = 1$) as follows:

$$P \propto A^{\frac{1}{2}}. \tag{A.1}$$

Within Euclidean geometry, Eq. (A.1) always holds because the dimension of area is $[L^2]$, while that of perimeter is $[L]$.

In the literature, various formulas have been developed to determine the perimeter of an ellipse. Most, if not all, of these are approximations. The two most widely applied models were proposed by Ramanujan[61] and are in the following forms:

$$P \approx \pi \left[(d_{\max} + w) + \frac{3(w - d_{\max})^2}{10(d_{\max} + w) + \sqrt{d_{\max}^2 + 14 d_{\max} w + w^2}} \right], \tag{A.2}$$

$$P \approx \pi [3(d_{\max} + w) - \sqrt{(d_{\max} + 3w)(3d_{\max} + w)}]. \tag{A.3}$$

In addition to these relatively simple relationships, there exist several other models in series form.
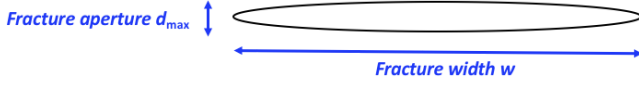
Fig. A.1 An elliptical fracture with a smooth boundary. d_{max} and w represent the fracture aperture (maximum opening) and the fracture width, respectively.
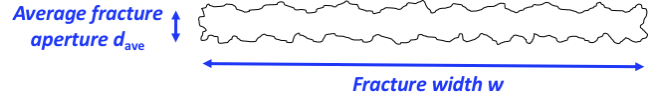

Fig. B.1 A rectangular fracture with a rough boundary. d_{ave} and w represent the average fracture aperture and the fracture width, respectively.

In contrast to the perimeter formula, the area of an ellipse is exact and can be calculated as

$$A = \frac{\pi}{4} w d_{max}. \tag{A.4}$$

Combining either Eq. (A.2) or (A.3) with Eqs. (A.1) and (A.4) does not yield any straightforward relationship between d_{max} and w for a smooth-walled elliptical fracture. This is simply because P is a complex function of d_{max} and w (see Eqs. (A.2) and (A.3)). Accordingly, there exists no simple relationship linking d_{max} to w. Only if $w \gg d_{max}$, both Eqs. (A.2) and (A.3) reduce to a linear proportionality between perimeter and width i.e. $P \propto w$. Given that $A \propto w d_{max}$, substituting P and A into Eq. (A.1) yields

$$d_{max} \propto w \tag{A.5}$$

which is the same as our Eq. (7). The difference between Eq. (A.5) and Eq. (7) is that the former holds if and only if $w \gg d_{max}$, while the latter does not depend on any assumption between w and d_{max}.

APPENDIX B

Here we assume a rectangular fracture with a fractal rough boundary (see Fig. B.1) and demonstrate that under specific circumstances fracture aperture scales linearly with its width. For this purpose, following Eq. (3), we assume that

$$P \propto A^{\frac{D_b}{2}}. \tag{B.1}$$

If $w \gg d_{max}$, one can approximate the fracture perimeter by

$$P \propto \varepsilon^{1-D_b} w^{D_b}. \tag{B.2}$$

This means that the fractal rough width of the fracture dominantly contributes to the perimeter. We further assume that the area can be represented by

$$A \propto w d_{ave}. \tag{B.3}$$

Combining Eqs. (B.2) and (B.3) with Eq. (B.1) gives

$$d_{ave} \propto w \tag{B.4}$$

which is similar to Eq. (7). One should note that any fracture whose geometry differs significantly from that given in Fig. B.1 and/or any failure in the assumptions $P \propto w^{D_b}$ (Eq. (B.2)) and $A \propto w d_{ave}$ (Eq. (B.3)) could cause significant deviations from Eq. (B.4).